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Algebraic cycles on generic Abelian varieties

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Abstract. We formulate a conjecture about the Chow groups of generic Abelian varieties and prove it in a few cases.

1. Introduction

In this paper we study the rational Chow groups of generic abelian varieties. More precisely we try to answer the following question:

For which integers d do there exist “interesting” cycles of codimension d on the generic abelian variety of dimension g ?

By “interesting” cycles we mean cycles which are not in the subring of the Chow ring generated by divisors or cycles which are homologically equivalent to zero but not algebraically equivalent to zero. As background we recall that G. Ceresa [5] has shown that for the generic abelian variety of dimension three there exist codimension two cycles which are homologically equivalent to zero but not algebraically equivalent to zero. He uses the fact that the generic (principally polarised) abelian variety is the Jacobian of a curve and then uses the Abel-Jacobi map to show that for the generic curve C the cycle $C - C^-$ is non-zero mod algebraic equivalence. Recently M. Nori has shown [18] that for $g > 3$ the image of a homologically trivial cycle of codimension d , $1 < d < g$ in the Intermediate Jacobian $IJ^d(X)$ is torsion modulo the largest abelian subvariety of $IJ^d(X)$. However in the same paper, using his Connectivity Theorem, Nori gave examples of cycles in certain complete intersections which are homologically trivial but not algebraically trivial and are in the kernel of the Abel-Jacobi map. This leads to the possibility that such cycles also exist in generic abelian varieties of dimension $g > 3$.

In Section 3 we study the cohomology of generic families of abelian varieties $\pi : X \rightarrow S$. An algebraic cycle of codimension d in X has a cycle class in $H^{2d}(X, \mathbb{Q})$ so knowing the Hodge structure of these groups would allow us to predict the existence (or nonexistence) of interesting cycles of codimension d . By the degeneration of the Leray spectral sequence at E_2 and the Lefschetz decomposition it is enough to consider the groups $H^m(S, \mathbb{P}_i)$ where \mathbb{P}_i is the local system on S corresponding to the i th primitive (rational) cohomology of the fibres of π . We

give sufficient conditions for $H^m(S, \mathbb{P}_i)$ to vanish or have a (pure) Hodge structure of type $((m+i)/2, (m+i)/2)$, the motivation being that if $H^m(S, \mathbb{P}_i) = 0$ for all $m+i = 2d$, $m < i$ then one should not expect “interesting” cycles of codimension d and if $H^m(S, \mathbb{P}_i)$ is nonzero and pure of type $((m+i)/2, (m+i)/2)$ for some m, i with $m < i$, then the Hodge conjecture would predict that there exist cycles of codimension $d = (i+m)/2$ which are not generated by divisors. More generally we consider $H^m(S, \mathbb{P}_i \otimes \mathbb{W})$ where \mathbb{W} is an arbitrary polarisable variation of Hodge structure (V.H.S.); this is used to show that certain cycles are non-zero mod algebraic equivalence. For the precise results see Theorem 3.4. We use results of M. Saito [20] to reduce these calculations to some computations of Lie algebra cohomology and then apply a theorem of B. Kostant [15]. This method was used by Nori for computing $H^1(S, \mathbb{P}_i)$. The same method was also used earlier by S. Zucker [21] in the case where S is a (compact) quotient of a Hermitian symmetric space D by a discrete (cocompact) subgroup Γ of the group of automorphisms of D . In this case, even when the quotient is not necessarily compact (but Γ is arithmetic), Borel [3] has proved stronger vanishing results by other methods. However these stronger vanishing results do not hold in our situation as shown by the results of Section 4.

Theorem 3.4 suggests the following

CONJECTURE 1.1. (1) All codimension d cycles in the generic abelian variety of dimension g are generated by divisors up to torsion, for $d < g/2$.

(2a) For $g > 2$ and even, there exist codimension $g/2 + 1$ cycles which are not generated by divisors.

(2b) For $g > 1$ and odd, there exist codimension $(g+1)/2$ cycles which are homologically trivial but not algebraically trivial.

We have not been able to verify (1) in any nontrivial case (i.e. $g > 3$). However in Section 4 we show that (2) is true for $g = 4, 5$ using that the generic principally polarised abelian variety of dimension ≤ 5 is a Prym variety. Using a degeneration argument inspired by that of Ceresa, we show that a certain component of the cohomology class of a generic double cover embedded in its Prym variety is nonzero (after “spreading out”) and in the case $g = 5$ we use Theorem 3.4 to prove

THEOREM 1.2. *The Griffiths groups of codimensions 3 and 4 of the generic abelian variety of dimension 5 are of infinite rank.*

By another degeneration argument we obtain the following

THEOREM 1.3. *The Griffiths group of codimension i cycles which are homologically equivalent to zero modulo the cycles which are algebraically equivalent to zero in the generic Prym variety of dimension $g \geq 5$ is non-torsion for $3 \leq i \leq g-1$.*

We note that all the cycles that we obtain are in the kernel of the Abel-Jacobi map. Thus the above results also show that for the generic Jacobian of genus $g \geq 11$

the subgroup of the Griffiths group of dimension 1 which maps to zero under the Abel-Jacobi map is nontorsion.

In an appendix we show how similar methods can be used to extend Ceresa’s theorem to all positive characteristics i.e. we prove the following

THEOREM 1.4. *Let C be a generic curve of genus $g \geq 3$ over a field of arbitrary characteristic. Then the cycle $C - C^-$ is not algebraically equivalent to zero in $J(C)$.*

2. Preliminaries

2.1. In this paper all schemes will be over \mathbb{C} unless explicitly stated otherwise and by open we shall mean open in the Zariski topology.

Let $\pi : X \rightarrow S$ be an abelian scheme of relative dimension g , where S is a smooth connected algebraic variety. We recall some facts about the singular cohomology of X . Deligne [7] has shown that the Leray spectral sequence

$$E_2^{p,q} = H^p(S, R^q\pi_*\mathbb{Q}) \Rightarrow H^{p+q}(X, \mathbb{Q})$$

degenerates at E_2 and moreover there is a canonical decomposition

$$H^n(X, \mathbb{Q}) \cong \bigoplus_{p+q=n} H^p(S, R^q\pi_*\mathbb{Q}) \quad (\text{in fact } R\pi_*\mathbb{Q} \cong \bigoplus_{i=0}^{2g} R^i\pi_*\mathbb{Q}[-i]). \quad (1)$$

Here $H^p(S, R^q\pi_*\mathbb{Q})$ is identified via the s.s. with the subspace of $H^n(X, \mathbb{Q})$ on which m_* acts by the constant m^{2g-q} for all m in \mathbb{Z} . Now let \mathcal{L} be a relatively ample line bundle on X . Via the s.s. we get an element L in $H^0(S, R^2\pi_*\mathbb{Q})$ (this is the image of $c_1(\mathcal{L}) \in H^2(X, \mathbb{Q})$ in $H^0(S, R^2\pi_*\mathbb{Q})$). Then we have the Lefschetz decomposition

$$R^q\pi_*\mathbb{Q} \cong \bigoplus_{i \geq 0} L^i \cdot \mathbb{P}_{q-2i},$$

where \mathbb{P}_j is the local system on S corresponding to the j th primitive (rational) cohomology of the fibres (hence $\mathbb{P}_j = 0$ if $j > g$). This gives us a further decomposition

$$H^p(S, R^q\pi_*\mathbb{Q}) \cong \bigoplus_{i \geq 0} H^p(S, L^i \cdot \mathbb{P}_{q-2i}).$$

Thus to compute the cohomology of X it is enough to compute $H^p(S, \mathbb{P}_j)$.

We note that these decompositions are functorial under pullbacks and they are valid over any algebraically closed field k if we replace singular cohomology $H^*(X, \mathbb{Q})$ with etale cohomology $H_{\text{et}}^*(X, \mathbb{Q}_l)$ where l is prime to the characteristic of k .

