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GIUSEPPE PARESCHI

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Gaussian maps and multiplication maps on certain projective varieties

GIUSEPPE PARESCHI
Dipartimento di Matématica, Universitá di Ferrara, Via Machiavelli 35, 44100 Ferrara, Italy

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In recent years there has been a considerable body of activity concerning the surjectivity (and the corank) of gaussian maps associated to line bundles on projective curves over an algebraically closed field of characteristic zero. The starting point was Wahl's discovery of the connection between gaussian maps

$$\gamma_{K_C,L}\colon \mathrm{Rel}(K_C, L) \to H^0(K_C^{\otimes 2} \otimes L)$$

(L a line bundle on a curve C) and deformation theory, leading to the striking result that if L is normally generated and $\gamma_{K_C,L}$ is surjective then $C \hookrightarrow \mathbf{P}(H^0(L))$ is not the hyperplane section of a normal surface other than a cone ([W1], [W3]). E.g. if C is a hyperplane section of a K3 surface then the map γ_{K_C,K_C} (usually referred to as the Wahl map) is not surjective. This is contrasted by a result of Ciliberto–Harris–Miranda ([CHM]) stating that the Wahl map of the general curve of genus $g = 10$ or $g \geqslant 12$ is surjective, and also by a result of Lazarsfeld ([L3]), yielding that there are Brill–Noether–Petri general curves of any genus which are hyperplane sections of K3 surfaces. Therefore the non surjectivity of the Wahl map is a non-trivial condition, apparently not depending on classical Brill–Noether theory. These and other reasons stimulated a growing interest on two complementary themes: on the one hand to understand the nature of the obstructions to the surjectivity of the Wahl map and on the other hand to study systematically all gaussian maps $\gamma_{L,M}$, where L and M are line bundles (say of high degree) on a given curve C. As mentioned, the problem has a special interest when $L = K_C$ (we refer to [W3] for a survey on these and other related questions).

Concerning the first question, a striking result has been proved by C. Voisin ([V]): given a Brill–Noether–Petri general curve C, if the Wahl map is not surjective then there is an unexpected family of non-normally generated line bundles. Specifically, the family in question is $\{K_C \otimes A^{\vee}\}_{A \in Y}$, where $Y = W^1_{[(g+3)/2]}$ is the variety of pencils of minimal degree on C. As we said, this is unexpected, since Voisin proves also that if C is general in $\mathcal{M}_g$, $g = 10$ or $\geqslant 12$, the general of those line bundles is normally generated, thus reproving the mentioned theorem in [CHM]. This method has been partially extended by Paoletti ([P]) to gaussian maps of type $\gamma_{K_C,L}$ on B-N-P general curves.

The starting point of this paper is that something similar to the first step of Voisin's argument holds in full generality:

THEOREM A. *Let C be any curve of genus $g \geqslant 1$ and let E and F be vector bundles on C. Assume that $Y \subset \mathrm{Pic}^{\mathrm{d}}(\mathrm{C})$ is a subvariety generating the jacobian as a group and such that the general line bundle A parametrized by Y is a base point free pencil. Under mild hypotheses, if for A general in Y the multiplication map*

$$m_{E\otimes A^\vee, F\otimes A^\vee}: H^0(E\otimes A^\vee)\otimes H^0(F\otimes A^\vee)\to H^0(E\otimes F\otimes A^{\otimes -2})$$

is surjective then the gaussian map $\gamma_{E,F}\colon \mathrm{Rel}(E,F) \to H^0(K_C\otimes E\otimes F)$ is surjective. The reader is referred to Theorems 3.1 and 3.2 below for precise, and in fact more general, statements.

As a particular case, when C is Brill–Noether–Petri general and $E = F = K_C$, taking $Y = W^1_{(g+3)/2}$ we recover Voisin's lemma ([V] 2.8) in the odd genus case. If the genus is even, we get a somehow weaker result, since, as $W^1_{(g+2)/2}$ is a finite set, in order to get a family of pencils generating the jacobian, we are forced to consider pencils of the subminimal degree $(g+4)/2$. This is balanced by the fact that here the same result works as well for curves satisfying the weaker Brill–Noether condition (Y is not required to be smooth).

The proof of Theorem A is very different from Voisin's one, even within the B-N-P condition. Surprisingly enough, the present argument (which was inspired by the reading of Kempf's works [K1], [K2], Chapter 6 and [K3]) relies on very general properties of the duality between $\mathrm{Pic}^0(\mathrm{C})$ and $\mathrm{Alb}(C)$. In fact, Theorem A is, via the classical base point free pencil trick, a corollary of the following theorem, valid for varieties of arbitrary dimension having immersive Albanese map:

THEOREM B. *Let X be a smooth irreducible projective variety such that Ω^1_X is generated by its global sections and let E, F', F'' be vector bundles on X. Moreover let Y be a nondegenerate (cf. Section 1.2) subvariety of $\mathrm{Pic}^0(\mathrm{X})$. Under mild hypotheses, if, for α general on Y, the multiplication map $\mathrm{Rel}(E,F'_\alpha)\otimes H^0(F''_{-\alpha}) \to \mathrm{Rel}(E\otimes F''_{-\alpha}, F'_\alpha)$ is surjective then the gaussian map $\gamma_{E,F'\otimes F''}$: $\mathrm{Rel}(E, F'\otimes F'') \to H^0(\Omega^1_X\otimes E\otimes F'\otimes F'')$ is surjective. More generally, if the multiplication map $\mathrm{Rel}^k(E,F'_\alpha)\otimes H^0(F''_{-\alpha}) \to \mathrm{Rel}^k(E\otimes F''_{-\alpha}, F'_\alpha)$ is surjective for α general in Y, then the k-th higher gaussian map $\gamma^k_{E,F'\otimes F''}\colon \mathrm{Rel}^k(E, F'\otimes F'') \to H^0(S^k\Omega^1_X\otimes E\otimes F'\otimes F'')$ is surjective.* The reader is referred to Theorems 1.3 and 1.9 below for precise and more general statements. Here F_α means F tensored with the line bundle corresponding to α via the choice of a Poincaré line bundle.

The previous theorems might have a wide range of applications. In this paper we work in the direction of Bertan–Ein Lazarsfeld's paper [BEL], tackling the following problem: *since, as it is easy to see, when the degrees of L and M are* *high enough the gaussian maps* $\gamma_{L,N}$ *are surjective, give (possibly optimal) explicit results.*

In fact, in view of Theorem B, one can deal with analogous questions in higher dimension as well (granting some knowledge about the surjectivity of multiplication maps between relations). Specifically, we start with an application to abelian varieties, generalizing earlier results of [W2] and [BEL] for elliptic curves:

THEOREM C. *Let* A *be an ample line bundle on an abelian variety* X *(over any algebraically closed field) and let* L *and* M *line bundles on* X*, algebraically equivalent respectively to a* l*-power and to a* m*-power of* A*. If* $l, m \geqslant 4$ *and* $l + m \geqslant 9$ *then the gaussian map* $\gamma_{L,M}\colon \mathrm{Rel}(L, M) \to H^0(\Omega^1_X \otimes L \otimes M)$ *is surjective. In particular, if* $l \geqslant 5$, $\gamma_{L,L}$ *is surjective. More generally, if* $l, m \geqslant 2(k+1)$ *and* $l + m \geqslant 4(k+1)+1$ *then the higher gaussian map* $\gamma^k_{L,M}\colon \mathrm{Rel}^k(L, M) \to H^0(\mathrm{S}^k\Omega^1_X \otimes L \otimes M)$ *is surjective; this is already sharp for elliptic curves and* $k = 1$.

Next, we turn to the case of curves (in characteristic 0). Here, optimal bounds, valid for any curve of given genus g, are known, basically from the works [W2] and [BEL]. Nevertheless, in analogy with the case of multiplication maps ([GL]), one still looks for more refined results, in function of the intrinsic geometry of the curve. Applying Theorem A we find:

D. *A lower bound on* $\deg(L)$*, as a function of the Cliffor index and/or the gonality of the curve* C*, ensuring the surjectivity of maps* $\gamma_{K_C,L}$, (Theorem 3.4). Such a bound coincides (essentially) with the one of [BEL] if $\mathrm{cliff}(C) \leqslant g/3$ and improves it otherwise.

E. *Other lower bounds on* $\deg(L)$ *and* $\deg(M)$*, as functions of the geometry of the curve via Clifford index and/or gonality, ensuring the surjectivity of gaussian maps of type* $\gamma_{L,M}$ *(*L *and* M *line bundles)* (Theorems 3.7, 3.8 and Prop. 3.9). Some results about the surjectivity of maps $\gamma_{L,L}$ seem to have a special interest.

F. *Explicit lower bounds on the slopes of two vector bundles* E *and* F *on a curve* C *ensuring the surjectivity of gaussian maps* $\gamma_{E,F}$ (Theorem 3.10, Cor. 3.11 and Prop. 3.12).

Since some of the theorems above are complicated to state, we refer directly to Section 3, where all this material is presented. It is worth observing that, as a particular case, we recover, with a unified proof, essentially all the previously known results in this direction. Finally, for Brill–Noether–Petri general curves, we slightly sharpened our methods to get:

THEOREM G. *Let* C *be any Brill–Noether–Petri general curve of genus* $g \geqslant 22$ *and* L *a line bundle on* C*. If* $\deg(L) \geqslant 2g+9$ *then the map* $\gamma_{K_C,L}$ *is surjective. Moreover, if* $\deg(L) \geqslant 2g+7$ *then the gaussian map* $\gamma_{L,L}$ *is surjective.*

Concerning the proofs, the leit-motif is very simple. One plugs into Theorem A:

(a) an estimate of the degrees d such that on the curve C there are *families of base point free pencils of degree d generating* $\mathrm{Jac}(C)$ (dealing with maps $\gamma_{K_C,L}$, a special role is played by *primitive* pencils, i.e. base point free pencils A such that also $K_C \otimes A^{\vee}$ is base point free);
(b) explicit results about the surjectivity of multiplication maps of global sections of line bundles.

Concerning point (b), an optimal theorem, due to Green–Lazarsfeld ([GL]), is available in the case that the two line bundles coincide. In the general case, in absence of references in the literature, we had to adapt the methods of [G], [L2] and [GL] to get somehow analogous results. This material is somehow separated from the theme of the present article, and it is in fact a prerequisite to it. Therefore we present it in an Appendix at the end.

Although the results mentioned in D, E and F above do not seem to be sharp, the proofs are very explicit and from them it appears that one could get close to optimal bounds by refining points (a) and (b) above. E.g. when the curve is Brill–Noether general this can be done easily and in this way one proves the stronger Theorem G.

The results above suggest that, dealing with gaussian maps of line bundles of high degree on curves, *up to a certain point* their surjectivity should be determined by a complicated interaction of factors which are nevertheless of a Brill–Noether theoretic nature. We will come back to this point in Section 3.

Throughout the paper we will work over an algebraically closed field of characteristic zero, with the exception of Section 2 where any characteristic is allowed.

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1. The main construction

(A) Notation and preliminaries

Let X be a smooth projective variety over an algebraically closed field and let Δ be the diagonal of $X \times X$. Given two vector bundles E and F on X, we consider the following exact sequence on $X \times X$

$$0 \to p_1^*(E) \otimes p_2^*(F) \otimes \mathcal{I}_\Delta \to p_1^*(E) \otimes p_2^*(F) \to p_1^*(E) \otimes p_2^*(F)_{|\Delta} \to 0.$$

Taking H^0's one obtains the *multiplication map*:

$$\begin{array}{ccc} H^0(E) \otimes H^0(F) & \xrightarrow{m_{E,F}} & H^0(E \otimes F) \\ \| & & \| \\ H^0(p_1^*(E) \otimes p_2^*(F)) & \longrightarrow & H^0(p_1^*(E) \otimes p_2^*(F)_{|\Delta}) \end{array}$$

The *vector space of relations* between E and F is

$$\mathrm{Rel}(E, F) := \ker(m_{E,F}) = H^0(p_1^*(E) \otimes p_2^*(F) \otimes \mathcal{I}_\Delta).$$

Then one considers on $X \times X$ the sequence

$$0 \to p_1^*(E) \otimes p_2^*(F) \otimes \mathcal{I}_\Delta^2 \to p_1^*(E) \otimes p_2^*(F) \otimes \mathcal{I}_\Delta$$
$$\to p_1^*(E) \otimes p_2^*(F) \otimes \frac{\mathcal{I}_\Delta}{\mathcal{I}_\Delta^2} \to 0.$$

Taking global sections we get the *gaussian map*

$$\begin{array}{ccc} \mathrm{Rel}(E, F) & \xrightarrow{\gamma_{E,F}} & H^0(\Omega_X^1 \otimes E \otimes F) \\ \| & & \| \\ H^0(p_1^*(E) \otimes p_2^*(F) \otimes \mathcal{I}_\Delta) & \longrightarrow & H^0(p_1^*(E) \otimes p_2^*(F) \otimes \mathcal{I}_\Delta/\mathcal{I}_\Delta^2) \end{array}$$

We refer to [W3] and [CHM] for other interpretations of this map, and also for the motivation of the name "gaussian".

