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Open surfaces with non-positive Euler characteristic

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1. Introduction

Recently, A. J. de Jong and J. H. M. Steenbrink (cf. [J-S]) have proved the following result (conjectured by W. Veys):

'Let $C \subset \mathbf{P}^2$ be a reduced algebraic curve over $\mathbf{C}$. Assume that the topological Euler characteristic $e(\mathbf{P}^2 - C) \leqslant 0$. Then every irreducible component of C is a rational curve.'

The aim of this paper is to prove a much more precise result valid for a very general class of non-complete algebraic surfaces.

The main results we prove are the following theorems.

THEOREM 1. *Let X be a smooth projective surface and D a non-empty connected curve on X such that $e(X - D) < -1$, then $X_{\min}$ is isomorphic to $\mathbf{P}^2$ or a ruled surface. Further, $X - D$ has a morphism to a curve of general type with general fibre isomorphic to $\mathbf{P}^1$ or $\mathbf{C}$.*

THEOREM 2. *Let X be a smooth projective surface and D a non-empty connected curve on X such that $e(X - D) = 0$ or -1. Then $X_{\min}$ is isomorphic to $\mathbf{P}^2$, ruled surface, hyperelliptic surface, abelian surface or an elliptic surface of Kodaira dimension* 1.

(1) *Suppose $X_{\min}$ is a hyperelliptic surface or an abelian surface.*

(i) *If $e(X - D) = 0$, then either D is the union of the exceptional curve for the morphism $X \to X_{\min}$ with a smooth elliptic curve or the union of all but one irreducible components of the exceptional curve.*

(ii) *If $e(X - D) = -1$, then D is the exceptional curve of the morphism $X \to X_{\min}$.*

(2) *If $\kappa(X_{\min}) = 1$, then D is contained in a fibre F_0 of the elliptic fibration on X.*

(i) *If* $e(X - D) = 0$ *and* $e(X_{\min}) = 0$, *then either* D *is the union of all but one (tip) irreducible components of* E *or* $D = F_0$. *Here,* E *is the exceptional curve for the morphism* $X \to X_{\min}$.

(ii) *If* $e(X - D) = 0$ *and* $e(X_{\min}) > 0$, *then* D *is the union of all but possibly one irreducible components of* F_0.

(iii) *If* $e(X - D) = -1$, *then* $e(X_{\min}) = 0$ *and* D *is the exceptional curve.*

THEOREM 3. *Let* X *be a smooth projective surface and let* D *be a non-empty connected curve on* X *such that* $e(X - D) \leqslant 0$. *Then the following assertions are true:*

(1) *There is a morphism* ϕ *from* $X - D$ *to a smooth curve* B *with general fibre isomorphic to* $\mathbf{P}^1$, $\mathbf{C}$, $\mathbf{C}^*$ *or an elliptic curve, except in the case when* $X_{\min}$ *is a simple abelian surface and* D *is the union of all but at most one irreducible components (which is a tip) of the exceptional curve.*

(2)(i) *If the general fibre of* ϕ *is* $\mathbf{P}^1$, *then all the irreducible components of* D *are rational.*

(ii) *If the general fibre is* $\mathbf{C}$, *then* D *has at most one irrational irreducible component.*

(iii) *If the general fibre is an elliptic curve, then* D *has at most one irrational irreducible component, in which case it is a smooth elliptic curve.*

(iv) *If the general fibre is* $\mathbf{C}^*$, *then* D *has at most two irrational irreducible components. Moreover, if* $q(X) = 0$ *then* D *has at most one irrational irreducible component, in which case it is a hyperelliptic curve.*

Our proofs are quite different from [J-S]. They depend in an essential way on the theory of non-complete algebraic surfaces developed by Iitaka, Kawamata, Fujita, Miyanishi, Kobayashi, Tsunoda and other Japanese mathematicians. An inequality of Miyaoka-Yau type proved by R. Kobayashi plays an important role in our proof. For the proofs of the theorems above, we also need results of P. Deligne on the degeneration of Hodge spectral sequence for non-complete algebraic varieties. The proof of theorem 1 imitates Castelnuovo's argument for the projective case as given in [B, Chapter 10].

We get the following striking consequence from our proofs. It can be regarded as a generalization of Castelnuovo's criterion of ruledness (for relatively minimal surfaces) as surfaces with negative Euler characteristic (cf. [B], Chapter 10).

'Let V *be a smooth quasi-projective surface which is connected at infinity. Assume that either* $e(V) < -1$, *or* V *is affine and* $e(V) < 0$, *then there is a morphism from* V *to a curve of general type with general fibre isomorphic to* $\mathbf{C}$ *or* $\mathbf{P}^1$.*'* (cf. Sect. 5).

2. Notations and preliminaries

All algebraic varieties considered in this paper are defined over the field of complex numbers $\mathbf{C}$.

For any topological space X, $e(X)$ denotes its topological Euler characteristic.

Given a connected, smooth, quasiprojective variety V, $\overline{\kappa}(V)$ denotes the logarithmic Kodaira dimension of V as defined by S. Iitaka (cf. [I]). It is easy to see that for a connected, smooth algebraic curve B, $\overline{\kappa}(B) = 1$ if and only if $e(B) < 0$. Hence we will call any connected, smooth curve B with $e(B) < 0$ a *curve of general type*. This agrees with the usual definition.

By a $(-n)$-curve on a smooth algebraic surface we mean a smooth rational curve with self-intersection $-n$. By a *normal crossing divisor* on a smooth algebraic surface we mean a reduced algebraic curve C such that every irreducible component of C is smooth, no three irreducible components pass through a common point and all intersections of the irreducible components of C are transverse.

For a smooth projective surface X, X_{min} denotes a relatively minimal smooth projective surface birationally isomorphic to X.

Following Fujita, we call a divisor Δ on a smooth projective surface Y *pseudo-effective* if $H \cdot \Delta \geqslant 0$ for every ample divisor H on Y.

For the convenience of the reader, we now recall some basic definitions which are used in the results about Zariski-Fujita decomposition of a pseudo-effective divisor (cf. [F], Sect. 6;[M-T], Chapter 1]).