2.2. For an algebraic variety we shall denote by $A^p(X)$ the *rational* (i.e. $\otimes \mathbb{Q}$) Chow group of codimension p cycles modulo rational equivalence on X and by $\text{Griff}^p(X)$ the subquotient of $A^p(X)$ consisting of cycles which are homologically equivalent to zero modulo those which are algebraically equivalent to zero. When X, S, π are as in 2.1 Beauville [2] and Deninger and Murre [9] have shown that there exists a functorial decomposition

$$A^p(X) = \bigoplus_{s=p_1}^{p_2} A_{(s)}^p(X),$$

where $p_1 = \text{Max}(p - g, 2p - 2g)$, $p_2 = \text{Min}(2p, p + d)$, $d = \dim S$ and m^* and m_* act on $A_{(s)}^p$ by multiplication by m^{2p-s} and $m^{2g-2p+s}$ respectively. This decomposition is compatible via the cycle class map with the decomposition (1). Now suppose that $S = \text{Spec } \mathbb{C}$. Then $p_1 = p - g$ and $p_2 = p$ and there is also an induced decomposition

$$\text{Griff}^p(X) = \bigoplus_{s=p-g}^p \text{Griff}_{(s)}^p(X).$$

It is easy to see that these decompositions are preserved by specialisation. The Fourier transform of Mukai-Beauville induces isomorphisms [2]

$$A_{(s)}^p(X) \rightarrow A_{(s)}^{g-p+s}(\hat{X})$$

and similarly for $\text{Griff}_{(s)}^p(X)$, where \hat{X} is the dual abelian variety of X . In our applications X will have a canonical principal polarisation which we shall use to identify X and \hat{X} .

3. Cohomology computations

3.1. VARIATIONS OF HODGE STRUCTURE ON THE SIEGEL SPACE

In this section we recall how rational representations of $\text{Sp}(2g)$ give rise to variations of Hodge structure on the Siegel space and explain the relationship with Lie algebra cohomology. For the omitted proofs we refer the reader to the paper of Zucker [21] where arbitrary Hermitian symmetric spaces are considered. See also the book of Faltings and Chai [13], where a theorem of Bernstein-Gelfand-Gelfand is used instead of the theorem of Kostant that we use.

Let $G = \text{Sp}(2g)$ as an algebraic group over \mathbb{Q} , P the maximal parabolic subgroup of G consisting of matrices of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

in G . Let T be the maximal torus of G consisting of diagonal matrices. $G(\mathbb{C})/P(\mathbb{C})$ is a compact complex manifold and can be identified with the space of maximal

isotropic (or Lagrangian) subspaces of \mathbb{C}^{2g} . The base point, i.e. the coset $P(\mathbb{C})$ corresponds to the subspace of \mathbb{C}^{2g} whose last g coordinates are 0. Consider the set $D \subset \check{D} = G(\mathbb{C})/P(\mathbb{C})$ consisting of those subspaces $V \in \check{D}$ s.t. $i(v, \bar{v}) > 0 \forall v \in V$, where $(,)$ is the standard symplectic form on \mathbb{C}^{2g} . D is an open subset of \check{D} on which $G(\mathbb{R})$ acts transitively. It is well known that D can be identified with the Siegel space H_g .

Let (V, λ) be an irreducible representation of G and let

$$h = \text{diag}(1/2, \dots, 1/2, -1/2, \dots, -1/2) \in \text{Lie } T.$$

Using h we define a decreasing filtration on V as follows:

$$F^p V = \text{sum of eigenspaces of } h \text{ on } V \text{ with eigenvalues } \geq p, p \text{ in } \frac{1}{2}\mathbb{Z}.$$

It is easy to see that for

$$\mathfrak{g} = \text{Lie } G, \mathfrak{g} \cdot F^p V \subset F^{p-1} V. \tag{2}$$

Consider the (trivial) vector bundle $\mathbb{V} = \check{D} \times V_{\mathbb{C}}$ on \check{D} . This is a homogenous $G(\mathbb{C})$ bundle in the obvious way and we can define a decreasing filtration $F^p \mathbb{V}$ by letting $F^p \mathbb{V}$ be the unique homogenous subbundle of \mathbb{V} with fibre $F^p V_{\mathbb{C}}$ at the basepoint. \mathbb{V} has a canonical flat connection ∇ and it follows from equation (2) that

$$\nabla(F^p \mathbb{V}) \subset \Omega_D^1 \otimes F^{p-1} \mathbb{V}. \tag{3}$$

It is shown in [21] that with this filtration and connection \mathbb{V} restricted to D is a polarisable V.H.S. of weight 0 except for the fact that we allow p to take half-integral values. Note that for $\lambda =$ standard representation of G we just get the V.H.S. corresponding to H^1 of the universal family of abelian varieties on D (except for a twist) and in general for $\lambda = \lambda_i, 0 \leq i \leq g$, the i th fundamental representation of G we get the V.H.S. corresponding to the i th primitive cohomology of the universal family.

It follows from (3) that we have subcomplexes of the de Rham complex of \mathbb{V}

$$0 \rightarrow \mathbb{V} \rightarrow \Omega_D^1 \otimes \mathbb{V} \rightarrow \Omega_D^2 \otimes \mathbb{V} \rightarrow \dots$$

of the form

$$0 \rightarrow F^p \mathbb{V} \rightarrow \Omega_D^1 \otimes F^{p-1} \mathbb{V} \rightarrow \Omega_D^2 \otimes F^{p-2} \mathbb{V} \rightarrow \dots$$

and hence quotient complexes

$$(C^\cdot)^p : 0 \rightarrow Gr^p \mathbb{V} \rightarrow \Omega_D^1 \otimes Gr^{p-1} \mathbb{V} \rightarrow \Omega_D^2 \otimes Gr^{p-2} \mathbb{V} \rightarrow \dots.$$

In the first and second complex the maps are just \mathbb{C} -linear but in the third one they are homogenous \mathcal{O}_D -linear maps.

3.2. LIE ALGEBRA COHOMOLOGY AND KOSTANT’S THEOREM

For V as above and for any $m \geq 0$ we will now calculate the largest p s.t. the m th cohomology sheaf of the complex $(C^\cdot)^p$, $\mathcal{H}^m((C^\cdot)^p)$ is nonzero.

Since the maps in the complex $(C^\cdot)^p$ are homogenous \mathcal{O}_D -linear maps we can restrict to any point $x \in \check{D}$ to calculate the degrees in which the cohomology sheaves vanish. We will use the basepoint x_0 of $\check{D} = G(\mathbb{C})/P(\mathbb{C})$. The tangent space to x_0 can be canonically identified with $\mathfrak{n} \cong \mathfrak{g}/\mathfrak{p}$ where \mathfrak{n} is the subalgebra of \mathfrak{g} consisting of matrices of the form

$$\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}.$$

With this identification it is easy to see that the map

$$Gr^p V \rightarrow \mathfrak{n}^* \otimes Gr^{p-1} V$$

obtained by restricting the complex to the basepoint is nothing but the map induced by the action of \mathfrak{n} on V . Identifying $Gr^p V$ with a subspace of V and taking the direct sum over all p of the complexes $((C^\cdot)^p)_{x_0}$ we get a complex

$$V \rightarrow \mathfrak{n}^* \otimes V \rightarrow \wedge^2 \mathfrak{n}^* \otimes V \rightarrow \dots \tag{4}$$

and it follows easily from the above discussion that this is the complex that computes the Lie algebra cohomology of V thought of as a representation of \mathfrak{n} (see [21] for details).

Let $\mathfrak{m} \subset \mathfrak{g}$ be the Levi subalgebra consisting of matrices of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

\mathfrak{m} acts on \mathfrak{n} by the adjoint action and also acts on V and hence acts on all the terms of the complex (4). It is known that the action commutes with the differentials and hence \mathfrak{m} also acts on the cohomology groups of the complex (4). We note that h acts on $\wedge^i \mathfrak{n}^* \otimes Gr^{p-i} V \subset \wedge^i \mathfrak{n}^* \otimes V$ by multiplication by p . Hence it suffices to know the cohomology groups of the complex (4) as \mathfrak{m} -modules. This is achieved by using the theorem of Kostant stated below.