Higher gaussian maps are a natural generalization of gaussian maps. To define them, we consider the vector space of *higher relations*

$$\mathrm{Rel}^k(E,F) := H^0(p_1^*(E)\otimes p_2^*(F)\otimes \mathcal{I}_\Delta^k).$$

Considering the exact sequence

$$0\to p_1^*(E)\otimes p_2^*(F)\otimes \mathcal{I}_\Delta^{k+1}\to p_1^*(E)\otimes p_2^*(F)\otimes \mathcal{I}_\Delta^k$$
$$\to p_1^*(E)\otimes p_2^*(F)\otimes \frac{\mathcal{I}_\Delta^k}{\mathcal{I}_\Delta^{k+1}}\to 0$$

and taking global sections, one defines the *kth higher gaussian map*

$$\begin{array}{ccc}
\mathrm{Rel}^k(E,F) & \xrightarrow{\gamma^k_{E,F}} & H^0(\mathrm{S}^k\Omega^1_X\otimes E\otimes F)\\
\| & & \|\\
H^0(p_1^*(E)\otimes p_2^*(F)\otimes \mathcal{I}_\Delta^k) & \longrightarrow & H^0(p_1^*(E)\otimes p_2^*(F)\otimes \mathcal{I}_\Delta^k/\mathcal{I}_\Delta^{k+1})
\end{array}$$

Therefore $\mathrm{Rel}^k(E,F) = \ker(\gamma^{k-1}_{E,F})$. Note that, in this perspective, the multiplication map can be seen as the "0th gaussian map" $\gamma^0_{E,F} := m_{E,F}$.

In the course of the proof of Theorem 2.5 below, on higher gaussian maps on abelian varieties, we will use the following additional notation and facts: let us introduce the following coherent sheaf on X

$$R^k_{E,F} := p_{1*}(p_1^*(E)\otimes p_2^*(F)\otimes \mathcal{I}_\Delta^k)\cong p_{1*}(p_2^*(F)\otimes \mathcal{I}_\Delta^k)\otimes E.$$

Then, by induction, $\mathrm{Rel}^k(E,F)\cong H^0(R^k_{E,F})\cong H^0(R^k_{F,E})$ and there is a complex

$$0\to R^k_{E,F}\to R^{k-1}_{E,F}\to \mathrm{S}^{k-1}\Omega^1_X\otimes E\otimes F\to 0 \tag{1}$$

exact on the left and in the middle. The $(k-1)$th higher gaussian map is obtained taking H^0 of the third arrow in (1). By induction one can also prove that: *if for any h, with $0\leqslant h\leqslant k-1$, the vector bundles $\mathrm{S}^h\Omega^1_X\otimes E\otimes F$ are generated by their global sections and the higher gaussian maps $\gamma^h_{E,F}$ are surjective then (1) is exact on the right too. In particular the sheaves $R^h_{E,F}$'s are locally free.* We leave this to the reader.

(B) Precise statement and proof of Theorem B

Given three coherent sheaves, L, M and N, on X, we will consider the two natural multiplication maps of relations with global sections

$$m_L(M,N)\colon \mathrm{Rel}(L,M)\otimes H^0(N)\to \mathrm{Rel}(L,M\otimes N),$$

$$m_M(L,N)\colon \mathrm{Rel}(L,M)\otimes H^0(N)\to \mathrm{Rel}(L\otimes N,M).$$

As it is easy to see, they fit in the commutative diagram

$$\begin{array}{ccc} \mathrm{Rel}(L,M)\otimes H^0(N) & \xrightarrow{m_L(M,N)} & \mathrm{Rel}(L,M\otimes N) \\ \Big\downarrow{\scriptstyle m_M(L,N)} & & \Big\downarrow{\scriptstyle \gamma_{L,M\otimes N}} \\ \mathrm{Rel}(L\otimes N,M) & \xrightarrow{\gamma_{L\otimes N,M}} & H^0(\Omega^1_X\otimes L\otimes M\otimes N) \end{array}$$

Let us fix once for all a Poincaré line bundle $\mathcal{P}$ on $X\times \mathrm{Pic}^0\mathrm{X}$. We will adopt the following notation: given a point $\alpha\in \mathrm{Pic}^0\mathrm{X}$ corresponding via $\mathcal{P}$ to a line bundle L_α, we will denote by E_α the sheaf $E\otimes L_\alpha$.

Now let L, M', M'' be three vector bundles on X and set $M:=M'\otimes M''$. By the above there is a commutative diagram

$$\begin{array}{ccc} \bigoplus_{\alpha\in \mathrm{Pic}^0\mathrm{X}}\mathrm{Rel}(L,M'_{-\alpha})\otimes H^0(M''_\alpha) & \xrightarrow{\sum m^1_\alpha} & \mathrm{Rel}(L,M) \\ \Big\downarrow{\scriptstyle \oplus m^2_\alpha} & & \Big\downarrow{\scriptstyle \gamma_{L,M}} \\ \bigoplus_{\alpha\in \mathrm{Pic}^0\mathrm{X}}\mathrm{Rel}(L\otimes M''_\alpha,M'_{-\alpha}) & \xrightarrow{\sum \gamma_\alpha} & H^0(\Omega^1_X\otimes M\otimes L) \end{array}$$

where $m^1_\alpha := m_L(M'_{-\alpha},M''_\alpha), m^2_\alpha := m_{M'_{-\alpha}}(L,M''_\alpha)$ are multiplication maps of relations with global sections and $\gamma_\alpha := \gamma_{L\otimes M''_\alpha, M'_{-\alpha}}, \gamma_{L,M}$ are gaussian maps. This proves

LEMMA 1.1. *Let L, M, M', M'' be vector bundles on X such that $M = M'\otimes M''$. Assume that there exists a subset $Y\subset \mathrm{Pic}^0\mathrm{X}$ such that*

(a) *the map $m_{M'_{-\alpha}}(L,M''_\alpha)\colon \mathrm{Rel}(L,M'_{-\alpha})\otimes H^0(M''_\alpha)\to \mathrm{Rel}(L\otimes M''_\alpha, M'_{-\alpha})$ is surjective for any $\alpha\in Y$;*
(b) *the map $\Sigma_{\alpha\in Y}\gamma_{L\otimes M''_\alpha,M'_{-\alpha}}\colon \oplus_{\alpha\in Y}\mathrm{Rel}(L\otimes M''_\alpha,M'_{-\alpha})\to H^0(\Omega^1_X\otimes L\otimes M)$ is surjective.*

Then the gaussian map $\gamma_{L,M}\colon \mathrm{Rel}(L,M) \to H^0(\Omega^1_X \otimes L \otimes M)$ *is surjective.*

The main content of the paper will be to find subsets $Y \subset \mathrm{Pic}^0\mathrm{X}$ satisfying the hypotheses of the previous lemma. The basic Lemma 1.2 below will provide a criterion in order to find in a natural way subsets $Y \subset \mathrm{Pic}^0\mathrm{X}$ satisfying condition (b) of Lemma 1.1. Before stating it we need some additional notation and hypotheses. First of all, from this point, with the exception of Section 2, we will work over an algebraically closed field *of characteristic zero*. In the sequel Y will be a subvariety (i.e. an *irreducible and reduced closed subscheme*) of $\mathrm{Pic}^0\mathrm{X}$. Taking the H^1 of the canonical surjection $\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}} \to \mathcal{O}_Y$ and dualizing one gets a map

$$\phi_Y\colon H^1(\mathcal{O}_Y)^\vee \to H^0(\Omega^1_X),$$

where $H^1(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}})^\vee$ is identified to $H^0(\Omega^1_X)$ via duality between abelian varieties:

$$H^1(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}}) \cong T_{0,\mathrm{Pic}^0(\mathrm{Pic}^0\mathrm{X})} \cong T_{0,\mathrm{Alb}X} = H^0(\Omega^1_X)^\vee.$$

Let us denote V_Y the image of the map ϕ_Y.

Moreover, given a sheaf E on X, we will denote $Y^+(E)$ and $Y^-(E)$ the loci of $\alpha \in Y$ where respectively $h^0(E_\alpha)$ and $h^0(E_{-\alpha})$ jump. If F is another sheaf on X we will denote $m(Y,E,F)$ the locus where the multiplication map $m_{E_\alpha,F_{-\alpha}}$: $H^0(E_\alpha)\otimes H^0(F_{-\alpha}) \to H^0(E\otimes F)$ is not surjective. Let also $m(Y,E,F)_1$ denote the union of all components of $m(Y,E,F)$ of codimension one in Y. Finally, we will say that a certain property holds for α *general in* Y if holds on an open set of Y.

LEMMA 1.2. *Assume that* Ω^1_X *is globally generated and let* E *and* F *be two vector bundles on* X *such that* $H^1(\Omega^1_X \otimes E \otimes F) = 0$. *Suppose that* Y *is a Cohen–Macaulay subvariety of* $\mathrm{Pic}^0\mathrm{X}$ *such that the jump locus* $Y^+(E) \cup Y^-(F)$ *has codimension* $\geqslant 2$ *in* Y *and*

(a) *the multiplication map* $V_Y \otimes H^0(E\otimes F) \to H^0(\Omega^1_X \otimes E \otimes F)$ *is surjective;*
(b) *the multiplication map* $m_{E_\alpha,F_{-\alpha}}\colon H^0(E_\alpha) \otimes H^0(F_{-\alpha}) \to H^0(E\otimes F)$ *is surjective (and not injective) for* α *general in* Y. *Then for any open set* $U \subset Y$ *meeting each component of* $m(Y,E,F)_1$ *the map*

$$\sum_{\alpha\in U} \gamma_{E_\alpha,F_{-\alpha}}\colon \bigoplus_{\alpha\in U} \mathrm{Rel}(E_\alpha,F_{-\alpha}) \to H^0(\Omega^1_X \otimes E \otimes F)$$

is surjective.

The next theorem is a corollary of the two previous lemmas.

THEOREM 1.3. *Let* X *be a smooth irreducible projective variety such that* Ω^1_X *is globally generated. Let* L *and* M *be two vector bundles on* X *such that* $H^1(\Omega^1_X \otimes$

$L \otimes M) = 0$ and assume that $M = M' \otimes M''$. Let $Y \subset \mathrm{Pic}^0 X$ be a CM subvariety such that the jump locus $Y^+(L \otimes M'') \cup Y^-(M')$ has codimension $\geqslant 2$ in Y and $U \subset Y$ be an open set meeting each component of $m(Y, L \otimes M'', M')_1$ such that

(a) *the multiplication map $V_Y \otimes H^0(L \otimes M) \to H^0(\Omega^1_X \otimes L \otimes M)$ is surjective;*
(b) *the multiplication map $m_{L\otimes M'', M'}: H^0(L \otimes M''_\alpha) \otimes H^0(M'_{-\alpha}) \to H^0(L \otimes M)$ is surjective (and not injective) for α general in Y;*
(c) *the map $m_{M'_{-\alpha}}(L, M''_\alpha): \mathrm{Rel}(L, M'_{-\alpha}) \otimes H^0(M''_\alpha) \to \mathrm{Rel}(L \otimes M''_\alpha, M'_{-\alpha})$ is surjective for any α in U.*

Then the gaussian map $\gamma_{L,M}: \mathrm{Rel}(L, M) \to H^0(\Omega^1_X \otimes L \otimes M)$ is surjective.