Let (Y, D) be a pair of a nonsingular surface Y and a normal crossing divisor D. A connected curve T consisting of irreducible curves in D (a connected curve in D, for short) is a *twig* if the dual graph of T is a linear chain and T meets $D - T$ in a single point at one of the end points of T; the other end of T is called a *tip* of T. A connected curve R (resp. F) in D is a *club* (resp. an *abnormal club*) if R (resp. F) is a connected component of D and the dual graph of R (resp. F) is a linear chain (resp. the dual graph of the exceptional curves of a minimal resolution of singularities of a non-cyclic quotient singularity). A connected curve B in D is *rational* (resp. *admissible*) if each irreducible component of B is rational (resp. if none of the irreducible components of B is a (-1)-curve and the intersection matrix of B is negative definite). An admissible rational twig T is *maximal* if T is not contained in an admissible rational twig with more irreducible components.

Let $\{T_\lambda\}$ (resp. $\{R_\mu\}$ and $\{F_\nu\}$) be the set of all admissible rational maximal twigs (resp. admissible rational maximal clubs and admissible rational maximal abnormal clubs). Then there exists a decomposition of D into a sum of effective $\mathbf{Q}$-divisors, $D = D^\# + Bk(D)$, such that $\operatorname{Supp}(Bk(D)) = (\bigcup_\lambda T_\lambda) \cup (\bigcup_\mu R_\mu) \cup (\bigcup_\nu F_\nu)$ and $(K_Y + D^\#) \cdot Z = 0$ for every irreducible component Z of $\operatorname{Supp}(Bk(D))$. The divisor $Bk(D)$ is called the *bark* of D, and we say that $K_Y + D^\#$ is produced by the *peeling* of D. For details of how $Bk(D)$ is obtained from D, see [M-T].

The Zariski-Fujita decomposition of $K_Y + D$, in case $K_Y + D$ is pseudo-effective, is as follows:

There exist $\mathbf{Q}$-divisors P, N such that $K_Y + D \approx P + N$ where, $\approx$ denotes numerical equivalence, and

(a) P is numerically effective (nef, for short)

(b) N is effective and the intersection form on the irreducible components of N is negative definite

(c) $P \cdot D_i = 0$ for every irreducible component D_i of N.

N is unique and P is unique upto numerical equivalence. If some multiple of $K_Y + D$ is effective, then P is also effective.

The following result from [F, Lemma 6.20] is very useful.

LEMMA 1. *Let (Y, D) be as above. Assume that all the maximal rational twigs, maximal rational clubs and maximal abnormal rational clubs of D are admissible. Let $\overline{\kappa}(Y - D) \geqslant 0$. As above, let $P + N$ be the Zariski decomposition of $K_Y + D$. If $N \neq Bk(D)$, then there exists a (-1)-curve L, not contained in D, such that one of the following holds:*

(i) L is disjoint from D

(ii) $L \cdot D = 1$ and L meets an irreducible component of $Bk(D)$

(iii) $L \cdot D = 2$ and L meets two different connected components of D such that one of the connected components is a maximal rational club R_ν of D and L meets a tip of R_ν

Further, $\overline{\kappa}(Y - D - L) = \overline{\kappa}(Y - D)$.

The following results proved by Kawamata will be used frequently:

LEMMA 2 (cf. [Ka1]). *Let Y be a smooth quasi-projective algebraic surface and $f : Y \to B$ be a surjective morphism to a smooth algebraic curve such that a general fibre F of f is irreducible. Then $\overline{\kappa}(Y) \geqslant \overline{\kappa}(B) + \overline{\kappa}(F)$.*

LEMMA 3 (cf. [Ka2]). *Let Y be a smooth quasi-projective algebraic surface with $\overline{\kappa}(Y) = 1$. Then there is a Zariski-open subset U of Y which admits a morphism $f : U \to B$ onto a smooth algebraic curve B such that a general fibre of f is isomorphic to either $\mathbf{C}^*$ or an elliptic curve. (We call such a fibration a $\mathbf{C}^*$-fibration or an elliptic fibration respectively).*

The next result is proved by M. Suzuki in [S], Sect. 9.

LEMMA 4. *Let S be a smooth affine surface with a morphism $g : S \to B$ onto a smooth algebraic curve B such that a general fiber of g is connected. Then*

$$e(S) = e(B) \cdot e(G) + \Sigma(e(G_i) - e(G)),$$

where, G is a general fibre of g and G_i are the singular fibres of g. Each term in the summation is non-negative and $e(G_i) = e(G)$ implies that $(G_i)_{\mathrm{red}}$ is diffeomorphic to G.

M. G. Zaidenberg (cf. [Z], Lemma 3.2) has strengthened this result by proving that when all the fibres of a fibration are diffeomorphic and if there is a multiple fibre, then the general fibre is isomorphic to $\mathbf{C}$ or $\mathbf{C}^*$.

We recall the following results of Deligne about the computation of the cohomology using the logarithmic de Rham complex from the degeneration of the Hodge spectral sequence.

Given a smooth projective variety X over $\mathbf{C}$ and a divisor D with normal crossings, we can define the logarithmic de Rham complex of X, denoted by $\Omega_X^*\langle D\rangle$. We recall that $\Omega_X^0\langle D\rangle = \mathcal{O}_X\langle D\rangle = \mathcal{O}_X$. A section of $\Omega_X^1\langle D\rangle$ is a meromorphic 1-form which is holomorphic in $X - D$ and at a point p of D it has the form $\Sigma_1^r g_i\, \mathrm{d}z_i/z_i + \Sigma_{r+1}^n g_i\, \mathrm{d}z_i$ where, $z_1, \ldots, z_n$ are the local holomorphic coordinates at p and D is defined by the equation $z_1 \cdots z_r = 0$ near p and g_i are suitable holomorphic functions in a neighborhood of p. The sheaf of logarithmic forms of higher degree is obtained by taking the exterior powers of logarithmic 1-forms. We define the Hodge filtration F^p of this complex as the subcomplex $(F^p\Omega_X^*\langle D\rangle)^n = 0$ if $n < p$ and $= \Omega_X^n\langle D\rangle$ if $n \geqslant p$. Recall the following theorem and its corollaries due to Deligne (cf. [D], Corollary 3.2.13 and 3.2.14).

