COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 98, n° 3 (1995), p. 305-330
<http://www.numdam.org/item?id=CM_1995__98_3_305_0>

Compositio Mathematica **98**: 305–330, 1995.

Degenerations of moduli of stable bundles over algebraic curves

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Received 27 September 1993; accepted in final form 28 July 1994

0. Introduction

Let X be a smooth projective curve of genus $g \geqslant 2$ over $\mathbf{C}$. For an odd integer d, let $M(2,d)$ (resp. $M(2,\xi)$) denote the space of isomorphism classes of rank two semistable bundles of degree d (resp. degree d with determinant ξ), which is nonsingular and projective. Consider a family of smooth projective curves X_t degenerating to a singular one X_0. Then the space $M_t(2,d)$ (resp. $M_t(2,\xi_t)$) over X_t will subsequently degenerate to a variety $M_0(2,d)$ (resp. M_0). This limit moduli is in no way canonical, depending on what objects over X_0 to be considered. One way to construct such a $M_0(2,d)$ (resp. M_0) is to use torsion free sheaves over the singular curve X_0, as studied by Newstead [8] and Seshadri [11]. Another, introduced by Gieseker [4], utilizes vector bundles over X_0, together with bundles over certain semistable models of X_0. The second method has certain advantages. Indeed, when X_0 is an irreducible curve with a single node, Gieseker has constructed the moduli $M_0(2,d)$ which is irreducible and has only normal crossing singularities.

In this paper we continue Gieseker's work to study the limit of $M_t(2,d)$ and $M_t(2,\xi_t)$ when X_0 consists of two smooth irreducible components meeting at a simple node. Assume that X_0 is obtained by identifying $p \in X_1$ and $q \in X_2$. We first show (Section 1) that the resulting $M_0(2,d)$ has also two smooth irreducible components, intersecting transversally along a divisor (Remark 1.4). Next we prove (Corollary 1.6) that the same is true for M_0 (which will be our main object of study). Denote the two components of M_0 by W_1 and W_2. Then, by interpreting a point in M_0 in terms of semistable bundles over X_1 and X_2, we explicitly build up two smooth projective varieties U_1 and U_2 from the moduli spaces of semistable bundles over X_1 and X_2 (Sections 2 and 3). The natural maps $\alpha_i\colon U_i \to W_i$ $(i = 1,2)$ turn out to be locally free $\mathbf{P}^1$-bundles (Theorems 3.6 and 5.1). Finally, these maps α_i enable us to derive certain properties of W_i, especially the corresponding degeneration of the generalized theta divisor Θ_t in $M_t(2,\xi_t)$ (Theorems 3.15 and 5.3).

The construction of U_1 and U_2 is based on a proposition (Proposition 1.1) that relates Hilbert semistability of a bundle E on X_0 to the semistability of the restrictions $E|_{X_1}$ and $E|_{X_2}$. (For the definition of Hilbert semistability, see [5].) It states that a vector bundle E of degree d over X_0 is Hilbert semistable if and only if $E_i = E|_{X_i}$ are semistable with appropriate degrees $(d_1, d_2) = (\deg(E_1), \deg(E_2))$. There are two choices for such (d_1, d_2) for odd d, corresponding to the fact that M_0 has two components W_1 and W_2. Suppose W_1 corresponds to one of the choices $(d_1, d_2) = (e_1, e_2)$, and assume $(e_1, e_2) = (-1, 0)$ for simplicity. Let B be a generic bundle in W_1, and write $\det(B|_{X_1}) = \xi$ and $\det(B|_{X_2}) = \eta$. Denote by $M_{i,\sigma}$ the moduli of rank two semistable bundles with determinant σ over X_i. There exists a universal bundle E over $X_1 \times M_{1,\xi}$, but none over $X_2 \times M_{2,\eta}$ [9]. However, starting from a universal bundle F' over $X_2 \times M_{2,\eta(q)}$, we can use the Hecke operation to produce a family of semistable bundles F over X_2 with determinant η, parameterized by $N_2 = \mathbf{P}(F_q'^*)$. This operation is defined as follows. A point t in N_2 corresponds to a pair (G, γ), where G is a bundle in F' and γ is a quotient $G_q \to \mathcal{O}_q \to 0$. The bundle F_t is then the modification $\mathrm{Ker}(G \xrightarrow{\gamma} \mathcal{O}_q)$. Since G is stable with det(G) $= \eta(q)$, F_t is semistable with determinant η. Now a Hilbert semistable bundle over X_0 can be obtained by gluing a bundle B_1 in $M_{1,\xi}$ with a bundle B_2 in N_2 along the two fibers $B_{1|p}$ and $B_{2|q}$. This allows us to construct a projective bundle $V_1 = \mathbf{P}(\mathrm{Hom}(E_p, F_q)) \to M_{1,\xi} \times N_2$, where E and F are pullbacks to $X_i \times M_{1,\xi} \times N_2$. V_1 contains all the gluing data, hence there is a natural rational map $\alpha: V_1 \to W_1$. The locus $Z_1 \subset V_1$ where α is not defined comes from the strictly semistable bundles parameterized in N_2. Indeed, if a family of gluing data degenerates to a rank one map $\phi_0: B_{1|p} \to B_{2|q}$, the cokernel of ϕ_0 provides a quotient $\gamma_0: B_{2|q} \to \mathcal{O}_q \to 0$. To produce a Hilbert semistable bundle, we need to modify B_2 again by γ_0. When γ_0 coincides with a semistabilizing quotient of B_2, the modification will be an unstable bundle over X_2, which will subsequently give a bundle which is not Hilbert semistable.

To describe Z_1, we further assume that $g_1 = 1$ for simplicity. So $M_{1,\xi}$ is a single point. Let L be a Poincare bundle over $X_2 \times J_2, J_2 = \mathrm{Jac}(X_2)$, and $p_J: X_2 \times J_2 \to J_2$ the second projection. Let $H = R^1 p_{J*}(L^2(-q \times J_2))$ and consider $\mathbf{P}(H) \xrightarrow{\nu} J_2$. A point in $\mathbf{P}(H)$ over $j \in J_2$ represent a nontrivial extension of j^{-1} by j. Thus $\mathbf{P}(H)$ parameterizes a family of nontrivial extensions given by the bundle $\mathcal{E}$ over $X_2 \times \mathbf{P}(H)$:

$$0 \to \nu^\# L \otimes p_2^* \tau_\nu^* \to \mathcal{E} \xrightarrow{\beta} \nu^\#(L^{-1}(q \times J_2)) \to 0,$$

where τ_ν denotes the tautological subline bundle of $\nu^* H$, $p_2: X_2 \times \mathbf{P}(H) \to \mathbf{P}(H)$, and $\nu^\# = (1 \times \nu)^*$. $\mathcal{E}$ defines a map $\mathbf{P}(H) \xrightarrow{\alpha_h} M_{2,\xi}$, which lifts to a map $\psi_0: \mathbf{P}(H) \to N_2$. The lifting is induced by a bundle $\mathcal{E}'$ (plus certain quotient) over $X_2 \times \mathbf{P}(H)$, given by the following extension:

$$0 \to \nu^\# L \otimes p_2^* \tau_\nu^* \to \mathcal{E}' \to \nu^\# L^{-1} \to 0,$$

which is a modification of the previous one by a natural quotient. $\mathcal{E}'$ is a family of strictly semistable bundles, and $\psi_0(\mathbf{P}(H)) \subset N_2$ will be the strictly semistable locus in N_2. Let E be the pullback of E and consider $\pi_h: Z_h = \mathbf{P}(\mathrm{Hom}(E_p, (\nu^\# L \otimes p_2^* \tau_\nu^*)_q) \to \mathbf{P}(H)$. Then Z_h admits a map ψ_h to V_1, and $Z_1 = \psi_h(Z_h)$. We verify that ψ_h is actually an embedding.

Let T_1 be the preimage in Z_h of the locus where $\mathcal{E}'$ is an extension of line bundles of order two. We then show that the induced map $Z_1 \to N_2$ ramifies along T_1. Hence we first blow up T_1, then blow up the strict transformation of Z_1. These two blowings up will resolve the rational map α. The resulting morphism can be further blown down twice. The first is to blow down the strict transformation of the first exceptional divisor in another direction; the second is essentially to contract along the direction $\nu: \mathbf{P}(H) \to J_2$. The final space we obtain is U_1, and the natural map $U_1 \to W_1$ will be a locally free $\mathbf{P}^1$-bundle. The construction for U_2 and the natural map $\alpha_2: U_2 \to W_2$ are similar.

1. Moduli of Hilbert semistable bundles and geometric realizations

Let X_1 and X_2 be two smooth projective curves of genus $g_1 \geqslant 1$ and $g_2 \geqslant 1$ with fixed points $p \in X_1$ and $q \in X_2$ respectively. Assume that $\pi: X \to C$ is a family of curves of genus $g \geqslant 2$ with both X and C smooth and projective, such that for some $0 \in C, X_0 = \pi^{-1}(0)$ is the singular curve with one node, obtained by identifying $p \in X_1$ with $q \in X_2$, but for $0 \neq t \in C, X_t = \pi^{-1}(t)$ is smooth. As mentioned in the introduction, we will use the theory of Hilbert stability, developed by Gieseker-Morrison [5], to construct a moduli $M_0(2, d)$ over X_0. Such $M_0(2, d)$ respects the degeneration of the curves X_t, and a generic point in it represents a Hilbert semistable bundle over X_0.

Points in $M_0(2, d)$ are characterized by the following two propositions. They can be verified, in one direction, through computations analogous to those carried out in the end of [5], and in the other, by arguments parallel to ([4], Proposition 3.1). Let $X_0' = X_1 \cup X_2 \cup \mathbf{P}^1$ such that $X_1 \cap \mathbf{P}^1 = p, X_2 \cap \mathbf{P}^1 = q$, and no other intersections. Write $c_i = \frac{2g_i - 1}{2(g-1)} d$ and assume d is large.

PROPOSITION 1.1 (Bundles of Type I). *A rank two bundle E of degree d over X_0 is Hilbert semistable if and only if*

(i) *for* $i = 1, 2, E_i = E_{|X_i}$ *is semistable over* X_i, *and*
(ii) $d_i = \deg(E_i)$ *satisfies the inequality* $c_i - 1 \leqslant d_i \leqslant c_i + 1$. □

PROPOSITION 1.2 (Bundles of Type II). *A rank two bundle E' of degree d over X_0' is Hilbert semistable if and only if*

(i) $E'_{|\mathbf{P}^1} = \mathcal{O} \oplus \mathcal{O}(1)$, *and for* $i = 1, 2, E_i' = E'_{|X_i}$ *is semistable,*
(ii) $d_i' = \deg(E_i')$ *satisfies the inequality* $c_i - 1 \leqslant d_i' \leqslant c_i$, *and*

(iii) *E' has the following property: E_1' (resp. E_2') has no semistabilizing quotient identified with the trivial quotient of $E_{\mathbf{P}^1}$ over p (resp. q).* □

PROPOSITION 1.3. *There exists a smooth projective variety $M(2,d)$ and a map $M(2,d) \overset{\varpi}{\to} C$, such that $\varpi^{-1}(t) = M_t(2,d)$ for all $t \neq 0$, and $M_0(2,d) = \varpi^{-1}(0) \subset M(2,d)$ is a divisor with normal crossing singularities.*

Proof. All arguments in ([4], Sect. 4) hold true for our context. □

REMARK 1.4. Since d is odd and $d_1 + d_2 = d$, (d_1, d_2) has exactly two solutions by Proposition 1.1. So the moduli space $M_0(2,d)$ has two components, denoted by $W_i(2,d), i = 1, 2$. Because the inequalities in both propositions are strict for odd d, every Hilbert semistable bundle over X_0 or X_0' is actually Hilbert stable (which will be simply referred to as stable). Bundles of Type I constitute a Zariski open subset of each component, and those of Type II correspond to the boundary. $W_1(2,d)$ and $W_2(2,d)$ naturally glue along these boundaries to form $M_0(2,d)$, since the boundary points in both $W_1(2,d)$ and $W_2(2,d)$ have the same degree distribution by Proposition 1.2 and since X_0' has two ways to deform to X_0 by smoothing away the two nodes separately. Furthermore, the normal crossing property implies that $W_1(2,d)$ and $W_2(2,d)$ are smooth along the boundaries. Since $W_i(2,d)$ $(i = 1,2)$ are clearly smooth away from the boundaries, they are smooth everywhere.