Proof of Lemma 2.2. In the first place let us globalize (according to Kempf, [K2] Chapter 6), at least "generically", the multiplication maps

$$m_{E_\alpha, F_{-\alpha}}: H^0(E_\alpha) \otimes H^0(F_{-\alpha}) \to H^0(E \otimes F).$$

On the product $X \times X \times \mathrm{Pic}^0 X$ let us consider the three projections on the intermediate factors p_{12}, p_{13} and p_{23}. Then $p_{13}^*(\mathcal{P}) \otimes p_{23}^*(\mathcal{P}^\vee)_{|\Delta \times \mathrm{Pic}^0 X}$ is trivial. Let us denote also $\Delta_Y := \Delta \times Y$ and $\mathcal{I}_{\Delta_Y} := \mathcal{I}_{\Delta_Y | X \times X \times Y}$, the ideal sheaf of Δ_Y in $X \times X \times Y$. Setting

$$\mathcal{L} := p_{13}^*(p_1^*(E) \otimes \mathcal{P}) \otimes (p_{23}^*(p_2^*(F) \otimes \mathcal{P}^\vee)_{|X \times X \times Y},$$

we have on $X \times X \times Y$ the exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{I}_{\Delta_Y} \to \mathcal{L} \to p_{12}^*(p_1^*(E) \otimes p_2^*(F))_{|\Delta_Y} \to 0.$$

Applying p_{3*} (where now we mean the projection from $X \times X \times Y$ onto Y) one gets a sequence on Y

$$0 \to p_{3*}(\mathcal{L} \otimes \mathcal{I}_{\Delta_Y}) \to p_{3*}(\mathcal{L}) \to H^0(E \otimes F) \otimes \mathcal{O}_Y \to \tau \to 0, \tag{1}$$

where τ is some sheaf on Y. We have that $p_{3*}(\mathcal{L})$ and $p_{3*}(\mathcal{L} \otimes \mathcal{I}_{\Delta_Y})$, as direct images of torsion free sheaves, are (non zero) torsion free sheaves on Y. Moreover, as Y is assumed to be reduced, off the jump locus $Y^+(E) \cup Y^-(F)$ we have that $p_{3*}(\mathcal{L})$ is locally free on U and, for any $\alpha \in U, p_{3*}(\mathcal{L})(\alpha) \cong H^0(E_\alpha) \otimes H^0(F_{-\alpha})$ (Künneth formula). Moreover the map $p_{3*}(\mathcal{L})(\alpha) \to H^0(E \otimes F)$ is the multiplication map $m_{E_\alpha, F_{-\alpha}}$ and $p_{3*}(\mathcal{L} \otimes \mathcal{I}_{\Delta_Y})(\alpha) \cong \mathrm{Rel}(E_\alpha, F_{-\alpha})$. Therefore, thanks to hypothesis (b), τ is a torsion sheaf on Y, whose support is contained on $Y(E) \cup Y(F) \cup m(Y, E, F)$.

Now let us globalize generically the gaussian maps

$$\gamma_{E_\alpha, F_{-\alpha}}: \mathrm{Rel}(E_\alpha, F_{-\alpha}) \to H^0(\Omega^1_X \otimes E \otimes F).$$

We consider the map

$$f\colon \mathcal{L}\otimes\mathcal{I}_{\Delta_Y}\to\mathcal{L}\otimes\mathcal{I}_{\Delta_Y}/\mathcal{I}^2_{\Delta_Y}.$$

Since we have natural isomorphisms

$$\begin{aligned}\mathcal{L}\otimes\mathcal{I}_{\Delta_Y}/\mathcal{I}^2_{\Delta_Y} &\cong p_{12}^*(p_1^*(E)\otimes p_2^*(F))\otimes\mathcal{N}^\vee_{\Delta_Y|X\times X\times Y}\\ &\cong p_{12}^*(p_1^*(E)\otimes p_2^*(F)\otimes\mathcal{N}^\vee_{\Delta|X\times X}),\end{aligned}$$

(where $\mathcal{N}^\vee$ means conormal sheaf), applying p_{3_*} one gets a map

$$p_{3_*}(f) =: \tilde{\gamma}\colon p_{3_*}(\mathcal{L}\otimes\mathcal{I}_{\Delta_Y})\to H^0(E\otimes F\otimes\Omega^1_X)\otimes\mathcal{O}_Y$$

and, as above, on some non empty open set $W\subset U$ the map

$$\tilde{\gamma}(\alpha)\colon p_{3_*}(\mathcal{L}\otimes\mathcal{I}_{\Delta_Y})(\alpha)\cong \mathrm{Rel}(E_\alpha,F_{-\alpha})\to H^0(\Omega^1_X\otimes E\otimes F)$$

coincides with the gaussian map $\gamma_\alpha=\gamma_{E_\alpha,F_{-\alpha}}$.

Since τ is a torsion sheaf on Y, dualizing (1) we get that $H^0(E\otimes F)^\vee\otimes\mathcal{O}_Y$ sits naturally as a subsheaf of $p_{3_*}(\mathcal{L})^\vee$. Let $\mathcal{W}$ be the quotient:

$$0\to H^0(E\otimes F)^\vee\otimes\mathcal{O}_Y\to p_{3_*}(\mathcal{L})^\vee\to\mathcal{W}\to 0.$$

Again, dualizing sequence (1) we get the exact sequence

$$0\to\mathcal{E}xt^1(\tau,\mathcal{O}_X)\to\mathcal{W}\to p_{3_*}(\mathcal{L}\otimes\mathcal{I}_{\Delta_Y})^\vee \tag{2}$$

Next, we will construct a canonical lifting

$$\dot{\gamma}^\vee\colon H^0(\Omega^1_X\otimes E\otimes F)^\vee\otimes\mathcal{O}_Y\to\mathcal{W}$$

of the map $\tilde{\gamma}^\vee$. To this purpose, let us denote $\Delta^{(2)}$ the first infinitesimal neighborhood of Δ in $X\times X$ and $\Delta_Y^{(2)} := \Delta^{(2)}\times Y$. There is a natural isomorphism between the ideal sheaf $\mathcal{I}_{\Delta_Y|\Delta_Y^{(2)}}$ of Δ_Y in $\Delta_Y^{(2)}$ and the conormal sheaf $\mathcal{N}^\vee_{\Delta_Y|X\otimes X\otimes Y}=\mathcal{I}_{\Delta_Y}/\mathcal{I}^2_{\Delta_Y}$. Therefore on $X\times X\times Y$ we can consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{L}\otimes\mathcal{I}_{\Delta_Y} & \to & \mathcal{L} & \to & \mathcal{L}_{|\Delta_Y} & \to & 0\\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \mathcal{L}\otimes\mathcal{I}_{\Delta_Y}/\mathcal{I}^2_{\Delta_Y} & \to & \mathcal{L}_{|\Delta_Y^{(2)}} & \to & \mathcal{L}_{|\Delta_Y} & \to & 0\end{array} \tag{3}$$

Applying p_{3*} and using the hypothesis $H^1(\Omega^1_X \otimes E \otimes F) = 0$ one gets

$$\begin{array}{ccccccccc} 0 & \longrightarrow & p_{3*}(\mathcal{L} \otimes \mathcal{I}_{\Delta_Y}) & \longrightarrow & p_{3*}(\mathcal{L}) & \longrightarrow & H^0(E \otimes F) \otimes \mathcal{O}_Y & \longrightarrow & \tau \\ & & \downarrow{\scriptstyle \tilde{\gamma}} & & \downarrow & & \| & & \\ 0 & \longrightarrow & H^0(\Omega^1_X \otimes E \otimes F) \otimes \mathcal{O}_Y & \longrightarrow & p_{3*}(\mathcal{L}_{|\Delta^{(2)}_Y}) & \longrightarrow & H^0(E \otimes F) \otimes \mathcal{O}_Y & \longrightarrow & 0 \end{array}$$

Dualizing one gets a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(E \otimes F)^\vee \otimes \mathcal{O}_Y & \longrightarrow & p_{3*}(\mathcal{L}_{|\Delta^{(2)}_Y})^\vee & \longrightarrow & H^0(\Omega^1_X \otimes E \otimes F)^\vee \otimes \mathcal{O}_Y & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \\ 0 & \longrightarrow & H^0(E \otimes F)^\vee \otimes \mathcal{O}_Y & \longrightarrow & p_{3*}(\mathcal{L})^\vee & \longrightarrow & \mathcal{W} & \longrightarrow & 0 \end{array} \tag{4}$$

This induces a natural map

$$\dot{\tilde{\gamma}}^\vee \colon H^0(\Omega^1_X \otimes E \otimes F)^\vee \otimes \mathcal{O}_Y \to \mathcal{W}$$

which is our canonical lifting of $\tilde{\gamma}^\vee$.

CLAIM. $H^0(\dot{\tilde{\gamma}}^\vee)\colon H^0(\Omega^1_X \otimes E \otimes F)^\vee \to H^0(\mathcal{W})$ *is injective.*

Let us first show that the Claim implies the statement of Lemma 1.2. Let us observe first of all that, since Y is CM, $\mathcal{E}xt^1(\tau, \mathcal{O}_Y)$ is supported on the one codimensional components of the support of τ, i.e., by hypothesis, $m(Y, E, F)_1$. Moreover $\mathcal{E}xt^1(\tau, \mathcal{O}_Y)$, as a sheaf on its support, is torsion free and $p_{3*}(\mathcal{L} \otimes \mathcal{I}_{\Delta_Y})^\vee$ is torsion free on Y.

The Claim is equivalent to the injectivity of the map

$$\prod_{\alpha \in Y} \dot{\tilde{\gamma}}^\vee_\alpha \colon H^0(\Omega^1_X \otimes E \otimes F)^\vee \to \prod_{\alpha \in Y} \mathcal{W}_\alpha$$

(where now the subscript α means "stalk at α"). Let $W \subset Y$ be an open set meeting every component of $m(Y, E, F)_1$. By the above and sequence (2) this is equivalent to the injectivity of the map

$$\prod_{\alpha \in W} \dot{\tilde{\gamma}}^\vee(\alpha) \colon H^0(\Omega^1_X \otimes E \otimes F)^\vee \to \prod_{\alpha \in W} \mathcal{W}(\alpha)$$

But on some open subset $U \subset W$ we have that $\mathcal{W}(\alpha) \cong \mathrm{Rel}(E_\alpha, F_{-\alpha})^\vee$ and $\dot{\tilde{\gamma}}^\vee(\alpha) = \tilde{\gamma}(\alpha)^\vee = \gamma^\vee_{E_\alpha, F_{-\alpha}}$. Therefore the injectivity of the above map is equivalent to the injectivity of

$$\prod_{\alpha \in U} \gamma^\vee_{E_\alpha, F_{-\alpha}} \colon H^0(\Omega^1_X \otimes E \otimes F)^\vee \to \prod_{\alpha \in U} \mathrm{Rel}(E_\alpha, F_{-\alpha})^\vee$$

$$\cong \left(\bigoplus_{\alpha \text{ gen} \in Y} \mathrm{Rel}(E_\alpha, E_{-\alpha}) \right)^\vee$$

i.e., dualizing, to the statement of Lemma 1.2. □

We will show that the map $H^0(\dot{\tilde{\gamma}}^\vee)$: $H^0(\Omega^1_X \otimes E \otimes F)^\vee \to H^0(\mathcal{W})$ is injective. This implies that the map $H^0(\tilde{\gamma}^\vee)$ is injective and hence the Claim.