Let $\mathfrak{t} = \text{Lie } T$. This is a Cartan subalgebra of \mathfrak{g} . Let $\Delta = \{x_1 - x_2, x_2 - x_3, \dots, x_{g-1} - x_g, 2x_g\} \subset \mathfrak{t}^*$, where x_i is the $g+i$ th diagonal entry of an element of \mathfrak{t} . Δ is a base for the root system Φ of \mathfrak{g} with respect to \mathfrak{t} . (Note: The somewhat non-standard choice of positive roots is made in order to be consistent with Kostant’s paper since we are computing the cohomology of \mathfrak{n} instead of the nilpotent radical of \mathfrak{p} .) Let Φ_+ and Φ_- be the sets of positive and negative roots respectively of \mathfrak{g} with respect to Δ . Let Φ'_+ be the subset of Φ_+ consisting of elements which are not positive roots for \mathfrak{m} w.r.t. the base $\{x_1 - x_2, \dots, x_{g-1} - x_g\}$. Let $W = W_{\mathfrak{g}}$ be the Weyl group of \mathfrak{g} and let $W' = \{w \in W \mid w(\Phi_-) \cap \Phi_+ \subset \Phi'_+\}$. W' is a set

of coset representatives of $W_m \backslash W_g$ (see [15] Prop. 5.13). Let $\delta =$ half the sum of the positive roots = sum of the fundamental dominant weights. We now state the special case of Kostant’s theorem ([15] Thm 5.14) that we will use.

THEOREM 3.1 (Kostant). *Let $V = V_\lambda$ be an irreducible representation of \mathfrak{g} with highest weight λ . Then $H^m(\mathfrak{n}, V_\lambda)$ is the direct sum of representations of \mathfrak{m} with highest weight $w(\lambda + \delta) - \delta$, the sum being over all $w \in W'$ of length m with each representation occurring with multiplicity one.*

Thus the set of eigenvalues of h on $H^m(\mathfrak{n}, V_\lambda)$ is precisely

$$\{p \mid p = (w(\lambda + \delta) - \delta)(h) \text{ for some } w \in W' \text{ of length } m\}.$$

We now describe W' explicitly : The set $\{x_1, \dots, x_g\}$ is a basis of \mathfrak{t}^* so we can describe an element of W by its action on the x_i ’s. Hence we can represent an element $w \in W$ by a g -tuple of elements of \mathfrak{t}^* . It is known that the elements of W are precisely those g -tuples (y_1, \dots, y_g) where $y_i = w(x_i) = \pm x_{\sigma(i)}$ for some $\sigma \in S_g$.

LEMMA 3.2. *W' is the set of all g -tuples as above s.t. (1) if $y_i = x_{\sigma(i)}, y_j = x_{\sigma(j)}$ and $i < j$ then $\sigma(i) < \sigma(j)$, (2) if $y_i = -x_{\sigma(i)}, y_j = -x_{\sigma(j)}$ and $i < j$ then $\sigma(i) > \sigma(j)$ and (3) if $y_i = x_{\sigma(i)}, y_j = -x_{\sigma(j)}$ then $\sigma(i) < \sigma(j)$.*

Proof. It follows from the definition that W' contains this set. Since the cardinalities of both sets is the same i.e. 2^g it follows that they must be equal.

Note that the length of such an element $w \in W'$ is $\sum(g - i + 1)$ where the sum is over all i s.t. $y_i = -x_j$ for some j (use that for any $w \in W$ $\text{length}(w) = |w(\Phi^+) \cap \Phi^-|$). Also note that $(w(\delta) - \delta)(h) = \text{length}(w)$.

Let $\lambda = \sum m_i \cdot \lambda_i, \lambda_i = x_1 + \dots + x_i, m_i \geq 0$ be a dominant weight. If $m_i = 0$ for all $i \geq g - m$ then it is clear from our description of W' that $w(\lambda) = \lambda$ for all $w \in W'$ of length m . Hence

$$(w(\lambda + \delta) - \delta)(h) = - \sum m_i \cdot i/2 + m. \tag{5}$$

Now for simplicity assume that $\lambda = \lambda_i$ for some $i, 0 \leq i \leq g$. To calculate the largest eigenvalue of h acting on $H^m(\mathfrak{n}, V_\lambda)$ we need to calculate

$$\max\{w(\lambda_i)(h) \mid w \in W', \text{length}(w) = m\}.$$

From our description it follows that the elements of W' of length m correspond to partitions of m of distinct terms, each term being $\leq g$. The maximum will be achieved for those partitions which have the most terms greater than $g - i$. It is then an easy exercise to see that this number is

$$r = r(g, m, i) \stackrel{\text{defn}}{=} \max\{q \in \mathbb{Z} \mid q(g - i) + q(q + 1)/2 \leq m\}. \tag{6}$$

Thus the required maximum is equal to

$$r + m - i/2. \quad (7)$$

3.3. VANISHING RESULTS

We now show how the results of Section 3.2 lead to vanishing results for the cohomology of generic families of abelian varieties.

Let S be a smooth connected algebraic variety over \mathbb{C} and let $(\mathbb{V}, F^\cdot, \nabla)$ be a polarisable V.H.S. on S . We denote by \mathcal{V} the associated local system. It is well known that the de Rham complex :

$$(C^\cdot): 0 \rightarrow \mathbb{V} \rightarrow \Omega_S^1 \otimes \mathbb{V} \rightarrow \Omega_S^2 \otimes \mathbb{V} \rightarrow \dots$$

is a resolution of \mathcal{V} and hence $H^m(S, \mathcal{V}) = \mathbb{H}^m(S, C^\cdot)$. As before we have a filtration of C^\cdot by subcomplexes :

$$F^p(C^\cdot): 0 \rightarrow F^p \mathbb{V} \rightarrow \Omega_S^1 \otimes F^{p-1} \mathbb{V} \rightarrow \dots$$

and we also have the associated graded complexes where now the maps are \mathcal{O}_S linear. Suppose the V.H.S. is of weight i . Then it is a theorem of Saito [20] that $H^m(S, \mathcal{V})$ has a canonical M.H.S. of weights $\geq m + i$. We will use this to find conditions under which $H^m(S, \mathcal{V}) = 0$.

The map $F^p(C^\cdot) \rightarrow C^\cdot$ induces a map $\mathbb{H}^\cdot(S, F^p(C^\cdot)) \rightarrow \mathbb{H}^\cdot(S, C^\cdot)$. We denote its image by $G^p \mathbb{H}^\cdot(S, C^\cdot)$. It follows from the construction of the M.H.S. on $H^m(S, \mathcal{V})$ by Saito [20] that $F^p \mathbb{H}^m(S, C^\cdot) \subset G^p \mathbb{H}^m(S, C^\cdot)$ where by $F^p \mathbb{H}^m(S, C^\cdot)$ we mean the Hodge filtration of the canonical M.H.S. on $\mathbb{H}^m(S, C^\cdot)$. Hence $\mathbb{H}^m(S, F^p(C^\cdot)) = 0$ implies that $F^p \mathbb{H}^m(S, C^\cdot) = 0$. Since $\mathbb{H}^m(S, C^\cdot)$ has a M.H.S. of weights $\geq i + m$ we see that

- (a) $F^p \mathbb{H}^m(S, C^\cdot) = 0$ for all $p \geq (i + m)/2$ implies that $\mathbb{H}^m(S, C^\cdot) = 0$.
- (b) $F^p \mathbb{H}^m(S, C^\cdot) = 0$ for all $p > (i + m)/2$ implies that $\mathbb{H}^m(S, C^\cdot)$ is pure of weight $i + m$ and type $((i + m)/2, (i + m)/2)$.

$F^p(C^\cdot)$ is filtered by $F^{p+1}(C^\cdot), F^{p+2}(C^\cdot), \dots$. An easy spectral sequence argument (see [18] Sect. 7.5) shows that if $\mathcal{H}^j(Gr^d(C^\cdot)) = 0$ for all $j \leq m$ and $d \geq p$ then $\mathbb{H}^m(S, F^p(C^\cdot)) = 0$. Thus with \mathbb{V} as above we conclude

- (a) $\mathcal{H}^j(Gr^d(C^\cdot)) = 0$ for all $j \leq m$ and $d \geq (i + m)/2 \Rightarrow H^m(S, \mathcal{V}) = 0$.
- (b) $\mathcal{H}^j(Gr^d(C^\cdot)) = 0$ for all $j \leq m$ and $d > (i + m)/2 \Rightarrow H^m(S, \mathcal{V})$ is pure of type $((i + m)/2, (i + m)/2)$.

