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On overconvergent isocrystals and F-isocrystals of rank one

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0. Introduction

Let K be a field of characteristic zero complete for a non-archimedean absolute value, let $\mathcal{V}$ be the associated valuation ring and k its residue field of characteristic p. In this article we prove (cf. Section 4) that an overconvergent isocrystal of rank one defined on an open subscheme X of $\mathbf{P}^1_k$ has a Frobenius structure (i.e. it is an overconvergent F-isocrystal) when it has exponents (4.4.4) in each residue class of $Z = \mathbf{P}^1_k \backslash X$ of the type $z/(p^s - 1)$, where $s \in \mathbf{N}$ and $z \in \mathbf{Z}$ (i.e. in $\mathbf{Z}_p \cap \mathbf{Q}$).

It is known [Ba-Ct] that in dimension one the notion of overconvergent isocrystals can be translated into the notion of "convergence on the generic disk" for classical p-adic differential equations. Using this point of view we are able to prove the local existence of a Frobenius structure (Section 2) in each residue class of $Z = \mathbf{P}^1_k \backslash X$. The problem of connecting the Frobenius structures in the different residue classes is then solved in Section 3 using a method due to Dwork [Dw2].

1. Notation. Arithmetic properties of differential operators

Throughout this article K will denote a field of characteristic 0, complete under a non-archimedean absolute value $| - |$; $\mathcal{V}$ denotes its valuation ring and $\mathcal{M} \subset \mathcal{V}$ the maximal ideal, we indicate by k the residue field of finite characteristic p, which we suppose to be perfect. Moreover, the absolute value is normalized by $| p |= p^{-1}$.

In this paragraph we will deal with arithmetic properties of the coefficients of a linear differential operator whose solutions converge on the generic disk of $D(0, 1^-)$.

1.1. Consider a first order differential operator L

$$\frac{d}{dx} + f(x) \tag{1.1.2}$$

where $f(x) \in K(x)$ has 0 as the only pole in $D(0, 1^-)$. By Mittag–Leffler decomposition we may write:

$$f(x) = f^+(x) + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}, \tag{1.1.3}$$

where $a_i \in K$ and $f^+(x) \in K(x)$ has no poles in $D(0, 1^-)$.

1.2. We will say that (1.1.2) has the property of *convergence on the generic disk of* (or *converges on the generic disk of*) $D(0, 1^-)$ if its solution at the generic point t converges in the whole open disk $D(t, 1^-)$.

1.3. Let $K[x]$ be the ring of polynomials in K in the indeterminate x. We consider $R \subseteq R'$ two K-algebras which contain $K[x]$ and are endowed with a derivative $\frac{d}{dx}$ which extends $\frac{d}{dx}$ of $K[x]$. We indicate $\mathcal{D}_R = R[\frac{d}{dx}]$. Every element of $\mathcal{D}_R$ acts on R' and, of course, R. We will say that two elements $L, L' \in \mathcal{D}_R$ are *equivalent over* R' if there exists $M \in R'^{\times}$ (invertible elements of R'), such that

$$L' = M^{-1} \cdot L \cdot M.$$

1.4. For an operator as in (1.1), under the hypothesis of convergence on the generic disk of $D(0, 1^-)$, we are interested in the arithmetic properties of the coefficients of the singular part of $f(x)$ at 0, i.e. in the notation (1.1.3): $f(x) - f^+(x)$.

We denote by $\mathcal{O}(D(0, 1^-))$ the K-algebra of the analytic functions on $D(0, 1^-)$. And $\mathcal{O}(D(0, 1^-))[\frac{1}{x}]$ denotes the K-algebra of analytic functions on $D(0, 1^-) \setminus \{0\}$ with meromorphic pole at 0. We then have

LEMMA 1.4.1 (cf. [Ro1, Lemme 5.3]). *Consider a differential operator L as in (1.1):*

$$L = \frac{d}{dx} + f^+(x) + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \frac{a_n}{x^n},$$

which converges on the generic disk of $D(0, 1^-)$. Then L is equivalent on $\mathcal{O}(D(0, 1^-))[\frac{1}{x}]$ to L' where

$$L' = \frac{d}{dx} + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \frac{a_n}{x^n}.$$

In particular L' has the property of convergence at the generic disk of $D(0, 1^-)$.

We then obtain

PROPOSITION 1.4.2. *Consider*

$$L = \frac{d}{dx} + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \frac{a_n}{x^n}$$

$a_i \in K$ and suppose it has the property of convergence on the generic disk of $D(0, 1^-)$, then

(i) $|a_i| \leqslant 1, i = 1, \ldots, n$.
(ii) $a_1 \in \mathbf{Z}_p$.

Proof. The first assertion is an easy consequence of Dwork–Frobenius theorem [Ch1, 4.8.1]. For the second, see for example [Ro1, Lemme 5.4]. Q.E.D.

OBSERVATION 1.4.3. *Consider L as in the previous proposition, then also*

$$L' = \frac{d}{dx} + \frac{a_2}{x^2} + \cdots \frac{a_n}{x^n}$$

has the property of convergence on the generic disk of $D(0, 1^-)$.

Proof. By 1.4.2, $a_1 \in \mathbf{Z}_p$. We have that $(x - t)^{a_1}$ converges in $D(t, 1^-)$ [Ro2], [Ch2]. Q.E.D.

1.5. We denote by π the element which, in case, belongs to an opportune extension of K such that $\pi^{p-1} = -p$, hence $|\pi| = p^{-(1/p-1)}$. We now give a first estimate result for the coefficients

PROPOSITION 1.5.1. *Let*

$$L = \frac{d}{dx} + \frac{a_2}{x^2} + \cdots \frac{a_n}{x^n}$$

be a differential operator, with $a_i \in K$ ($a_n \neq 0$), which has the property of convergence on the generic disk of $D(0, 1^-)$. Then $|a_n| \leqslant |\pi|$. In particular if $n \not\equiv 1$, $\operatorname{mod} p$, then also

$$L' = \frac{d}{dx} + \frac{a_2}{x^2} + \cdots \frac{a_{n-1}}{x^{n-1}}$$

has the property of convergence on the generic disk of $D(0, 1^-)$.

Proof. One may consider the solution of L at the generic point t:

$$\exp\left(\sum_{j\geqslant 2}^{n} \frac{a_j}{j-1}\frac{1}{x^{j-1}} - \sum_{j\geqslant 2}^{n} \frac{a_j}{j-1}\frac{1}{t^{j-1}}\right)$$

which has value 1 at t. We can also develop the Taylor series of the argument of the exp in the neighborhood of t, using $x = y + t$

$$\exp\left(-\left(\sum_{j\geqslant 2}^{n} \frac{a_j}{t^j}\right) y + \sum_{l\geqslant 2} (-1)^l \left(\sum_{j\geqslant 2}^{n} \frac{a_j}{j-1}\binom{j-2+l}{l}\frac{1}{t^{j+l-1}}\right) y^l\right).$$

Now we can take the expansion of exp, we obtain a series:

$$\sum_{N\geqslant 1} A_N y^N, \tag{1.5.1.1}$$

where

$$A_N = (-1)^N \left(\sum_{j=2}^{n} \frac{a_j}{j-1} \binom{j-2+N}{N} \frac{1}{t^{j+N-1}} + \cdots + \frac{1}{N!} \left(\sum_{j=2}^{n} \frac{a_j}{t^j} \right)^N \right).$$

By hypothesis we know that (1.5.1.1) converges for $|y| < 1$. The degree of A_N, as polynomial in $\frac{1}{t}$, is exactly nN: the coefficient of the highest power is

$$\frac{a_n^N}{N!}.$$

The fact that t is the generic point and the fact that (1.5.1.1) converges for $|y| < 1$ allow us to write

$$\lim_{N \to +\infty} \left| \frac{a_n^N}{N!} \right| \epsilon^N = 0$$

for each $\epsilon < 1$. We deduce $|a_n| \leqslant |\pi|$.