Let X be a smooth projective variety over $\mathbf{C}$ and D be a divisor with normal crossings. Then the spectral sequence associated to the Hodge filtration,

$$E_1^{p,q} = H^q(X, \Omega_X^p\langle D\rangle) \Rightarrow H^{p+q}(X - D, \mathbf{C})$$

degenerates at E_1.

COROLLARY 1. *If ω is a meromorphic p-form holomorphic on $X - D$ and having logarithmic poles along D, then $\omega\,|_{X-D}$ is closed and the corresponding cohomology class is 0 if and only if $\omega = 0$.*

COROLLARY 2. $H^n(X - D, \mathbf{C}) \cong \bigoplus_{p+q=n} H^q(X, \Omega_X^p\langle D\rangle)$

3. Kobayashi's inequality

Let (X, D) be a pair of a smooth projective surface X and a connected curve D with $e(X - D) \leqslant 0$. The first important step in the proofs of Theorems 1 and 2 is the following result which is probably well-known to the experts:

PROPOSITION 1. *Let (Y, C) be a pair of a smooth projective surface Y and a connected normal crossing divisor C on Y. Assume that $\overline{\kappa}(Y - C) = 2$. Then there is a Zariski-open subset V of $Y - C$ and a smooth projective compactification Z of V with $D := Z - V$ either a single point or a normal crossing divisor, such that the following properties hold:*

(i) *if D is a single point, then Z is a minimal surface and if D is a divisor, then (Z, D) is NC-minimal (the definition appears below)*

(ii) $e(Z - D) \leqslant e(Y - C)$

(iii) *if D is a divisor, then there is no log exceptional curve of the second kind on Z (the definition appears below).*

Proof. We can assume that any (-1)-curve in C meets at least three other irreducible components of C.

Since $|n(K_Y+C)| \neq \phi$ for some $n \geqslant 1$, K_Y+C is pseudo effective and hence there is a Zariski decomposition

$$K_Y + C \approx P + N.$$

Following Fujita, we say that (Y, C) is *NC-minimal* if $N = Bk(C)$. We will first reduce to the situation when (Y, C) is NC-minimal.

So suppose that $N \neq Bk(C)$. Then by Lemma 1, there is a (-1)-curve L on Y satisfying the properties stated in lemma 1.

Let $Y = Y_1, C = C_1$ and $\psi\colon Y_1 \to Y_2$ be the contraction of L. Then $\psi(C)$ is a normal crossing divisor C_2 and $e(Y_2 - C_2) \leqslant e(Y_1 - C_1) - 1$. By a suitable succession of contractions, we can assume that any (-1)-curve in C_2 meets at least three other irreducible components of C_2. It is possible that in this process $\psi(C)$ is contracted to a point p. Then $e(Y_2 - \{p\}) \leqslant -1$ and $e(Y_2) \leqslant 0$. By Castelnuovo's theorem mentioned earlier, Y_2, and hence also Y_1, cannot be of general type. Further, $\overline{\kappa}(Y_2 - \psi(C_1)) = 2 = \overline{\kappa}(Y_2) = \overline{\kappa}(Y_1 - C_1)$. Hence proposition 1 follows in this case easily by considering the morphism $Y_2 \to Y_{2_{\min}}$. If C_2 is not reduced to a point, we start with the pair (Y_2, C_2) and repeat the argument above. In finitely many steps, we reach an NC-minimal pair (Y_r, C_r) (or, Proposition 1 is proved).

Now we assume that $N = Bk(C)$. We will use some fundamental results of Kawamata (cf. [Ka2]). Let $C^{\#}$ denote the divisor $C - Bk(C)$. It is known that $R = \oplus_{n=0}^{\infty} H^0(Y, n(K_Y + C))$ is a finitely generated algebra over $\mathbf{C}$. Let $\operatorname{Proj} R = Y_c$. Then Y_c is called the *log canonical model* of Y. There is a morphism $\Phi\colon Y \to Y_c$ which is the minimal resolution of singularities of Y_c. An irreducible curve E on Y is contracted to a point by Φ iff $(K_Y + C^{\#}) \cdot E = 0$ and the intersection matrix of $E + Bk(C)$ is negative definite. If $E \subset C$, by contracting E to a point $Y - C$ is unchanged. Suppose that E is not contained in C.

Case 1. $E \cap C = \phi$. Then $K_Y \cdot E = 0$ and $E^2 < 0$ implies that E is a (-2)-curve.

Case 2. $E \cap C \neq \phi$. Then $K_Y \cdot E < 0, E^2 < 0$ and hence E is a (-1)-curve. Such a curve is called a *log exceptional curve of the second kind* (w.r.t. Φ).

Assume that there is a log exceptional curve of the second kind w.r.t. Φ. In this case, the precise nature of $E \cap C$ is known (cf.[M-T], Sect. 1.8). E meets at most two irreducible components C_1, C_2 of C, $E \cdot C_i = 1$ for all i and $C_1 \cap C_2 = \phi$. Then $C \cup E$ is a normal crossing divisor and the image of C under the contraction $\psi\colon Y \to Y_1$ of E to a point is a normal crossing divisor $\psi(C)$. Then $e(Y_1 - \psi(C)) \leqslant e(Y - C)$ and $\overline{\kappa}(Y_1 - \psi(C)) = \overline{\kappa}(Y - C)$.

We start with the pair $(Y_1, \psi(C))$ instead of (Y, C). By alternately constructing NC-minimal model and then contracting log exceptional curves of the second kind (which by definition lie outside C), we find a smooth projective surface Z and a normal crossing divisor D on Z satisfying the required properties.