Now let $\pi : \mathcal{A} \rightarrow S$ be a polarised abelian scheme. Let $u : \tilde{S} \rightarrow S$ be the universal cover of S . Then $u^*(R^i \pi_* \mathbb{Z})$ is a trivial local system on \tilde{S} so we can choose a symplectic basis which gives rise to a (complex analytic) classifying map $\tilde{S} \rightarrow D$. We assume that this map is a local homeomorphism. Let \mathbb{P}_i be the

local system on S corresponding to the i th primitive cohomology of the fibres of $\pi : \mathcal{A} \rightarrow S$ (it is a direct summand of $R^i\pi_*\mathbb{Q}$) and let \mathbb{W} be any other polarisable V.H.S. on S . We assume that \mathbb{W} is of positive weight k and that $F^0\mathbb{W} = \mathbb{W}$. We give below sufficient conditions for $H^m(S, \mathbb{P}_i \otimes \mathbb{W})$ to vanish or to have a pure H.S. of type $((m + i + k)/2, (m + i + k)/2)$.

Let $\mathbb{V} = \mathbb{P}_i \otimes \mathbb{W}$, then \mathbb{V} is a polarisable V.H.S. of weight $i + k$. Thus with notation as above we need :

- (a) $\mathcal{H}^j(Gr^d(C^\cdot)) = 0$ for all $j \leq m$ and $d \geq (i + k + m)/2$, or
- (b) $\mathcal{H}^j(Gr^d(C^\cdot)) = 0$ for all $j \leq m$ and $d > (i + k + m)/2$.

We now use a filtration of $Gr^d(C^\cdot)$ to get a condition that depends only on k .

$$F^p\mathbb{V} = F^p(\mathbb{P}_i \otimes \mathbb{W}) = F^p\mathbb{P}_i \otimes F^0\mathbb{W} + \dots + F^0\mathbb{P}_i \otimes F^p\mathbb{W}$$

and so the images of the complexes

$$0 \rightarrow (F^p\mathbb{P}_i \otimes F^0\mathbb{W} + \dots + F^{p-j}\mathbb{P}_i \otimes F^j\mathbb{W}) \rightarrow (F^{p-1}\mathbb{P}_i \otimes F^0\mathbb{W} + \dots + F^{p-j-1}\mathbb{P}_i \otimes F^j\mathbb{W}) \otimes \Omega_S^1 \dots$$

in $Gr^p(C^\cdot)$ are subcomplexes and hence we get an increasing filtration of $Gr^p(C^\cdot)$. Thus to show that $\mathcal{H}^t(Gr^p(C^\cdot)) = 0$ for all $t \leq m$ it is enough to check that $\mathcal{H}^t(Gr_j(Gr^p(C^\cdot))) = 0$ for $0 \leq j \leq k$. $Gr_j(Gr^p(C^\cdot))$ is the complex

$$0 \rightarrow (F^{p-j}\mathbb{P}_i \otimes F^j\mathbb{W}) \rightarrow (F^{p-j-i}\mathbb{P}_i \otimes F^j\mathbb{W}) \otimes \Omega_S^1 \rightarrow \dots$$

Thus we see that $\mathcal{H}^j(Gr^p(\tilde{C}^\cdot)) = 0$ for all $j \leq m$ and $p \geq d - k \Rightarrow \mathcal{H}^j(Gr^p(C^\cdot)) = 0$ for all $j \leq m$ and $p \geq d$ (here \tilde{C}^\cdot is the de Rham complex associated to \mathbb{P}_i). We have thus obtained the following

PROPOSITION 3.3. (a) $\mathcal{H}^j(Gr^p(\tilde{C}^\cdot)) = 0$ for all $j \leq m$ and $p \geq (i + m - k)/2 \Rightarrow H^m(S, \mathbb{P}_i \otimes \mathbb{W}) = 0$.

(b) $\mathcal{H}^j(Gr^p(\tilde{C}^\cdot)) = 0$ for all $j \leq m$ and $p > (i + m - k)/2 \Rightarrow H^m(S, \mathbb{P}_i \otimes \mathbb{W})$ has a pure H.S. of type $((i + k + m)/2, (i + k + m)/2)$.

We now use the results of 3.2 to see when this condition is satisfied. To check that $\mathcal{H}^j(Gr^p(\tilde{C}^\cdot)) = 0$ is a local (in the analytic topology) question so it is enough to do it on \tilde{S} . We have the classifying map $\tilde{S} \rightarrow D$ and the V.H.S. \mathbb{P}_i on \tilde{S} is just the pullback of the V.H.S. \mathbb{V}_{λ_i} on D associated to the representation of G with dominant weight λ_i . We are thus reduced to checking exactness on D which was the result of 3.2 except for a shift in the filtration. There we considered \mathbb{V}_{λ_i} as a V.H.S. of weight 0 and we saw that the largest p s.t. $\mathcal{H}^j(Gr^p(\tilde{C}^\cdot)) \neq 0$ was $r + j - i/2$ (cf. (6,7)). Since \mathbb{P}_i is of weight i we must shift the filtration by $i/2$. Thus the largest p s.t. $\mathcal{H}^j(Gr^p(\tilde{C}^\cdot)) \neq 0$ is $r + j$ and so we have the main result of this section.

THEOREM 3.4. *Let \mathbb{W} be a polarisable V.H.S. of weight k with $F^0\mathbb{W} = \mathbb{W}$. Then with $r = r(g, m, i)$ as in (3.7) we have*

- (a) $H^m(S, \mathbb{P}_i \otimes \mathbb{W}) = 0$ if $r + (m + k)/2 < i/2$
- (b) $H^m(S, \mathbb{P}_i \otimes \mathbb{W})$ is pure of type $((m+i+k)/2, (m+i+k)/2)$ if $r + (m+k)/2 = i/2$.

We now list several special cases of the formula :

- (1) Suppose \mathbb{W} is the trivial V.H.S. (i.e. \mathbb{Z}_S) so $\mathbb{P}_i \otimes \mathbb{W} \cong \mathbb{P}_i$ and $k = 0$. Then $m < i < g - m \Rightarrow r(g, m, i) = 0$ and hence $H^m(S, \mathbb{P}_i) = 0$.
- (2) If g is even $i = g/2 + 2$, $m = g/2$ then $r = 1$ and we see that $H^m(S, \mathbb{P}_i)$ is pure of type $((i + m)/2, (i + m)/2)$.
- (3) If g is odd $i = (g + 3)/2$, $m = (g - 1)/2$ then $r = 1$ and again $H^m(S, \mathbb{P}_i)$ is pure of type $((i + m)/2, (i + m)/2)$.
- (4) If g is odd $i = (g + 3)/2$, $m = (g - 3)/2$ then $r = 0$. Hence if \mathbb{W} is a V.H.S. of weight 1 then $H^m(S, \mathbb{P}_i \otimes \mathbb{W}) = 0$.

The above special cases along with the lemma below give the motivation for Conjecture 1.1 of the introduction.

LEMMA 3.5. *Let α be a codimension d algebraic cycle in the generic abelian variety s.t. (after “spreading out”) the component of its cohomology class in $H^m(S, \mathbb{P}_i)$, for some $i + m = 2d$, $i > m$ remains nonzero when restricted to all open subsets of S . Then α is not in the subring of the Chow ring generated by divisors.*

Proof. This follows immediately by considering the cohomology classes of all the cycles involved plus the fact that $H^0(S, \mathbb{P}_2) = 0$.

REMARK 3.6. We have restricted ourselves to fundamental dominant weights since we have not been able to obtain simple conditions for the theorem to hold for all weights except in the case of small values of m (see [18] for the case $m = 1$). However the methods used easily show that for $\lambda = \sum_{i=1}^g m_i \cdot \lambda_i$, $H^m(S, \mathbb{P}_\lambda) = 0$ if $m < \sum_{i=1}^g m_i \cdot i$ and $m_i = 0$ for all $i \geq g - m$ (cf. 5 of Section 3.2).

One can also use Lemma 4.1 to show that in general $H^i(S, \mathbb{P}_i) \neq 0$ for $1 \leq i \leq g$ by computing the cohomology class of an explicit cycle (see [12] for details).

4. Cycles in Prym varieties

In this section we will show that Conjecture 1.1, (2) is true for $g = 4, 5$ (the cases $g = 1, 2$ being trivial and the case $g = 3$ being (essentially) the result of Ceresa. See also the appendix).