If $n \not\equiv 1 \bmod p$, then one may consider:

$$\tilde{L} = \frac{d}{dx} + \frac{a_n}{x^n} \tag{1.5.1.2}$$

and write the expansion at t of the solution ($n \geqslant 2$)

$$\exp\left(\frac{a_n}{n-1} \sum_{N \geqslant 1} (-1)^N \binom{n-2+N}{N} \frac{1}{t^{n-1+N}} y^N \right). \tag{1.5.1.3}$$

Apply the previous method to (1.5.1.3). It turns out that the Taylor expansion at t is

$$\sum_{N>0} \frac{(-1)^N}{N!} \left(\frac{a_n}{n-1} \frac{1}{t^n} \right)^N y^N$$

which is convergent for $|y| < 1$: in fact $|\frac{a_n}{n-1}| \leqslant |\pi|$. Finally L' has the property of convergence on the generic disk: in fact its solution at the generic point t is the quotient of the solutions of (1.5.1.2) (i.e. (1.5.1.3)) and of L which converged in $D(t, 1^-)$. Q.E.D.

REMARK 1.5.1.4. Using the same methods, if in the statement of the previous theorem the hypothesis of convergence on the generic disk of $D(0, 1^-)$ had been replaced by the hypothesis that the solution converges in the closed disk $D(0, 1^+)$ then the conclusion would be $|a_n| < |\pi|$.

For the case $n \equiv 1$, mod p, we have a similar result under further assumptions.

PROPOSITION 1.5.2. *Suppose that L is the linear differential operator*

$$L = \frac{d}{dx} + \frac{a}{x^{p+1}}$$

$a \in K$, *has the property of convergence on the generic disk of* $D(0, 1^-)$, *then*

$$\left|\frac{a}{p}\right| \leqslant |\pi| .$$

Proof. Consider as usual the solution at the generic point t:

$$\exp\left(\frac{a}{px^p} - \frac{a}{pt^p}\right) .$$

One may then consider $x = y + t$ and the expansion of the argument of exp

$$\exp\left(\sum_{l\geqslant 1}(-1)^l\left(\frac{a}{p}\binom{p+l-1}{l}\frac{1}{t^{p+l}}\right)y^l\right) .$$

By developing exp we have the series $\sum_{N\geqslant 1} A_N y^N$ with

$$A_N = (-1)^N\frac{a}{p}\binom{p+N-1}{N}\frac{1}{t^{N+p}} + \cdots + \frac{1}{N!}a^N\frac{1}{t^{pN+N}} .$$

As in the Proposition 1.5.1 one may then conclude that $|a| \leqslant |\pi|$. Consider now the coefficient A_N with $N = ps$ for $s \in \mathbf{N}$, $s \neq 0, 1$. In this case A_{ps} is a polynomial in $\frac{1}{t}$ given by monomials of degrees which range from $p+ps$ to $psp+ps = p^2s+ps$. Let us consider the coefficient in A_{ps} of $\frac{1}{t^{ps+ps}}$: it may be written as (up to sign)

$$\frac{1}{s!}\left[\frac{a^s}{p^s}\binom{p+p-1}{p}^s + \sum_{(i_j)\in\mathbf{J}^*}\alpha_{(i_j)}\frac{a^s}{p^s}\prod_{j=1}^{s}\binom{p+i_j-1}{i_j}\right],$$

where $\mathbf{J}^*$ is the set of s-uples, $(i_j) \in \mathbf{N}^{*s}$ such that $i_1 + \ldots + i_s = ps$ but excluding the case $(p, \ldots, p)$; $\alpha_{(i_j)} \in \mathbf{Z}$ and it depends on (i_j). We notice then that for each $l \in \mathbf{N}$:

$$\left|\binom{p+l-1}{l}\right| < 1$$

if $l \not\equiv 0 \bmod p$. We then have

$$\left|\frac{1}{s!}\left[\frac{a^s}{p^s}\binom{p+p-1}{p}^s + \sum_{(i_j)\in\mathbf{J}^{ast}}\alpha_{(i_j)}\frac{a^s}{p^s}\prod_{j=1}^{s}\binom{p+i_j-1}{i_j}\right]\right| = \left|\frac{1}{s!}\frac{a^s}{p^s}\right| .$$

In particular, t being a generic element, we have:

$$\lim_{s\to+\infty}\left|\frac{1}{s!}\frac{a^s}{p^s}\right|\epsilon^{ps}=0$$

for each $\epsilon<1$. We conclude that $|\frac{a}{p}|\leqslant|\pi|$. Q.E.D.

REMARK 1.5.3. The same proposition holds in the case

$$L=\frac{d}{dx}+\frac{a}{x^{pr+1}}$$

$r\in\mathbf{N}$, $r\geqslant 2$. In particular the argument now involves the monomial of degree prs in place of that of degree ps.

REMARK 1.5.4. The previous results can be easily generalized to the case of systems (i.e. when the $a_i\in M_n(K)$, $n\in\mathbf{N}$): we will get information on the eigenvalues of the associated matrices.

2. Convergence on the generic disk and overconvergent Frobenius

2.1. Consider now the following differential operator

$$L=\frac{d}{dx}+\frac{a_2}{x^2}+\cdots+\frac{a_n}{x^n} \tag{2.1.1}$$

$a_i\in K$. It may be seen as an operator in $\mathbf{P}^1_K$ with its only singularity at $x=0$. In particular one may take, x_∞, the coordinate at ∞ and re-write (2.1.1) by means of this coordinate:

$$L=x_\infty^2\frac{d}{dx_\infty}-a_2x_\infty^2-a_3x_\infty^3-\cdots-a_nx_\infty^n\ . \tag{2.1.2}$$

Then L has the property of convergence on the generic disk of $D(0,1^-)$ if and only if it has the same property for the generic disk of $D_{x_\infty}(0,1^-)$. If this is the case the fact that L has no singularities in $D_{x_\infty}(0,1^-)$ allows us to conclude by transfer [Dw1] that the solution of L at $x_\infty=0$ converges in the whole open disk $D_{x_\infty}(0,1^-)$.

REMARK 2.1.3. Of course dividing (2.1.2) by x_∞^2 does not change its properties, and for the purposes of this paper, it is equivalent to refer to the operator L at ∞ after division by x_∞^2 i.e. to the operator

$$\frac{d}{dx_\infty}-a_2-a_3x_\infty-\cdots-a_nx_\infty^{n-2}\ , \tag{2.1.4}$$

which we will once again indicate by L.

2.2. By a Frobenius automorphism of the field K, with perfect residue field k, we mean a continuous automorphism (hence isometric) $\sigma: K \to K$ which lifts the usual Frobenius automorphism of k. In particular for each $x \in \mathcal{V}$ we have

$$|x^\sigma - x^p| < 1.$$

We will denote by σ^s the s-iterated map. Connected to σ we have also an automorphism of the field $K(x)$. For $f(x) \in K(x)$ we will indicate $f^\sigma(x) \in K(x)$ the element obtained applying σ to the coefficients of $f(x)$. On $K(x)$ we then define the map $\varphi: K(x) \to K(x)$, as $\varphi(f(x)) = f^\sigma(x^p)$ (i.e. we substitute x^p to x in $f^\sigma(x)$) and we call it Frobenius. We may also iterate φ

$$\varphi^h(f(x)) = f^{\sigma^h}(x^{p^h}).$$

By continuity we extend the map φ and its iterates to the field E of analytic elements [Ch1] (the completion under the Gauss norm of $K(x)$), and also to the ring of "functions" $W = W(0,1)$,

$$W = \left\{\sum_{n\in\mathbf{Z}} a_n x^n, a_n \in K \mid \exists N \in \mathbf{R}\ |a_n| \leqslant N,\ \forall n \geqslant 0\,;\ \lim_{n\to-\infty} |a_n| = 0\right\} \tag{2.2.1}$$

[Ch3], by setting for $g(x) = \sum_{n\in\mathbf{Z}} a_n x^n \in W$, $h \in \mathbf{N}$, $h \geqslant 1$

$$\varphi^h(g(x)) = g^{\sigma^h}(x^{p^h}) = \sum_{n\in\mathbf{Z}} a_n^{\sigma^h} x^{np^h}.$$

In particular the Frobenius automorphism of W stabilizes $W^o(0,1)$,

$$W^o(0,1) = \left\{\sum_{n\in\mathbf{N}} a_n x^n,\ a_n \in K \mid \exists N \in \mathbf{R}\ |a_n| \leqslant N,\ \forall n \geqslant 0\right\}.$$

Note that $E \subset W$ (cf. [Ch3, Section 5]).