By diagram (4) it is enough to show that the coboundary map

$$\delta: H^0(\Omega^1_X \otimes E \otimes F)^\vee \to H^0(E \otimes F)^\vee \otimes H^1(\mathcal{O}_Y)$$

of the top row of (4) is injective. This in turn follows from hypothesis (a) and the following

LEMMA 1.4. *Up to multiplication for scalar coefficients, the map δ is the dual of the composed map*

$$\begin{array}{ccc} H^0(E \otimes F) \otimes H^1(\mathcal{O}_Y)^\vee & \xrightarrow{id \otimes \phi_Y} & H^0(E \otimes F) \otimes H^0(\Omega^1_X) \\ & & \Big\downarrow m_{E\otimes F, \Omega^1_X} \\ & & H^0(E \otimes F \otimes \Omega^1_X). \end{array}$$

Proof. Let us denote $\mathcal{Q} := p_{13}^*(p_1^*(E) \otimes \mathcal{P}) \otimes p_{23}^*(p_2^*(F) \otimes \mathcal{P}^\vee)$ and let us consider the sequence on $X \times X \times \mathrm{Pic}^0\mathrm{X}$ (analogous to the bottom row of (3))

$$0 \to \mathcal{Q} \otimes \mathcal{N}^\vee_{\Delta \times \mathrm{Pic}^0\mathrm{X}} \to \mathcal{Q}_{|\Delta^{(2)} \times \mathrm{Pic}^0\mathrm{X}} \to \mathcal{Q}_{|\Delta \times \mathrm{Pic}^0\mathrm{X}} \to 0. \tag{5}$$

Applying p_{3*} and using, as before, that $H^1(\Omega^1_X \otimes E \otimes F) = 0$ one gets the exact sequence

$$0 \to H^0(\Omega^1_X \otimes E \otimes F) \otimes \mathcal{O}_{\mathrm{Pic}^0\mathrm{X}} \to p_{3*}(\mathcal{Q}_{|\Delta^{(2)} \times \mathrm{Pic}^0\mathrm{X}})$$
$$\to H^0(E \otimes F) \otimes \mathcal{O}_{\mathrm{Pic}^0\mathrm{X}} \to 0. \tag{6}$$

Setting $g := h^1(\mathcal{O}_X) = \dim(\mathrm{Pic}^0\mathrm{X})$, we have the Serre duality isomorphisms $H^g(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}}) \cong k, H^{g-1}(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}}) \cong H^1(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}})^\vee$, and the isomorphism $H^1(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}})^\vee \cong H^0(\Omega^1_X)$. Thus Lemma 1.4 is immediately implied by the following Lemma 1.5, whose proof is a straightforward application of the duality theory on abelian varieties as started by Mumford ([M]) and developed by Kempf ([K1]) and Mukai ([Mu]). For the reader's convenience, we will outline a proof in the next section.

LEMMA 1.5. *Via the identifications above, the coboundary map*

$$H^0(E \otimes F) \otimes H^{g-1}(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}}) \to H^0(\Omega^1_X \otimes E \otimes F) \otimes H^g(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}})$$

associated to (6) coincides (up to scalar factors) with the multiplication map

$$m_{E\otimes F,\Omega^1_X}: H^0(E \otimes F) \otimes H^0(\Omega^1_X) \to H^0(\Omega^1_X \otimes E \otimes F)$$

Throughout the rest of the paper we will use the following terminology: we will say that a subvariety Y of $\mathrm{Pic}^0\mathrm{X}$ is *nondegenerate* if $V_Y = H^0(\Omega^1_X)$ i.e. if the map ϕ_Y is surjective or, equivalently, if $H^1(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}}) \hookrightarrow H^1(\mathcal{O}_Y)$. We will say moreover that Y is *weakly nondegenerate* if V_Y is a base point free subspace of $H^0(\Omega^1_X)$.

REMARKS. For a better understanding of Theorem 1.3, a few comments about its hypotheses are in order.

(a) When X is e.g. a curve or an abelian variety and L and M are e.g. ample line bundles then the condition on the vanishing of $H^1(\Omega^1_X \otimes L \otimes M)$ is obvious.

(b) In order to apply Theorem 1.3 one needs subvarieties of $\mathrm{Pic}^0\mathrm{X}$ which are (at least) weakly nondegenerate. E.g. if X is a complex curve of genus g and Y is the support of a Brill–Noether variety $W^r_d(X)$ such that $\rho(d, g, r) > 0$ then Y is nondegenerate ([FL], Remark 1.9). If Y is nondegenerate condition (a) of Theorem 1.3 becomes simply that the multiplication map $H^0(\Omega^1_X) \otimes H^0(L \otimes M) \to H^0(\Omega^1_X \otimes L \otimes M)$ should be onto. Again, this is obvious for abelian varieties, while for line bundles on curves this is true under the mild hypothesis $\deg(L) + \deg(M) \geqslant 2g + 3$ ([G] or [EKS]), see also the Appendix below).

(c) We recall that a line bundle L on X is said to be *normally generated* or, *to verify property* N_0 if the multiplication map

$$m_{L,L}: H^0(L) \otimes H^0(L) \to H^0(L^{\otimes 2})$$

is surjective. Moreover (at least in characteristic zero), one can say that L *is normally presented* or that it *verifies property* (N_1) if the map

$$m_L(L, L): \mathrm{Rel}(L, L) \otimes H^0(L) \to \mathrm{Rel}(L^{\otimes 2}, L)$$

is surjective (see [M1], [G1] and [L3] for more about this terminology). Clearly (N_1) is stronger than (N_0). With this in mind it is not difficult to convince themselves that condition (b) of Theorem 1.3, which is “of type (N_1)” for the triples $(L, M'_{-\alpha}, M''\alpha)$ is much harder to realize than condition (a) of the same Theorem, which is “of type (N_0)” on the pairs $(L \otimes M''_\alpha, M'_{-\alpha})$. This is why Theorem 1.3 can be roughly stated as Theorem B of the Introduction. As we will see in Chapter 3 below, when X is a curve, in many cases one can reduce – via the classical base point free pencil trick – the question of the failure of the surjectivity of the maps

$m_{M'_{-\alpha}}(L, M''_\alpha)$ to the question of the failure of the surjectivity of multiplication maps

$$m_{E_\alpha,F_{f(\alpha)}}: H^0(E_\alpha) \otimes H^0(F_{f(\alpha)}) \to H^0(E \otimes F_{\alpha+f(\alpha)})$$

for suitable families $(E_\alpha, F_{f(\alpha)})$ and, in this way, one can vastly extend some of Voisin's results ([V]) to different contexts.

(d) Theorem 1.3 extends verbatim to the case when Y reduced but not irreducible, replacing the sentence "α general in Y" with "α general in each component of Y".

(e) If the loci $Y(E), Y(F)$ and $m(Y, E, F)$ are empty it is not necessary to assume Y to be CM.

(C) Appendix: sketch of proof of Lemma 1.5

As mentioned above, Lemma 1.5 is an elementary consequence of the duality theory (the "Fourier functor" of [Mu]) between Alb$X$ and Pic^{0}X. However, since we have not found a comfortable reference for the specific statement we need, for the reader's convenience we sketch a proof here.

Let us recall the setup. We have a variety X such that the cotangent bundle Ω^1_X is generated by its global sections. Let us choose a Poincaré line bundle $\mathcal{P}$ on $X \times \mathrm{Pic}^0\mathrm{X}$. Let also $\Delta := \Delta^{(1)}$ be the diagonal in $X \times X$ and $\Delta^{(2)}$ its first infinitesimal neighborhood. Let p_i denote the projections on $X \times X \times \mathrm{Pic}^0\mathrm{X}$ and p_{ij} the projections on the intermediate factors. Finally let E and F be locally free sheaves on X.

We denote $\mathcal{Q} := p_{13}^*(p_1^*(E) \otimes \mathcal{P}) \otimes p_{13}^*(p_2^*(F) \otimes \mathcal{P}^\vee)$. Applying p_{3*} to the sequence

$$0 \to \mathcal{Q} \otimes \mathcal{I}_{\Delta \times \mathrm{Pic}^0\mathrm{X}|\Delta^{(2)} \times \mathrm{Pic}^0\mathrm{X}} \to \mathcal{Q} \otimes \mathcal{O}_{\Delta^{(2)} \times \mathrm{Pic}^0\mathrm{X}} \to \mathcal{Q}_{|\Delta \times \mathrm{Pic}^0\mathrm{X}} \to 0 \qquad (1)$$

we get (using the isomorphisms $\mathcal{Q}_{|\Delta \times \mathrm{Pic}^0\mathrm{X}} \cong p_{12}^*(p_1^*(E) \otimes p_2^*(F))$, $\mathcal{Q} \otimes \mathcal{I}_{\Delta \times \mathrm{Pic}^0\mathrm{X}|\Delta^{(2)} \times \mathrm{Pic}^0\mathrm{X}} \cong p_{12}^*(p_1^*(E) \otimes p_2^*(F) \otimes \mathcal{N}^\vee_{\Delta|X \times X})$ and the fact that $H^1(\Omega^1_X \otimes E \otimes F)$ is supposed to vanish) the exact sequence

$$\begin{aligned} 0 &\to H^0(E \otimes F \otimes \Omega^1_X) \to p_{3*}(\mathcal{Q}_{|\Delta^{(2)} \times \mathrm{Pic}^0\mathrm{X}}) \\ &\to H^0(E \otimes F) \otimes \mathcal{O}_{\mathrm{Pic}^0\mathrm{X}} \to 0. \end{aligned} \qquad (2)$$

Let us remark, by the way, that extension (2) globalizes extensions of vector spaces

$$0 \to H^0(\Omega^1_X \otimes E \otimes F) \to H^0(P^1(E_\alpha) \otimes F_{-\alpha}) \to H^0(E \otimes F) \to 0,$$

where $P^1(E)$ is the first jet bundle associated to E. Lemma 1.5 is a particular case of

THEOREM 1.6. *Up to scalar multiplication, the coboundary maps*

$$H^0(E\otimes F)\otimes H^{g-j-1}(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}})\to H^0(\Omega^1_X\otimes E\otimes F)\otimes H^{g-j}(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}})$$

of the long cohomology sequence associated to (2) coincide, via the usual identifications, with the Koszul maps $H^0(E\otimes F)\otimes\Lambda^{j+1}H^0(\Omega^1_X)\to H^0(E\otimes F\otimes\Omega^1_X)\otimes\Lambda^jH^0(\Omega^1_X)$.

Sketch of proof. Although the statement is probably an easy consequence of Mukai's theory ([Mu]), we will follow closely Kempf's treatment ([K1]). Let us fix an Albanese map $a\colon X\to \mathrm{Alb}X$ and consider on the product $\mathrm{Alb}X\times\mathrm{Pic}^0\mathrm{X}$, a Poincaré line bundle $\tilde{\mathcal{P}}$ compatible with $\mathcal{P}$ via a. Let π_1 and π_2 be the two projections on $\mathrm{Alb}X\times\mathrm{Pic}^0\mathrm{X}$. The key starting point is Mumford's Theorem, stating that

$$R^k\pi_{i_*}(\tilde{\mathcal{P}})=0 \text{ for } k<g \quad\text{and}\quad R^g\pi_{i_*}(\tilde{\mathcal{P}})=\mathcal{O}_0,$$

where $\mathcal{O}_0$ denotes the skyscraper sheaf of rank one at the zero point in the abelian variety in question ([M2], Chapter 13). As a consequence one gets, for example, that given a scheme over $\mathrm{Alb}X$, $l\colon S\to\mathrm{Alb}X$, then

$$R^j\pi_{S_*}((l\times\mathrm{id}_{\mathrm{Pic}^0\mathrm{X}})^*(\tilde{\mathcal{P}}))\cong\mathcal{T}or_{g-j}^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_S,\mathcal{O}_0),\tag{3}$$

where $\mathcal{O}_S$ is seen as an $\mathcal{O}_{\mathrm{Alb}X}$-module via l ([K_1] Cor. 2.2).