FIXING DETERMINANTS

Let (e_1, e_2) and (h_1, h_2) be the two choices for (d_1, d_2). Then $|e_i - h_i| = 1, i = 1, 2$. One can assume $e_1 = h_1 - 1$ and $e_2 = h_2 + 1$, and arrange $W_1(2,d)$ to correspond to (e_1, e_2) and $W_2(2,d)$ to (h_1, h_2). Let J_i^k be the k-th Jacobian of $X_i, i = 1, 2$.

PROPOSITION 1.5. *There exists a natural surjective map* $\det_1 : W_1(2,d) \to J_1^{e_1} \times J_2^{e_2}$ *(resp.* $\det_2 : W_2(2,d) \to J_1^{h_1} \times J_2^{h_2}$*), and all the fibers of* $\det_1$ *(resp.* $\det_2$*) are isomorphic.*

Proof. Suppose $E \in W_1(2,d)$. If E is of Type I, then define $\det_1(E) = (\det(E_1), \det(E_2))$. If E is of Type II, define $\det_1(E) = (\det(E_1), \det(E_2)(q))$. One sees that $\det_1$ is a morphism. Assume now M_1 and M_2 are two fibers of $\det_1$ and let M_1° and M_2° be their Type I loci. One finds a line bundle L over X_0 which induces a map $M_1^\circ \to M_2^\circ$ by assigning to $E \in M_1^\circ$ the bundle $E \otimes L \in M_2^\circ$. This map can be extended to Type II bundles by similarly tensoring L', where L' is the pull back of L to X_0' through the standard map $X_0' \to X_0$. One checks that the resulting map $M_1 \to M_2$ is an isomorphism. The surjectivity follows from Proposition 1.1. The claims for $\det_2$ are derived by parallel arguments. □

COROLLARY 1.6. *The fibers of* $\det_1$ *(resp.,* $\det_2$*) are smooth and transversal to the Type II locus of $W_1(2,d)$ (resp., $W_2(2,d)$). Hence $M_0 = W_1 \cup W_2$, with W_i smooth and meeting transversally along the divisor of Type II bundles. Here M_0 and W_i are as in the introduction.*

Proof. This follows directly from the smoothness of $W_1(2,d)$ (resp. $W_2(2,d)$), $J_1^{e_1} \times J_2^{e_2}$ (resp. $J_1^{h_1} \times J_2^{h_2}$), and the Type II loci. □

We assume e_1 is odd in the sequel for convenience. Then e_2 is even, and the bundle E_2 (resp. E_1) as in Proposition 1.1 is semistable (resp. stable). Divide Type I into three classes:

I_{st}: E_2 is stable.
I_{sp}: $E_2 = L \oplus M$, where L and M are line bundles of degree $e_2/2$.
I_{ns}: E_2 is a nontrivial extension: $0 \to L \to E \to M \to 0$, with L and M as above.

GEOMETRIC REALIZATIONS

The construction of the spaces U_1 and U_2 employs the method of geometric realization introduced in [4], which we now review and modify in order to serve our context. Let S be a smooth curve and $R \in S$ a fixed point. Let E and F be two vector bundles over S. Call an isomorphism ϕ from E to F over $U = S \backslash R$ a rational isomorphism. For such a ϕ, there is a unique $r \in \mathbf{Z}$ so that ϕ induces a morphism $\phi'\colon E(rR) \to F$ which is nonzero at R. There also exists a unique $s \in \mathbf{Z}$ so that $(\mathrm{coker}(\phi'))_R = \mathcal{O}_R/m_R^s$. We say (r,s) is the type of ϕ.

Now suppose that E (resp. F) is a rank two bundle over $X_1 \times S$ (resp. $X_2 \times S$), which is a semistable family of degree e_1 (resp. e_2) over X_1 (resp. X_2). Let ϕ be a rational isomorphism of type (r,s) between $E_p = E_{|p\times S}$ and $F_q = F_{|q\times S}$. Then $\phi\colon (E_p)_{|U} \cong (F_q)_{|U}$ glues E_U to F_U to yield a stable family of Type I bundles over X_0, parameterized by U. We will extend this U-family to a stable S-family; the latter is called the geometric realization of ϕ. (When dim $S > 1$ and $U \subset S$ a Zariski open subset, we will also refer to each step of extending the stable U-family as a geometric realization.) Notice that we may assume $r = 0$, since we can always replace the family E by $E \otimes \mathcal{O}_{X_1\times S}(r(X_1 \times R))$ when performing the geometric realization. One notational remark: If E is a vector bundle over $X \times T$, then $E_Y = E_{|Y\times T}$ and $E_V = E_{X\times V}$ for $Y \subset X$ and $V \subset T$.

LEMMA 1.7 (Case (0, 1)). *Suppose $s = 1$. One then has an exact sequence $0 \to E_p \xrightarrow{\phi} F_q \xrightarrow{\beta} Q_R \to 0$. Distinguish two subcases:*

(a) *If F_R has no semistabilizing quotient coinciding with $\beta|_R$, then the geometric realization of ϕ gives a bundle of Type II at $R \in S$.*
(b) *If F_R has a semistabilizing quotient $F_R \to M \to 0$ coinciding with $\beta|_R$, then the geometric realization of ϕ gives a bundles of Type I at $R \in S$.*

Proof. (b) Modify F by the $(X_2 \times R)$-supported $M\colon 0 \to F' \to F \to M \to 0$. Then $F'_q \cong \ker(F_q, Q_R)$, which provides an isomorphism $\phi'\colon E_p \cong F'_q$. Using ϕ'

as decent data, one produces a stable family of Type I bundles over X_0, since F'_R is evidently semistable.

(a) Blow up $X_2 \times S$ at $q \times R$ to form a surface X': $X' \xrightarrow{\pi} X_2 \times S$. Let $D_2 = \pi^{-1}(q \times R)$, and let X_2 and $\overline{q \times S}$ be the proper transformations of $X_2 \times R$ and $q \times S$ respectively. Modify $\pi^*(F)$ by $\pi^*(Q_R)$ over X': $0 \to F' \to \pi^*(F) \to \pi^*(Q_R) \to 0$, where $\pi^*(Q_R) = \mathcal{O}_{D_2}$. Write $F'_q = F'|_{\overline{q \times S}}$. Then $F'_q \cong \ker(F_q, Q_R)$, whence ϕ': $E_p \cong F'_q$. Since $F'_{|D_2} = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(1)$ and $F'_{|X_2}$ is stable, gluing E and F' through ϕ': $E_p \cong F'_q$ forms a stable family over S, whose fiber over R is clearly of Type II. □

LEMMA 1.8 (Case (0, 2)). *Suppose $s = 2$. Then one has an exact sequence:* $0 \to E_p \xrightarrow{\phi} F_q \xrightarrow{\beta} Q_{2R} \to 0$. *Suppose F_R has a semistabilizing quotient $F_R \to M \to 0$ coinciding with $\beta \otimes \mathcal{O}_R$. Then it reduces to the case* (0, 1).

Proof. Modify F by the $(X_2 \times R)$-supported M to attain F': $0 \to F' \to F \to M \to 0$. Then F'_q fits in the diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_p & \xrightarrow{\phi'} & F'_q & \xrightarrow{\beta'} & N_R & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_p & \xrightarrow{\phi} & F_q & \xrightarrow{\beta} & Q_{2R} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Q_{2R} \otimes \mathcal{O}_R & = & Q_{2R} \otimes \mathcal{O}_R & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

Hence replacing F by F' transfers the problem to the geometric realization of ϕ' in the first row, which is of type (0,1). □

REMARK 1.9. Lemmas 1.7 and 1.8 work for the derivation of U_1, due to the choice of degrees. If one starts with the pair (h_1, h_2), the only modification one needs is to interchange the roles of X_1 and X_2.

2. Basic constructions

Let X_1 and X_2 be as in the setting of Section 1 and let A be a line bundle over X over C such that for any $t \neq 0, \deg(A_t) = d$, where $A_t = A_{|X_t}$. For the clarity of exposition, we assume $e_1 = -1$ and $e_2 = 0$, since one can conveniently translate the construction to appropriate (e_1, e_2) by tensoring line bundles. So $\deg(A_t) = -1$ for any $t \in C, t \neq 0$. We choose A such that $A_{0|X_1} = \mathcal{O}_{X_1}(-p)$ and $A_{0|X_2} = \mathcal{O}_{X_2}$. Let the corresponding component in M_0 be W_1. Now modify A over X by $A_0|_{X_1}$ to produce a new line bundle A': $0 \to A' \to A \to A_0|_{X_1} \to 0$, so that $A'_0|_{X_1} = \mathcal{O}_{X_1}$ and $A'_0|_{X_2} = \mathcal{O}_{X_2}(-q)$. Then the corresponding component in M_0 is W_2.

This section is the first step to establish U_1 and U_2 under the above assumptions. We will focus on U_1, since the same construction works for U_2 (see Remark 2.14). We will work on the case $g(X_1) = 1$ and $g(X_2) = g > 1$; other cases can be obtained by easy generalization. Hence we assume that E' stands for the unique stable rank two bundle over X_1 with $\det(E') = A_{0|X_1}$.

Denoting $A_{0|X_2}(q) = \mathcal{O}_{X_2}(q)$ by ξ, one has a moduli space $M_{2,\xi}$ of rank two stable bundles over X_2 with determinant ξ. Choose a Poincare bundle F' over $X_2 \times M_{2,\xi}$ such that $\det(F'_q)$ is the ample generator of $\mathrm{Pic}(M_{2,\xi})$. Consider $N_2 = \mathbf{P}(F'^*_q) \xrightarrow{\pi_0} M_{2,\xi}$. Then one obtains a vector bundle F through the following exact sequence over $X_2 \times N_2$: $0 \to F \to \pi_0^\# F' \to \tau_0^* \to 0$, with τ_0^* supported at $q \times N_2$. Here τ_0^* is the dual of the tautological subline bundle of $\pi_0^*(F'^*_q)$. Since F' is a stable family, F represents a family of semistable bundles over X_2, parameterized by N_2. Moreover, $\det(F_v) = \mathcal{O}_{X_2}$ for all $v \in N_2$. Hence F defines a map $\rho_0: N_2 \to M_{2,0}$, where $M_{2,0}$ denotes the moduli space of rank two semistable bundles over X_2 with trivial determinant (modulo S-equivalence). The two maps π_0 and ρ_0 are related as in the following diagram:

$$\begin{array}{ccc} & N_2 & \\ {}^{\pi_0}\swarrow & & \searrow^{\rho_0} \\ M_{2,\xi} & & M_{2,0}. \end{array}$$

Write $E = \pi_{X_1}^* E'$, where $\pi_{X_1}: X_1 \times N_2 \to X_1$ is the first projection. Introduce $V_1 = \mathbf{P}(\mathrm{Hom}(E_p, F_q)) \xrightarrow{\pi_1} N_2$, and let τ_1 be the tautological subline bundle. One then has an exact sequence over V_1 :

$$0 \to \pi_1^* E_p \otimes \tau_1 \xrightarrow{\phi_1} \pi_1^* F_q \xrightarrow{\beta_1} Q_D \to 0, \tag{2.1}$$

with D the rank dropping locus of $\phi_1: \mathcal{O}(D) = \bigwedge^2 \phi_1$.