DEFINITION 2.2.2. A linear differential operator of the type

$$L = \frac{d}{dx} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} \tag{2.2.3}$$

$a_i \in K$ has a strong overconvergent Frobenius structure if it is equivalent for a certain $s \in \mathbf{N}$ to

$$\varphi^s(L) = \frac{d}{dx} + p^s x^{p^s - 1}\left(\frac{a_2^{\sigma^s}}{x^{2p^s}} + \cdots + \frac{a_n^{\sigma^s}}{x^{np^s}}\right) \tag{2.2.4}$$

on the K-algebra of the analytic functions on $\{P \in \mathbf{P}^1_K \mid |x(P)| \geqslant \lambda\}$, for $\lambda \in \mathbf{R}$, $\lambda < 1$.

REMARK 2.2.5. To have a strong overconvergent Frobenius structure for a differential operator L as in (2.2.3) is equivalent to the assertion that (cf. (2.1))

$$L = \frac{d}{dx_\infty} - a_2 - a_3 x_\infty - \cdots - a_n x_\infty^{n-2}$$

is equivalent to

$$\frac{d}{dx_\infty} - p^s x^{p^s-1}(a_2^{\sigma^s} + a_3^{\sigma^s} x_\infty^{p^s} + \cdots + a_n^{\sigma^s} x_\infty^{n-2p^s})$$

on the ring of analytic functions on $\{P \in \mathbf{P}_K^1 \mid |x_\infty(P)| \leqslant \frac{1}{\lambda}\}$, $\lambda \in \mathbf{R}$, $\lambda < 1$.

REMARK 2.2.6. Suppose that a linear differential operator of the type

$$L = \frac{d}{dx} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n},$$

$a_i \in K$, has a strong overconvergent Frobenius structure. Then

$$L = \frac{d}{dx} + \frac{a}{x} + \left(\frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}\right)$$

also has a strong overconvergent Frobenius structure if and only if $a = \frac{z}{p^s-1}$ for $z \in \mathbf{Z}$, $s \in \mathbf{N}$ [Be1], [Ro2].

2.3. Using the definition and remarks in 2.2 we can prove

THEOREM 2.3.1. *Consider the operator of the type*

$$L = \frac{d}{dx} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}$$

where $a_i \in K$. Suppose it has the property of convergence on the generic disk of $D(0, 1^-)$. Then L has an overconvergent strong Frobenius structure.

Proof. Using 2.1 and Remark 2.2.5 we can study the problem from the point of view of the coordinate at infinity. In this setting, we have to prove that if an operator of the type

$$L = \frac{d}{dx} + a_2 + a_3 x + \cdots + a_n x^{n-2} \tag{2.3.1.1}$$

has the property of convergence on the generic disk of $D(0, 1^-)$ then it is equivalent for a certain $s \in \mathbf{N}$ to the differential operator

$$\varphi^s(L) = \frac{d}{dx} + p^s x^{p^s-1}(a_2^{\sigma^s} + a_3^{\sigma^s} x^{p^s} + \cdots + a_n^{\sigma^s} x^{(n-2)p^s}) \tag{2.3.1.2}$$

on the ring of analytic functions on $\{P \in \mathbf{A}_K^1 \mid |x(P)| \leqslant \frac{1}{\lambda}\}$, for $\lambda \in \mathbf{R}$, $\lambda < 1$. Notice that $\varphi^s(L)$ (2.3.1.2) has the property of convergence on the generic disk of $D(0, 1^-)$ [Ch1, 4.7.2].

Consider now a solution, g, of L as in (2.3.1.1): by the fact that such an operator has no singularities on $D(0,1^-)$ and by the property of convergence on the generic disk we conclude that $g(x) \in \mathcal{O}(D(0,1^-))$ (transfer principle [Dw1]). But, actually, $g(x) \in W^o(0,1) \subset W$ (2.2.1), [Ch1, 4.3.7 and 5.1.7]. We then obtain that L has a strong overconvergent Frobenius structure if and only if there exists $s \in \mathbf{N}$ such that the quotient

$$\frac{g(x)}{\varphi^s(g(x))} \tag{2.3.1.3}$$

defines an analytic function on $\{P \in \mathbf{A}^1_K \mid |x(P)| \leqslant \frac{1}{\lambda}\}$ where $\lambda \in \mathbf{R}$, $\lambda < 1$. (The fact that (2.2.1.3) is invertible is then a consequence of the fact that it satisfies a differential operator without singularities in $\{P \in \mathbf{A}^1_K \mid |x(P)| \leqslant \frac{1}{\lambda}\}$).

Formally one has

$$g(x) = \exp\left(a_2 x + \cdots + \frac{a_n}{n-1} x^{n-1}\right)$$

on the other hand ($h \in \mathbf{N}$)

$$\begin{aligned} g(x)^{p^h} &= \exp\left(a_2 x + \cdots + \frac{a_n}{n-1} x^{n-1}\right)^{p^h} \\ &= \exp\left(p^h a_2 x + \cdots + p^h \frac{a_n}{n-1} x^{n-1}\right), \end{aligned}$$

and if we choose h such that

$$\left| p^h \frac{a_i}{i-1} \right| < |\pi|$$

for $i = 2, \ldots, n$, then $g(x)^{p^h}$ converges in a closed disk $D(0, \frac{1}{\lambda'}^+)$, $\lambda' < 1$. In particular $g(x)^{p^h}$ belongs to E, the field of analytic elements. We then conclude that $g(x) \in W$ is actually algebraic over E.

We now strictly follow the articles [Ch3] and [Ch4]. The fact that $E[g(x)] \subset W$ is a finite extension field of E implies that the Frobenius stabilizes $E[g(x)]$, in fact

$$\varphi(E[g(x)]) = E[g(x)] \cap \varphi(W)$$

[Ch3, Theorem 5.2]. We then apply [Ch3, Proposition 7.1] and argue that $E[g(x)]$ is a semisimple object in the abelian category, $MC(E)$, whose objects are pairs (M, ∇) where M is a finite dimensional E-vector space and ∇ is a connection i.e. a K-linear map from $\mathrm{Der}(E) = E[\frac{d}{dx}]$ to $\mathrm{End}_K(M)$.