Let us denote a_1-a_2 the composed map

$$X\times X\xrightarrow{a\times a}\mathrm{Alb}X\times\mathrm{Alb}X\to\mathrm{Alb}X.$$

$$(x,y)\mapsto x-y$$

The second main point is that there is a natural isomorphism

$$p_{13}^*(\mathcal{P})\otimes p_{23}^*(\mathcal{P}^\vee)\cong((a_1-a_2)\otimes\mathrm{id}_{\mathrm{Pic}^0\mathrm{X}})^*(\mathcal{P}).\tag{4}$$

When X is an abelian variety itself this is a consequence of the see-saw principle (see e.g. [K1], 5.1 or [Mu] p. 156), and the general case follows easily from the universal property of the Albanese variety (see [K1], 5.2 for the case of curves). Therefore one has natural isomorphisms

$$p_{13}^*(\mathcal{P})\otimes p_{23}^*(\mathcal{P}^\vee)_{|\Delta^{(i)}\otimes\mathrm{Pic}^0\mathrm{X}}\cong((a_1-a_2)_{|\Delta^{(i)}})\times\mathrm{id}_{\mathrm{Pic}^0\mathrm{X}})^*(\mathcal{P})\tag{5}$$

for $i=1,2$. Since $p_1^*(E)\otimes p_2^*(F)$ are locally free, by (3), (5) and projection formula one gets natural isomorphisms

$$\begin{array}{c}R^jp_{12*}(p_{13}^*(p_1^*(E)\otimes\mathcal{P})\otimes p_{23}^*(p_2^*(F)\otimes\mathcal{P}^\vee)_{|\Delta^{(i)}\times\mathrm{Pic}^0\mathrm{X}}))\\ \Big\| \\ p_1^*(E)\otimes p_2^*(F)\otimes\mathcal{T}or_{g-j}^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_{\Delta^{(i)}},\mathcal{O}_0)\end{array}\tag{6}$$

for $i = 1, 2$, where $\mathcal{O}_{\Delta(i)}$ is viewed as an $\mathcal{O}_{\mathrm{Alb}X}$-module via $a_1 - a_2$.

Now, using the functorially of the isomorphisms (3), one gets that the long cohomology sequence obtained applying p_{12*} to (1)

$$\cdots \to R^{g-j}p_{12*}(\mathcal{Q}_{|\Delta^{(2)}}) \to R^{g-j}p_{12*}(\mathcal{Q}_{|\Delta})$$

$$\to R^{g-j+1}p_{12*}(\mathcal{Q} \otimes N_{\Delta\times \mathrm{Pic}^0 X}) \to \cdots \tag{7}$$

is canonically isomorphic to the long exact sequence of $\mathcal{T}or$'s

$$\cdots \to p_1^*(E) \otimes p_2^*(F) \otimes \mathcal{T}or_j^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_{\Delta^{(2)}}, \mathcal{O}_0)$$

$$\to E \otimes F \otimes \mathcal{T}or_j^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_\Delta, \mathcal{O}_0)$$

$$\to E \otimes F \otimes \Omega_X^1 \otimes \mathcal{T}or_{j-1}^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_\Delta, \mathcal{O}_0) \to \cdots \tag{8}$$

obtained applying $\otimes_{\mathcal{O}_{\mathrm{Alb}X}} \mathcal{O}_0$ to the sequence of $\mathcal{O}_{\mathrm{Alb}X}$-modules (via $a_1 - a_2$):

$$0 \to E \otimes F \otimes \mathcal{N}_\Delta^\vee \to p_1^*(E) \otimes p_2^*(F) \otimes \mathcal{O}_{\Delta^{(2)}} \to E \otimes F \otimes \mathcal{O}_\Delta \to 0.$$

As a is immersive, Δ is the scheme theoretic fibre of the map $a_1 - a_2$ at the point 0 and therefore $\mathcal{T}or_j^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_\Delta, \mathcal{O}_0) \cong \mathcal{T}or_j^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_0, \mathcal{O}_0) \otimes \mathcal{O}_\Delta$. Since, as it is well known, $\mathcal{T}or_j^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_0, \mathcal{O}_0) \cong \Lambda^j H^0(\Omega_X^1) \otimes \mathcal{O}_{\mathrm{Alb}X}$, we get a canonical isomorphism

$$\mathcal{T}or_j^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_\Delta, \mathcal{O}_0) \cong \Lambda^j H^0(\Omega_X^1) \otimes \mathcal{O}_\Delta.$$

Moreover, taking H^0's in (8) the maps

$$H^0(E \otimes F \otimes \mathcal{T}or_j^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_\Delta, \mathcal{O}_0))$$

$$\to H^0(E \otimes F \otimes \Omega_X^1 \otimes \mathcal{T}or_{j-1}^{\mathcal{O}_{\mathrm{Alb}X}}(\mathcal{O}_\Delta \mathcal{O}_0))$$

coincide, via the identifications above, with the Koszul maps

$$H^0(E \otimes F) \otimes \Lambda^j H^0(\Omega_X^1) \to H^0(E \otimes F \otimes \Omega_X^1) \otimes \Lambda^{j-1} H^0(\Omega_X^1). \tag{9}$$

Therefore, taking H^0's in (7), the maps

$$H^0(R^{g-j}p_{12*}(\mathcal{Q}_{|\Delta})) \to H^0(R^{g-i+1}p_{12*}(\mathcal{Q} \otimes N_{\Delta\times \mathrm{Pic}^0 X}))$$

are canonically identified to the Koszul maps (9). Thus the Theorem follows considering the two functorial Leray spectral sequences

$$H^k(X \times X, R^j p_{12*}(\bullet)) \Longrightarrow H^{k+j}(X \times X \times \mathrm{Pic}^0 X, \bullet)$$

$$H^k(\mathrm{Pic}^0\mathrm{X}, R^j p_{3*}(\bullet)) \Longrightarrow H^{k+j}(X \times X \times \mathrm{Pic}^0\mathrm{X}, \bullet)$$

applied to (1). □

In the course of the next section, dealing with higher gaussian maps, we will need a generalization of Theorem 1.6, which is proved more or less in the same way. To this purpose, let us consider the exact sequence

$$0 \to \mathcal{Q} \otimes \frac{\mathcal{I}^k_{\Delta\times\mathrm{Pic}^0\mathrm{X}}}{\mathcal{I}^{k+1}_{\Delta\times\mathrm{Pic}^0\mathrm{X}}} \to \mathcal{Q} \otimes \frac{\mathcal{I}^{k-1}_{\Delta\times\mathrm{Pic}^0\mathrm{X}}}{\mathcal{I}^{k+1}_{\Delta\times\mathrm{Pic}^0\mathrm{X}}} \to \mathcal{Q} \otimes \frac{\mathcal{I}^{k-1}_{\Delta\times\mathrm{Pic}^0\mathrm{X}}}{\mathcal{I}^{k}_{\Delta\times\mathrm{Pic}^0\mathrm{X}}} \to 0. \tag{10}$$

Assume that $H^1(E \otimes F \otimes \mathrm{S}^k\Omega^1_X) = 0$. Then applying p_{3*} to (7) we get as above the exact sequence

$$0 \to H^0(E \otimes F \otimes \mathrm{S}^k\Omega^1_X) \otimes \mathcal{O}_{\mathrm{Pic}^0\mathrm{X}} \to p_{3*}\left(\mathcal{Q} \otimes \frac{\mathcal{I}^{k+1}_{\Delta\times\mathrm{Pic}^0\mathrm{X}}}{\mathcal{I}^{k-1}_{\Delta\times\mathrm{Pic}^0\mathrm{X}}}\right)$$
$$\to H^0(E \otimes F \otimes \mathrm{S}^{k-1}\Omega^1_X) \otimes \mathcal{O}_{\mathrm{Pic}^0\mathrm{X}} \to 0. \tag{11}$$

THEOREM 1.7. *Up to scalar factors the coboundary maps*

$$H^0(E \otimes F \otimes \mathrm{S}^{k-1}\Omega^1_X) \otimes H^{g-j-1}(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}})$$
$$\to H^0(E \otimes F \otimes \mathrm{S}^{k}\Omega^1_X) \otimes H^{g-j}(\mathcal{O}_{\mathrm{Pic}^0\mathrm{X}})$$

of (11) coincide, via the usual identifications, with the "Eagon-Northcott" maps

$$H^0(E \otimes F \otimes \mathrm{S}^{k-1}\Omega^1_X) \otimes \Lambda^{j+1} H^0(\Omega^1_X)$$
$$\to H^0(E \otimes F \otimes \mathrm{S}^{k}\Omega^1_X) \otimes \Lambda^{j} H^0(\Omega^1_X).$$

(D) Generalization to higher gaussian maps

The arguments of Section (b) above extend almost verbatim to higher gaussian maps. In this section we will state the results, and give only an outline of the proofs.

Let L, M and N be locally free sheaves on X. For any k one has the two natural multiplication maps of higher relations with global sections

$$m^k_L(M,N)\colon \mathrm{Rel}^k(L,M) \otimes H^0(N) \to \mathrm{Rel}^k(L, M \otimes N),$$

$$m^k_M(L,N)\colon \mathrm{Rel}^k(L,M) \otimes H^0(N) \to \mathrm{Rel}^k(L \otimes N, M).$$

E.g., starting from $m_L^1(M,N)$, one can construct inductively commutative diagrams

$$\begin{array}{ccc} \mathrm{Rel}^{k-1}(L,M)\otimes H^0(N) & \xrightarrow{\gamma^{k-1}_{L,M}} & H^0(\mathrm{S}^{k-1}\Omega^1_X\otimes L\otimes M)\otimes H^0(N) \\ \Big\downarrow{\scriptstyle m_L^{k-1}(M,N)} & & \Big\downarrow \\ \mathrm{Rel}^{k-1}(L,M\otimes N) & \xrightarrow{\gamma^{k-1}_{L,M\otimes N}} & H^0(\mathrm{S}^{k-1}\Omega^1_X\otimes L\otimes M\otimes N) \end{array}$$

inducing naturally the map $m_L^k(M,N)$. The maps $m_M^k(L,N)$ are defined in the same way. It is easy to check that the following diagram is commutative

$$\begin{array}{ccc} \mathrm{Rel}^{k}(L,M)\otimes H^0(N) & \xrightarrow{m_L^k(M,N)} & \mathrm{Rel}^k(L,M\otimes N) \\ \Big\downarrow{\scriptstyle m_M^k(L,N)} & & \Big\downarrow{\scriptstyle \gamma^k_{L,M\otimes N}} \\ \mathrm{Rel}^{k}(L\otimes N,M) & \xrightarrow{\gamma_{L\otimes N,M}} & H^0(\mathrm{S}^{k}\Omega^1_X\otimes L\otimes M\otimes N) \end{array}$$

Let us introduce the following notation: given two sheaves E and F on X let us denote $Y(E,F)^k$ the locus where $\mathrm{Rel}^k(E_a, F_{-\alpha})$ jumps, $\gamma^k(Y,E,F)$ the locus where the higher gaussian map $\gamma^k(E_\alpha, F_{-\alpha})$ is not surjective, and $\gamma^k(Y,E,F)_1$ be the union of all components of $\gamma^k(Y,E,F)$ of codimension one. The generalization of Theorem 1.3 is

THEOREM 1.8. *Let X be a smooth irreducible projective variety such that Ω^1_X is generated by its sections. Let L and M be two vector bundles such that $H^1(\mathrm{S}^k\Omega^1_X\otimes L\otimes M)=0$ and let $M=M'\otimes M''$. Assume that Y is a CM subvariety of* $\mathrm{Pic}^0\mathrm{X}$ *such that the jump locus $Y(L\otimes M'', M')^{k-1}$ has codimension $\geqslant 2$ in Y and that $U\subset Y$ is an open set meeting every component of $\gamma^{k-1}(Y, L\otimes M'', M')_1$ such that*

(a) $V_Y\otimes H^0(\mathrm{S}^{k-1}\Omega^1_X\otimes L\otimes M)\to H^0(\mathrm{S}^k\Omega^1_X\otimes L\otimes M)$ *is surjective;*
(b) *the higher gaussian map* $\gamma^{k-1}_{L\otimes M''_\alpha, M'_{-\alpha}}:\mathrm{Rel}^{k-1}(L\otimes M''_\alpha, M'_{-\alpha})\to H^0(\mathrm{S}^{k-1}\Omega^1_X\otimes L\otimes M)$ *is surjective for α general in Y;*
(c) *the map* $m^k_{M'_{-\alpha}}(L,M''_\alpha):\mathrm{Rel}^k(L,M'_{-\alpha})\otimes H^0(M''_\alpha)\to\mathrm{Rel}^k(L\otimes M''_\alpha, M'_{-\alpha})$ *is surjective for any α in U.*

Then the higher gaussian map $\gamma^k_{L,M}:\mathrm{Rel}^k(L,M)\to H^0(\mathrm{S}^k\Omega^1_X\otimes L\otimes M)$ *is surjective.*

Let us consider the commutative diagram

$$\begin{array}{ccc} \bigoplus_{\alpha\in\mathrm{Pic}^0X}\mathrm{Rel}^k(L,M'_{-\alpha})\otimes H^0(M''_\alpha) & \xrightarrow{\sum m^{1k}_\alpha} & \mathrm{Rel}^k(L,M) \\ \Big\downarrow{\scriptstyle \oplus m^{2k}_\alpha} & & \Big\downarrow{\scriptstyle \gamma^k_{L,M}} \\ \bigoplus_{\alpha\in\mathrm{Pic}^0X}\mathrm{Rel}^k(L\otimes M''_\alpha, M'_{-\alpha}) & \xrightarrow{\sum \gamma^k_\alpha} & H^0(S^k\Omega^1_X\otimes L\otimes M) \end{array} \tag{1}$$

where $m^{1k}_\alpha := m^k_L(M'_{-\alpha}, M''_\alpha), m^{2k}_\alpha := m^k_{M'_{-\alpha}}(L, M''_\alpha)$ are multiplication maps of higher relations with global sections and $\gamma^k_\alpha := \gamma^k_{L\otimes M''_\alpha, M'_{-\alpha}}, \gamma^k_{L,M}$ are higher gaussian maps. As in the previous section, Theorem 1.8 will follow from diagram (1) and the following generalization of Lemma 1.2.