4.1. We begin with an elementary

LEMMA 4.1. *Let X, S be smooth algebraic varieties and $\pi : X \rightarrow S$ be a smooth proper morphism. Let $a \in H^m(X, \mathbb{Q})$. Suppose there exists a subvariety T of S s.t. for all non-empty open subsets U of T , $i^*(a) \neq 0$, where $i : \pi^{-1}(U) \hookrightarrow X$ is the inclusion. Then for all non-empty open subsets V of S , $j^*(a) \neq 0$, where $j : \pi^{-1}(V) \hookrightarrow X$ is the inclusion.*

Proof. By induction it is clear that we may assume that T is an irreducible divisor which we may also assume to be smooth (by localising). Also it is clear that it is enough to prove the result for a (possibly) smaller open subset of V . Let $S' = S \setminus S \setminus (V \cup T)$. S' is a non-empty open subset of S , $U = S' \cap T$ is a non-empty open subset of T and $V' = S' \setminus T$ is a non-empty open subset of V . By further localisation if necessary we may assume that the normal bundle of U in S' is trivial. We have the exact Gysin sequence :

$$H^{m-2}(\pi^{-1}(U)) \xrightarrow{i_*} H^m(\pi^{-1}(S')) \xrightarrow{j^*} H^m(\pi^{-1}(V')).$$

The composition $i^* \circ i_* : H^{m-2}(\pi^{-1}(U)) \rightarrow H^m(\pi^{-1}(U))$ is cup product with the first Chern class of the normal bundle of $\pi^{-1}(U)$ in $\pi^{-1}(V')$ (see for example [10] Ch. 8) which is trivial. Hence $i^* \circ i_* = 0$ which implies that i^* factors through the image of $H^m(\pi^{-1}(S'))$ in $H^m(\pi^{-1}(V'))$. Since $i^*(a) \neq 0$ it follows that $j^*(a) \neq 0$ also.

We shall apply this lemma when $\pi : X \rightarrow S$ is a polarised abelian scheme and a is the cohomology class of some algebraic cycle α . As we have seen a has various components under the decomposition of $H^*(X, \mathbb{Q})$ in Section 2. Since these decompositions as well as the cycle class maps are functorial under pullbacks [14], it follows from the lemma that to show j^* of a certain component of a is nonzero it suffices to show that the corresponding component of the cohomology class of $i^*(\alpha)$ is nonzero. Again we note that the above lemma and discussion are valid over any algebraically closed field k after replacing singular cohomology with etale cohomology ([SGA5]).

4.2. CONSTRUCTION OF THE DEGENERATION

We now construct the abelian scheme to which we shall apply Lemma 4.1, based on the well known construction of Prym varieties. We follow the papers of Beauville [1] and Donagi-Smith [11].

Let \mathcal{M}_g^t , $g \geq 5$, be the open subset of $\overline{\mathcal{M}}_g$, the moduli space of stable genus g curves, whose geometric points correspond to curves, whose graphs are trees and let $S_0 = \mathcal{M}_g^{t,(n)}$ be the moduli space of such curves with level n -structure where $n \geq 3$ is some fixed integer. It is known that $\mathcal{M}_g^{t,(n)}$ is smooth over \mathbb{C} and there exists a universal family of curves $\Gamma^{t,(n)} \rightarrow \mathcal{M}_g^{t,(n)}$ ([19], [16]). Let C, C' be curves of genus $g-3, 2$ resp. and assume that there are no Hodge classes in $H^1(C, \mathbb{Q}) \otimes H^1(C', \mathbb{Q})$. Let E an elliptic curve (with a fixed origin 0). Let $q \in C'$,

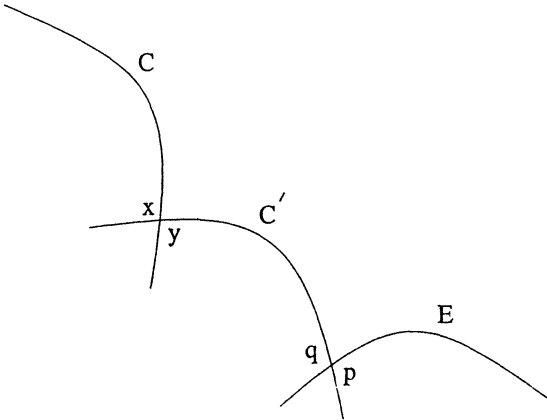


Fig. 1.

$p \in E$ be fixed and let $x \in C, y \in C'$. Consider the curve D which is the union of C, C' and E , glued together as in Figure 1.

This is a stable curve of genus g and as we let x vary in C and y vary in $C' - \{q\}$ we obtain a family of stable curves of genus $g, \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}' - \{q\}$. By choosing a level n -structure we get a map $h : (\mathcal{C} \times \mathcal{C}' - \{q\}) \rightarrow \mathcal{M}_g^{t,(n)}$ s.t. $h^*(\Gamma^{t,(n)}) = \mathcal{C}$ and this map is an embedding.

Let $T_0 = h(\mathcal{C} \times \mathcal{C}' - \{q\})$. Then there is a constant section of $\Gamma^{t,(n)}|_{T_0} \rightarrow T_0$ which is 0 on each fibre. Let $f_1 : S_1 \rightarrow S_0$ be an etale map such that there exists a section σ_1 of $\Gamma_1 = S_1 \times_{S_0} \Gamma^{t,(n)} \rightarrow S_1$ extending the section on (an open subset of) T_0 and let T_1 be the corresponding component of $f_1^{-1}(T_0)$. Consider $Y_1 = \text{Pic}(\Gamma_1/S_1)$ which is the algebraic space, smooth and locally of finite type over S_1 , representing the relative Picard functor. We refer to the book of Bosch et al. ([4] Chapters 8, 9) for the facts about the relative Picard functor that we shall use. Since $\Gamma_1 \rightarrow S_1$ has a section Y_1 also represents the functor of families of line bundles rigidified along the section.

Let $S_2 = \text{Ker}(2 \cdot : Y_1 \rightarrow Y_1) \setminus (\text{zero-section}) \hookrightarrow Y_1$ (this is a scheme) and let $f_2 : S_2 \rightarrow S_1$ the induced map. Then there exists a line bundle \mathcal{M} on Γ_2 , the pullback of Γ_1 to S_2 , of order two which is nontrivial on each fibre and hence gives rise to a family of double covers $\tilde{\Gamma}_2 \rightarrow \Gamma_2 \rightarrow S_2$. Let T_3 be a component of $T_2 = f_2^{-1}(T_1)$ such that the family $\tilde{\Gamma}_2|_{T_3} \rightarrow T_3$ consists of curves as in Figure 2, where \tilde{E} is an unramified double cover of E and b is a nonzero 2-torsion point of \tilde{E} s.t. $\tilde{E}/\{0, b\} \cong E$ and \tilde{p} is one of the points lying over p . Clearly $f_2|_{T_3} : T_3 \rightarrow T_1$ is an isomorphism. Let S_3 be a connected open subset of S_2 which contains T_3 and such that all the fibres of the family $\tilde{\Gamma}_3 = \tilde{\Gamma}_2|_{S_3} \rightarrow S_3$ are treelike curves. Finally let $f : S \rightarrow S_3$ be an etale map such that there is a section σ of $\tilde{\Gamma} = S \times_{S_3} \tilde{\Gamma}_3 \rightarrow S$ which extends the constant section corresponding to 0 over an open subset T of T_3 .

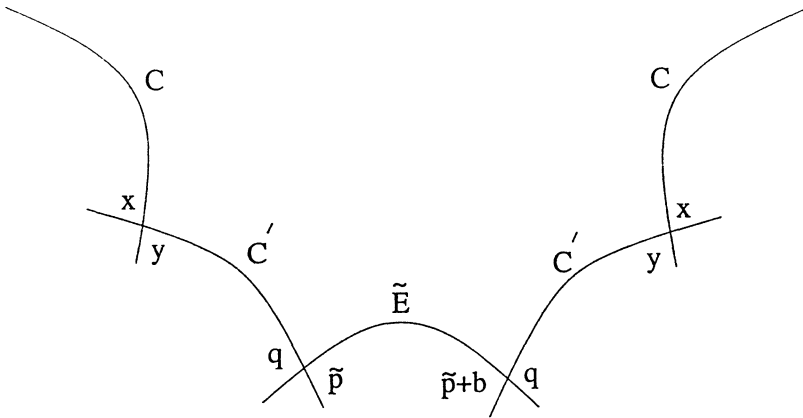


Fig. 2.