Exactly as in ([Ch3], end of Sect. 7) we conclude that the sub-E-vector spaces of $E[g(x)]$ generated by the various $\varphi^h(g(x))$, $h \in \mathbf{N}$

$$\varphi^h(g(x))E,$$

are sub-objects of $E[g(x)]$ in $MC(E)$: hence they are in finite number (up to isomorphism). There will exist $h, h' \in \mathbf{N}$ such that

$$\varphi^h(g(x))E \cong \varphi^{h'}(g(x))E,$$

i.e. there exists $a \in E^*$ such that $\varphi^h(g(x)) = a\varphi^{h'}(g(x))$. From the "faithful" action of the Frobenius [Ch5, 10.1] we conclude that for $s = |h - h'| \in \mathbf{N}$ we have [Ch4]

$$q(x) = \frac{g(x)}{\varphi^s(g(x))} \in E. \tag{2.3.1.4}$$

In particular one may notice that $q(x)$ is an analytic element in $D(0, 1^-)$ (it is a quotient of two invertible elements of $\mathcal{O}(D(0, 1^-))$) and

$$P(x) = \frac{q'}{q} = \frac{g(x)'}{g(x)} - \frac{\varphi^s(g(x))'}{\varphi^s g(x)} \in K[x].$$

To conclude the proof of the theorem one needs only prove the following

PROPOSITION 2.3.2. *If* $q(x) \in E$ *is analytic in* $D(0, 1^-)$ *and*

$$q(x)'/q(x) = P(x) \in K[x],$$

then $q(x)$ *is analytic in* $D(0, \frac{1}{\lambda})$ *with* $\lambda \in \mathbf{R}$ *and* $\lambda < 1$.

Proof. By [Mo, Proposition 1], we know that, actually, $q(x)$ is an analytic function in the closed disk $D(0, 1^+)$. We write

$$P(x) = b_0 + b_1 x + \cdots + b_n x^n$$

with $b_i \in K$. We may cancel in $P(x)$ the b_i's such that

$$\left|\frac{b_i}{i+1}\right| < |\pi|.$$

In fact, in this case, $\exp(-\frac{b_i}{i+1}x^{i+1})$ is then analytic in $D(0, r^-)$ for some $r > 1$, and we can replace $q(x)$ by

$$q(x)\exp\left(-\frac{b_i}{i+1}x^{i+1}\right).$$

If, after this simplification, $P(x)$ is zero, then

$$q(x) = c\prod_{i=0}^{n}\exp\left(\frac{b_i}{i+1}x^{i+1}\right)$$

with $c \in K^*$, and $|\frac{b_i}{i+1}| < |\pi|$ for each i, is convergent in $D(0, \frac{1}{\lambda'})$ with $\lambda' \in \mathbf{R}$ and $\lambda' < 1$. If after simplification $P(x)$ is not zero, consider the term of highest degree

b_m. By the fact that the solution converges in the closed disk $D(0,1^+)$ we know that $|b_m| < |\pi|$ (cf. 1.5.1.4). But our hypothesis about $P(x)$ implies that

$$\left|\frac{b_m}{m+1}\right| \geqslant |\pi| ,$$

we conclude that $m+1$ must be divisible by p. In [Ro1, 10.8], Robba introduced the following functions for each $h \in \mathbf{N}$

$$f_h(x) = \exp\left(\frac{\pi x^{p^h}}{p^h} + \gamma_1 \frac{x^{p^{h-1}}}{p^{h-1}} + \cdots + \gamma_h x\right),$$

where γ_i belong to a spherically complete extension of K [Ch1, 1.9.7] and

$$|\gamma_h| = p^{\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^h}}$$

and such that $f_h(x)$ converges in $D(0,1^-)$. We may then write $m = lp^h - 1$ for a $l, h \in \mathbf{N}$, $(l,p) = 1$. Consider $\beta \in K$ such that

$$\beta^{p^h} = -\frac{b_m}{\pi l},$$

and the function $f_h(\beta x^l)$, $h \in \mathbf{N}$. This function converges in $D(0, |\beta|^{-\frac{1}{l}})$: but $|b_m| < |\pi|$ hence $|\beta| < 1$ and

$$|\beta|^{-\frac{1}{l}} > 1 .$$

We may replace $q(x)$ by $q(x) f_h(\beta x^l)$ and get rid of the term b_m of highest degree of $P(x)$. We apply this method by induction and we conclude that $q(x) \in E$ is actually convergent in $D(0, \frac{1}{\lambda})$ with $\lambda \in \mathbf{R}$ and $\lambda < 1$.

This concludes the proof of the Proposition 2.3.2 and, hence, of Theorem 2.3.1. Q.E.D.

REMARK 2.3.3. Recently it has been proved that the γ_i's, which appear in the proof of the previous proposition, can be chosen to be algebraic over $\mathbf{Q}$ ([MA], unpublished and partial answers to the problem have been given by B. Dwork and D. Chinellato).

3. Frobenius structure

In this paragraph we will connect Frobenius structures with respect to different local coordinates: we will apply the results of this Section in Section 4. The problem *in the large* is the following: consider a series $f(x-a)$ where $a \in \mathcal{V}\backslash\mathcal{M}$. One would like to have information about the convergence set of the ratio

$$\frac{f(x-a)}{\varphi(f(x-a))},$$

where $\varphi(f(x-a))$ is the Frobenius transform of $f(x-a)$, that is, $\varphi(f(x-a)) = f^\sigma(x^p - a^\sigma)$. The idea is to change the variable $x-a$ to x and to study $f(x)$ and $f^\sigma(x^p) = \varphi(f(x))$. But the information that we then obtain from $f^\sigma(x^p)$ is related to $f^\sigma((x-a)^p)$ not to $f^\sigma(x^p - a^\sigma)$. Elaborating an idea of Dwork (cf. [Dw2]), we have

PROPOSITION 3.1. *Consider $a \in \mathcal{V} \backslash \mathcal{M}$ and suppose that the differential operator*

$$L = \frac{d}{dx} + \frac{a_2}{(x-a)^2} + \cdots + \frac{a_n}{(x-a)^n} \tag{3.1.1}$$

satisfies the property of convergence on the generic disk of $D(0, 1^-)$ (which is the same as that of $D(a, 1^-)$). Then there exists $s \in \mathbf{N}$ such that

$$\varphi^s(L) = \frac{d}{dx} + p^s x^{p^s - 1}\left(\frac{a_2^{\sigma^s}}{(x^{p^s} - a^{\sigma^s})^2} + \cdots + \frac{a_n^{\sigma^s}}{(x^{p^s} - a^{\sigma^s})^n}\right)$$

is equivalent to L on the ring of analytic functions on

$$\{P \in \mathbf{P}^1_K \mid |x(P) - a| > \lambda\}$$

for a $\lambda \in \mathbf{R}$, $\lambda < 1$.

Proof. As usual we can study the problem at "∞" and must check whether

$$\frac{d}{dx} - a_2 - a_3(x-a) - \cdots - a_n(x-a)^{n-2} \tag{3.1.2}$$

is equivalent to

$$\frac{d}{dx} - p^s x^{p^s-1}(a_2^{\sigma^s} + a_3^{\sigma^s}(x^{p^s} - a^{\sigma^s}) + \cdots + a_n^{\sigma^s}(x^{p^s} + a^{\sigma^s})^{n-2}) \tag{3.1.3}$$

on the analytic functions on $\{P \in \mathbf{A}^1_K \mid |x(P) - a| < \frac{1}{\lambda}\}$ for a $\lambda \in \mathbf{R}$, $\lambda < 1$.

We denote the solution of (3.1.2) by $u(x-a)$. Then by the hypothesis of convergence on the generic disk, we conclude that $u(x-a) \in W^o(a,1)$ [Ch1, 4.3.7, 5.1.7]. (The definition of $W^o(a,1)$ is analogous to that of $W^o(0,1)$ in (2.2), x is replaced by $x-a$ [Ch1].) We define the Frobenius action, φ, on $W^o(a,1)$ in the usual way: if $v(x-a) \in W^o(a,1)$ then $\varphi(v(x-a)) = v^\sigma(x^p - a^\sigma)$. So $\varphi^s(u(x-a)) = u^{\sigma^s}(x^{p^s} - a^{\sigma^s})$ is a solution of (3.1.3). We must show that there exists $s \in \mathbf{N}$ such that

$$\frac{u(x-a)}{\varphi^s(u(x-a))} = \frac{du(x-a)}{u^{\sigma^s}(x^{p^s} - a^{\sigma^s})}$$

converges for $\{P \in \mathbf{A}^1_K \mid |x(P) - a| < \frac{1}{\lambda}\}$ for a $\lambda \in \mathbf{R}$, $\lambda < 1$.