We want to determine the subvariety $Z_1 \subset V_1$ at which the geometric realization of ϕ_1 produces unstable bundles. Notice that a point $z \in V_1$ belongs to Z_1 if and only if $\beta_1|_z$ results from the restriction to $q \times z$ of a semistabilizing quotient $(\pi_1^\# F)_z \to M \to 0$. Thus to understand Z_1, we first need to locate the strictly semistable bundles in the family F.

Let L be a Poincare bundle over $X_2 \times J_2$, $J_2 = \operatorname{Jac}(X_2)$, and $p_J: X_2 \times J_2 \to J_2$ the second projection. Consider $H = R^1 p_{J_*}(L^2(-q \times J_2))$ and $\mathbf{P}(H) \xrightarrow{\nu} J_2$. A fiber $\mathbf{P}(H_j) = \mathbf{P}(H^1(X_2, j^2(-q)))$ over any $j \in J_2$ represents all nontrivial extensions: $0 \to j \to * \to j^{-1}(q) \to 0$. All such are accommodated in a universal extension over $X_2 \times \mathbf{P}(H)$: $0 \to \nu^\# L \otimes p_2^* \tau_\nu^* \to \mathcal{E} \xrightarrow{\beta} \nu^\#(L^{-1}(q \times J_2)) \to 0$, where τ_ν denotes the tautological subline bundle of $\nu^* H$, and $p_2: X_2 \times \mathbf{P}(H) \to \mathbf{P}(H)$ the second projection. $\mathcal{E}$ is a family of triangular bundles [7], parameterized by $\mathbf{P}(H)$. It supplies a map $\mathbf{P}(H) \xrightarrow{\alpha_h} M_{2,\xi}$, and a lifting $\psi_0: \mathbf{P}(H) \to N_2$. To define the lifting, it suffices to observe that for every $u \in \mathbf{P}(H)$, $\mathcal{E}_u$ is a stable bundle endowed with a linear form $\beta_{|q \times u}$ on $\mathcal{E}_{|q \times u}$. One can describe the map ψ_0 in more detail. Notice that a point $(E, \gamma: E \to \mathcal{O}_q \to 0)$ in N_2 can be interpreted equivalently as a semistable bundle F plus a quotient $\beta: F \to \mathcal{O}_q \to 0$, where F is the modification of E by γ and β is the canonical quotient corresponding to γ. Define a family $\mathcal{E}'$ over $X_2 \times \mathbf{P}(H)$ through the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nu^\# L \otimes p_2^* \tau_\nu^* & \longrightarrow & \mathcal{E}' & \longrightarrow & \nu^\# L^{-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nu^\# L \otimes p_2^* \tau_\nu^* & \longrightarrow & \mathcal{E} & \longrightarrow & \nu^\#(L^{-1}(q \times J_2)) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{S} & = & \mathcal{S} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}
\qquad (2.2)
$$

where $\mathcal{S} = \nu^{\#}(L^{-1}(q \times J_2))|_{q\times \mathbf{P}(H)}$. Consider the canonical quotient $\mathcal{E}' \to \mathcal{T} \to 0$ corresponding to $\mathcal{E} \to \mathcal{S} \to 0$. Then the map ψ_0 is induced from $\mathcal{E}'$ plus the quotient $\mathcal{E}' \to \mathcal{T}$.

Evidently, $\mathcal{E}'$ is a family of strictly semistable bundles, and $\mathcal{E}' = \psi_0^{\#}F$. Further, Lemma 7.3 of [7] claims that $\psi_0(\mathbf{P}(H)) \subset N_2$ is isomorphic to the strictly semistable locus in N_2.

Let $E_h = \pi_{X_1}^* E'$, where π_{X_1} is the first projection $X_1 \times \mathbf{P}(H) \to X_1$, and let $\pi_h: Z_h = \mathbf{P}(\mathrm{Hom}((E_h)_p, (\nu^{\#}L \otimes p_2^* \tau_\nu^*)_q) \to \mathbf{P}(H)$. Then Z_h admits a map ψ_h to V_1, and the destabilizing locus $Z_1 = \psi_h(Z_h)$. We want to show that ψ_h is actually an embedding. The first row in (2.2) provides a section $\theta_h \in H^0(\mathbf{P}(H), R^1 p_{2*}(\nu^{\#}L^2) \otimes \tau_\nu^*)$. The sheaf $R^1_{p_{2*}}(\nu^{\#}L^2)$ over $\mathbf{P}(H)$ is locally free of rank $g-1$ away from $\nu^{-1}(j), j^2 = 0$, and locally free of rank g over such $\nu^{-1}(j)$. Lemma 7.4 of [7] asserts that θ_h is generic. More specifically, θ_h vanishes at a unique point s_j when restricted to the fiber $\nu^{-1}(j)$ for any $j, j^2 \neq 0$. Furthermore, the same lemma shows that $\psi_0: \nu^{-1}(j) \to N_2$ is an embedding for all j and $\psi_0(\nu^{-1}(j))$ meets $\psi_0(\nu^{-1}(j^*))$ $(j^2 \neq 0)$ at the unique point where θ_h vanishes. But s_j and s_{j^*} correspond to two distinct destabilizing quotients of the same bundle $\mathcal{E}'_{s_j} = \mathcal{E}'_{s_{j^*}}$. Thus when lifted to V_1, $\psi_h(\pi_h^{-1}(\nu^{-1}(j)))$ does not meet $\psi_h(\pi_h^{-1}(\nu^{-1}(j^*)))$. Moreover, there is no other intersections between the ψ_h-images of two fibers of $\nu \circ \pi_h$. Consequently, we have proved the following proposition.

PROPOSITION 2.3. *The destablizing subvariety Z_1 in V_1 for the geometric realization of ϕ_1 is isomorphic to $Z_h \cong \mathbf{P}(H) \times \mathbf{P}^1$.* □

Before extending the morphism $V_1 \backslash Z_1 \to W_1$, we digress for a moment to describe the types of bundles parameterized by $V_1 \backslash Z_1$. By the above discussion, the zeroes of θ_h defines a section s of ν away from $j \in J_2, j^2 = 0$.

LEMMA 2.4. *The schematic closure θ of s in $\mathbf{P}(H)$ is isomorphic to the blowing up of J_2 simultaneously at all points of order two. (So $\theta_n =: \theta \backslash s = \bigcup_{j\in J_2, j^2=0} \mathbf{P}_j^{g-1}$, where $\mathbf{P}_j^{g-1}$ is the exceptional divisor over j.)*

Proof. by functoriality $R^1 p_{2*}(\nu^{\#}L^2) = \nu^*(R^1 p_{J*}(L^2))$. Choose the Poincare bundle L over $X_2 \times J_2$ such that $L_q = \mathcal{O}_{J_2}$ for simplicity. Taking direct image of the exact sequence: $0 \to L^2(-(q \times J_2)) \to L^2 \to L_q^2 \to 0$ produces another one over J_2: $0 \to \mathcal{O}_{J_2} \to R^1 p_{J*}(L^2(-(q \times J_2))) \to R^1 p_{J*}(L^2) \to 0$. Pulling back to $\mathbf{P}(H)$ then tensoring by τ_ν^*, one has

$$0 \to \tau_\nu^* \to \nu^*(R^1 p_{J*}(L^2(-(q \times J_2)))) \otimes \tau_\nu^* \to \nu^*(R^1 p_{J*}(L^2)) \otimes \tau_\nu^* \to 0.$$

Write $\nu^*(R^1p_{J_*}(L^2(-(q \times J_2)))) \otimes \tau_\nu^* = R$ and $\nu^*(R^1p_{J_*}(L^2)) \otimes \tau_\nu^* = T$. Then R is locally free of rank g and $T = R^1p_{2_*}(\nu^\# L^2) \otimes \tau_\nu^*$. The section θ_h induces a diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau_\nu^* & \longrightarrow & R' & \longrightarrow & \mathcal{O} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \sigma_h} & & \downarrow{\scriptstyle \theta_h} & & \\
0 & \longrightarrow & \tau_\nu^* & \longrightarrow & R & \longrightarrow & T & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T/\mathcal{O} & = & T/\mathcal{O} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}
$$

We claim that the nonlocally free support $\theta' = s\bigcup(\cup_{j \in J_2, j^2=0}\nu^{-1}(j))$ of $T/\mathcal{O}$ is reduced and irreducible, hence isomorphic to J_2 blown up at all points of order two. Indeed, the above diagram says that θ' equals the first degeneracy locus associated to σ_h, and σ_h is locally represented by a $2 \times g$ matrix. But $\sigma_h|_{\tau_\nu^*} = \text{id}$ implies that this matrix takes the form

$$
\begin{bmatrix} 1 & 0 & \dots & 0 \\ c_1 & c_2 & \dots & c_g \end{bmatrix}
$$

with respect to suitable bases. So θ' is cut out by at most $(g-1)$ functions, so every component of θ' has dimension $\geqslant (2g-1)-(g-1) = g$. In particular θ' has no $\nu^{-1}(j)$ as component, since $\nu^{-1}(j)$ has dimension $g-1$. Thus θ' is irreducible, g dimensional, and Cohen-Macaulay [1]. It follows that θ' has no embedded components, hence is reduced along each $\nu^{-1}(j)$. This shows that θ' can be identified with the blown up of J_2 at all $j, j^2 = 0$. But the irreducibility of θ and the inclusion $\theta \subset \theta'$ immediately imply $\theta = \theta'$. □

For the mentioned description of types, we also need to form $P_h = \mathbf{P}(\mathrm{Hom}((E_h)_p, \mathcal{E}'_q)) \xrightarrow{\rho_h} \mathbf{P}(H)$. Then we have an exact sequence analogous to (2.1) over P_h:

$$0 \to \rho_h^*((E_h)_p) \otimes \tau_h \xrightarrow{\phi_h} \rho_h^*(\mathcal{E}'_q) \xrightarrow{\beta_h} Q_{D_h} \to 0, \tag{2.5}$$

with τ_h the tautological subline bundle associated to ρ_h. There exists a natural lifting of ψ_0 to a map ψ_1:

$$\begin{array}{ccc} P_h & \xrightarrow{\psi_1} & V_1 \\ \Big\downarrow{\scriptstyle \rho_h} & & \Big\downarrow{\scriptstyle \pi_1} \\ \mathbf{P}(H) & \xrightarrow{\psi_0} & N_2 \end{array}$$

so that (2.5) is the pullback of (2.1) by ψ_1.

Let $\Delta = \psi_1(P_h), \Theta = \pi_1(\rho_h^{-1}(\theta))$, and $\Theta_n = \psi_1(\rho_h^{-1}(\theta_n))$. Then, under the geometric realization of $\phi_1, D\backslash Z_1 \subset \mathrm{II}, V_1\backslash(D \cup \Delta) \subset \mathrm{I}_{st}, \Delta\backslash(D \cup (\Theta\backslash\Theta_n)) \subset \mathrm{I}_{ns}$, and $(\Theta\backslash\Theta_n)\backslash D \subset \mathrm{I}_{sp}$.

Now we go back to resolve the rational map $V_1 \to W_1$. It will take two steps. First we blow up a subvariety $T_1 \subset Z_1$, then blow up the strict transformation of Z_1. Write $T_j = \psi_h((\nu \circ \pi_h)^{-1}(j))$ for $j \in J_2$. Then $T_1 = \cup_{j \in J_2, j^2=0} T_j$.