Now consider $\text{Pic}(\tilde{\Gamma}/S)$ as above. Let $\text{Pic}^0(\tilde{\Gamma}/S)$ be the open subspace of $\text{Pic}(\tilde{\Gamma}/S)$ which represents the functor of line bundles which are of degree zero on each component of each fibre (this is in fact an abelian scheme over S) and $\text{Pic}^{(i)}(\tilde{\Gamma}/S)$, $i \in \mathbb{Z}$, the open subspace of $\text{Pic}(\tilde{\Gamma}/S)$ corresponding to line bundles of total degree i on each fibre. Let $\tilde{\Gamma}_0$ be the open subset of $\tilde{\Gamma}$ at which the map $\tilde{\pi} : \tilde{\Gamma} \rightarrow S$ is smooth.

There is a natural morphism $\gamma_1 : \tilde{\Gamma}_0 \rightarrow \text{Pic}^{(1)}(\tilde{\Gamma}/S)$ and using the section σ we can translate to get a morphism $\gamma_0 : \tilde{\Gamma}_0 \rightarrow \text{Pic}^{(0)}(\tilde{\Gamma}/S)$. Let $H \subset \text{Pic}^{(0)}(\tilde{\Gamma}/S)$ be the “closure” of the identity in the generic fibre. This is a closed subgroupspace which is etale over S . Let $P(\tilde{\Gamma}/S)$ be the quotient of $\text{Pic}^{(0)}(\tilde{\Gamma}/S)$ by H . This is an abelian scheme over S and the composite map $\text{Pic}^0(\tilde{\Gamma}/S) \rightarrow \text{Pic}^{(0)}(\tilde{\Gamma}/S) \rightarrow P(\tilde{\Gamma}/S)$ is an isomorphism (since it is so on the generic fibre). Hence we get a morphism $\tilde{\Gamma}_0 \rightarrow \text{Pic}^0(\tilde{\Gamma}/S)$ and this extends to give a morphism $\gamma : \tilde{\Gamma} \rightarrow \text{Pic}^0(\tilde{\Gamma}/S)$ since $\tilde{\Gamma}$ is normal.

The involution on $\tilde{\Gamma}$ induces an involution ι on $\text{Pic}^0(\tilde{\Gamma}/S)$. Let

$$X = \text{Prym}(\tilde{\Gamma}/\Gamma) \stackrel{\text{dfn}}{=} \text{Im}(\text{Id} - \iota : \text{Pic}^0(\tilde{\Gamma}/S) \rightarrow \text{Pic}^0(\tilde{\Gamma}/S)).$$

It is proved in [1] that $\pi : X \rightarrow S$ is a principally polarised abelian scheme of relative dimension $g - 1$. Let $Y = ((\text{Id} - \iota) \circ \gamma)(\tilde{\Gamma}) \hookrightarrow X$. This a subvariety of X of codimension $g - 2$.

4.3. COMPUTATION OF THE COHOMOLOGY CLASS

We now describe $Y|_T \hookrightarrow X|_T$. First we describe the situation in the fibre over a point $(x, y) \in T$. $\tilde{\Gamma}|_{(x,y)}$ is as in Fig. 2 hence $\text{Pic}^0(\tilde{\Gamma}|_{(x,y)}) \cong J(C) \times J(C') \times \tilde{E} \times J(C') \times J(C)$. Following through the definitions it is easy to see how $\tilde{\Gamma}|_{(x,y)}$ is embedded in $\text{Pic}^0(\tilde{\Gamma}|_{(x,y)})$. Below we describe the images of each of the five

components of $\tilde{\Gamma}|_{(x,y)}$. For a point $x \in C$, let $i_x : C \hookrightarrow J(C)$ denote the usual embedding where $x \mapsto 0$ and similarly for C' .

- (1) C embedded as $i_x(C) \times [y - q] \times -\tilde{p} \times 0 \times 0$.
- (2) C' embedded as $0 \times i_q(C') \times -\tilde{p} \times 0 \times 0$.
- (3) \tilde{E} embedded as $0 \times 0 \times \tilde{E} \times 0 \times 0$.
- (4) C' embedded as $0 \times 0 \times b - \tilde{p} \times i_q(C') \times 0$.
- (5) C embedded as $0 \times 0 \times b - \tilde{p} \times [y - q] \times i_x(C)$.

The involution ι acts on $\text{Pic}^0(\tilde{\Gamma}|_{(x,y)}) \cong J(C) \times J(C') \times \tilde{E} \times J(C') \times J(C)$ by switching the factors symmetrically about \tilde{E} and fixes \tilde{E} pointwise. Thus $X|_{(x,y)}$ can be identified with $J(C) \times J(C')$ by projection onto the first two factors and the polarisation is the product of the natural principal polarisations on each factor (see [1]). We now describe $Y|_{(x,y)} \hookrightarrow X|_{(x,y)}$. Applying $\text{Id} - \iota$ to each of the five components described above and then projecting onto the first two factors we see that the components of $Y|_{(x,y)}$ are as follows

- (1) C embedded in $J(C) \times J(C')$ as $i_x(C) \times [y - q]$
- (2) C' embedded in $J(C) \times J(C')$ as $0 \times i_q(C')$
- (3) \tilde{E} maps to a point and hence is zero as a cycle.
- (4) C' embedded in $J(C) \times J(C')$ as $0 \times i_q(C')^-$
- (5) C embedded in $J(C) \times J(C')$ as $i_x(C)^- \times [q - y]$

The above description makes sense over all points of $C \times C'$. We now describe the cycle in $C \times C' \times J(C) \times J(C')$ and calculate (part of) its cohomology class. Note that in this case the decomposition of the cohomology of the total space given by the Kunneth formula for $(C \times C') \times (J(C) \times J(C'))$ is the same as that obtained by the action of m_* as in Section 2.1. We want to show that the component of the cohomology class of the cycle in the ‘‘primitive’’ part of $H^1(C) \otimes H^1(C') \otimes H^{2g-6}(J(C) \times J(C'))$ is nonzero.

- (1) $C \times C' \times C$ is embedded in $C \times C' \times J(C) \times J(C')$ by $(x_1, y, x_2) \mapsto (x_1, y, [x_2 - x_1], [y - q])$ and so this has (only) a nonzero component in $H^1(C) \otimes H^1(C') \otimes H^{2g-9}(J(C)) \otimes H^3(J(C'))$
- (2) $C \times C' \times C'$ is embedded in $C \times C' \times J(C) \times J(C')$ by $(x, y_1, y_2) \mapsto (x, y_1, 0, [y_2 - q])$ so its component in $H^1(C) \otimes H^1(C') \otimes H^{2g-6}(J(C) \times J(C'))$ is clearly zero.
- (4) As in (2) the component in $H^1(C) \otimes H^1(C') \otimes H^{2g-6}(J(C) \times J(C'))$ is zero.
- (5) The component in $H^1(C) \otimes H^1(C') \otimes H^{2g-6}(J(C) \times J(C'))$ is the same as that in (1) since -1_* acts by the identity on $H^{2g-6}(J(C) \times J(C'))$.

Thus we see that the only nonzero contribution in $H^1(C) \otimes H^1(C') \otimes H^{2g-6}(J(C) \times J(C'))$ comes from (1) and (5). We claim that this class is ‘‘primitive’’ in the sense that it is not in the image of $H^1(C) \otimes H^1(C') \otimes H^4(J(C) \times J(C'))$ under cup product with L^{g-5} , where L is the component of the polarisation in

$H^0(C \times C') \otimes H^2(J(C) \times J(C'))$). Using the fact mentioned earlier that the polarisation is just the product of the polarisations on the factors this is an easy exercise using the Kunneth formula which we leave for the reader (recall that $g \geq 5$). Finally we note that this class remains nonzero when restricted to nonempty open subsets of $C \times C'$ because there are no Hodge classes in $H^1(C) \otimes H^1(C')$. Applying Lemma 4.1 we see that we have proved the following

PROPOSITION 4.2. *For $g \geq 5$ the component of the cohomology class of Y in the “primitive” part of $H^2(S, R^{2g-4}\pi_*\mathbb{Q})$ remains nonzero when restricted to all nonempty open subsets of S .*

It is a classical fact that the map $S_g = S \rightarrow \mathcal{A}_{g-1}$ is dominant for $g = 5, 6$ (see [1] for a proof). Hence we obtain the following

PROPOSITION 4.3. *For $g = 4, 5$ there exists a smooth algebraic variety S , a principally polarised abelian scheme $\pi : X \rightarrow S$ of relative dimension g , an algebraic cycle Y in X of codimension $g - 1$ s.t. the classifying map $S \rightarrow \mathcal{A}_g$ is smooth and the component of the cohomology class of Y in the “primitive” part of $H^2(S, R^{2g-4}\pi_*\mathbb{Q})$ remains nonzero when restricted to all nonempty open subsets of S .*

This proposition along with Theorem 4.4 verifies (2) of Conjecture 1.1 for $g \leq 5$ since the case $g = 3$ was already known and the result is trivial for $g = 1, 2$.