First of all we change the variable x to $x-a$ in (3.1.2), then obtaining

$$\frac{d}{dx} - a_2 - a_3 x - \cdots - a_n x^{n-2}. \tag{3.1.4}$$

The operator (3.1.4) satisfies the property of convergence on the generic disk. It follows from Theorem 2.3.1 that there exists $s \in \mathbf{N}$ such that

$$\frac{u(x)}{\varphi^s(u(x))} \tag{3.1.5}$$

converges for $\{P \in \mathbf{A}^1_K \mid |x(P)| < \frac{1}{\lambda'}\}$ for some $\lambda' \in \mathbf{R}$, $\lambda' < 1$. Let us write

$$\frac{u(x-a)}{u^{\sigma^s}(x^{p^s} - a^{\sigma^s})} = \frac{u(x-a)}{u^{\sigma^s}((x-a)^{p^s})} \frac{u^{\sigma^s}((x-a)^{p^s})}{u^{\sigma^s}(x^{p^s} - a^{\sigma^s})} .$$

The first term on the right hand side is nothing but (3.1.5) after we have replaced x by $x - a$. Hence it converges for $\{P \in \mathbf{A}^1_K \mid |x(P) - a| < \frac{1}{\lambda'}\}$ for some $\lambda' \in \mathbf{R}$, $\lambda' < 1$. To prove the theorem we need to check the assertion of overconvergence for the last term. To this end we introduce the function of two variables

$$\chi(Y, Z) = \frac{u^{\sigma^s}(Y+Z)}{u^{\sigma^s}(Y)} . \tag{3.1.7}$$

We will exhibit the relationship between Z and Y in order to obtain convergence for (3.1.7). We know that $u^{\sigma^s}(Y)$ satisfies the following differential operator

$$\frac{d}{dY} - a_2^{\sigma^s} - a_3^{\sigma^s} Y - \cdots - a_n^{\sigma^s} Y^{n-2}$$

which has the property of convergence on the generic disk of $D(0, 1^-)$ (in the Y coordinate) (cf. [Ch1, 4.6.1]). We may expand (3.1.7) in the Taylor series:

$$\chi(Y, Z) = \sum_{l \geqslant 0} \frac{1}{l!} \frac{u^{\sigma^s (l)}(Y)}{u^{\sigma^s}(Y)} Z^l ,$$

where $\frac{u^{\sigma^s (l)}(Y)}{u^{\sigma^s}(Y)}$ is, by recurrence, a polynomial in Y of degree less than or equal to $(n-2)l$. By the hypothesis of convergence on the generic disk [Ch1, 4.3.7], there exists $N \in \mathbf{R}$, such that

$$\left| \frac{1}{l!} \frac{u^{\sigma^s (l)}(Y)}{u^{\sigma^s}(Y)} \right| \leqslant N$$

for each $l \in \mathbf{N}$, where $| - |$ is the Gauss norm i.e. the boundary norm [Ch1, 2.4.7]. So, for each $|\overline{Y}| < r$, $r > 1$, by the fact that

$$\frac{1}{l!} \frac{u^{\sigma^s (l)}(Y)}{u^{\sigma^s}(Y)}$$

is a polynomial of degree less than or equal to $(n-2)l$, whose coefficients are bounded by N, we obtain

$$\left| \frac{1}{l!} \frac{u^{\sigma^s (l)}(\overline{Y})}{u^{\sigma^s}(\overline{Y})} \right| \leqslant N r^{(n-2)l} .$$

We deduce that $\chi(Y, Z)$ converges for Z such that $|Z| < \frac{1}{r^{n-2}}$ and $|Y| < r$. And *viceversa* if $|Z| < \overline{\lambda} < 1$ then the series converges for $|Y| < (\overline{\lambda})^{-\frac{1}{n-2}}$.

To conclude the proof it now suffices to set $Y = x^{p^s} - a^{\sigma^s}$ and $Z = (x-a)^{p^s} - x^{p^s} - a^{\sigma^s}$ and an easy calculation shows that there exists $\lambda \in \mathbf{R}$, $\lambda < 1$, such that if

$$|x| \leqslant \frac{1}{\lambda},$$

then $|Z| \leqslant \overline{\lambda} < 1$ and $|Y| = |x^{p^s} - a^{\sigma^s}| < (\overline{\lambda})^{-\frac{1}{n-2}}$. Q.E.D.

4. Overconvergent F-crystals

The main result of this section is the following

THEOREM 4.1. *Let X be an open k-subscheme of $\mathbf{P}_k^1$. Then in the category of overconvergent isocrystals*

$$\mathrm{Isoc}^{\dagger}(X/K)$$

the objects, i.e. the overconvergent isocrystals, which have rank one and exponents of the type $\frac{z}{p^{\overline{s}}-1}$ for $z \in \mathbf{Z}$ and $\overline{s} \in \mathbf{N}$ at each residue class are overconvergent F-isocrystals.

The proof of this theorem will be given in 4.6.

4.2. Before going through the proof of the theorem we will recall some definitions (mainly from [Be2]): we will restrict ourselves to giving them in our particular setting (i.e dimension 1), even though they can be given in a general situation. We will always refer to Berthelot's notation.

Consider X, a k-open subscheme of $\mathbf{P}_k^1$; the projective k-line may be viewed as a compactification of X. The open subscheme X will be of the type

$$X = \mathbf{P}_k^1 \backslash \{\overline{a}_1, \ldots, \overline{a}_m\},$$

where we may suppose that among the $\overline{a}_i$'s there is $\infty = \overline{a}_m$. Following Berthelot's notation we will indicate $Z_k = \{\overline{a}_1, \ldots, \overline{a}_m\}$. We denote by $\hat{\mathbf{P}}_{\mathcal{V}}^1$ the formal projective $\mathcal{V}$-line, we then obtain the diagram

$$X \hookrightarrow \mathbf{P}_k^1 \longrightarrow \hat{\mathbf{P}}_{\mathcal{V}}^1,$$

where the first map is an open immersion while the second represents $\mathbf{P}_k^1$ as a closed subscheme of $\hat{\mathbf{P}}_{\mathcal{V}}^1$. Of course the generic fiber of $\hat{\mathbf{P}}_{\mathcal{V}}^1$ (in Raynaud's sense [Ra]) is $\mathbf{P}_K^1$.

Let $a_1, \dots, a_{m-1}$ be representatives in $\mathbf{A}^1_K$ of $\overline{a}_1, \dots, \overline{a}_{m-1}$: the a_i's will be elements of $\mathcal{V}$ in distinct residue classes. We then consider for each $\lambda \in (0,1)$ the following affinoid subset of $\mathbf{P}^1_K$:

$$\mathbf{V}_\lambda = \left\{ x \in \mathbf{A}^1_K \mid |x| \leqslant \frac{1}{\lambda} \right\} \Big\backslash \bigcup_{i=1}^{m-1} D(a_i, \lambda^-).$$

They form a cofinal system in the set of strict neighborhood of $]X[_{\hat{\mathbf{P}}^1_{\mathcal{V}}}$ (the tube of X in $\mathbf{P}^1_K$) [Be2, 1.2.1], [Ba-Ct]. We have the following inclusion for each $\lambda \in (0,1)$

$$j_\lambda \colon \mathbf{V}_\lambda \longrightarrow \mathbf{P}^1_K.$$

Let $\mathcal{E}$ be a sheaf defined in some strict neighborhood $\mathbf{V}_{\overline{\lambda}}$, we may associate to it a sheaf defined in $\mathbf{P}^1_K$ as

$$j^\dagger \mathcal{E} = \varinjlim_{\substack{\lambda \geqslant \overline{\lambda} \\ \lambda \to 1}} j_{\lambda *} j_\lambda{}^* (j_{\overline{\lambda} *} \mathcal{E}) = \varinjlim_{\lambda \to 1} j_{\lambda *} (\mathcal{E}_{|\mathbf{V}_\lambda}).$$