LEMMA 2.6. *T_1 can be characterized by the property that $d\psi_0$ fails to inject along $\pi_h(T_1)$. Moreover,* $\ker(d\psi_0)|_{T_1}$ *is a line bundle over T_1.*

Proof. A point in $\mathbf{P}(H)$ gives a bundle E which is an extension $0 \to j \to E \to j^* \to 0$. The subline bundle j deforms infinitesimally inside E if and only if $H^0(X_2, j^2) \neq 0$, or $j^2 = 0$. This will imply that $d\psi_0$ drops rank along T_1. The assertion that $\ker(d\psi_0)|_{T_1}$ is locally free of rank 1 is due to the fact that $H^0(X_2, j^2) = \mathbf{C}$ for $j^2 = 0$ (cf. Proposition 6.8, [7]). □

Blow up V_1 along T_1 to achieve V_2: $V_2 \xrightarrow{\pi_2} V_1$. Let $T_2 = \pi_2^{-1}(T_1)$ and Z_2 be the proper transformation of Z_1. The exact sequence (2.1) becomes: $0 \to E_p^{(1)} \to F_q^{(1)} \to Q_D^{(1)} \to 0$ when pulled back to V_2. It induces an exact sequence:

$$0 \to E_p^{(1)} \xrightarrow{\phi_2} F_q^{(2)} \xrightarrow{\beta_2} Q_D^{(1)} \otimes \mathcal{O}_D(-T_2) \to 0. \tag{2.7}$$

Let Q' be the invertible $(X_2 \times T_1)$-quotient $\pi_1^\#(F) \xrightarrow{\beta} Q' \to 0$ over $X_2 \times V_1$, such that $\beta|_{q\times T_1} = \beta_1|_{T_1}$. Let $Q_{X_2\times T_2} = \pi_2^\#(Q')$. Then $F_q^{(2)}$ is the restriction to $q \times V_2$ of the bundle modification over $X_2 \times V_2$:

$$0 \to F^{(2)} \to F^{(1)} \to Q_{X_2\times T_2} \to 0. \tag{2.8}$$

To examine the geometric realization of ϕ_2, one needs to inspect the splitting situation of $F^{(2)}$. We first state the following proposition.

PROPOSITION 2.9. *The unstable locus in V_2 for the geometric realization of ϕ_2 is Z_2.*

The proof requires a lemma. Let $S_0 = \psi_0(\mathbf{P}(H)) \subset N_2$. Let F be the bundle specified in the beginning of this section. Let $u \in N_2$ represents a semistable bundle F_u which is an extension: $0 \to M \to F_u \to M^{-1} \to 0$ for some $M \in \mathrm{Jac}(X_2)$. Suppose Y is a smooth curve in N_2 passing through u. Modify the family F_Y by $(X_2 \times u)$-supported $M^{-1}: 0 \to F'' \to F_Y \to M^{-1} \to 0$.

LEMMA 2.10. *If F''_u splits, then $T_{u,Y} \subset TC_{u,S_0}$, where TC denotes tangent cone.*

Proof. Suppose F''_u splits. Then $F_Y \to M^{-1} \to 0$ lifts to a quotient $F_Y \to M' \to 0$, where M' is a line bundle over $X_2 \times Y_\epsilon$. Here $Y_\epsilon = \mathrm{Spec}(\mathcal{O}_{u,Y}/m^2)$, $m =$ the maximal ideal of $\mathcal{O}_{u,Y}$ at u. By the property of ψ_0, the inclusion $Y_\epsilon \to N_2$ factors through $\mathbf{P}(H)$. □

Proof of Proposition 2.9. Let $\pi_T = \pi_2|_{T_2}: T_2 \to T_1$, which is a $\mathbf{P}^{2g}$-bundle. Restricting (2.8) to $X_2 \times T_2$ suggests the following exact sequence:

$$0 \to Q_{X_2\times T_2} \otimes \pi_T^{\#}\tau_T^{-1} \to F_{T_2}^{(2)} \overset{\beta_T}{\to} Q_{X_2\times T_2}^{-1} \to 0, \tag{2.11}$$

where τ_T is the tautological line bundle associated to π_T. This extension defines a section $s \in H^0(T_2, R^1p_{2*}(Q^2_{X_2\times T_2}) \otimes \tau_T^{-1})$ over T_2, where $p_2: X_2 \times T_2 \to T_2$ is the second projection. Clearly the sheaf $R^1p_{2*}(Q^2_{X_2\times T_2})$ is locally free of rank g. We claim that the section s is generic. Indeed, since $R^1p_{2*}(Q^2_{X_2\times T_2})$ is trivial along the fibers of $T_2 \to T_1$, zero$(s) = \mathbf{P}^r$-bundle over T_1 for some $r \geqslant g$. On the other hand, Lemmas 2.10 and 2.6 shows that $r \leqslant g$ by dimension counting. Hence zero$(s) = \mathbf{P}^g$-bundle, which means s is generic. Observe that the extension (2.11) splits at $y \in T_2$ if and only if $y \in$ zero(s). Since the locus where β_T in (2.11) coincides with β_2 in (2.7) over a point in T_1 is of codimension one in the splitting locus zero(s), the coinciding locus G inside zero(s) is a $\mathbf{P}^{g-1}$ bundle over T_1. On the other hand, $\mathrm{codim}(T_1, Z_1) = ((2g-1)+1)-(g) = g$ implies that $Z_2 \cap T_2$ is also a $\mathbf{P}^{g-1}$-bundle over T_1. The fact that $Z_2 \cap T_2 \subset G$ forces $Z_2 \cap T_2 = G$, confirming that G is identified with the exceptional divisor of Z_2 under π_2. Therefore, the unstable locus for the geometric realization of ϕ_2 is exactly Z_2. □

Now blow up V_2 along Z_2 to create V_3: $V_3 \overset{\pi_3}{\to} V_2$. Let $Z_3 = \pi_3^{-1}(Z_2)$ and T_3 be the strict transformation of T_2 in V_3. Pull back the exact sequence (2.7) to V_3 to yield another one:

$$0 \to E_p^{(2)} \overset{\phi_3}{\to} F_q^{(4)} \overset{\beta_3}{\to} Q_D^{(2)} \otimes \mathcal{O}_D(-T_3 - Z_3) \to 0. \tag{2.12}$$

PROPOSITION 2.13. *ϕ_3 realizes stable bundles over the entire V_3.*

Proof. We need to analyze the splitting situation of $F^{(4)}: 0 \to F^{(4)} \to F^{(3)} \to Q_{X_2 \times Z_3} \to 0$, where $F^{(3)} = \pi_3^\# F^{(2)}$ and $Q_{X_2\times Z_3}$ is interpreted similarly as $Q_{X_2\times T_2}$ in (2.8). When restricted to $X_2 \times Z_3$, we derive an extension analogous to (2.11) and an $s' \in H^0(Z_3, R^1p_{2*}(Q^2_{X_2\times X_3}) \otimes \tau_Z^{-1})$ over Z_3. Here $p_2: X_2 \times Z_3 \to Z_3$ is the second projection and τ_Z the tautological line bundle associated to $\pi_Z = \pi_3|_{Z_3}: Z_3 \to Z_2$.

First, we assume $y \in Z_2 \backslash T_2$. One argues as in Proposition 2.9 that the section s' is generic over such y. Since $R^1p_{2*}(Q^2_{X_2\times Z_3})$ is locally free of rank $g-1$ along the fiber over y, the splitting locus of $F^{(4)}$ in $\pi_Z^{-1}(y)$ equals a $\mathbf{P}^1$. But the coinciding locus is of codimension two inside the splitting locus for such y, so it is empty. Thus $\phi_3|_{\pi_Z^{-1}(y)}$ realizes stable bundles.

We now take $y \in Z_2 \cap T_2$. In order to understand zero(s') over such y, we study modifications of 1-dimensional family around y inside V_2. Take any smooth curve $Y \subset V_2$ passing through y. Since $\mathrm{codim}(Z_2, V_2) = (3g+1) - (2g) = g+1 = \mathrm{codim}(T_2 \cap Z_2, T_2)$, $\pi_Z^{-1}(y)$ is contained in the exceptional divisor of T_3 under π_3. Thus it suffices to choose Y inside T_2. Let $\pi_T(f)$ stands for a fiber of $\pi_T: T_2 \to T_1$. From the proof of Proposition 2.9, $Z_2 \cap \pi_T(f) = \mathbf{P}^{g-1}$ which has codimension $g+1$ in $\pi_T(f)$. So we can essentially limit Y inside $\pi_T(f)$. In other words, we have reduced to the case of examining the splitting possibilities when we blow up $\pi_T(f)$ along the $\mathbf{P}^{g-1}$. Write $s_T(f) = \mathrm{zero}(s)|_{\pi_T(f)}$, with s as in the proof of Proposition 2.9. Then $\mathrm{codim}(Z_2 \cap \pi_T(f), s_T(f)) = 1$. Observe that when restricting (2.11) to $X_2 \times s_T(f)$, the induced extension:

$$0 \to (Q_{X_2\times T_2} \otimes \pi_T^\# \tau_T^*)_{s_T(f)} \to F^{(2)}_{s_T(f)} \to (Q^{-1}_{X_2\times T_2})_{s_T(f)} \to 0$$

splits. We can then reverse this exact sequence:

$$0 \to (Q^{-1}_{X_2\times T_2})_{s_T(f)} \to F^{(2)}_{s_T(f)} \xrightarrow{\beta_t} (Q_{X_2\times T_2} \otimes \pi_T^\# \tau_T^*)_{s_T(f)} \to 0.$$

The destablizing property of β_2 from (2.7) over $Z_2 \cap \pi_T(f)$ means that $B_2|_{Z_2\cap\pi_T(f)}$ coincides with $\beta_f|_{q\times(Z_2\cap\pi_T(f))}$. Suppose we select $Y \subset \pi_T(f)$ such that Y is transversal to $s_T(f)$. Then (2.11) gives a diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{S} & \longrightarrow & F'_Y & \longrightarrow & (Q^{-1})_Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & (Q \otimes \pi_T^{\#}\tau_T^{*})_Y & \longrightarrow & F_Y^{(2)} & \longrightarrow & (Q^{-1})_Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & (Q \otimes \pi_T^{\#}\tau_T^{*})_y & = & (Q \otimes \pi_T^{\#}\tau_T^{*})_y & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0, & & & &
\end{array}
$$

where $Q = Q_{X_2 \times T_2}$ and $\mathcal{S} = (Q \otimes \pi_T^{\#}\tau_T^{*})_Y(-(X_2 \times y))$. The first row defines a section $s_Y \in H^0(R^1\pi_{Y*}((Q^2)_Y) \otimes \tau_T^{*}|_Y(-y))$. If s_Y vanishes at y, then it vanishes at y to the second order when considered as a section of $R^1\pi_{Y*}((Q^2)_Y) \otimes \tau_T^{*}|_Y$. But s_Y equals $s|_Y$ in $H^0(R^1\pi_{Y*}((Q^2)_Y) \otimes \tau_T^{*}|_Y)$, contradicting the fact that s has only simple zeroes. Therefore, F'_y does not split for such Y. When we take $Y \subset s_T(f)$, on the other hand, the resulting F'_y clearly splits. It follows from $\operatorname{codim}(Z_2 \cap \pi_T(f), s_T(f)) = 1$ that $F^{(4)}_{\pi_Z^{-1}(y)}$ splits in a single point not contained in D. One then concludes that $\phi_3|_{\pi_Z^{-1}(y)}$ is stable. This completes the proof of stability of ϕ_3 over V_3. □

Therefore, there exists a morphism $V_3 \to W_1$ induced by the geometric realization of ϕ_3. We will show in the next section that this morphism factors through two blowings down; the resulting morphism $\alpha_1 \colon U_1 \to W_1$ is a locally free $\mathbf{P}^1$-bundle.