Now any irreducible curve contracted by the morphism

$$Z \to Z_c = \operatorname{Proj}(\oplus H^0(Z, n(K_Z + D)))$$

is either contained in D or disjoint from D and Z_c has only rational double point singularities outside the image of D. Let $D_c = \Phi(D)$.

For a pair (Z, D) as above which satisfies the conditions (i) and (iii) in Proposition 1, R. Kobayashi has proved the following inequality.

LEMMA 5 (cf. [Ko], Theorem 2). *With the above notations, the following inequality holds*

$$0 < (K_{Z_c} + D_c)^2 \leqslant 3e(Z_c - D_c).$$

We apply all this to a pair (X, D) of a smooth projective surface and a reduced connected curve D on it. If $e(X - D) \leqslant 0$, then from the above considerations we deduce the following:

PROPOSITION 2. *Let X be a smooth projective surface and D a connected curve on X such that $e(X - D) \leqslant 0$. Then $\bar{\kappa}(X - D) \leqslant 1$.*

4. Proofs of theorems 1 and 2

In this section we prove the following two theorems stated in the introduction:

THEOREM 1. *Let X be a smooth projective surface and D a non-empty connected curve on X such that $e(X - D) < -1$. Then $X_{\min}$ is isomorphic to $\mathbf{P}^2$ or a ruled surface. Further, $X - D$ has a morphism to a curve of general type with general fibre isomorphic to $\mathbf{P}^1$ or $\mathbf{C}$.*

THEOREM 2. *Let X be a smooth projective surface and D a non-empty connected curve on X such that $e(X - D) = 0$ or -1. Then $X_{\min}$ is isomorphic to $\mathbf{P}^2$, ruled surface, hyperelliptic surface, abelian surface or an elliptic surface of Kodaira dimension* 1.

(1) *Suppose $X_{\min}$ is a hyperelliptic surface or an abelian surface.*

(i) *If $e(X - D) = 0$, then either D is the union of the exceptional curve for the morphism $X \to X_{\min}$ with a smooth elliptic curve or the union of all but one irreducible components of the exceptional curve.*

(ii) *If $e(X - D) = -1$, then D is the exceptional curve of the morphism $X \to X_{\min}$.*

(2) *If $\kappa(X_{\min}) = 1$, then D is contained in a fibre F_0 of the elliptic fibration on X.*

(i) *If $e(X - D) = 0$ and $e(X_{\min}) = 0$, then either D is the union of all but one (tip) irreducible components of E or $D = F_0$. Here, E is the exceptional curve for the morphism $X \to X_{\min}$.*

(ii) *If $e(X - D) = 0$ and $e(X_{\min}) > 0$, then D is the union of all but possibly one irreducible components of F_0.*

(iii) *If $e(X - D) = -1$, then $e(X_{\min}) = 0$ and D is the exceptional curve.*

The proofs of these theorems use a generalization of the arguments of the classical proof of Castelnuovo as given in [B]. We use the notation that $h^i(\mathcal{F}) =$

$\dim H^i(\mathcal{F})$ for any sheaf of abelian groups $\mathcal{F}$ on X and $b_i(X) = i$th Betti number of X. Let $q(X) = h^0(\Omega^1_X)$, $q(X - D) = h^0(\Omega^1_X\langle D\rangle)$, $p_g(X) = h^0(\Omega^2_X)$ and $p_g(X - D) = h^0(\Omega^2_X\langle D\rangle)$.

Now let (X, D) be a pair of a smooth projective surface X and a non-empty curve D on it. From Corollary 2 we conclude that:

(i) $b_0(X - D) = b_0(X) = 1$,

(ii) $b_1(X - D) = q(X - D) + q(X)$,

(iii) $b_2(X - D) = p_g(X - D) + h^1(\Omega^1_X\langle D\rangle) + p_g(X)$,

(iv) $b_3(X - D) = h^1(\Omega^2_X\langle D\rangle) + h^2(\Omega^1_X\langle D\rangle)$,

(v) $b_4(X - D) = 0$.

Now we begin with the proofs of the two theorems of this section.

Case 1. First we consider the case when $e(X - D) < 0$.

Without loss of generality we will assume that D is a divisor with normal crossings. By Corollary 2 we obtain

$$\begin{aligned} &1 - b_1(X - D) + b_2(X - D) - b_3(X - D) \\ &\quad = 1 - q(X - D) - q(X) + p_g(X - D) \\ &\qquad + p_g(X) + h^1(\Omega^1_X\langle D\rangle) - b_3(X - D) \leqslant -1, \end{aligned}$$

i.e.

$$\begin{aligned} p_g(X - D) \leqslant\, & q(X - D) + q(X) - p_g(X) \\ & - h^1(\Omega^1_X\langle D\rangle) + b_3(X - D) - 2. \end{aligned} \tag{\#}$$

Consider the linear map obtained by taking wedge products of 1-forms:

$$\wedge^2 H^0(X, \Omega^1_X\langle D\rangle) \to H^0(X, \Omega^2_X\langle D\rangle)$$

The kernel of this map has codimension $\leqslant p_g(X - D)$. The decomposable vectors of the form $\omega_1 \wedge \omega_2$ in $\wedge^2 H^0(X, \Omega^1_X\langle D\rangle)$ form a cone of dimension $2q(X - D) - 3$ (being the cone over the Grassmanian $G(2, H^0(X, \Omega^1_X\langle D\rangle))$).

Case 1.1. In addition to $e(X - D) < 0$, assume that $p_g(X - D) \leqslant 2q(X - D) - 4$. Then this cone and the kernel have a nontrivial intersection. But this implies that there are two linearely independent global sections ω_1 and ω_2 of $\Omega^1_X\langle D\rangle$ such that $\omega_1 \wedge \omega_2 = 0$. Then there exists a rational function $g \in \mathbf{C}(X)$ such that $\omega_2 = g\omega_1$. By Corollary 1 we have $d\omega_1 = d\omega_2 = 0$. Then $dg \wedge \omega_1 = 0$ and hence there is a rational function $f \in k(X)$ such that $\omega_1 = f dg$, and $df \wedge dg = 0$. Consider the rational map $\phi = (f, g)\colon X - D \to \mathbf{C}^2$. Then the image of $X - D$ under ϕ is a curve as f and g are algebraically dependent (because $df \wedge dg = 0$). Note that $\omega_1 = \phi^*(x\,\mathrm{d}y)$ and $\omega_2 = \phi^*(xy\,\mathrm{d}y)$ where x, y are the coordinates on $\mathbf{C}^2$.