It is then clear that if $\lambda_2 > \lambda_1$ and if $\mathcal{E}$ is a sheaf defined in $\mathbf{V}_{\lambda_1}$ then

$$j^\dagger \mathcal{E} = j^\dagger \mathcal{E}_{|\mathbf{V}_{\lambda_2}}.$$

In particular one can take the structural sheaf of $\mathbf{P}^1_K$, $\mathcal{O}$, and consider $j^\dagger \mathcal{O}$. Of course

$$j^\dagger \mathcal{O} = j^\dagger \mathcal{O}_{\mathbf{V}_\lambda}$$

(where $\mathcal{O}_{\mathbf{V}_\lambda}$ is the structural sheaf in $\mathbf{V}_\lambda$). The sheaf $j^\dagger \mathcal{O}$ is a sheaf of rings: we may introduce the category of $j^\dagger \mathcal{O}$-modules whose objects are sheaves in $\mathbf{P}^1_K$ which are $j^\dagger \mathcal{O}$-modules (cf. [Be2, 2.1] for general statements). It is known that if G is a coherent $j^\dagger \mathcal{O}$-module [Be2, 2.1.9], then there exists a strict neighborhood $\mathbf{V}_\lambda$ and a coherent $\mathcal{O}_{\mathbf{V}_\lambda}$-module, $\mathcal{E}$, such that

$$j^\dagger \mathcal{E} = G$$

[Be2, 2.1.10]. We may also define [Be2, 2.2.2] a connection, ∇, (automatically integrable: we are in dimension 1) relative to K on a coherent $j^\dagger \mathcal{O}$-module G (we will then refer to (G, ∇) as a coherent connection $j^\dagger \mathcal{O}$-module) as a K-linear homomorphism

$$\nabla \colon G \longrightarrow G \otimes_{\mathcal{O}_{\mathbf{P}^1_K}} \Omega^1_{\mathbf{P}^1_K/K}$$

which satisfies the usual Leibnitz's rule. Every coherent connection $j^\dagger \mathcal{O}$-module is also the image by $j^\dagger$ of a coherent module endowed with a connection defined in a strict neighborhood of the tube of X in $\mathbf{P}^1_K$ [Be2, 2.2.3]: namely if (G,∇) is a coherent connection $j^\dagger\mathcal{O}$-module, relative to K, then there exists a strict neighborhood $\mathbf{V}_\lambda$ of the tube of X in $\hat{\mathbf{P}}^1_V$ and a coherent $\mathcal{O}_{\mathbf{V}_\lambda}$-module $\mathcal{E}$ endowed with a "usual" connection ∇_o such that $j^\dagger(\mathcal{E},\nabla_o) = (G,\nabla)$. In particular, by the fact that $\mathbf{P}^1_K$ is smooth and everything is defined over K, of characteristic 0, we conclude that $\mathcal{E}$ is locally free.

4.3. Under the previous hypotheses, we may now give the following [Be2, 2.3.6]

DEFINITION 4.3.1. The category $\text{Iso}^\dagger(X/K)$ is the category whose objects are coherent connection (relative to K, automatically integrable) $j^\dagger\mathcal{O}$-modules, (G,∇), which are overconvergent along Z_k. Namely this means that, if $(G,\nabla) = j^\dagger(\mathcal{E},\nabla_o)$, for each $\eta < 1$ there exists $\overline{\lambda} \in (0,1)$ such that, for each section $e \in \Gamma(\mathbf{V}_{\overline{\lambda}},\mathcal{E})$:

$$\lim_{l\to+\infty} \left\| \frac{1}{l!}(\partial^l)e \right\|_{\mathbf{V}_{\overline{\lambda}}} \eta^l = 0,$$

(where $\| - \|_{\mathbf{V}_{\overline{\lambda}}}$ denotes any Banach norm on $\Gamma(\mathbf{V}_{\overline{\lambda}},\mathcal{E})$. And ∂ denotes the usual derivation) [Be1, 2.2.14], [Ba-Ct].

The objects of $\text{Iso}^\dagger(X/K)$ will be called *overconvergent isocrystals.*

REMARK 4.3.2. It is proven in [Be2] that the category is independent of the compactification of X.

4.4. By the general arguments in 4.2 we know that if (G,∇) is a coherent connection $j^\dagger\mathcal{O}$-module, then it is of the type $j^\dagger(\mathcal{E},\nabla_o) = (G,\nabla)$, where $\mathcal{E}$ is locally free sheaf on a strict neighborhood $\mathbf{V}_\lambda$. We define that the *rank of* (G,∇) as the rank of $\mathcal{E}$.

The affinoid sets $\mathbf{V}_\lambda$ are known in the literature as affinoid connected sets of $\mathbf{P}^1_K$. We have

PROPOSITION 4.4.1 (Fresnel-van der Put). *Every locally free sheaf of rank 1 on an affinoid connected set of* $\mathbf{P}^1_K$ *is free.*

Proof. The K-affinoid algebra of the holomorphic functions on the affinoid connected set is a principal ideal ring [F-vdP, II.4.13], and the locally free modules over a principal ring are free [F-vdP, III.8.3]. Q.E.D.

Let us consider a coherent connection $j^\dagger\mathcal{O}$-module (G,∇), suppose it of rank 1: by means of the discussion in 4.2 we know that (G,∇) is actually $j^\dagger(\mathcal{E},\nabla_o)$, where $\mathcal{E}$ is a free $\mathcal{O}_{\mathbf{V}_\lambda}$-module of rank 1 and the connection can be defined after choosing a basis of $\mathcal{E}$, $\underline{e}$ and for each $a \in \mathcal{O}(\mathbf{V}_\lambda)$, by

$$\nabla_o(a\underline{e}) = \underline{e} \otimes da + fa\underline{e} \otimes dx, \tag{4.4.2}$$

where $f \in \mathcal{O}(\mathbf{V}_\lambda)$. By the Mittag–Leffler decomposition [F-vdP] f can be written

$$f = \sum_{l=0}^{\infty} b_l x^l + \sum_{j=1}^{m-1} \left(\sum_{l=1}^{\infty} b_{il} \frac{1}{(x - a_j)^l} \right), \tag{4.4.3}$$

where $b_l, b_{il} \in K$, and

$$\lim_{l \to \infty} |b_l| \lambda^{-l} = 0,$$

and for $j = 1, \ldots, m-1$

$$\lim_{l \to \infty} |b_{jl}| \lambda^{-l} = 0.$$

We may now give the following

DEFINITION 4.4.4. In the foregoing hypotheses and notation (4.4), for $j = 1, \ldots, m-1$ we define b_{1j} as the exponents of the coherent connection $j^\dagger \mathcal{O}$-module of rank 1, (G, ∇).

REMARK. We will refer to

$$-\sum_{j=1}^{m-1} b_{1j}$$

as the exponent at ∞. Notice that the definition is independent of the choice of the representative a_j of $\overline{a}_j$, $j = 1, \ldots, m-1$.

We also recall that for a coherent connection $j^\dagger \mathcal{O}$-module of rank one (hence free), (G, ∇), associated to f, multiplication by an invertible element g of $\mathcal{O}_{\mathbf{V}_{\overline{\lambda}}}(\mathbf{V}_{\overline{\lambda}})$ ($\overline{\lambda} \in (0,1)$) gives an isomorphic connexion (G, ∇') with the same underlined $j^\dagger \mathcal{O}$-module but associated to $f + \frac{g'}{g}$. In fact the two give isomorphic overconvergent isocrystals. We may then state:

PROPOSITION 4.4.5. *Consider a coherent connection $j^\dagger \mathcal{O}$-module of rank one (G, ∇): it is free and it admits a basis, $\underline{e}$, in some $\mathbf{V}_{\overline{\lambda}}$, for some $\overline{\lambda} \in (0,1)$, such that the connection may be represented by*

$$\nabla(\underline{e}) = f(x) \underline{e}\, dx, \tag{4.4.5.1}$$

where $f(x) \in K(x)$ and has only poles in the a_i's.