We can easily see that a point in $D \subset V_3$ represents a Type II bundle, and a point in $V_3 \backslash D$ features Type I. For bundles of Type I in Z_3, $\operatorname{zero}(s') \backslash D \subset \mathrm{I}_{sp}$, and $Z_3 \backslash (D \cup \operatorname{zero}(s')) \subset \mathrm{I}_{ns}$. Away from Z_3, ϕ_3 is isomorphic to ϕ_2. Thus the types over $V_3 \backslash Z_3$ coincide with that for ϕ_2, as mentioned immediately after the proof of Proposition 2.9.

REMARK 2.14. For the second component U_2, we consider the following:

(i) The smooth moduli $U_{X_2}(2, \mathcal{O}_{X_2}(-q))$ and a universal bundle E over $X_2 \times U_{X_2}(2, \mathcal{O}_{X_2}(-q))$. No modifications will happen to E, as one can see from the construction of U_1.

(ii) The moduli $M_{X_1}(2, \mathcal{O}(p))$ (a single point) and the unique bundle F' over X_1 parameterized by $M_{X_1}(2, \mathcal{O}(p))$. The Hecke operation and all the subsequent modifications are applied to this F'.

If $U_{X_2}(2, \mathcal{O}_{X_2}(-q))$ were a single point, then the construction parallels the one we have already discussed. But the magnitude of $U_{X_2}(2, \mathcal{O}_{X_2}(-q))$ does not introduce any new difficulty, because E is essentially fixed during the whole process. In other words, one obtains a family of those constructions parameterized by $U_{X_2}(2, \mathcal{O}_{X_2}(-q))$.

3. Blowings down and related computations

In this section we first blow down V_3 twice to obtain U_1, the show that the natural map $\alpha_1: U_1 \to W_1$ is a $\mathbf{P}^1$-bundle and compute the relative differential sheaf Ω_{α_1}. We will also state the variations for U_2. In the end, we describe the corresponding degeneration of the generalized theta divisor Θ_t in $\operatorname{Pic}(M_t(2, A_t))$.

The strict transformation T_3 of the first exceptional divisor T_2 under π_3 gains a ruling by blowing up T_2 along G. Contracting T_3 along this ruling constitutes the first blowing down. The second basically contracts Z_3 along the direction $\nu: \mathbf{P}(H) \to J_2$.

LEMMA 3.1. *Let $\tilde{G}$ denote the exceptional divisor of $\pi_3|_{T_3}: T_3 \to T_2$. Then $\tilde{G} = G \times_{T_1} G'$ where $G' \xrightarrow{\beta'} T_1$ is a $\mathbf{P}^g$-bundle. Moreover, there exists a map $T_3 \xrightarrow{\gamma'} G'$ which is a $\mathbf{P}^g$-bundle.*

Proof. We illustrate these by defining G', β' and γ'. Since $Z_1 \cong \mathbf{P}(H) \times \mathbf{P}^1$ and $T_j \cong \nu^{-1}(j) \times \mathbf{P}^1$, it follows that $N_{T_1/Z_1} \cong \mathcal{O}^{\oplus g}$. Hence $G = T_1 \times \mathbf{P}^{g-1}$. Let s be any trivial section of the projection $G \to T_1$. Then take $G' = \mathbf{P}(N_{G/T_2}|_s)$ and $\beta': G' \to s \cong T_1$. One checks that $\tilde{G} = G \times_{T_1} G'$.

The map $T_3 \to T_1$ naturally factors through G':

$$\begin{array}{ccc} T_3 & \longrightarrow & G' \\ & {\scriptstyle \pi_3} \searrow \quad \swarrow {\scriptstyle \beta'} & \\ & T_1. & \end{array}$$

Then define γ' to be the horizontal map $T_3 \to G'$, which will have the desired property. □

PROPOSITION 3.2. *V_3 can be blown down along $T_3 \xrightarrow{\gamma'} G'$ to a smooth parameterizing variety V_4: $V_3 \xrightarrow{\pi_4} V_4$.*

Proof. We first show $N_{T_3/V_3}|_{\gamma'^{-1}(g)} = \mathcal{O}(-1)$ for every $g \in G'$. From the natural identities: $N_{T_3/V_3} = K_{T_3} \otimes K_{V_3}^{-1}, K_{T_3} = \pi_3^* K_{T_2} \otimes \mathcal{O}_{T_3}(g\tilde{G})$, and $K_{V_3} = \pi_3^* K_{V_2} \otimes \mathcal{O}_{V_3}(gZ_3)$, it follows that $N_{T_3/V_3} = \pi_3^*(K_{T_2} \otimes K_{V_2}^{-1})$. Similarly, $K_{T_2} \otimes K_{V_2}^{-1} = \pi_T^*(K_{T_1} \otimes K_{V_1}^{-1} \otimes M) \otimes \mathcal{O}_{T_2}(-\sigma_T)$, where σ_T is the tautological divisor associated to $T_2 \xrightarrow{\pi_T} T_1$ and M a line bundle on T_1. Thus

$$N_{T_3/V_3} = (\pi_T \circ \pi_3)^*(K_{T_1} \otimes K_{V_1}^{-1} \otimes M) \otimes \pi_3^*(\mathcal{O}_{T_2}(-\sigma_T)).$$

It follows from $\sigma_T|_{\gamma'^{-1}(g)} = 1$ and $(\pi_T \circ \pi_3)^*(K_{T_1} \otimes K_{V_1}^{-1} \otimes M)|_{\gamma'^{-1}(g)} = 0$ that $N_{T_3/V_3|\gamma'^{-1}(g)} = \mathcal{O}(-1)$.

We now prove that every fiber of γ' represents a single stable bundle over X_0. Choose any $t \in T_1$ and a fiber of γ' over a point in $\beta'^{-1}(t)$. This fiber is represented by a $P \subset \pi_T(f) = \pi_T^{-1}(t), P = \mathbf{P}^g$. If P intersects $s_T(f)$ transversally, then a diagram similar to the one in the proof of Proposition 2.13 shows that $F_P^{(4)}$ is a family of nontrivial extensions of a line bundle R by R^{-1}, with $R \in \mathrm{Jac}(X_2)$ and $R^2 = \mathcal{O}$. Since $h^1(X_2, R^2) = g$, there exists a universal extension over $X_2 \times \mathbf{P}^{g-1}, \mathbf{P}^{g-1} = \mathbf{P}(H^1(X_2, R^2))$. Hence one has a map $P \to \mathbf{P}^{g-1}$, which has to be constant because $P = \mathbf{P}^g$. It follows that P parameterizes a unique nontrivial extension, denoted by F'. On the other hand, Lemma 4.2 (see Section 4) shows that the moduli derived from the original E' over X_1 and this F' has image Q_0 in W_1, where Q_0 is the blowing down of $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2))$ along the (-2)-curve C_0. Recall that the Type II locus $C_1 \cong \mathbf{P}^1$ in Q_0 is ample. One then simply argues that the induced map $P \to Q_0$ has to be constant.

The first paragraph of the proof says we can blow down V_3 smoothly, and the second asserts that the resulting V_4 remains to parameterize stable bundles over X_0.

Let Z_4 be the image of Z_3 in V_4. Since $\pi_3^{-1}(G) = \tilde{G} = T_3 \cap Z_3$ and $\tilde{G} = G \times_{T_1} G', Z_4$ is the blowing down of Z_3 along $\tilde{G} \to G'$. One can show as in Proposition 3.2 that Z_4 is smooth. Moreover, the blowing down $\pi_4 \colon Z_4 \to Z_3$ covers that of $Z_2 \to Z_1$. Namely one has a commutative diagram:

$$\begin{array}{ccc} Z_3 & \xrightarrow{\pi_4} & Z_4 \\ \Big\downarrow{\scriptstyle \pi_3} & & \Big\downarrow{\scriptstyle \pi_4''} \\ Z_2 & \xrightarrow{\pi_2} & Z_1. \end{array}$$

The map π_4'' is a $\mathbf{P}^g$-bundle. Recall that $Z_1 = \mathbf{P}(H) \times \mathbf{P}^1$.

LEMMA 3.3. *$Z_4 = Z_1 \times_{(J_2\times \mathbf{P}^1)} G''$ where $G'' \xrightarrow{\beta''} J_2 \times \mathbf{P}^1$ is a $\mathbf{P}^g$-bundle. Furthermore, the map $Z_4 \to G''$, denoted by γ'', is a $\mathbf{P}^{g-1}$-bundle.*

Proof. For any $j \in J_2$ and $t \in \mathbf{P}^1$, $\pi_4''^{-1}(\nu^{-1}(j)\times t) = (\nu^{-1}(j)\times t)\times \mathbf{P}^g_{(j,t)}$. Such $\mathbf{P}^g_{(j,t)}$ fits together to give G''. The rest follows. □

PROPOSITION 3.4. *V_4 can be smoothly blown down along $Z_4 \xrightarrow{\gamma''} G''$ to a parameterizing variety U_1: $V_4 \xrightarrow{\pi_5} U_1$.*

Proof. For fixed $(j,t)\in J_2\times \mathbf{P}^1, (h\times t)\times \mathbf{P}^g_{(j,t)} \subset Z_4$ parameterizes the same family of stable bundles over X_0 for all $h\in \nu^{-1}(j)$. So it suffices to show that $N_{Z_4/V_4}|_{\nu^{-1}(j)\times t} = \mathcal{O}(-1)$, since $\nu_{t,j} := \nu^{-1}(j)\times t = \gamma''^{-1}(g)$ for some $g\in G''$. It can be further reduced to computing $N_{Z_3/V_3}|_{\nu_{t,j}} = \mathcal{O}(-1)$ for any $j, j^2 \neq 0$, due to the fact that π_4 blows down along T_3, which is away from such $\nu_{t,j}$. From $N_{Z_3/V_3} = \mathcal{O}(Z_3)\otimes \mathcal{O}_{Z_3} = K_{Z_3}\otimes K_{V_3}^{-1}$, one computes

$$N_{Z_3/V_3} = (\pi_2\circ\pi_3)^*(K_{Z_1}\otimes K_{V_1}^{-1})|_{Z_3}\otimes \omega_{\pi_Z}\otimes \mathcal{O}(-(g+1)\tilde{G})\otimes \mathcal{O}(-gZ_3).$$