Claim: ϕ is morphism. Let C be the normalization of the closure of the image of ϕ in $\mathbf{P}^2$. Let $\tilde{\phi}\colon \widetilde{X - D} \to C$ be the morphism obtained by resolving the indeterminacies of ϕ. Let α_1 and α_2 be the 1-forms on C obtained by pulling back $x\,\mathrm{d}y$ and $xy\,\mathrm{d}y$ respectively. Then α_1 and α_2 are linearly independent meromorphic forms as they pull back to linearly independent holomorphic forms on $\widetilde{X - D}$. Now if ϕ is not a morphism, then the exceptional divisor E of $\widetilde{X - D} \to X - D$ must map onto C. This implies that both α_i are holomorphic on C, as their pull backs are holomorphic in a neighbourhood of E which maps onto C. But if C has two linearly independent global holomorphic 1-forms, it must have genus $\geqslant 2$. As every component of E is rational, the map from E to C is constant, contradicting the assumption that E dominates C. Hence ϕ is a morphism.

Now as each ω_i has only simple poles, it follows by an easy local computation that each α_i has only logarithmic singularities. Let $C' = C - \cup\mathrm{Sing}(\alpha_i)$. Then by corollary 2, $b_1(C') \geqslant 2$ (i.e. C' is a curve of general type), and ϕ maps $X - D$ into C'. By taking the Stein factorization of ϕ, we may assume that the general fibre of ϕ is irreducible. The base of this new map will be clearly of general type.

Proof of Theorem 1: Let the pair (X, D) be as in the statement of Theorem 1. By assumption, $e(X - D) < 0$. From Proposition 2 we already know that $\overline{\kappa}(X - D) \leqslant 1$. This implies that $\overline{\kappa}(X) \leqslant 1$. Then by the classification of relatively minimal projective surfaces it follows that X has a minimal model $X_{\min}$ of the form:

(i) minimal rational surface,

(ii) ruled surface of genus $g \geqslant 1$,

(iii) Enriques surface,

(iv) $K3$ surface,

(v) hyperelliptic surface,

(vi) Abelian surface,

(vii) minimal elliptic surface with Kodaira dimension 1.

We now analyse each minimal surface occuring in the above list.

We will first show that if $q(X) = 0$ then the inequality $p_g(X - D) \leqslant 2q(X - D) - 4$ is satisfied.

We have, $e(X-D) = 1 - b_1(X-D) + b_2(X-D) - b_3(X-D) \leqslant -1$ i.e., $1 \leqslant 1 + b_2(X-D) - b_3(X-D) < b_1(X-D) = q(X-D)$, as $q(X) = 0$. By duality, $H_3(X-D) \cong H^1(X,D)$. The long exact cohomology sequence (with **Q**-coefficients) for the pair (X,D) shows that $H^1(X,D) = (0)$, because $H^1(X) = (0)$. Hence $q(X-D) \leqslant 2q(X-D) - 2$. By applying this inequality to (#), we obtain the required inequality. We will now rule out the possibilities that $X_{\min}$ is a $K3$ surface or an Enriques surface. If X is a $K3$ surface, then $q(X) = 0$ and by the above arguments there is a morphism φ from $X - D$ onto a curve of general type B. Lemma 2 implies that the general fibre of φ is an elliptic curve, since a $K3$ surface does not have a continuous family of rational curves. This implies that the morphism φ extends to X and D is contained in a single fibre. Hence B is isomorphic to $\mathbf{P}^1$ or $\mathbf{C}$. But this contradicts the assumption that B is a curve of general type.

Suppose that X is an Enriques surface. There is a 2-sheeted cover X' of X which is a $K3$ surface. Then the inverse image D' of D can have at most two connected components. Consider the long exact relative cohomology sequence (with **Q**-coefficients) of the pair (X', D'). By duality, $H^1(X', D') \cong H_3(X' - D')$ and $H^1(X') = 0$. Hence $b_3(X' - D') \leqslant 1$. Since $p_g(X') = 1$, by (#) we still get $q(X' - D') > 1$ and by the above arguments, there is a morphism from $X' - D'$ to a curve of general type. But again D' is contained in a union of two fibres implies that the image contains $\mathbf{C}^*$, a contradiction.

Next we show that $X_{\min}$ cannot be a hyperelliptic or an abelian surface. In case $X_{\min}$ is hyperelliptic or an abelian surface, $e(X) = b_2(E)$ where E is the exceptional curve of the morphism $X \to X_{\min}$. If $e(X-D) < 0$, we have $e(X) = b_2(E) < e(D) \leqslant 1 + b_2(\mathrm{Rat}\, D) + r$, where Rat D is the union of rational curves in D and $r \leqslant 0$ is the sum of $b_2(D_i) - b_1(D_i)$ over irrational irreducible curves D_i in D. But since $X_{\min}$ does not contain any rational curves, we see that Rat $D \subset E$. Above inequality implies that $r = 0$ and $D = E$. Then $e(X-D) = -1$, contradicting the assumption of theorem 1 that $e(X-D) < -1$.

Finally we show that $X_{\min}$ cannot have Kodaira dimension 1. Assume that $\kappa(X) = 1$. By Proposition 2 it follows that $\overline{\kappa}(X-D) = 1$. Lemma 3 implies that $X - D$ has a Zariski-open subset which has an elliptic fibration (it cannot have a $\mathbf{C}^*$-fibration!). As above, this extends to a morphism φ from X to $\overline{B}$ and D is contained in a single fibre, say F_0. Let $\phi := \varphi|_{X-D}$. We use now the following result which will be needed again.