LEMMA 1.9. *Let X be as in Theorem1.8 and let E and F be vector bundles on X such that $H^1(S^k\Omega^1_X\otimes E\otimes F)=0$. Assume that Y is a CM subvariety of* Pic^0X *such that the jump locus $Y(E,F)^{k-1}$ has codimension $\geqslant 2$ in Y and*

(a) *$V_Y\otimes H^0(S^{k-1}\Omega^1_X\otimes E\otimes F)\to H^0(S^k\Omega^1_X\otimes E\otimes F)$ is surjective;*

(b) *the higher gaussian map $\gamma^{k-1}_{E_\alpha,F_{-\alpha}}: \mathrm{Rel}^{k-1}(E_\alpha,F_{-\alpha})\to H^0(S^{k-1}\Omega^1_X\otimes E\otimes F)$ is surjective for any α in U.*

Then for any open set $U\subset Y$ meeting every component of $\gamma^{k-1}(Y,E,F)_1$ the map

$$\sum_{\alpha\,\mathrm{gen}\,\in Y}\gamma_{E_\alpha,F_{-\alpha}}: \bigoplus_{\alpha\,\mathrm{gen}\,\in Y}\mathrm{Rel}^k(E_\alpha,F_{-\alpha})\to H^0(S^k\Omega^1_X\otimes E\otimes F)$$

is surjective.

Let us sketch the proof of Lemma 1.9. To start with, one globalizes the higher gaussian maps as before: applying p_{3*} to the exact sequence

$$0\to\mathcal{L}\otimes\mathcal{I}^k_{\Delta_Y}\to\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y}\to\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y}/\mathcal{I}^k_{\Delta_Y}\to 0$$

one gets

$$0\to p_{3*}(\mathcal{L}\otimes\mathcal{I}^k_{\Delta_Y})\to p_{3*}(\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y})$$
$$\xrightarrow{\tilde\gamma^{k-1}} H^0(S^{k-1}\Omega^1_X\otimes E\otimes F)\otimes\mathcal{O}_Y\to\tau\to 0. \tag{2}$$

As Y is reduced, off the jump locus $Y(E,F)^{k-1}$ one has that $p_{3*}(\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y})$ is locally free and $p_{3*}(\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y})(\alpha)\cong\mathrm{Rel}^{k-1}(E_\alpha,F_{-\alpha})$. Moreover the map

$$\tilde\gamma^{k-1}(\alpha): p_{3*}(\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y})(\alpha)\cong\mathrm{Rel}^{k-1}(E_\alpha,F_{-\alpha})$$
$$\to H^0(S^{k-1}\Omega^1_X\otimes E\otimes F)$$

is the $(k-1)$th gaussian map and $p_{3*}(\mathcal{L}\otimes\mathcal{I}^k_{\Delta_Y})(\alpha)\cong \mathrm{Rel}^k(E_\alpha, F_{-\alpha})$. Let us consider the next map

$$f\colon \mathcal{L}\otimes\mathcal{I}^k_{\Delta_Y}\to\mathcal{L}\otimes\mathcal{I}^k_{\Delta_Y}/\mathcal{I}^{k+1}_{\Delta_Y}.$$

By the same reason applying p_{3*} we get a map

$$p_{3*}(f)=:\gamma^k\colon p_{3*}(\mathcal{L}\otimes\mathcal{I}^k_{\Delta_Y})\to H^0(\mathrm{S}^k\Omega^1_X\otimes E\otimes F)\otimes\mathcal{O}_Y$$

which is the globalization of kth higher gaussian maps $\gamma^k_{E_\alpha,F_{-\alpha}}$. Let us consider the commutative exact diagram

$$\begin{array}{ccccccc}
\to H^0(\mathrm{S}^{k-1}\Omega^1_X\otimes E\otimes F)^\vee\otimes\mathcal{O}_Y & \longrightarrow & p_{3*}\left(\mathcal{L}\otimes\dfrac{\mathcal{I}^{k-1}_{\Delta_Y}}{\mathcal{I}^{k+1}_{\Delta_Y}}\right)^\vee & \longrightarrow & H^0(\mathrm{S}^k\Omega^1_X\otimes E\otimes F)^\vee\otimes\mathcal{O}_Y & \longrightarrow \\
\Vert & & \downarrow & & & \\
\to H^0(\mathrm{S}^{k-1}\Omega^1_X\otimes E\otimes F)^\vee\otimes\mathcal{O}_Y & \longrightarrow & p_{3*}(\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y})^\vee & \longrightarrow & \mathcal{W}^k & \longrightarrow
\end{array}\tag{3}$$

where:

(i) the bottom row is the first short exact sequence obtained dualizing (2) (we remark that, due to hypothesis (b), τ is a torsion sheaf on Y),

(ii) the top row is obtained from the sequence of $\mathcal{O}_{X\times X\times Y}$-modules

$$0\to\mathcal{L}\otimes\mathcal{I}^k_{\Delta_Y}/\mathcal{I}^{k+1}_{\Delta_Y}\to\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y}/\mathcal{I}^{k+1}_{\Delta_Y}\to\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y}/\mathcal{I}^{k}_{\Delta_Y}\to 0$$

applying p_{3*} and dualizing (we have 0 on the right since $H^1(\mathrm{S}^k\Omega^1_X\otimes E\otimes F)$ is supposed to vanish),

(iii) the middle vertical arrow is the dual of $p_{3*}(\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y}\to\mathcal{L}\otimes\mathcal{I}^{k-1}_{\Delta_Y}/\mathcal{I}^{k+1}_{\Delta_Y})$.

By means of diagram (3) one can lift canonically the map $\tilde{\gamma}^{k\vee}$ to a map

$$\dot{\tilde{\gamma}}^{k\vee}\colon H^0(\mathrm{S}^k\Omega^1_X\otimes E\otimes F)^\vee\otimes\mathcal{O}_Y\to\mathcal{W}^k.$$

Arguing as in the proof of lemma 1.2 it is sufficient to prove that $\dot{\tilde{\gamma}}^{k\vee}$ is injective at the global sections level. This is in turn implied by

LEMMA 1.10. *Let us consider the coboundary map*

$$\delta\colon H^0(\mathrm{S}^k\Omega^1_X\otimes E\otimes F)^\vee\to H^0(\mathrm{S}^{k-1}\Omega^1_X\otimes E\otimes F)^\vee\otimes H^1(\mathcal{O}_Y)$$

associated to the top row of (3). Then, via the identification $H^0(\Omega^1_X) \cong H^1(\mathcal{O}_{\mathrm{Pic}^0 X})^\vee$, *the map* δ *coincides, up to proportionality, with the dual of the composed map*

$$\begin{array}{ccc} H^0(S^{k-1}\Omega^1_X \otimes E \otimes F) \otimes H^1(\mathcal{O}_Y)^\vee & \xrightarrow{\mathrm{id}\otimes\phi_Y} & H^0(S^{k-1}\Omega^1_X \otimes E \otimes F) \otimes H^0(\Omega^1_X) \\ & & \downarrow \\ & & H^0(S^k\Omega^1_X \otimes E \otimes F) \end{array}$$

where the vertical arrow is the natural map.

Lemma 1.10 is proved as Lemma 1.5 using Theorem 1.7 of the previous section. □

2. Application I: gaussian maps on abelian varieties

In this section we will show how, plugging into Theorem 1.3 results about multiplication maps on abelian varieties due to Mumford–Koizumi–Sechiguchi and Kempf, one obtains sharp results about the surjectivity of gaussian maps of line bundles on abelian varieties.

Clearly when X is an abelian variety Theorems 1.3 and 1.8 work over any algebraically closed field of any characteristic. We will use the following terminology. Let X be an abelian variety (defined over an algebraically closed field k). Given an ample line bundle A over X and another line bundle L over X we will say that the *type* of L with respect to A (denoted $t_A(L)$) is l if L is algebraically equivalent to $A^{\otimes l}$.

THEOREM 2.1. *Let* X *be an abelian variety over an algebraically closed field and let* A, L *and* M *be ample line bundles on* X. *If* $t_A(L), t_A(M) \geqslant 4$ *and* $t_A(L) + t_A(M) \geqslant 9$ *then the gaussian map* $\gamma_{L,M}$ *is surjective. As a particular case, if* $t_A(L) \geqslant 5$ *then the map* $\gamma_{L,L}$ *is surjective.*

Proof. We apply Theorem 1.3 taking as Y the full $X^\vee$ and writing $M = M' \otimes M''$ with $t_A(M'') = 2$. The statement will follow as soon as we check that the hypotheses of 1.3 are fulfilled. The fact that $H^1(\Omega^1_X \otimes L \otimes M) = 0$ is obvious as well as condition 1.3(a). Since $t_A(M') \geqslant 2$ and $t_A(L \otimes M'') \geqslant 6$, by a theorem of Mumford *et al.* ([K2] Theorem 6.8(c)) the multiplication map $m_{L\otimes M''_\alpha, M'_{-\alpha}}$ is surjective for any $\alpha \in X^\vee$ and this settles condition (b). Concerning (c), we have that either $t_A(L) \geqslant 5, t_A(M') \geqslant 2$ or $t_A(L) \geqslant 4$ and $t_A(M') \geqslant 3$. Therefore, by a theorem of Kempf ([K2], Theorem 6.14), the map $m_{M'_{-\alpha}}(L, M''_\alpha)$ is surjective. □

REMARK. The statement of Theorem 2.1 for elliptic curves was originally proved in Wahl's work [W] (it is also a particular case of [BEL], Theorem 1) and, as pointed out in [W], it is sharp: e.g. if $L = \mathcal{O}_E(3p)$ then $\gamma_{L,L^{\otimes 2}}$ is not surjective. Moreover if $L = \mathcal{O}_E(4p)$ then $\gamma_{L,L}\colon \Lambda^2 H^0(L) \to H^0(L^{\otimes 2})$ can't be surjective for dimension reasons.

The next result (Theorem 2.2) is a generalization of Theorem 2.1 to higher gaussian maps. In the course of the proof we will need the following theorem of Kempf, generalizing to vector bundles a classical result of Mumford on multiplication maps of line bundles on abelian varieties ([M1], see also [K2], Lemma 4.6):

THEOREM. (Kempf) *Let X be an abelian variety over an algebraically closed field and let E and F be vector bundles on X such that $H^j(E_\alpha) = H^j(F_\alpha) = 0$ for any $\alpha \in$ Pic^{0}X and for any $j > 0$. Then the map*

$$\sum_{\alpha \text{ gen } \in \mathrm{Pic}^0 X} m_{E_\alpha, F_{-\alpha}}\colon \bigoplus_{\alpha \text{ gen } \in \mathrm{Pic}^0 X} H^0(E_\alpha) \otimes H^0(F_{-\alpha}) \to H^0(E \otimes F)$$

is surjective (where gen $\in$ Pic^{0}X *means "in an open set of* Pic^{0}X*)".*

As Lazarsfeld pointed out, the above Theorem, although not explicitly stated, is implicitly proved in [K3] (see also [K2] p. 52).