Hence $(g+1)N_{Z_3/V_3} = (\pi_2\circ\pi_3)^*(K_{Z_1}\otimes K_{V_1}^{-1})|_{Z_3}\otimes\omega_{\pi_Z}\otimes\mathcal{O}(-(g+1)\tilde{G})$. Restricting to $\nu_{t,j}$ gives $(g+1)N_{Z_3/V_3}|_{\nu_{t,j}} = (\pi_2\circ\pi_3)^*(K_{Z_1}\otimes K_{V_1}^{-1})|_{\nu_{t,j}}$. Thus to show $N_{Z_3/V_3}|_{\nu_{t,j}} = \mathcal{O}(-1)$, it is equivalent to show $\det(N_{Z_1/V_1})|_{\nu_{t,j}} = \mathcal{O}(-g-1)$. By the following Lemma 3.5, $\det(N_{Z_1/V_1})|_{\nu_{t,j}} = \det(N_{\nu_{t,j}/V_1}) = K_{\nu_{t,j}}\otimes K_{V_1}^{-1} = \mathcal{O}(-g)\otimes(\Theta_{2,\xi}^3\otimes\Theta_{2,0}^2\otimes\tau_1^{-4})|_{\nu_{t,j}}$. Consequently, we can complete the proof by verifying that $\Theta_{2,\xi}|_{\nu_{t,j}} = 1, \Theta_{2,0}|_{\nu_{t,j}} = 0$, and $\tau_1|_{\nu_{t,j}} = 1$. First $\Theta_{2,0}|_{\nu_{t,j}} = 0$ stands because when considering $\nu_{t,j}$ as sitting inside N_2, $\rho_0(\nu_{t,j})$ is a single point in $M_{2,0}$. Next after identifying $\nu_{t,j}$ with its image in $M_{2,\xi}$, Lemma 6.22 (i) of [7] shows that $\det(N_{\nu_{t,j}/M_{2,\xi}}) = -(g-2)$. But $\det(N_{\nu_{t,j}/M_{2,\xi}}) = K_{\nu_{t,j}}\otimes K_{M_{2,\xi}}^{-1} = \mathcal{O}(-g)\otimes\Theta_{2,\xi}^2$, whence $\Theta_{2,\xi}|_{\nu_{t,j}} = 1$. Finally, the universality of (V_1,π_1) and the definition of Z_1 (hence of $\nu_{t,j}$) lead to $\tau_1|_{\nu_{t,j}} = 1$. □

LEMMA 3.5. *Let $\Theta_{2,0}$ and $\Theta_{2,\xi}$ be the ample generators of $\mathrm{Pic}(M_{2,0})$ and $\mathrm{Pic}(M_{2,\xi})$ respectively. Denote also by $\Theta_{2,0}$ and $\Theta_{2,\xi}$ their natural pullbacks. Then $K_{V_1} = \Theta_{2,\xi}^{-3}\otimes\Theta_{2,0}^{-2}\otimes\tau_1^4$.*

Proof. It is known that $K_{N_2} = \pi_0^*\Theta_{2,\xi}^{-1}\otimes\rho_0^*\Theta_{2,0}^{-2}$ [2]. From the exact sequence over V_1: $0\to\tau_1\to\pi_1^*\mathrm{Hom}(E_p,F_q)\to\tau_1\otimes\Omega_{\pi_1}^\vee\to 0$, one computes $\omega_{\pi_1} = \Theta_{2,\xi}^{-2}\otimes\tau_1^4$. Thus $K_{V_1} = \pi_1^*K_{N_2}\otimes\omega_{\pi_1} = \Theta_{2,\xi}^{-3}\otimes\Theta_{2,0}^{-2}\otimes\tau_1^4$. □

THEOREM 3.6. *The natural map* $\alpha_1 \colon U_1 \to W_1$ *is a locally free* $\mathbf{P}^1$*-bundle. So one has a diagram:*

$$\begin{array}{ccccccccc} V_1 & \longleftarrow & V_2 & \longleftarrow & V_3 & \longrightarrow & V_4 & \longrightarrow & U_1 \\ {\scriptstyle \pi_1}\downarrow & & & & & & & & \downarrow{\scriptstyle \alpha_1} \\ N_2 & & & & & & & & W_1. \end{array}$$

We need to establish two lemmas for its proof. Let $M_{2,0}^{\circ} \subset M_{2,0}$ *and* $N_2^{\circ} \subset N_2$ *be the open subsets representing stable bundles over* X_2 *with trivial determinant, and let* $V_1^{\circ} = \pi_1^{-1}(N_2^{\circ})$*. Denote by* Δ_U *the final proper transformation of* $\Delta \subset V_1$ *in* U_1*, and write* $U_1^{\circ} = U_1 \backslash \Delta_U, \Delta_W = \alpha_1(\Delta_U)$*, and* $W_1^{\circ} = W_1 \backslash \Delta_W$*. Notice that* $\operatorname{codim}(\Delta, V_1) = \operatorname{codim}(\Delta_U, U_1) = g - 1$*. Since* Δ_U *represents exactly the bundles over* X_0 *coming from strictly semistable bundles over* $X_2, V_1^{\circ} = V_1 \backslash \Delta \cong U_1 \backslash \Delta_U = U_1^{\circ}$*. So one has a diagram:*

$$\begin{array}{ccccc} & & V_1^{\circ} & & \\ & {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \rho_1} & \\ N_2^{\circ} & & & & W_1^{\circ} \\ & {\scriptstyle \rho_0}\searrow & & \swarrow{\scriptstyle \pi_W} & \\ & & M_{2,0}^{\circ}. & & \end{array}$$

LEMMA 3.7. $\operatorname{Pic}(V_1) \cong \operatorname{Pic}(U_1)$.

Proof. When $g > 2, \operatorname{Pic}(V_1) = \operatorname{Pic}(V_1^{\circ}) = \operatorname{Pic}(U_1^{\circ}) = \operatorname{Pic}(U_1)$, since $\operatorname{codim}(\Delta, V_1) = \operatorname{codim}(\Delta_U, U_1) = g - 1 > 1$. When $g = 2, \Delta$ and Δ_U are divisors in V_1 and U_1 respectively. However, $V_1 \backslash Z_1 \cong U_1 \backslash G''$. It then follows from $\operatorname{codim}(Z_1, V_1) = 3$ and $\operatorname{codim}(G'', U_1) = 2$ that $\operatorname{Pic}(V_1) \cong \operatorname{Pic}(U_1)$. □

LEMMA 3.8. *Every reduced fiber of the restriction* $\alpha_\Delta = \alpha_1|_{\Delta_U} \colon \Delta_U \to \Delta_W$ *is isomorphic to* $\mathbf{P}^1$.

Proof. The proof of this lemma will be the content of Section 4. □

Proof of Theorem 3.6. The Hecke correspondence and the isomorphism $V_1^{\circ} \cong U_1^{\circ}$ indicate that the map $\alpha_1|_{U_1^{\circ}} \colon U_1^{\circ} \to W_1^{\circ}$ is a $\mathbf{P}^1$-bundle. This and Lemma 3.8 imply that every reduced fiber of α_1 is isomorphic to $\mathbf{P}^1$. By Lemma

3.5, $K_{V_1} = \Theta_{2,\xi}^{-3} \otimes \Theta_{2,0}^{-2} \otimes \tau_1^4$. Restricting to a generic fiber f of ρ_1 produces $-2 = K_{V_1}|_f = \Theta_{2,\xi}^{-3}|_f + \tau_1^4|_f$. Computing from the map ρ_0, one obtains $\Theta_{2,\xi}|_f = 2$, whence $\tau_1|_f = 1$. It follows from $\mathrm{Pic}(V_1) \cong \mathrm{Pic}(U_1)$ and $\rho_1 \cong \alpha_1|_{U_1^\circ}$ that τ_1 in $\mathrm{Pic}(U_1)$ also has degree one over a generic fiber of α_1. But α_1 is obviously flat, since all its fibers have the same dimension (one) and since U_1 and W_1 are both smooth. So τ_1 has degree one over every fiber of α_1, hence all fibers of α_1 are actually reduced. Furthermore, α_1 is a locally free $\mathbf{P}^1$-bundle due to the existence of such a line bundle τ_1 [10]. □

RELATIVE DIFFERENTIAL SHEAVES

To compute the sheaf of relative differentials, we treat the case of $g > 2$ which is easy to visualize, but the assertions will stand for $g = 2$ (Remark 3.11). When $g > 2$, $\mathrm{Pic}(V_1) = \mathrm{Pic}(V_1^\circ) = \mathrm{Pic}(U_1^\circ) = \mathrm{Pic}(U_1)$. Under these identifications, $\Omega_{\alpha_1} = \Omega_{p_1}$.

LEMMA 3.9. *Using the notation in Lemma 3.5, one has*

(a) $\Omega_{\rho_1} = \pi_1^* \Omega_{\rho_0}$.
(b) $\Omega_{\rho_0} = \pi_0^* \Omega_{2,\xi}^{-1} \otimes \rho_0^* \Theta_{2,0}^2$, hence $\Omega_{\rho_1} = \Theta_{2,\xi}^{-1} \otimes \Theta_{2,0}^2$.

Proof. (a) Equivalently we need to show that the above diagram is a fiber product. Suppose that a scheme T admits two maps $T \xrightarrow{t_N} N_2^\circ$ and $T \xrightarrow{t_W} W_1^\circ$ such that $\rho_0 \circ t_N = \pi_W \circ t_W$. Then the map t_W says that T represents gluing data derived from stable bundles over X_2; whereas the map t_N indicates that the gluing data actually come from bundles parameterized in N_2°. The universality of (V_1°, π_1) then provides a lifting of (t_N, t_W). Therefore V_1° is the fiber product of π_W and ρ_0.

(b) One has $\omega_{M_{2,0}} = \Theta_{2,0}^{-4}$ [3], where $\omega_{M_{2,0}}$ denotes the dualizing sheaf of $M_{2,0}$. Since $K_{N_2} = \pi_0^* \Theta_{2,\xi}^{-1} \otimes \rho_0^* \Theta_{2,0}^{-2}$, it follows that $\Omega_{\rho_0} = K_{N_2} \otimes \rho_0^* \omega_{M_{2,0}}^{-1} = \pi_0^* \Theta_{2,\xi}^{-1} \otimes \pi_0^* \Theta_{2,0}^2$. □

PROPOSITION 3.10.

(a) $\mathrm{Pic}(W_1) = \langle \Theta_{2,0}, D_w \rangle$, *where* $D_w = \alpha_1(D)$.
(b) $K_{W_1} = -4\Theta_{2,0} - 2D_w$.

Proof. (a) ρ_1 is a locally free $\mathbf{P}^1$-bundle by Theorem 3.6. Since $D = \Theta_{2,\xi} - 2\tau_1$ by (3.1), $\mathrm{Pic}(V_1) = \langle \Phi_{2,0}, \Phi_{2,\xi}, \tau_1 \rangle = \langle \Theta_{2,0}, D, \tau_1 \rangle$. But $\mathrm{Pic}(V_1) = \mathrm{Pic}(U_1) = \langle \alpha_1^*(\mathrm{Pic}(W_1)), \tau_1 \rangle$, whence $\mathrm{Pic}(W_1) = \langle \Theta_{2,0}, D_w \rangle$. (b) Suppose $K_{W_1} = a\Theta_{2,0} +$

bD_w. Then $\rho_1^* K_{W_1} = a\Theta_{2,0} + bD = a\Theta_{2,0} + b(\Theta_{2,\xi-2\tau_1})$. On the other hand, $\rho_1^* K_{W_1} = K_{V_1} \otimes \Omega_{\rho_1}^\vee$. It follows from Lemma 3.9 and coefficients comparison that $a = -4, b = -2$. □

REMARK 3.11. When $g = 2, M_{2,0} \cong \mathbf{P}^3[6]$. Identifying $\Theta_{2,0}$ with $\mathcal{O}(1)$, the formulas $\Omega_{\alpha_1} = \Theta_{2,\xi}^{-1} \otimes \Theta_{2,0}^2, Pic(W_1) = \langle \Theta_{2,0}, D_w \rangle$, and $K_{W_1} = -4\Theta_{2,0} - 2D_w$ still hold true.

For the second component U_2 we start with (cf. Remark 2.14).

(i) a universal bundle E over $X_2 \times M_{2,-\xi}$ such that $\det(E_q) = \Theta_{2,-\xi}$, where $M_{2,-\xi}$ and $\Theta_{2,-\xi}$ are interpreted similarly as for $M_{2,\xi}$ and $\Theta_{2,\xi}$ respectively;
(ii) a bundle F over $X_1 \times N_1$ which is a semistable family with trivial determinant. Here $N_1 = \mathbf{P}^1$ is derived similarly as N_2 by the Hecke operation.