LEMMA 6. *Let Y be a smooth projective surface with a surjective morphism φ to a smooth curve B. Let C be a connected curve in Y and $\phi := \varphi|_{Y-C}$. Then we have:*

(i) *If the general fibre of ϕ is $\mathbf{C}^*$, then for any fibre F of ϕ, $e(F) \geqslant 0$.*

(ii) *If the general fibre of ϕ is an elliptic curve, then for any fibre F of ϕ, $e(F) \geqslant -1$ and equality holds if and only if the fibre of φ containing C is the union of C and a smooth elliptic curve and C can be contracted to a smooth point.*

Proof: First we note that for any two curves $C_1 \subset C_2$, an easy Mayer-Vietoris sequence argument shows that $e(C_2 - C_1) = e(C_2) - e(C_1)$.

In case (i), let F_0 be the fibre of φ containing F and C_v the union of vertical components of C contained in F_0. Note that F is $F_0 - C_v$ with at most two points deleted and C_v has at most two connected components. For simplicity, we give the argument when C_v is non-empty and connected. In this case, $e(F_0 - C_v) = e(F_0) - e(C_v) = b_2(F_0) - b_2(C_v) \geqslant 1$. On the other hand F is obtained from $F_0 - C_v$ by removing at most one point, hence $e(F) \geqslant 0$.

In case (ii), C is contained in a fibre F_0 of φ. Now $e(F) = e(F_0) - e(C) = b_2(F_0) - b_2(C) - b_1(F_0) + b_1(C)$. Now $b_1(F_0) \leqslant 2$ with equality if and only if F_0 is obtained from a smooth elliptic curve by blowing up. Since $b_2(F_0) > b_2(C)$, it follows that $e(F) \geqslant -1$ with equality if and only if $b_1(F_0) = 2$, $b_1(C) = 0$ and $b_2(F_0) - b_2(C) = 1$ and hence C is the exceptional curve. This completes the proof of Lemma 6.

From Lemma 6 we see that for every fibre F_s of ϕ, other than $F_0 - D$, $e(F_s) \geqslant 0$. The proof of Suzuki's result (cf. Lemma 4) easily implies that $e(X - D) = \Sigma F_i$ where, F_i are all the "singular" fibres of ϕ (defined suitably). Also, $e(F_0 - D) \geqslant -1$ and equality holds iff F_0 is the union of D and a smooth elliptic curve and D can be contracted to a smooth point. In any case, $e(X - D) \geqslant -1$. This contradiction shows that $X_{\min}$ cannot have Kodaira dimension 1.

To complete the proof of Theorem 1, it remains to show that when X is $\mathbf{P}^2$ or ruled there is a morphism from $X - D$ onto a curve of general type with general fibre isomorphic to $\mathbf{P}^1$ or $\mathbf{C}$. First we show that there is a morphism from $X - D$ onto a curve of general type.

In case X is rational, $q(X) = 0$ and we have already shown above that there is such a morphism. If X is ruled over a curve of genus > 1, then clearly the restriction map to $X - D$ is such a morphism. Suppose that $X_{\min}$ is ruled over a curve of genus 1. Then $e(X_{\min}) = 0$ and $e(X) = b_2(E)$ where, E is the exceptional curve for the map $X \to X_{\min}$. Using the notation in the proof for hyperelliptic and abelian surface case, from $e(X - D) < -1$ we get $e(X) = b_2(E) < b_2(\text{Rat } D) + r$ with $r \leqslant 0$. The only rational curves in $X_{\min}$ are the fibres of the ruling, hence Rat D is contained in a finite union of fibres. We deduce easily that Rat D contains a full fibre of the ruling and hence $X - D$ maps onto a curve of general type.

Let now $\phi : X - D \to B$ be the given morphism with B of general type. Applying Lemma 2, we see that a general fibre F of ϕ has logarithmic Kodaira dimension $-\infty$ or 0. We claim that $\overline{\kappa}(F) \neq 0$. For otherwise, as above ϕ is either a $\mathbf{C}^*$-fibration or an elliptic fibration. As above, using Lemma 6 this implies that $e(X - D) \geqslant -1$, a contradiction. Hence ϕ is either a $\mathbf{P}^1$-fibration or a $\mathbf{C}$-fibration. This completes the proof of Theorem 1.

Proof of Theorem 2: In case $q(X) = 0$ and $e(X) \leqslant e(D)$, we have $2 + b_2(X) \leqslant 1 + b_2(D) - b_1(D)$ and hence $b_2(X) < b_2(D)$. This easily implies that there is a non-constant morphism from $X - D \to C^*$. When $X_{\min}$ is a $K3$ or Enrique surface the proof of Theorem 1 works immediately if $e(X-D)\leqslant -1$. Suppose that $e(X-D)=0$. Now we notice that for a $K3$ surface, $p_g(X)=1$ and again $q(X-D)\geqslant 2$ if $e(X-D)=0$. This again yields $p_g(X-D)\leqslant 2q(X-D)-4$, and hence a morphism to a general type curve. The rest of the proof of Theorem 1 works and we see that $X_{\min}$ cannot be a $K3$ surface. The proof for the Enriques case is similar and $X_{\min}$ cannot be an Enriques surface.

When $X_{\min}$ is hyperelliptic or abelian surface we see that the union of the rational components, Rat D, of D is again contained in E. Hence we obtain $e(X)=b_2(E)=e(D)$ or $e(D)-1$. Again, $e(D)\leqslant 1+b_2(\text{Rat } D)+r$. Hence either $r=-1$ with Rat $D=E$ or $r=0$ with $b_2(E)\leqslant 1+b_2(\text{Rat } D)$.

Clearly $r=-1$ implies that there is an irrational component of D which is homeomorphic to an elliptic curve. It is easy to see that this component is also smooth. Similarly the other case implies that there are no irrational curves in D and either $D=E$ or E has one more irreducible component than D which is a tip component of E.