THEOREM 2.2. *If $t_A(L) + t_A(M) \geqslant 1 + 4(k+1)$ and $t_A(L), t_A(M) \geqslant 2(k+1)$ then the higher gaussian map $\gamma^k_{L,M}\colon \mathrm{Rel}^k(L,M) \to H^0(S^k\Omega^1_X \otimes L \otimes M)$ is surjective. In particular, if $t_A(L) \geqslant 1 + 2(k+1)$ then the higher gaussian map $\gamma^k_{L,L}$ is surjective.*

Proof. For $k = 1$ the statement is just Theorem 1.1. Let us assume the statement true for any $h < k$ i.e.

($*$) *Let h be an integer $< k$ and let E and F be ample line bundles on X such that $t_A(E), t_A(F) \geqslant 2(h+1)$ and $t_A(E) + t_A(F) \geqslant 1 + 4(h+1)$. Then the higher gaussian map $\gamma^h_{E,F}$ is surjective.*

Let L and M be line bundles as in the statement of Theorem 2.2. To prove the surjectivity of $\gamma^k_{L,M}$ we apply Theorem 1.8 taking as Y the full $X^\vee$ and writing $M = M' \otimes M''$ with $t_A(M'') = 2$. Then condition 1.8(a) is obvious. Condition 1.8(b) follows by induction, since $t_A(L) + t_A(M') > 1 + 4k$ and $t_A(L \otimes M'') + t_A(M') = t_A(L) + t_A(M)$. Therefore we need to show that condition 1.8(c) holds. Since $t_A(L) \geqslant 1 + 2(k+1)$ and $t_A(M') \geqslant 2k$, or $t_A(L) \geqslant 2(k+1)$ and $t_A(M') \geqslant 2k+1$, this will follow from the following

CLAIM. *Assume that hypothesis ($*$) holds and let L, M, N be ample line bundles such that $t_A(N) \geqslant 2$ and $t_A(L) \geqslant 1 + 2(k+1), t_A(M) \geqslant 2k$ or $t_A(L) \geqslant 2(k+1), t_A(M) \geqslant 1 + 2k$. Then the map $m^k_M(L,N)\colon \mathrm{Rel}^k(L,M) \otimes H^0(N) \to \mathrm{Rel}^k(L \otimes N, M)$ is surjective.*

Proof of the Claim. Let us write $L = L' \otimes L''$ with $t_A(L'') = 2$. We have the commutative diagram

$$\begin{array}{ccc} \bigoplus_{\alpha \in X^\vee} \operatorname{Rel}^k(L'_{-\alpha}, M) \otimes H^0(L''_\alpha) \otimes H^0(N) & \xrightarrow{\sum m_\alpha^{1k} \otimes \mathrm{id}} & \operatorname{Rel}^k(L, M) \otimes H^0(N) \\ \Big\downarrow{\scriptstyle \oplus(\mathrm{id} \otimes m_\alpha)} & & \Big\downarrow{\scriptstyle m_M^k(L,N)} \\ \bigoplus_{\alpha \in X^\vee} \operatorname{Rel}^k(L'_{-\alpha}, M) \otimes H^0(L''_\alpha \otimes N) & \xrightarrow{\sum m_\alpha^{2k}} & \operatorname{Rel}^k(L \otimes N, M) \end{array} \tag{1}$$

where $m_\alpha^{1k} := m_M^k(L'_{-\alpha}, L''_\alpha)$, $m_\alpha^{2k} := m_M^k(L'_{-\alpha}, L''_\alpha \otimes N)$ and m_α is the multiplication map $m_{L''_\alpha, N}$. Since $t_A(L'') = t_A(N) = 2$, it follows from another theorem of Mumford *et al.* ([K1], Theorem 6.8(b)) that the map

$$m_\alpha = m_{L''_\alpha, N}\colon H^0(L''_\alpha) \otimes H^0(N) \to H^0(L''_\alpha \otimes N)$$

is surjective for α general in $X^\vee$. Therefore, by diagram (1), the Claim is enough to prove that the map

$$\sum_{\alpha \text{ gen} \in X^\vee} m_M^k(L'_{-\alpha}, L''_\alpha \otimes N)\colon \bigoplus_{\alpha \text{ gen} \in X^\vee} \operatorname{Rel}^k(L'_{-\alpha}, M) \otimes H^0(L''_\alpha \otimes N)$$
$$\to \operatorname{Rel}^k(L \otimes N, M) \tag{2}$$

is surjective. Note that $t_A(L') + t_A(M) \geqslant 1 + 4k$ and $t_A(L'), t_A(M) \geqslant 2k$. For any $\alpha \in X^\vee$ let us consider complex (1) of Section 1(a) above relative to the line bundles $L'_{-\alpha}$ and M

$$0 \to R^k_{L'_{-\alpha}, M} \to R^{k-1}_{L'_{-\alpha}, M} \to S^{k-1}\Omega^1_X \otimes L'_{-\alpha} \otimes M \to 0.$$

Because of hypothesis $(*)$ the gaussian maps $\gamma^h_{L'_{-\alpha}, M}$ are surjective for $h \leqslant k-1$. Therefore such a complex is exact (cf. Section 1(a)) and the sheaf $R^k_{L'_{-\alpha}, M}$ is a vector bundle. Moreover we have that

$$h^i(R^k_{L'_{-\alpha}, M}) = 0 \quad \text{for any} \quad i > 0 \quad \text{and} \quad \alpha \in X^\vee. \tag{3}$$

This is proved inductively: since, again by Mumford's theorems, the multiplication map $m_{L'_{-\alpha}, M}\colon H^0(L'_{-\alpha}) \otimes H^0(M) \to H^0(L'_{-\alpha} \otimes M)$ is surjective, from the exact sequence

$$0 \to R^1_{L'_{-\alpha}, M} \to H^0(L'_{-\alpha}) \otimes M \to L'_{-\alpha} \otimes M \to 0$$

we get, that $h^i(R^1_{L'_{-\alpha},M}) = 0$ for any $i > 0$. Then one keeps going: by $(*)$ the maps

$$\gamma^h_{L'_{-\alpha},M}: H^0(R^h_{L'_{-\alpha},M}) \to H^0(S^h\Omega^1_X \otimes L'_{-\alpha} \otimes M)$$

are surjective and the sequences

$$0 \to R^{h-1}_{L'_{-\alpha},M} \to R^h_{L'_{-\alpha},M} \to S^h\Omega^1_X \otimes L'_{-\alpha} \otimes M \to 0$$

are exact for any $h < k$. Then (3) follows easily.

Now let us go back to our map (2). Since $R^k_{L'_{-\alpha},M} = p_{1*}(p_2^*(M) \otimes \mathcal{I}_\Delta) \otimes L_{-\alpha}$ we have that $R^k_{L'_{-\alpha},M} = (R^k_{L',M})_{-\alpha}$. For the same reason $R^k_{L\otimes N,M} = R^k_{L'\otimes M} \otimes L'' \otimes N$ and the map (2) is identified to the sum of multiplication maps of vector bundles

$$\sum_{\alpha \text{ gen} \in X^\vee} m_{(R^k_{L',M})_{-\alpha}, L''\otimes N_\alpha}: \bigoplus_{\alpha \text{ gen} \in X^\vee} H^0((R^k_{L',M})_{-\alpha}) \otimes H^0(L'' \otimes N_\alpha)$$

$$\to H^0(R^k_{L',M} \otimes L'' \otimes N).$$

But now, by (3) and the above Kempf's Theorem this map is surjective. This proves the Claim and hence Theorem 2.2. □

As a byproduct, we have the following generalization of the aforementioned Theorem 6.14 of [K1]:

PROPOSITION 2.3. *Let A, L, M, N be ample line bundles on an abelian variety X. If $t_A(N) \geqslant 2$ and either $t_A(L) \geqslant 1 + 2(k+1), t_A(M) \geqslant 2k$ or $t_A(L) \geqslant 2(k+1), t_A(M) \geqslant 1 + 2k$ then the map $m^k_M(L, N): \mathrm{Rel}^k(L, M) \otimes H^0(N) \to \mathrm{Rel}^k(L \otimes N, M)$ is surjective.*

The proof is as the one of the previous Claim and it is left to the reader.

3. Application II: gaussian maps on curves

(A) Precise statement and proof of Theorem A

As mentioned in Remark (c) at the end of Chapter 1, the main problem in order to apply Theorem 1.3 is that, dealing with maps

$$m_L(M, N): \mathrm{Rel}(L, M) \otimes H^0(M) \to \mathrm{Rel}(L, M \otimes N),$$

optimal, or close to optimal, results about their surjectivity, as a function of the geometry of the variety X and of the (suitably defined) "positivity" of L, M and N, are in general not available, even in the case of curves over the complex field (see e.g. $[GL]$ for a conjecture about the aforementioned condition (N_1) for a line bundle L, which is in turn equivalent to the surjectivity of the map $m_L(L, L)$).

However, in the case of curves, one can partially remove this obstacle using the classical "base point free pencil trick": *If A is a base point free line bundle on a curve C such that $h^0(A) = 2$ and F is a coherent sheaf on A then* $\text{Rel}(A, F) \cong H^0(F \otimes A^\vee(B_A))$*, where B_A is the base divisor of A. Moreover, if E is another coherent sheaf on C, under the above identification, the map*

$$m_A(F, E)\colon \text{Rel}(A, F) \otimes H^0(E) \to \text{Rel}(A, E \otimes F)$$

is the multiplication map

$$m_{F\otimes A^\vee(B_A),E}\colon H^0(F \otimes A^\vee(B_A)) \otimes H^0(E) \to H^0(F \otimes E \otimes A^\vee(B_A)).$$

In this way one reduces the problem to the surjectivity of multiplication maps of global sections of line bundles. One is then led to consider families of line bundles $Y \subset \text{Pic}^{\text{d}}(\text{C})$. Given a bundle M an C we will denote Y^+ and $Y^-(M)$ the loci of line bundles A in Y where respectively $h^0(A)$ and $h^0(M \otimes A^\vee)$ jump. Let also $m(Y, M)$ be the locus where the multiplication map $m_{A,M\otimes A^\vee}$ is not surjective and $m(Y, M)_1$ be the union of all components of $m(Y, M)$ of codimension one. Then, decomposing M as $(M \otimes A^\vee) \otimes A$, Theorem 1.3 becomes

THEOREM 3.1. *Let X be a smooth irreducible projective curve of genus $g \geqslant 1$ and let L and M two vector bundles on C. Assume that there exists a CM subvariety $Y \subset \text{Pic}^{\text{d}}(Y)$ such that the general line bundle A parametrized by Y is a base point free pencil and the jump locus $Y^+ \cup Y^-(L \otimes M)$ has codimension $\geqslant 2$ in Y and such that*

(a) *the multiplication map $m_{V_Y,L\otimes M}\colon V_Y \otimes H^0(L \otimes M) \to H^0(K_X \otimes L \otimes M)$ is surjective* (here we identify $\text{Pic}^{\text{d}}(\text{X})$ and $\text{Pic}^0(\text{X})$ via a translation)*;*
(b) *the multiplication map $m_{L\otimes M\otimes A^\vee,A}\colon H^0(L\otimes M\otimes A^\vee)\otimes H^0(A) \to H^0(L\otimes M)$ is surjective for A general in Y.*
Assume moreover that $U \subset Y$ is an open set meeting every component of the locus $m(Y, M)_1$ and such that
(c) *the multiplication map $m_{L\otimes A^\vee,M\otimes A^\vee}\colon H^0(L \otimes A^\vee) \otimes H^0(M \otimes A^\vee) \to H^0(L \otimes M \otimes A^{\otimes -2})$ is surjective for any A in U.*

Then the gaussian map $\gamma_{L,M}\colon \text{Rel}(L, M) \to H^0(K_X \otimes L \otimes M)$ is surjective. □

We will use the following notation and terminology: in the sequel C will always be a smooth projective irreducible curve of genus $g \geqslant 1$ defined over the complex field. We will say that $Y \subset \text{Pic}^{\text{d}}(\text{C})$ is a *(weakly) nondegenerate family of base point free pencils* if a translate of Y in $\text{Pic}^0(\text{C})$ is (weakly) nondegenerate and the general line bundle parametrized by Y is a base point free pencil and the locus Y^+

of line bundles A in Y such that $h^0(A) > 2$ has codimension $\geqslant 2$. Furthermore we will denote Y^B the locus of line bundles parametrized by Y which are not base point free, and, as usual, Y_1^B the union of all components of Y_B of codimension one in Y.