Let $V_1' = \mathbf{P}(\mathrm{Hom}(E_q, F_p)) \xrightarrow{\pi_1'} N_1 \times M_{2,-\xi}$. Here E and F denote the natural pullbacks by abuse of notation. One has a diagram which summarizes the blowings up and down:

$$\begin{array}{ccccccc} V_1' & \longleftarrow & V_2' & \longleftarrow & V_3' & \longrightarrow & U_2 \\ \Big\downarrow{\scriptstyle \pi_1'} & & & & & & \Big\downarrow{\scriptstyle \alpha_2} \\ N_1 \times M_{2,-\xi} & & & & & & W_2. \end{array}$$

REMARK 3.12. We only need one blowing down for the derivation of U_2. As mentioned earlier, the second blowing down for U_1 is basically the contraction of Z_3 along the direction $\nu: \mathbf{P}(H) \to J_2$. But the U_2 the corresponding bundle H over J_1 is a line bundle, which implies that the map $\nu: \mathbf{P}(H) \to J_1$ is an isomorphism.

PROPOSITION 3.13.

(a) $\mathrm{Pic}(W_2) = \langle \mu_w', \Theta_{2,-\xi}, D_w' \rangle$. *Here* μ_w' *and* $\Theta_{2,-\xi}$ *are the image of* $\pi_1'^*(p_1^*\mathcal{O}_{\mathbf{P}^1}(1))$ *and* $\pi_1'^*(p_2^*\Theta_{2,-\xi})$ *in* $\mathrm{Pic}(W_2)$ *respectively, with* $p_1: \mathbf{P}^1 \times M_{2,-\xi} \to \mathbf{P}^1$ *and* $p_2: \mathbf{P}^1 \times M_{2,-\xi} \to M_{2,-\xi}$. D_w' *is the Type II locus or the divisor at infinity in* W_2.
(b) $K_{W_2} = -4\mu_w' - 2\Theta_{2,-\xi} - 2D_w'$. □

DEGENERATION OF THE THETA DIVISORS

LEMMA 3.14. *Let* ω_{M_0} *be the dualizing sheaf of* M_0. *Then* $\omega_{M_0}|W_1 = K_{W_1}(D_w) = -4\Theta_{2,0} - D_w$ *and* $\omega_{M_0}|W_2 = K_{W_2}(D_w') = -4\mu_w' - 2\Theta_{2,-\xi} - D_w'$. □

THEOREM 3.15. *Let ω_ϖ be the relative dualizing sheaf of $M \xrightarrow{\varpi} C$. Then $\omega_\varpi^\vee \otimes \mathcal{O}_M(W_1) = \Theta_C^2 \otimes \varpi^* L$, where L is a line bundle over C and Θ_C a line bundle over M over C such that $\Theta_C|_t = \Theta_t$ is the ample generator of* $\mathrm{Pic}(M_t)$ *for $t \neq 0$. (Therefore Θ_C gives a degeneration of the generalized theta divisor.) The line bundle $\omega_\varpi^\vee \otimes \mathcal{O}_M(W_2)$ also has such property.*

Proof. By Lemma 3.14 and the fact that $K_{M_t} = \Theta_t^{-2}$ for all $t \neq 0$ [9], $\omega_\varpi^\vee \otimes \mathcal{O}_M(W_1)$ is divisible over every fiber of ϖ. □

4. Proof of Lemma 3.8

The proof of Lemma 3.8 is based on the following local analysis. Since the bundle E' over X_1 is fixed for the construction, it suffices to discuss the difference between strict semistable bundles parameterized by N_2.

Case 4.A. Let E' be the unique rank two stable bundle over X_1 with $\det(E') = A_0|_{X_1}$. Let $F' = L \oplus M$ with $M = L^{-1}, L \in \mathrm{Jac}(X_2)$ and $L^2 \neq \mathcal{O}_{X_2}$. Applying the construction in Section 2, one obtains the space $V_1 = \mathbf{P}(\mathrm{Hom}(E'_p, F'_q))$ and an exact sequence:

$$0 \longrightarrow E_p \otimes \tau_1 \xrightarrow{\phi_1} F_q \xrightarrow{\beta_1} Q_D \longrightarrow 0,$$

where E (resp. F) is the pullback of E' (resp. F') to $X_1 \times V_1$ (resp. $X_2 \times V_1$). There exist two distinguished disjoint lines $l, m \subset D$, corresponding to $\mathbf{P}(\mathrm{Hom}(E'_p, L_q))$ and $\mathbf{P}(\mathrm{Hom}(E'_p, M_q))$ respectively, such that $l \cup m$ represents exactly the unstable locus for descending ϕ_1. Blow up V_1 along $l \cup m$ to form V_3: $V_3 \xrightarrow{\pi_3} V_1$ (this notation is chosen for coherence). Let $Z_l = \pi_3^{-1}(l), Z_m = \pi_3^{-1}(m)$, and $Z = Z_l \cup Z_m$. Then Section 2 shows that V_3 admits a morphism to W_1.

LEMMA 4.1. *The image of V_3 inside W_1 is isomorphic to $Q = \mathbf{P}^1 \times \mathbf{P}^1$. Moreover, the map $V_3 \to Q$, denoted by α_Q, is a $\mathbf{P}^1$-bundle.*

Proof. The group $G = \mathbf{C}^* \times \mathbf{C}^*$ of automorphisms of F' acts naturally on $\mathrm{Hom}(E'_p, F'_q)$. This action induces a free PG action on $V_1 \backslash (l \cup m) = V_1^\circ$. The geometric quotient of V_1° by PG can be identified with $Q = \mathbf{P}^1 \times \mathbf{P}^1$. Indeed, if we fix a basis $\{f_1, f_2\}$ for F'_q such that f_1 and f_2 generate L_q and M_q respectively, then each orbit in V_1° represents two ordered lines (e_1, e_2) in E'_p by assigning e_i to f_i. Hence such an orbit corresponds to a point in $\mathbf{P}(E'_p) \times \mathbf{P}(E'_p) = \mathbf{P}^1 \times \mathbf{P}^1$. If the two lines e_1 and e_2 are distinct, one obtains a Type I bundle. When they coincide, i.e., representing a point in the diagonal of $\mathbf{P}^1 \times \mathbf{P}^1$, they provide a bundle of Type II.

We can be more precise. Tensoring the above exact sequence by τ_1^{-1}, followed by restricting to V_1°, one can descend $(\phi_1 \otimes \tau_1^{-1})|_{V_1^\circ}$ to a map $\overline{\phi_1^\tau}$ over Q. So we have an exact sequence:

$$0 \longrightarrow \overline{E_p} \xrightarrow{\overline{\phi_1^\tau}} \overline{F_q^\tau} \xrightarrow{\overline{\beta_1^\tau}} \overline{Q_D^\tau} \longrightarrow 0.$$

Here the superscript "τ" denotes the corresponding twisting by τ_1^{-1}. One checks that the geometric realization of $\overline{\phi_1^\tau}$ is stable.

The natural map α_Q is just the fiberwise compactification of the projection $V_1^\circ \to Q$, which has fiber $\mathbf{C}^*$. □

Case 4.B. Replace F' in Case 4.A by a nontrivial extension $0 \to L \to F' \to L \to 0$, with $L^2 = \mathcal{O}_{X_2}$. We still write the extension as $0 \to L \to F' \to M \to 0$ with $L = M$ for convenience. Then, unlike the above case, one locates a single distinguished line $l \subset D$, corresponding to $\mathbf{P}(\mathrm{Hom}(E_p', L_q))$, such that l constitutes the unstable locus when descending ϕ_1.

Blow up V_1 along l to create V_2: $V_2 \xrightarrow{\pi_2} V_1$. Let $Z_l = \pi_2^{-1}(l)$ The main difference, however, is that we need to further blow up V_2 along $D \cap Z_l =: m$ to achieve V_3: $V_3 \xrightarrow{\pi_3} V_2$. Let $Z_m = \pi_3^{-1}(m)$, and denote the strict transformation of Z_l again by Z_l. Then one has a morphism $V_3 \to W_1$.

LEMMA 4.2.

(a) $Z_m \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2))$. *Assume Q_0 represents the blowing down of Z_m along the (-2)-curve C_0. Then Q_0 is isomorphic to the image of V_3 in W_1.*

(b) V_3 *admits a map α_{Z_m} to Z_m with fiber $\mathbf{P}^1$. Moreover, the section $C_1 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}) \subset Z_m$ corresponds to bundles of Type II, and $Z_m \backslash C_1$ of Type I.*

(c) V_3 *can be also blown down along Z_l to a singular variety V_0. Moreover, V_0 admits a map α_{Q_0} to Q_0 with fiber $\mathbf{P}^1$.*

(d) *The two composite maps $V_3 \xrightarrow{Bl_{Z_l}} V_0 \xrightarrow{\alpha_{Q_0}} Q_0$ and $V_3 \xrightarrow{\alpha_{Z_m}} Z_m \xrightarrow{Bl_{C_0}} Q_0$ coincide.*

Proof. (a) One computes directly that $N_{m/V_2} = \mathcal{O} \oplus \mathcal{O}(2)$, so $Z_m = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(2))$. Denote the quotient map $F' \to M$ by δ. Then the automorphism group of F' is $G = \{\lambda I + \mu\delta | \lambda \in \mathbf{C}^*, \mu \in \mathbf{C}\}$. G acts on $\mathrm{Hom}(E_p', F_q')$ naturally, and induces a free PG action on $V_1 \backslash l = V_1^\circ$. The orbit space V_1° / PG can be identified with the geometric bundle $\mathbf{L}$ of $\mathcal{O}_{\mathbf{P}^1}(2)$. To demonstrate this, we choose a basis $\{f_1, f_2\}$ for F_q' such that f_1 generates L_q and f_2 is linearly independent of f_1. Assigning to f_1 a line $e_1 \in \mathbf{P}(E_p')$, the choices for assigning to f_2 correspond effectively to the maps in $\mathrm{Hom}(e_1, e_1^\vee)$. Here $e_1^\vee$ is the quotient of E_p': $0 \to e_1 \to E_p' \to e_1^\vee \to 0$. The totality of such assignments is $\mathrm{Hom}(\gamma, \gamma^\vee) = \mathcal{O}_{\mathbf{P}^1}(2)$, where γ is the tautological line bundle over $\mathbf{P}^1 = \mathbf{P}(E_p')$. This shows that V_1° / PG coincides with $\mathbf{L}$.

Clearly $V_3\backslash Z_l \xrightarrow{\alpha_L} \mathbf{L}$ is the fiberwise compactification of $V_1^{\circ} \to \mathbf{L}$, which has fiber $\mathbf{C}$, and $Z_m\backslash C_0$ provides a section of $\alpha_{\mathbf{L}}$. Hence $Z_m\backslash C_0 \cong \mathbf{L}$, and Z_m compactifies $\mathbf{L}$. On the other hand, Z_l hence $Z_m \cap Z_l = C_0$ represents the single stable bundle obtained by gluing E' (over X_1) to $\mathcal{O}_{X_2} \oplus \mathcal{O}_{X_2}$ (over X_2) along the fibers over p and q. Therefore the blowing down of Z_m along C_0 parameterizes all the different stable bundles arising from the bundles E' over X_1 and F' over X_2.

(b) The blowings up show that α_{Z_m} is just the union of $V_3\backslash Z_l \to \mathbf{L}$ and $Z_l \to C_0$, where the fiber of $Z_l \to C_0$ is the ruling l of Z_l. Further, one can readily check that $D \cap Z_m = C_1$. Hence C_1 exactly locates bundles of Type II in Z_m.