Proof. By the previous discussions we know there exists $\lambda \in (0,1)$ such that $(G, \nabla) = j^\dagger(\mathcal{E}, \nabla_o)$, where $\mathcal{E}$ is a free $\mathcal{O}_{\mathbf{V}_\lambda}$-module of rank one (4.4.1). The connection ∇_o is represented in a basis $\underline{e}'$ of $\mathcal{E}$ by:

$$\nabla_o(a\underline{e}') = \underline{e}' \otimes da + fa\underline{e}' \otimes dx \tag{4.4.5.2}$$

where $f \in \mathcal{O}(\mathbf{V}_\lambda)$. Using the Mittag–Leffler decomposition [F-vdP] f can be written

$$f = \sum_{l=0}^{\infty} b_l x^l + \sum_{j=1}^{m-1} \left(\sum_{l=1}^{\infty} b_{jl} \frac{1}{(x-a_j)^l} \right), \tag{4.4.5.3}$$

where $b_l, b_{jl} \in K$ and

$$\lim_{l \to \infty} |b_l| \lambda^{-l} = 0,$$

and for $j = 1, \ldots, m-1$

$$\lim_{l \to \infty} |b_{jl}| \lambda^{-l} = 0.$$

We fix one of the elements which form the decomposition, say j; the proof for the others is similar.

We know that

$$\lim_{l \to \infty} \frac{|b_{jl}|}{\lambda^l} = 0 ,$$

it follows that the primitive of $\sum_{l=2}^{\infty} b_{jl} \frac{1}{(x-a_j)^l}$,

$$\sum_{l=2}^{\infty} - \frac{b_{jl}}{(l-1)(x-a_j)^{l-1}} ,$$

converges for $|x - a_j| > \lambda$ (strictly). Consider $1 > \lambda_j > \lambda$: in particular there will exist $n(j, \lambda_j) \in \mathbf{N}$, such that if $l \geqslant n(j, \lambda_j)$:

$$\left| - \frac{b_{jl}}{l-1} \right| \lambda_j^{1-l} < |\pi| .$$

It follows that

$$\zeta_j(x) = \sum_{l \geqslant n(j,\lambda_j)} - \frac{b_{jl}}{l-1} \frac{1}{(x-a_j)^{l-1}}$$

is analytic for $|x - a_j| \geqslant \lambda_j$, and

$$\sup_{|x-a_j| \geqslant \lambda_j} |\zeta_j(x)| < |\pi| .$$

But since $|\pi|$ is the radius of convergence of the exponential map, we obtain that

$$\varphi_j(x) = \exp(\zeta_j(x))$$

is invertible analytic in the domain $\{P \in \mathbf{P}^1_K \mid |x(P) - a_j| \geqslant \lambda_j\}$.

We apply the same procedure for each $j = 1, \dots, m-1$ and ∞. We obtain an invertible function

$$g(x) = \prod_{j=1}^{m-1} \exp(\zeta_j(x)) \cdot \exp(\zeta_\infty(x))$$

which is analytic and invertible in a strict neighborhood $\mathbf{V}_{\overline{\lambda}}$, where $\overline{\lambda} = \max(\lambda_1, \dots, \lambda_{m-1}, \lambda_\infty) < 1$. Then, $g(x)$ is the invertible element required by the proposition. Q.E.D.

REMARK 4.4.6. As a result of this section we obtain that a coherent connection $j^\dagger\mathcal{O}$-module of rank 1 in X (which may be supposed not to contain ∞), (G, ∇), may be viewed as $j^\dagger(\mathcal{E}, \nabla_o)$, where $\mathcal{E}$ is a free module of rank one in $\mathcal{O}_{\mathbf{V}_\lambda}$ for some $\lambda \in (0,1)$. $\mathcal{E}$ admits a basis $\underline{e}$ such that the connection is defined by

$$\nabla_o(\underline{e}) = f(x)\underline{e} \otimes dx,$$

where $f(x) \in K(x)$ and has only one singular point in each residue class $\overline{a}_1, \dots, \overline{a}_{m-1}, \infty$ ($\mathbf{P}^1_k \backslash X = \{\overline{a}_1, \dots, \overline{a}_m, \infty\}$).

4.5. We recall some other definitions. Suppose σ is an continuous automorphism $\sigma\colon K \to K$ which induces the Frobenius in the residue field, k, which is supposed to be perfect. An object, (G, ∇), of $\mathrm{Isoc}^\dagger(X/K)$ is said to be an *overconvergent F^s_σ-isocrystal* if there exists $s \in \mathbf{N}$ such that:

$$(F^{s*}_\sigma G, F^{s*}_\sigma \nabla) \simeq (G, \nabla) \tag{4.5.1}$$

(as $\mathrm{Isoc}^\dagger(X/K)$ objects), where F^{s*}_σ is the s-times iterated absolute Frobenius.

The object $F^{s*}_\sigma G$ is obtained from G by first applying an extension of scalars using σ, then by taking the inverse image in $\mathrm{Isoc}^\dagger(X/K)$ by means of the relative Frobenius in X, and, finally, iterating this procedure s-times (cf. [Be2, 2.3.7], [Be1, 4.1]).

REMARK 4.5.2. In our setting (i.e. X open subscheme of $\mathbf{P}^1_k$) we may be more explicit. In fact consider $\sigma : K \to K$ which is a lifting of the pth-power automorphism of the perfect field k. As we said X may be viewed as $X = \mathbf{P}^1_k \backslash \{\overline{a}_1, \dots, \overline{a}_{m-1}, \overline{a}_m\}$, by scalar extension, we can consider $X^{(p)} = \mathbf{P}^1_k \backslash \{\overline{a}^p_1, \dots, \overline{a}^p_{m-1}, \overline{a}^p_m\}$ (note that ∞ is sent to itself by this scalar extension). On the other hand an overconvergent isocrystal on (X/K), (G, ∇) may be represented (4.4.6) by an operator of the type

$$\frac{d}{dx} - \sum_{i=1}^{m} f_{a_i}(x),$$

where $f(x) = \sum_{i=1}^m f_{a_i}(x) \in K(x)$ and the $f_{a_i}(x)$'s are the residual parts of $f(x)$ along a_i's; the a_i's are lifting of $\overline{a}_i$'s in K. After the scalar extension by σ we obtain a system $(G^\sigma, \nabla^\sigma)$ defined in some strict neighborhood of $X^{(p)}$ in $\mathbf{P}^1_K$ of the type (for some $\lambda \in (0,1)$)

$$\mathbf{P}^1_K \Big\backslash \bigcup_{i=1}^{m} D(a_i^\sigma, \lambda^-)$$

by

$$\frac{d}{dx} - \sum_{i=1}^{m} f^\sigma_{a_i^\sigma}(x)$$

(this is obtained from $f(x)$ simply by applying σ to the coefficients). Of course the a_i^σ's are liftings in characteristic 0 of the $\overline{a}_i^p$'s.