Finally consider the case $\kappa(X)=1$. Recall from the proof of Theorem 1 for the case of Kodaira dimension 1 that $X-D$ has an elliptic fibration ϕ which extends to X and D is contained in a fibre, say F_0. In case $e(X-D)=-1$, by Lemma 6 first we see that $e(X_{\min})=0$ and then it follows that D must be the exceptional curve for the map $X\to X_{\min}$.

Consider the case $e(X-D)=0$. In this case it follows that $e(F_0-D)=0$ and all other fibres of ϕ are smooth elliptic curves, if taken with reduced structure. If $e(X_{\min})=0$, then following the proof of Theorem 1 for the hyperelliptic case, first we see that Rat D is contained in E and $r=0$ or -1. When $r=0$ we have $D=E-$ a tip component of E. When $r=-1$, we have $D=F_0$.

If $e(X_{\min})>0$ then we know that $e(F_0)=e(X)=e(X_{\min})+b_2(E)\geqslant 12+b_2(E)$, by Noether's formula. Then from [K] it follows that F_0 is the total transform of a singular fibre of type I_b (with $b\geqslant 12$) or I_b^* (with $b\geqslant 6$), as these are the only possible singular fibres with Euler characteristic $\geqslant 12$ in a minimal degeneration of elliptic curves.

Suppose F_0 is obtained from I_b. Then $e(F_0)=b_2(F_0)=1-b_1(D)+b_2(D)$. From this we deduce that either $b_1(D)=0$ or $b_1(D)=1$. In the first case, F_0-D is isomorphic to $\mathbf{C}^*$ and in the second case $D=F_0$.

Assume now that F_0 is obtained from I_b^*. A similar calculation as above shows that in this case $D=F_0$. This completes the proof of Theorem 2.

The following table summarizes the description of open surfaces with connected boundary and having nonpositive Euler characteristic.

$X_{\min}$	$e(X-D)=0,-1$	$e(X-D)<-1$
1. Rational Surfaces	maps to general type curve or to $\mathbf{C}^*$	maps to general type curve with general fibre $\mathbf{C}$ or $\mathbf{P}^1$
2. Ruled surfaces of genus 1	maps to general type curve or D = exceptional curve or exceptional curve – a tip component or elliptic curve + exceptional curve	maps to general type curve with general fibre $\mathbf{C}$ or $\mathbf{P}^1$
3. Ruled surface of genus $\geqslant 2$	maps to a curve of general type	maps to general type curve with general fibre $\mathbf{C}$ or $\mathbf{P}^1$
4. $K3$ & Enriques surface	not possible	not possible
5. Hyperelliptic surface or Abelian surface	D = exceptional curve or elliptic curve + exceptional curve or exceptional curve – one irreducible component	not possible
6. Properly elliptic	1. $e(X-D)=0$. If $e(X_{\min})=0$ then D = a fibre or D = the exceptional curve – a tip. If $e(X_{\min})>0$ then D = a fibre or D = a fibre – one irreducible component	not possible
	2. $e(X-D)=-1$ then $e(X_{\min})=0$ and D = exceptional curve	not possible
7. General type	not possible	not possible

5. The affine case and the curves in D

In this section we consider affine surfaces with $e(X - D) \leqslant 0$.

From the results of Section 4, it follows that X is either $\mathbf{P}^2$ or a ruled surface because in all the other cases $X - D$ contains complete curves. This is not possible if $X - D$ is affine.

Case 1. Suppose X is rational or ruled over a curve of genus > 1.

In this case we have proved that $e(X - D) < 0$ implies that there is a morphism from $X - D$ to a curve of general type. Applying Lemma 4, we deduce that the general fibre of this morphism is isomorphic to $\mathbf{C}$. Using Iitaka's easy addition formula (cf.[I], Theorem 4), this immediately implies that $\overline{\kappa}(X - D) = -\infty$.

Suppose now that $e(X - D) = 0$. If X is rational then $b_2(D) > b_2(X)$ and hence there is a morphism from $X - D$ onto $\mathbf{C}^*$ with connected general fibre or to a curve of general type. The second case implies as above that the general fibre of this morphism is $\mathbf{C}$ or $\mathbf{C}^*$.

If the general fibre is $\mathbf{C}^*$, then Lemma 4 implies that all the fibres are irreducible and isomorphic to $\mathbf{C}^*$, if taken with reduced structure.

Assume that $X - D$ maps to $\mathbf{C}^*$. Again Lemma 4 implies that all the fibres are irreducible and mutually diffeomorphic, if taken with reduced structure. Zaidenberg's strengthening of Lemma 4 implies that either $X - D \to \mathbf{C}^*$ is a C^∞ fibre bundle or the general fibre of this map is isomorphic to $\mathbf{C}$ or $\mathbf{C}^*$.

Case 2. Suppose that X is ruled over an elliptic curve E. Then there is a morphism $\phi\colon\ X - D \to B$ where, either $B = E$ or B is a curve of general type.

If $B = E$, then Lemma 4 implies that $e(X - D) = 0$. Again as above, all fibres are mutually diffeomorphic, if taken with reduced structure.

If $\overline{\kappa}(X - D) = -\infty$, then from Lemma 2 we see that $\overline{\kappa}(F) = -\infty$ where, F is a general fibre of ϕ.

If $\overline{\kappa}(X - D) = 0$, then by Lemma 2 we see that F is isomorphic to $\mathbf{C}^*$.

Finally, if $\overline{\kappa}(X - D) = 1$, then by Lemma 3 we deduce that $X - D$ admits a $\mathbf{C}^*$-fibration, say g. Then the image of a general fibre of g under ϕ is a point. This easily implies that $\phi = g$.

We have therefore shown that in case $B = E, \phi$ is either a $\mathbf{C}$-fibration or a $\mathbf{C}^*$-fibration.

If B is of general type, then the fibre is again $\mathbf{C}$ or $\mathbf{C}^*$. Assume that $e(X - D) = 0$. In this case, if the general fibre is $\mathbf{C}$ then by Lemma 4 there must be at least one singular fibre of ϕ which is a disjoint union of curves isomorphic to $\mathbf{C}$, with at least 2 irreducible components (cf. [M], Chapter I, Section 6). If the general fibre is $\mathbf{C}^*$, then all the fibres are $\mathbf{C}^*$, if taken with reduced structure.