(c) By the adjunction formula and the formula for canonical line bundles under blowing up, $N_{Z_l/V_3} = \mathcal{O}_{Z_l}(-2l)$. Here again we consider l as a ruling on Z_l. Hence V_3 can be blown down by contracting the fibering $Z_l \to l$ to yield a singular V_0. The natural map α_{Q_0} is a $\mathbf{P}^1$-bundle away from l, the image of Z_l. But l has to be mapped to the vertex of Q_0. $l = \mathbf{P}^1$ and the commutativity (see (d)) assure that α_{Q_0} is a $\mathbf{P}^1$-bundle everywhere.

(d) Obvious. □

REMARK 4.3. When $g(X_2) = 1$, Cases 4.A and 4.B show that W_1 admits a map to $\mathbf{P}^1$. Its fibers are isomorphic to Q, except at four points where the fibers are Q_0.

Case 4.C. Replace F' in Case 4.A by a nontrivial extension of L^{-1} by L.

LEMMA 4.4. *Blowing up one line in V_1 will yield an effectively parameterizing space V_3; in other words, $V_3 \to W_1$ is an embedding.* □

LEMMA 4.5. *Let $L \in J_2$ be not of order two and $Y_{\mathit{eff}} = \mathbf{P}(H^1(X_2, L^2)) \cong \mathbf{P}^{g-2}$. From the universal extension $\mathcal{F}$ over $X_2 \times Y_{\mathit{eff}}$, we create $V_{\mathit{eff}} = \mathbf{P}(\mathrm{Hom}(E_p, \mathcal{F}_q)) \to Y_{\mathit{eff}}$, where E is the pull back of E' to $X_1 \times Y_{\mathit{eff}}$. Then the corresponding geometric realization is unstable at $Z_{\mathit{eff}} \cong Y_{\mathit{eff}} \times \mathbf{P}^1$. Blow up V_{eff} along Z_{eff} to form V'_{eff}. Then V'_{eff} parameterizes stable bundles, and can be smoothly blown down along Z'_{eff}, the exceptional divisor, in the direction of $Z'_{\mathit{eff}} \to \mathbf{P}^1 \times \mathbf{P}^1$ to an effectively parameterizing space $\overline{V}_{\mathit{eff}}$.*

Proof. The blowing up comes from Case 4.C; the blowing down from Case 4.A, since all points $y \in Z'_{\mathit{eff}}$ correspond to the same trivial extension $0 \to L \to L \oplus L^{-1} \to L^{-1} \to 0$. □

Now recall the map $\nu\colon \mathbf{P}(H) \to J_2$ and the diagram (3.2). Let $t \in J_2$ and $Y_t = \nu^{-1}(t)$. From E' over X_1 and $\mathcal{E}'_{Y_t}$ over $X_2 \times Y_t$, we form $V_t = \mathbf{P}(\mathrm{Hom}(E_p, \mathcal{E}'_q)) \xrightarrow{\pi_t} Y_t$, which induces an exact sequence: $0 \to \pi_t^* E_p \otimes \tau_t \xrightarrow{\phi_t} \pi_t^* \mathcal{E}'_q \xrightarrow{\beta_t} Q_{D_t} \to 0$. Suppose first that t is not of order two. Let $y_0 \in Y_t$ corresponds to the unique point representing the trivial extension of t^{-1} by t. Then the geometric realization of ϕ_t yields unstable bundles at $Z_t \cong Y_t \times \mathbf{P}^1$ and $Z_0 \cong \mathbf{P}^1 \subset \pi_t^{-1}(y_0)$, $Z_t \cap Z_0 = \emptyset$. Blow up V_t along Z_t and Z_0 simultaneously to obtain V_t'. Let Z_0' and Z_t' be the two (disjoint) exceptional divisors.

LEMMA 4.6.

(a) *V_t' parameterizes stable bundles.*
(b) *V_t' can be blown down along Z_t' to a smooth variety $\overline{V}_t$.*
(c) *Every reduced fibers of the induced map $\overline{V}_t \xrightarrow{\alpha_t} W_1$ over its image is isomorphic to $\mathbf{P}^1$.*

Proof. (a) and (b) follow from Sections 2 and 3. (c) $Y_t \backslash y_0$ admits a map to Y_{eff}, which has fiber $\mathbf{C}$. It follows that for every line $l \subset Y_t$ through y_0, $l \backslash y_0$ represents a single bundle over X_2. Any lifting of such an $l \backslash y_0$ in V_t' extends over to Z_0'. So $Z_0' \to \overline{V}_{\mathit{eff}}$ is surjective. Both being $\mathbf{P}^g$ bundles over $\mathbf{P}^1$ shows they are isomorphic. Thus away from the closure of I_{sp}, α_t: $\overline{V}_t \to \overline{V}_{\mathit{eff}}$ is a $\mathbf{P}^1$-bundle. On the other hand, the closure of I_{sp} in $\overline{V}_t$ is $\overline{\pi_t^{-1}(y_0)}$, the proper transformation of $\pi_t^{-1}(y_0)$ in $\overline{V}_t$, and the closure of I_{sp} in $\overline{V}_{\mathit{eff}}$ is isomorphic to blowing down image of Z'_{eff}, which is $\mathbf{P}^1 \times \mathbf{P}^1$. By Case 4.A, $\overline{\pi_t^{-1}(y_0)} \to \mathbf{P}^1 \times \mathbf{P}^1$ is also a $\mathbf{P}^1$-bundle. Therefore every reduced fiber of α_t equals a $\mathbf{P}^1$. □

When $t \in J_2$ is of order two, change the subscript t to n. The unstable locus for the geometric realization of ϕ_n is $Z \cong Y_n \times \mathbf{P}^1$. Blow up V_n along Z to achieve V_n', then the unstable locus for the new geometric realization is $D \cap Z' \stackrel{\text{def}}{=} T$. Blow up V_n' along T to obtain V_n''. Let Z'' be the strict transformation of Z'. Then $Z'' \cong Z'$.

LEMMA 4.7.

(a) *V_n'' represents stable bundles over X_0.*
(b) *V_n'' can be blown down along $Z'' \to Z$ to a singular variety S''.*
(c) *S'' can be (small) contracted along $Z = Y_n \times \mathbf{P}^1 \to \mathbf{P}^1$ to a variety $\bar{S}$.*
(d) *Every reduced fibers of the induced map $\bar{V}_n \xrightarrow{\alpha_n} W_1$ over its image is isomorphic to $\mathbf{P}^1$.*

Proof. (a), (b) and (c) follow from Sections 2 and 3. (d) is a global version of Case 4.B. □

Proof of Lemma 3.8. Lemmas 4.6 and 4.7 show that a fiber of α_Δ: $\Delta_U \to \Delta_W$ is either a fiber of α_t or that of α_n. Hence every reduced fiber of α_Δ is isomorphic to $\mathbf{P}^1$. □

5. Generalizations

All constructions can be easily extended to cases of $g(X_1) > 1$ and $g(X_2) > 1$, and all assertions have more symmetrical flavor. We only sketch the final picture here. To describe the first component W_1, we begin with $M_{1,\zeta}$ which is the moduli space of rank two stable bundles over X_1 with determinant $\zeta = \mathcal{O}_{X_1}(-p)$. Take a Poincare bundle E over $X_1 \times M_{1,\zeta}$ such that $\det(E_p) = \Theta_{1,\zeta}$, the ample generator of $\mathrm{Pic}(M_{1,\zeta})$. Retain the data $M_{2,\xi}$, $M_{2,0}$, N_2 and so on for X_2, and form $\pi_1 \colon V_1 \to M_{1,\zeta} \times N_2$ as before.

THEOREM 5.1. The rational map $\rho_1 \colon V_1 \to W_1$ can be resolved by two blowings up to a morphism $V_3 \to W_1$. Furthermore, V_3 can be blown down twice to a smooth variety U_1 and the resulting map $\alpha_1 \colon U_1 \to W_1$ is a locally free $\mathbf{P}^1$-bundle. □

PROPOSITION 5.2.

(1) $\mathrm{Pic}(W_1) = \langle \Theta_{1,\zeta}, \Theta_{2,0}, D_w \rangle$, *where* D_w *is the divisor of Type II locus in* W_1.
(2) $K_{W_1} = -2\Theta_{1,\zeta} - 4\Theta_{2,0} - 2D_w$. □

For the second component W_2, we start with the moduli space $M_{1,-\zeta}$ and $M_{2,-\xi}$. But this time we need to form the Hecke triangle over X_1:

$$\begin{array}{ccc} & N_1 & \\ \swarrow & & \searrow \\ M_{1,-\zeta} & & M_{1,0}. \end{array}$$

But the derivation of U_2 is almost identical to the case in Theorem 5.1.

PROPOSITION 5.2′.

(1) $\mathrm{Pic}(W_2) = \langle \Theta_{1,0}, \Theta_{2,-\xi}, D'_w \rangle$, *where* D'_w *is the divisor of Type II locus in* W_2.
(2) $K_{W_2} = -4\Theta_{1,0} - 2\Theta_{2,-\xi} - 2D'_w$. □

THEOREM 5.3. *The generalized theta divisor* Θ_t *in* $\mathrm{Pic}(M_t)$ *degenerates correspondingly to a* Θ_0 *over* M_0, *whose restrictions are* $\Theta_0|_{W_1} = \Theta_{1,\zeta} + 2\Theta_{2,0} + \delta D_w$ *and* $\Theta_0|_{W_2} = 2\Theta_{1,0} + \Theta_{2,-\xi} + (1-\delta)D'_w$ *with* $\delta = 0$ *or* 1. □

REMARK 5.4. For cases $g(X_i) \geqslant 1, i = 1, 2$, all statements in this section hold true with the following conventions:

(i) If $N_i = \mathbf{P}^1$, then replace two blowings down by one in Theorem 5.1 (see Remark 4.3) and $\Theta_{i,0}$ by μ_w or μ'_w (see Proposition 3.13).

(ii) If $M_{1,\zeta}$ or $M_{2,-\xi}$ is a single point, think of $\Theta_{1,\zeta}$ or $\Theta_{2,-\xi}$ as being trivial.

References

1. Arbarello, E., Cornalba, M., Griffiths, P. A. and Harris, J., Geometry of algebraic curves I, Springer-Verlag, 1985.
2. Bertram, A. and Szenes, A., Hilbert polynomials of moduli spaces of rank two bundles II, Preprint, November 1991.
3. Drezet, J.-M. and Narasimhan, M. S., Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques, *Invent. Math.* 97, 1989.
4. Gieseker, D., A degeneration of the moduli space of stable bundles, *J. Diff. Geom.* 19, 1984.
5. Gieseker, D. and Morrison, I., Hilbert stability of rank two bundles on curves, *J. Diff. Geom.* 19, 1984.
6. Narasimhan, M. S. and Ramanan, S., Moduli of vector bundles on a compact Riemann surface, *Ann. Math.* 89, 1969.
7. Narasimhan, M. S. and Ramanan, S., Geometry of Hecke cycles I, C. P. Ramanujam – *A Tribute*, Springer, Berlin, 1978.
8. Newstead, P. E., Introduction to moduli problems and orbit spaces, *Tata Lecture Notes*, Narosa, 1978.
9. Ramanan, S., The moduli spaces of vector bundles over an algebraic curve, *Math. Ann.* 200, 1973.
10. Serre, J. P., Espaces fibrés algébriques, Exposé 1, *Seminaire Chevalley*, 1958.
11. Seshadri, C. S., Fibrés vectoriel sur les courbes algébriques, *Astérisques* 96, 1982.