We then have the relative Frobenius (which, in our case, is just the pth-power of the coordinates) $X \to X^{(p)}$, which can also be lifted to characteristic 0 by the pth-power on the coordinates, F. We then obtain ($s = 1$ in (4.5.1)) the definition

$$(F_\sigma^* G, F_\sigma^* \nabla) := (F^*(G^\sigma), F^*(\nabla^\sigma)) . \tag{4.5.3}$$

In particular $(F_\sigma^* G, F_\sigma^* \nabla)$ is connected with the differential operator:

$$\frac{d}{dx} - px^{p-1}\left(\sum_{i=1}^{m} f^\sigma_{a_i^\sigma}(x^p)\right)$$

defined in some strict neighborhood of X in $\mathbf{P}^1_K$. Furthermore $F_\sigma^* G$ is still a free rank one $j^\dagger\mathcal{O}$-module. It is then clear how to iterate this procedure.

4.6. We now give a proof of Theorem 4.1.

Proof. Let (G, ∇) be an object of $\mathrm{Isoc}^\dagger(X/K)$ of rank one. We know from 4.4 that (G, ∇) may be viewed as

$$(G, \nabla) = (j^\dagger\mathcal{E}, j^\dagger\nabla_o), \tag{4.6.1}$$

where $\mathcal{E}$ is a free module defined in some strict neighborhood, $\mathbf{V}_\lambda$ ($\lambda \in (0,1)$), of the tube of X and ∇_o is viewed as expressed in a global basis, $\underline{e}$, of $\mathcal{E}$ by:

$$\nabla_o(\underline{e}) = f(x)\underline{e} \otimes dx, \tag{4.6.2}$$

where $f(x) \in K(x)$ with just one pole in each residue class given by $Z_k = \{\overline{a}_1, \ldots, \overline{a}_m\}$.

This setting has been studied by [Ba-Ct, Sect. 3] and, in particular, it was shown that the condition of overconvergence is equivalent to the condition of convergence on the generic disk of $D(0, 1^-)$ for the linear differential equation associated to the horizontal sections of ∇_o (4.6.2)

$$\frac{d}{dx}Y = -f(x)Y. \tag{4.6.3}$$

We must show that, under our hypotheses, there exists an $s \in \mathbf{N}$ such that the coherent connection $j^\dagger \mathcal{O}$-module $(F_\sigma^{s*}G, F_\sigma^{s*}\nabla)$, which is connected in a suitable strict neighborhood of the tube of X in $\mathbf{P}^1_K$ (and after a choice of a basis) to the differential equation

$$\frac{d}{dx}Y' = -p^s x^{p^s-1} f^\sigma(x^p)Y' \tag{4.6.4}$$

is isomorphic (as connection $j^\dagger \mathcal{O}$-module) to (G, ∇). This turns out to be equivalent to showing that there exists a strict neighborhood $\mathbf{V}_{\overline{\lambda}}$ ($\overline{\lambda} < 1$) of the tube of X in $\mathbf{P}^1_K$ such that the two equations (4.6.4) and (4.6.3) are equivalent on $\mathcal{O}(\mathbf{V}_{\overline{\lambda}})$: i.e. there exists an invertible element of $\mathcal{O}(\mathbf{V}_{\overline{\lambda}})$, $\theta(x)$, which gives $Y' = \theta(x)Y$.

This will be done by writing (4.6.3) in the form

$$\frac{d}{dx}Y = -\sum_{i=1}^{m} f_{a_i}(x - a_i)Y, \tag{4.6.5}$$

where each $f_{a_i}(x - a_i)$ is the residual part at a_i in the Mittag–Leffler decomposition of $f(x)$.

We will then show that for each $i = 1, \dots, m$, there exists an $s(i) \in \mathbf{N}$ such that

$$\frac{d}{dx}Y = -f_{a_i}(x - a_i)Y, \tag{4.6.5$_i$}$$

and

$$\frac{d}{dx}Y' = -p^{s(i)} x^{p^{s(i)}-1} f_{a_i}^{\sigma^{s(i)}}(x^{p^{s(i)}} - a_i^{\sigma^{s(i)}})Y' \tag{4.6.6$_i$}$$

are equivalent over the ring of holomorphic functions in a set of the type

$$T_i = \{P \in \mathbf{P}^1_K \mid |x(P) - a_i| \geqslant \lambda_i\}, \tag{4.6.7}$$

with $\lambda_i < 1$ if $a_i \neq \infty$ (If $a_i = \infty$ we have $T_i = \{P \in \mathbf{A}^1_K \mid |x(P)| \leqslant \frac{1}{\lambda_i}\}$, still with $\lambda_i < 1$).

We then take $\overline{\lambda} = \max_{i=1,\dots,m} \lambda_i$ and $s = \operatorname{lcm}(s(1), \dots, s(m))$ to say that the two equations (4.6.3) and (4.6.4) are equivalent on the holomorphic functions on $\mathbf{V}_{\overline{\lambda}}$.

The condition of convergence at the generic disk of $D(0, 1^-)$ for (4.6.3), i.e. for the operator

$$\frac{d}{dx} + f(x), \tag{4.6.8}$$

is equivalent to the same condition for the disk $D(a_i, 1^-)$. It is known, then, that (4.6.8) is equivalent on $\mathcal{O}(D(a_i, 1^-))$ to

$$\frac{d}{dx} + f_{a_i}(x) \tag{4.6.9}$$

which still has the property of convergence at the generic disk of $D(a_i, 1^-)$ (cf. [Ro1, Lemme 5.3], 1.4.1). We explicitly write the differential equation connected with (4.6.9)

$$\frac{d}{dx}Y = -f_{a_i}(x)Y = \left(\frac{b_{1i}}{x-a_i} + \frac{b_{2i}}{(x-a_i)^2} + \cdots + \frac{b_{ki}}{(x-a_i)^k}\right)Y \quad (4.6.10)$$

where by hypothesis $b_{1i} \in 1/(p^{\bar{s}_i} - 1)\mathbf{Z}$, for a certain $\bar{s}_i \in \mathbf{N}$. From (4.6.4) we have

$$\frac{d}{dx}Y' = p^s x^{p^s-1}\left(\frac{b_{1i}^{\sigma^s}}{x^{p^s} - a_i^{\sigma^s}} + \frac{b_{2i}^{\sigma^s}}{(x^{p^s} - a_i^{\sigma^s})^2} + \cdots + \frac{b_{ki}^{\sigma^s}}{(x^{p^s} - a_i^{\sigma^s})^k}\right)Y'.(4.6.11)$$

We have to show that the two operators (4.6.10) and (4.6.11) are equivalent on the K-algebra, $\mathcal{O}(T_i)$, of analytic functions defined in a set of the type T_i, (4.6.7), i.e. there exists a $\theta \in \mathcal{O}(T_i)^*$ such that (4.6.11) is transformed in (4.6.10) by $Y = \theta Y'$. In particular $b_{1i} \in \mathbf{Z}_p$ hence also

$$\frac{d}{dx}Y = \left(\frac{b_{2i}}{(x-a_i)^2} + \cdots + \frac{b_{ki}}{(x-a_i)^k}\right)Y \quad (4.6.12)$$

satisfies the hypothesis of convergence on the generic disk of $D(a_i, 1^-)$. We then apply Proposition 3.1 to conclude that (4.6.12) is equivalent for a certain $\tilde{s} \in \mathbf{N}$ to

$$\frac{d}{dx}Y' = p^{\tilde{s}} x^{p^{\tilde{s}}-1}\left(\frac{b_{2i}^{\sigma^{\tilde{s}}}}{(x^{p^{\tilde{s}}} - a_i^{\sigma^{\tilde{s}}})^2} + \cdots + \frac{b_{ki}^{\sigma^{\tilde{s}}}}{(x^{p^{\tilde{s}}} - a_i^{\sigma^{\tilde{s}}})^k}\right)Y', \quad (4.6.13)$$

in a set of the type $\{P \in \mathbf{P}^1_K \mid |x(P) - a_i| > \frac{1}{\lambda_i}\}$. Using Remark 2.2.6 for b_{1j}, we take $s(i) = \mathrm{lcm}(\bar{s}_i, \tilde{s})$. Q.E.D.

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