Finally consider the case $e(X - D) < 0$. Then it follows that the general fibre is $\mathbf{C}$.

Combining the observations above with those in the previous section we get the following result which can be regarded as a generalization of Castelnuovo's theorem mentioned earlier:

'Let (X, D) be a pair of a smooth projective surface and a connected curve D on X such that either $e(X - D) < -1$, or $X - D$ is affine with $e(X - D) < 0$, then there is a morphism from $X - D$ to a curve of general type with general fibre $\mathbf{C}$ *or* $\mathbf{P}^1$.

The following lemma then explores the possibilities for components of D.

LEMMA 7. *Let X be a smooth projective surface and $D \subset X$ be a (not necessarily connected) curve. Let $f : X - D \to B$ be a rational map with general fibre isomorphic to either* $\mathbf{C}$ *or* $\mathbf{C}^*$. *Then every irreducible component of D, except possibily for 2 irreducible components, is rational. Moreover, if f does not extend to a morphism on X then every irreducible component of D is rational.*

Proof: Without loss of generality we may assume that f extends to a morphism $\tilde{f}$ from a blow up $\tilde{X}$ of X at one point and not defined on X. Then the exceptional curve maps onto the base and hence the base B is rational. As the general fibre of f is $\mathbf{C}$ or $\mathbf{C}^*$, it follows that at most one more irreducible component of D, say C_1, can map onto the base. Then C_1 is a cross-section of $\tilde{f}$ and hence rational. All the other irreducible components of D are contained in the fibres of $\tilde{f}$ and hence rational, as the general fibre is $\mathbf{P}^1$. It follows that if the morphism f does not extend to X, then D cannot contain any non-rational irreducible components. If D contains an irrational curve C_1, then C_1 is a cross-section or a $2-$section for $\tilde{f}$ and the other cross-section is the only other irrational irreducible component of D.

The Theorem 3 below stated as in the introduction follows easily from the arguments given so far. We leave the verifications to the reader.

THEOREM 3. *Let X be a smooth complex projective surface and let D be a non-empty connected curve on X such that $e(X - D) \leqslant 0$. Then the following assertions are true:*

(1) *There is a morphism ϕ from $X - D$ to a smooth curve B with general fibre isomorphic to* $\mathbf{P}^1$, $\mathbf{C}$, $\mathbf{C}^*$ *or an elliptic curve, except in the case when $X_{\min}$ is a simple abelian surface and D is the union of all but at most one irreducible components (which is a tip) of the exceptional curve.*

(2)(i) *If the general fibre of ϕ is* $\mathbf{P}^1$, *then all the irreducible components of D are rational.*

(ii) *If the general fibre is* $\mathbf{C}$, *then D has at most one irrational irreducible component.*

(iii) *If the general fibre is an elliptic curve, then D has at most one irrational irreducible component, in which case it is a smooth elliptic curve.*

(iv) *If the general fibre is* $\mathbf{C}^*$, *then D has at most two irrational irreducible components. Moreover, if $q(X) = 0$ then D has at most one irrational irreducible component, in which case it is a hyperelliptic curve.*

COROLLARY 3. *Let $D \subset \mathbf{P}^2$ be a reduced curve such that* $e(\mathbf{P}^2 - D) \leqslant 0$. *Then every irreducible component of D is rational.*

6. Examples

In this section we give some examples of pairs (X, D) of a non-singular projective surface X and a connected curve D such that $e(X - D) = 0$ and which illustrate various possibilities occuring in our results.

EXAMPLE 1. *D can have two distinct irreducible non-rational components*: Let F be a smooth projective curve of genus $\geqslant 1$. Let $X = F \times \mathbf{P}^1$ and $\pi\colon X \to F$ be the projection. Let D be the union of two fibres and two disjoint sections of π. Then D is connected and π restricted to $X - D$ has $\mathbf{C}^*$ as fibres. Hence $e(X - D) = 0$, and D has two non-rational components.

EXAMPLE 2. *Even when X is rational D can have an arbitrarily high genus component:* Let $X = \mathbf{P}^1 \times \mathbf{P}^1$ and D_1 be a general smooth curve of type $(2, d)$. Then the genus $g(D_1) = d - 1$. Since D_1 is ample, we may assume that the degree two map from D_1 onto $\mathbf{P}^1$ by projection has only simple ramification points. The Riemann-Hurwitz formula then implies that the number of ramification points is $2d$. Let D_2 be the curve obtained by taking all the fibres of the projection that are tangent to D_1, and $D = D_1 \cup D_2$. Then $e(X - D) = 0$ and D has a component of genus $d - 1$.

The following example was shown to us by K. Paranjape.

EXAMPLE 3. *In Theorem 2(ii), F_0 can be of type I_b:* Consider the pencil of cubic curves in $\mathbf{A}^2$ given by the equations $y^2 = 4x^3 - \lambda x - \lambda$. Let $X \to \mathbf{P}^1$ be the associated complete pencil. Then it is easy to see that this pencil has exactly 3 singular fibres corresponding to $\lambda = 0, 27$ and ∞. The morphism $j\colon \mathbf{P}^1 \to \mathbf{P}^1$ is $j(\lambda) = \lambda/(\lambda - 27)$, hence an isomorphism. Hence there is exactly one singular fibre of this pencil which has a monodromy of infinite order corresponding to $(\lambda = 27)$. The other singular fibres are of type II and III or III^*. The order of the monodromy is 6 in the first case and 4 in the other case. Let $C \to \mathbf{P}^1$ be a morphism totally ramified over $0, 1, 27$ and ∞ of order 12 and unramified outside these points. Let $\tilde{X} \to C$ be the desingularization of the fibre product of X and C over $\mathbf{P}^1$. The only fibre of this map with nontrivial monodromy is the fibre over the point lying over 27. This implies that all the other singular fibres can be blown down to a smooth fibre of a relatively minimal elliptic fibration. Hence the minimal model has exactly one singular fibre of type I_b.

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