4.4. THE GRIFFITHS GROUP

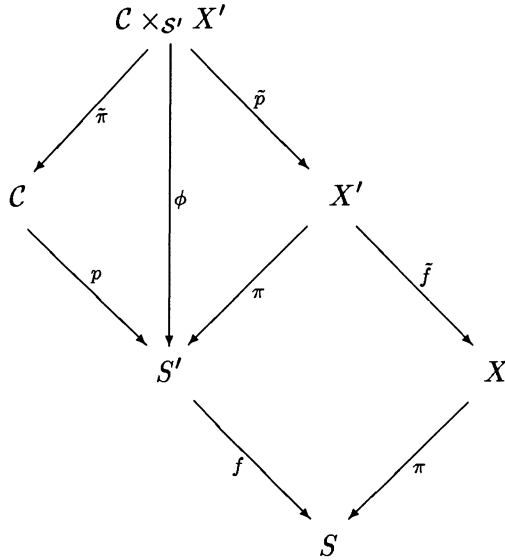
In this section we prove slightly more precise versions of the theorems stated in the introduction.

THEOREM 4.4. *$\text{Griff}_{(2)}^i(A)$ is infinite dimensional for the generic abelian variety A of dimension 5, ($i = 3, 4$).*

Proof. It is enough to prove the result for codimension 4 cycles since the result for codimension 3 will then follow from the Fourier transform (see Section 2.2.).

Let S, X, Y be as above. Let $\alpha = [Y]$ be the class of Y in $A^4(X)$. Let β be the component of α in $A_{(2)}^4(X)$ (hence β restricted to each fibre is homologically trivial). By the discussion in section 2 and Proposition 4.3 it follows that the component of the cohomology class of β in the “primitive” part of $H^2(S, R^6\pi_*\mathbb{Q})$ is nonzero. We will show that β restricted to the fibre over a generic geometric point of S is not algebraically equivalent to zero.

Suppose it is. Then by standard methods it follows that there exists a smooth algebraic variety S' an etale map $f : S' \rightarrow S$, a connected algebraic variety C and a smooth proper map of relative dimension one $p : C \rightarrow S'$, algebraic cycles $\gamma \in A^1(C), \delta \in A^4(C \times_{S'} X')$ s.t. $\tilde{f}^*(\beta) = \tilde{p}_*(\tilde{\pi}^*(\gamma) \cdot \delta)$ where $X', \tilde{p}, \tilde{\pi}, \tilde{f}$ are as in the diagram overleaf



We will get a contradiction by calculating the cohomology classes of both sides. The class of a relatively ample line bundle on C gives a decomposition $Rp_*\mathbb{Q} \cong \bigoplus_{i=0}^2 R^i p_*\mathbb{Q}[-i]$ and as in Section 2, m_* gives a decomposition $R\pi_*\mathbb{Q} \cong \bigoplus_{i=0}^{10} R^i \pi_*\mathbb{Q}[-i]$ in the derived category. The Kunnetth formula gives an isomorphism $R\phi_*\mathbb{Q} \cong Rp_*\mathbb{Q} \otimes R\pi_*\mathbb{Q}$ and we also have the canonical trace isomorphism $\text{tr}: R^2 p_*\mathbb{Q} \rightarrow \mathbb{Q}_{S'}$. The pushforward map $H^*(C \times_{S'} X', \mathbb{Q}) \xrightarrow{\tilde{p}_*} \mathcal{H}^{*-e}(X', \mathbb{Q})$ is induced by the map

$$R\phi_*\mathbb{Q} \cong Rp_*\mathbb{Q} \otimes R\pi_*\mathbb{Q} \xrightarrow{\text{tr} \otimes \text{Id}} \mathbb{Q}_{S'} \otimes R\pi_*\mathbb{Q} \cong R\pi_*\mathbb{Q}.$$

Since γ is of degree zero in the fibres the component of its cohomology class in $H^0(S', R^2 p_*\mathbb{Q})$ is zero. Since $H^0(S', \mathbb{P}_4) = 0 = H^1(S', R^1 p_*\mathbb{Q} \otimes \mathbb{P}_4)$ by Theorem 8, it follows that the component of the cohomology class of $\tilde{\pi}^*(\gamma) \cdot \delta$ in $H^2(S', R^2 p_*\mathbb{Q} \otimes L \cdot \mathbb{P}_4)$ is zero. This implies that the component of the cohomology class of $\tilde{p}_*(\tilde{\pi}^*(\gamma) \cdot \delta)$ in $H^2(S', L \cdot \mathbb{P}_4)$ is zero. This is a contradiction.

By looking at all abelian varieties isogenous to a generic abelian variety and the pushforward of the cycles by the isogeny we obtain infinitely many cycles that are nonzero mod algebraic equivalence. It follows from a result of A. Borel [3] that for S a quotient of the Siegel space H_5 by a torsion free arithmetic subgroup of $\text{Sp}(10, \mathbb{Q})$, $H^2(S, \mathbb{P}_4) = 0$. This shows that the cycle that we have constructed cannot be defined over the moduli space of abelian varieties with any level structure. Then an argument due to Nori (see [17] for details) shows that infinitely many of these cycles are linearly independent and hence for the generic 5-dimensional abelian variety the Griffiths group of codimension 3 and 4 are of infinite rank.

By a degeneration argument similar to that in Ceresa's paper [5] we obtain the following two consequences:

THEOREM 4.5. $\text{Griff}_{(2)}^i(P) \neq 0$ for the generic Prym variety P of dimension $g \geq 5$, $3 \leq i \leq g - 1$.

Proof. We prove by induction that $\text{Griff}_{(2)}^i(P) \neq 0$ for $4 \leq i \leq g - 1$ (the case $i = 3$ will then follow by Fourier transform) by showing that for the generic Prym variety $P = \text{Prym}(\tilde{C}/C)$, the component in $\text{Griff}_{(2)}^i(P)$ of the cycle class of the image of $W_{g-i}(\tilde{C})$ in P is nonzero.

The case $g = 5$, $i = 4$ was proved in Theorem 4.4. Now assume that the theorem is proved for the generic Prym of dimension $g \geq 5$. We will prove it for dimension $g + 1$ by degenerating a generic Prym to the Prym variety of a certain reducible curve. Let D be a generic curve of genus g and let \tilde{D} be an unramified double cover of D . Let E be any elliptic curve. Consider the (reducible) curve B which has two irreducible components D and E meeting transversally at $p \in D$ and $0 \in E$. Let p_1 and p_2 be the two points of \tilde{D} lying over p and let \tilde{B} be the double cover of B with three irreducible components: \tilde{D} and two copies of E meeting \tilde{D} transversally in p_1 and p_2 . Then $\text{Prym}(\tilde{B}/B) \cong \text{Prym}(\tilde{D}/D) \times E$. Consider $W_{g+1-i}(\tilde{B}) \hookrightarrow \text{Pic}^0(\tilde{B})$, where by $W_r(\tilde{B})$ we mean as usual the image of $\text{Sym}^r \tilde{B}$ in $\text{Pic}^0(\tilde{B})$. It follows from the inductive hypothesis by an easy calculation that the component in $\text{Griff}_{(2)}^i(\text{Prym}(\tilde{B}/B))$ of the cycle class of image of $W_{g+1-i}(\tilde{B})$ in $\text{Prym}(\tilde{B}/B)$ is nonzero. The theorem then follows by specializing the generic Prym $P = \text{Prym}(\tilde{C}/C)$ to $\text{Prym}(\tilde{B}/B)$ and noting that the appropriate cycles also specialize.

COROLLARY 4.6. $\text{Griff}_{(2)}^{g-1}(J) \neq 0$ for the generic Jacobian J of dimension $g \geq 11$

Proof. It follows from Theorem 4.4 that if \tilde{C} is an unramified double cover of a generic curve C of genus six, then the component of $[\tilde{C}]$ in $\text{Griff}_{(2)}^{10}(\text{Jac}(\tilde{C}))$ is nonzero (since its image in $\text{Griff}_{(2)}^4 \text{Prym}(\tilde{C}/C)$ is nonzero). The corollary follows by specialising a generic curve of genus 11 to one as above and then copying the argument of Ceresa for higher genus.

This answers, in part, a question raised in a paper of Colombo and van Geemen ([6], Introduction).

Appendix

A. Ceresa's theorem in positive characteristics

We sketch how the methods used in this paper can be used to prove the following

THEOREM A.1. *Let C be a generic curve of genus $g \geq 3$ over a field of arbitrary characteristic. Then the cycle $C - C^-$ is not algebraically equivalent to zero in $J(C)$*

Proof. The argument is a modification of the argument of Ceresa [5] in characteristic zero. Let $k = \overline{\mathbb{F}}_p$ and let $l \neq p$ be a prime. As in [5] it is enough to prove the result for $g = 3$.

Let C be a curve of genus 2 and E an elliptic curve over k . Let $q \in C$ and let D be the stable curve of genus 3 obtained by gluing C and E at q . As we vary $q \in C$ we get a family of stable curves of genus 3 parametrised by C . By choosing a level n -structure we get a map $C \rightarrow \mathcal{M}_3^{t,(n)}$ which is an embedding if $n \geq 3$. By fixing a point $p \in C$ we get a section of the universal family $\Gamma_3^{t,(n)}$ over $C - \{P\}$ and we can find an étale map $f: S \rightarrow M_3^{t,(n)}$ s.t. there exists a section of $\Gamma = \Gamma_3^{t,(n)} \times_{M_3^{t,(n)}} S \rightarrow S$ extending the section over $C - \{p\}$. As in Section 4.2 we then obtain an embedding of Γ in $X = \text{Pic}^0(\Gamma/S)$ whose image we call Y . An elementary computation using the Kunneth formula shows that the component of the cohomology class of Y in the primitive part of $H_{\text{ét}}^1(C, R^3\pi_*\mathbb{Q}_l|_C) \neq 0$ and hence by Lemma 4.1 it follows that the component of the cohomology class of Y in $H_{\text{ét}}^1(S, R^3\pi_*\mathbb{Q}_l) \neq 0$ when restricted to any nonempty open subset of S . Let $\alpha = [Y] \in A^2(X)$ and let $\beta = \alpha - [-1]^*(\alpha)$. We claim that when restricted to a geometric generic fibre β is homologically equivalent to zero but nonzero mod algebraic equivalence. Proceeding as in Theorem 4.4 and keeping track of Tate twists we see that it is enough to prove the following

LEMMA A.2. *Let B be a smooth algebraic variety over \mathbb{F}_q , $q = p^m$ and let $\pi: Z \rightarrow B$ be a principally polarised abelian scheme of relative dimension 3 s.t. the classifying map $B \rightarrow \mathcal{A}_3$ is étale. Then $H^0(B, \mathbb{W} \otimes \mathbb{P}_3(2)) = 0$, where $\mathbb{W} = R^1\pi'_*\mathbb{Q}_l$ for $\pi': Z' \rightarrow B$ is any abelian scheme and as before \mathbb{P}_3 is the primitive part of $R^3\pi_*\mathbb{Q}_l$.*

Proof. Ordinary abelian varieties form a Zariski open subset of \mathcal{A}_3 , hence after a finite extension of the base field if necessary we may assume that there exists a closed point $b \in B$ with residue field \mathbb{F}_q s.t. Z_b is an ordinary abelian variety. There is then a (split) exact sequence $1 \rightarrow \pi_1^g \rightarrow \pi_1^a \rightarrow \text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \rightarrow 1$. \mathbb{P}_3 and \mathbb{W} then correspond to representations W and P_3 of π_1^a well defined upto conjugacy and $H^0(B, \mathbb{W} \otimes \mathbb{P}_3(2)) \cong (W \otimes P_3(2))^{\pi_1^a}$. It is known that the action of π_1^g on P_3 is absolutely irreducible; in this case it can be proved as follows: by the theory of Lefschetz pencils applied to plane curves of deg 4, hence genus 3 it follows that the Zariski closure of the image of π_1^g in $\text{GL}(H^1)$ is $\text{Sp}(6)$ (see [8] Sect. 5), the desired result then follows from the representation theory of the symplectic group. Thus if $(W \otimes P_3(2))^{\pi_1^a} \neq 0$ it follows that for each eigenvalue of (geometric) Frobenius λ on P_3 there exists an eigenvalue of Frobenius μ on W s.t. $\lambda \cdot \mu = q^2$. Let ν be the valuation of $\overline{\mathbb{Q}}_p$ normalised so that $\nu(q) = 1$. Since X_b is ordinary there exist eigenvalues of Frobenius $\lambda_1, \lambda_2, \lambda_3$ on $H^1((X_b)_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$ s.t. $\nu(\lambda_i) = 1$,

$i = 1, 2, 3$. Then $\lambda = \lambda_1 \cdot \lambda_2 \cdot \lambda_3$ is an eigenvalue of Frobenius on P_3 . By comparing valuations we see that there cannot exist a μ s.t. $\lambda \cdot \mu = q^2$ (since μ must be an algebraic integer).

REMARK A.3. If $p \geq 3$ one can also modify Nori's argument in [17] to show that $\text{Griff}^2(X)$ is of infinite rank, where X is a generic abelian variety of dimension three (see [12] for details).

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References

1. Beauville, A.: Prym varieties and the Schottky problem. *Invent.-Math.*, **41**:149–196, 1977.
2. Beauville, A.: Sur l'anneau de Chow d'une variété abélienne. *Math.-Ann.*, **273**:647–651, 1986.
3. Borel, A.: *Stable real cohomology of arithmetic groups. Manifolds and Lie groups, Progr. Math. 14.* Birkhauser, 1981.
4. Bosch, A., Lütkebohmert, W. and Raynaud, M.: *Neron Models.* Springer Verlag, 1990.
5. Ceresa, G.: C is not algebraically equivalent to C^- in its Jacobian. *Ann.-of Math.*, **117**:285–291, 1983.
6. Colombo, E. and van Geemen, B.: Note on curves in a Jacobian. *Compo.-Math.*, **88**:333–353, 1993.
7. Deligne, P.: Theoreme de Lefschetz et Criteres de Degenerescence de Suites Spectrales. *Inst.-Hautes-Etudes-Sci.-Publ.-Math.*, **35**:19–68, 1968.
8. Deligne, P.: La conjecture de Weil I. *Inst.-Hautes-Etudes-Sci.-Publ.-Math.*, **43**:273–307, 1974.
9. Deninger, C. and Murre, J.: Motivic decomposition of abelian schemes and the Fourier transform. *J.-Reine-Angew.-Math.*, **422**:201–219, 1991.
10. Dold, A.: *Lectures on Algebraic Topology.* Springer Verlag, 1972.
11. Donagi, R. and Smith, R.: The Structure of the Prym map. *Acta.-Math.*, **146**:25–102, 1981.
12. Fakhruddin, N.: *Algebraic cycles on generic Abelian varieties.* PhD thesis, The University of Chicago, 1995.
13. Faltings, G. and Chai, C-L.: *Degeneration of Abelian Varieties.* Springer Verlag, 1990.
14. Fulton, W.: *Intersection Theory.* Springer Verlag, 1984.
15. Kostant, B.: Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann.-of-Math.*, **74**:329–387, 1961.
16. Mostafa, S.: Die Singularitäten der Modulmannigfaltigkeiten ... *J.-reine-Angew.-Math.*, **343**:81–98, 1983.
17. Nori, M.: Cycles on the generic Abelian threefold. *Proc.-Ind.-Acad.-Sci.*, **99**:191–196, 1989.
18. Nori, M.: Algebraic cycles and Hodge theoretic connectivity. *Invent.-Math.*, **111**:349–373, 1993.
19. Popp, H.: *Moduli Theory and Classification Theory of Algebraic Varieties. LNM 620.* Springer Verlag, 1977.
20. Saito, M.: Mixed Hodge modules. *Publ.-Res.-Inst.-Math.-Sci.(Kyoto University)*, **26**:221–333, 1990.
21. Zucker, S.: Locally homogeneous variations of Hodge structure. *Enseign.-Math.*, **27**:243–276, 1981.