THEOREM 3.2. *Let L and M be line bundles on C such that* $\deg(L) + \deg(M) \geqslant 2g + 2d - 1$. *Assume that $Y \subset \mathrm{Pic^d(C)}$ is a nondegenerate CM subvariety of base point free pencils and let $U \subset Y$ be an open set meeting every component of the locus Y_1^B. Under these hypotheses if the gaussian map $\gamma_{L,M}$ is not surjective then the multiplication map $m_{L\otimes A^\vee, M\otimes A^\vee}$ is not surjective for A general in Y.*

Proof. The statement follows at once from the previous Theorem since if $\deg(L \otimes M) \geqslant 2g + 3$ then the map $m_{K_C, L\otimes M}$ is surjective ([G], 4.e.4, cf. also App. B below). Moreover under the present hypotheses the jump locus $Y^-(L\otimes M)$ is empty and the locus $m(Y, L \otimes M)$ concides with the locus of line bundles with base points Y^B since, by the base point free pencil trick, the multiplication map $m_{L\otimes M\otimes A^\vee}$ is surjective if A is base point free and $\deg(L) + \deg(M) \geqslant 2g + 2d - 1$. □

REMARKS. (a) Condition (a) of Theorem 3.1 is *never* satisfied if the subspace V_Y of $H^0(K_C)$ has base points, e.g. if Y is an elliptic curve, or if C has a ramified map onto an irrational curve $C \to \Gamma$ and Y is a variety of pencils pulled back from Γ.

(b) In analogy with what pointed out Remark (c) at the end of Section 1.2, for degree reasons the surjectivity of the multiplication map $m_{L\otimes M\otimes A^\vee, A}$ is a much weaker condition than the surjectivity of the multiplication map $m_{L\otimes A^\vee, M\otimes A^\vee}$. Therefore, as a rough formulation, one gets Theorem A as stated in the introduction.

(c) If C is a Brill–Noether–Petri general curve of genus $\geqslant 7$, $L = M = K_C$ and $Y = W^1_{(g+3)/2}$, Lemma 5.1 and Theorem 5.2 are proved in Voisin's paper [V] (Cor. 2.8). This has been generalized by Paoletti ([P]) to the case where $L = K_C$ and M is a different line bundle and also to the even genus case taking $Y = W^1_{(g+4)/2}(C)$. Now these statements appear as particular cases of a much more general picture. One should also note that, even in the case of general curves, the present proof is totally different, and applies as well to curves satisfying the weaker *Brill–Noether* condition (cf. below). But the even genus case is subtler: for a Brill–Noether–Petri general curve C of even genus $\geqslant 10$, Voisin ([V] Prop. 3.2) proves the considerably stronger statement that if γ_{K_C,K_C} is not surjective then the multiplication map $m_{K_C\otimes A^\vee, K_C\otimes A^\vee}$ is not surjective for any A in $Y = W^1_{(g+2)/2}$ (note that in this case Y is a finite set and our methods do not apply at all).

(B) Preliminaries about gonality, Clifford index, and a lemma

An important point in order to apply Theorem 3.2 is to find a suitable nondegenerate family of base point free pencils. To this purpose, let us recall some terminology and basic facts about linear systems on curves.

A line bundle A on a curve C is said to be *primitive* if both A and $K_C \otimes A^\vee$ are base point free. If A is a primitive pencil then obviously $\deg(A) \leqslant g - 1$.

The *gonality* (denoted gon(C)) of a curve C is the minimum degree of a (necessarily complete) g^1_d on C. By the existence theorem of Kempf–Kleiman–Laksov ([ACGH]) we have that

$$\operatorname{gon}(C) \leqslant \left[\frac{g+3}{2}\right], \tag{1}$$

The bound is achieved e.g. by Brill–Noether general curves. A $g^1_{\operatorname{gon}(C)}$ – say A – on C is necessarily base point free. Moreover it is also primitive unless C is isomorphic to a smooth plane curve. Indeed if $K_C \otimes A^\vee$ has a base point p then $h^0(A(p)) = 3$ and $A(p)$ has to be very ample since otherwise there is a g^1_d with $d < \operatorname{gon}(C)$. Let us define the invariant $h(C)$ as the *minimal dimension of an irreducible component of the variety* $W^1_{\operatorname{gon}(C)}$. By [FHL] we have that $h(C) \leqslant 1$.

The *Clifford index of a line bundle* A on a curve C is the integer $\operatorname{cliff}(A) := \deg(A) - 2(h^0(A) - 1)$. The *Clifford index of* C itself is the minimum of the Clifford indexes of all line bundles A on C *contributing to the Clifford index*, i.e. such that $h^0(A) \geqslant 2$ and $h^1(A) \geqslant 2$. It is also said that A *computes the Clifford index* if A contributes to the Clifford index and $\operatorname{cliff}(A) = \operatorname{cliff}(C)$. It is known that

$$\operatorname{gon}(C) - 3 \leqslant \operatorname{cliff}(C) \leqslant \operatorname{gon}(C) - 2, \tag{2}$$

the inequality on the right being obvious, while the one on the left is a result of Coppens–Martens ([CM], Theorem 2.3). Moreover we have that

$$0 \leqslant \operatorname{cliff}(C) \leqslant [(g-1)/2]. \tag{3}$$

The bound on the left is the easy part of Clifford's theorem, while the one on the right follows from (2) and the quoted existence theorem of Kempf et al.

The gonality and the Clifford index keep track in a quantitative way of how special is the geometry of the line bundle on C: the smaller are gon(C) and cliff(C), the more exceptional are the line bundles on C. E.g. Clifford's theorem asserts that the lower bound in (3) is attained if and only if $\operatorname{gon}(C) = 2$, i.e. if C is hyperelliptic. On the other extreme the upper bound is attained by Brill–Noether general curves. Moreover, it should be said that for any integer c satisfying the constraints (3) there are curves C such that $\operatorname{gon}(C) + 2 = \operatorname{cliff}(C) = c$ ([B], [CM]).

Finally, we will say that $Y \subset \operatorname{Pic}^d(C)$ is a *(weakly) nondegenerate family of primitive pencils* if Y is a (weakly) nondegenerate family of base point free pencils such that the general element of Y is also primitive. In the sequel we will need the following lemma, relating the existence of non degenerate families of primitive pencils of a given degree with the invariants $\mathrm{gon}(C)$, $\mathrm{cliff}(C)$ and $h(C)$.

LEMMA 3.3 *Let C be a curve of Clifford index $\geqslant 2$. Then there exists an integer h with $h \leqslant g+3+h(C)-\mathrm{gon}(C)$ such that C has a nondegenerate CM family of primitive pencils of degree h.*

Proof. To start with, let us prove the statement under the additional hypotheses that C is not isomorphic to a smooth plane curve and $\mathrm{gon}(C) \geqslant 5$. In thus case we have

$$\mathrm{gon}(C) < g+3+h(C)-\mathrm{gon}(C) \leqslant g-1$$

(note that if g is odd and $\mathrm{gon}(C) = (g+3)/2$ then $h(C) = 1$). Let U be an irreducible component of minimal dimension of $W^1_{\mathrm{gon}(C)}$. Let $k = g+3+h(C)-\mathrm{gon}(C)$ and consider the irreducible subvariety $V_k := U + W^0_{k-\mathrm{gon}(C)} \subset J^k(C)$. Since

$$\begin{aligned}\rho(k,g,2) &= 2k-g-2\\ &= h(C)+k-\mathrm{gon}(C)+1\\ &= \dim(V_k)+1,\end{aligned}$$

we have that V_k is strictly contained in an irreducible component, say Y_k, of W^1_k. Then either the general element of Y_k is a base point free pencil or, by an easy dimension count, there is an h with $\mathrm{gon}(C)+1 \leqslant h \leqslant k$ such that W^1_h has an irreducible component Y_h strictly containing $U + W^0_{h-\mathrm{gon}(C)}$ and such that the general element of Y_h is base point free. We claim that Y_k is a nondegenerate family of primitive pencils. First of all Y_h is nondegenerate since $W^0_{h-\mathrm{gon}(C)}$ generates the jacobian as a group. We claim that the general element of Y_h is primitive as well. If A is in $U + V_h \subset Y_h$ then $K_C \otimes A^\vee$ is of the form $K_C \otimes B^\vee(-p_1 - \cdots - p_n)$, where B is in U and $n = h - \mathrm{gon}(C)$. As we are supposing that C is not isomorphic to a smooth plane curve, B is primitive, i.e. $K_C \otimes B^\vee$ is base point free. Therefore, for a general choice of the points $p_1, \ldots, p_n$ also $K_C \otimes A^\vee$ is. In the same way it is easy to see that the jump locus Y_h^+ of line bundles $A \in Y_h$ such that $h^0(A) > 2$ has codimension $\geqslant 2$ (we leave this to the reader). If Y_h is CM, e.g. if it has the right dimension $\rho(h,g,1)$, Y_h satisfies the conditions of the present Lemma. Otherwise $\dim Y_h > \rho(h,g,1)$ and one can find a CM subvariety of Y_h containing $U + W^0_{\mathrm{gon}(C)-h}$ satisfying the requested conditions. This proves the Lemma under the additional hypotheses above. Next, let us take care of the other cases. If C is a smooth plane curve of degree $\geqslant 6$ the statement is proved with a similar argument (left to the reader). If $\mathrm{gon}(C) = 4$ and C is not bielliptic then the argument above works since then $\dim W^1_{\mathrm{gon}(C)} = 0$. If $\mathrm{gon}(C) = 4$ and $h(C) = 1$ then C is bielliptic. In this case it is known ([We], [CS]) that there is a component W of $W^1_{g-1}(C)$ such that the general pencil parametrized by W is primitive. It is also possible to show (e.g. going through the proof of [CS] Prop. 3.3) that W is CM and nondegenerate. □

REMARKS AND PROBLEMS

(a) The case of curves of Clifford dimension $\geqslant 3$ is interesting (we refer to the paper [ELMS] for definitions, basic facts and further references about this subject). For such curves the variety $W^1_{\mathrm{gon}(C)}$ is already a nondegenerate family (of dimension 1) of primitive pencils. Indeed by a result of Coppens–Martens ([CM] Thm. 3.2), using a previous result of [ELMS] (Thm. 3.7), for such curves $\dim(W^1_{\mathrm{gon}(C)}) = 1$. Going through the proofs of the quoted results it is also possible to check that $W^1_{\mathrm{gon}(C)}$ is non degenerate. Some results in the present paper suggest that: *if $W^1_{\mathrm{gon}(C)}(C)$ has positive dimension and it is nondegenerate then either $g(C)$ is odd and C is Brill–Noether general or C should have Clifford dimension $\geqslant 2$.*

(b) It seems likely that the lower bound of Lemma 3.3 is not sharp. One may expect the right bound to be $g + 1 - \mathrm{cliff}(C)$. Such a bound coincides with the one of Lemma unless $h(C) = 1$ and $\mathrm{cliff}(C) = \mathrm{gon}(C) - 2$. In this case the difference between the bound of the Lemma and the expected one is 1. Note that for Brill–Noether general curves of odd genus and bielliptic curves the expected bound is valid. As we will see, the above discrepancy reflects on the bound for the surjectivity of the maps $\gamma_{K_C,L}$.

(c) Curves of Clifford index $\leqslant 1$ have no (weakly or not) nondegenerate family of primitive pencils. On the other hand, if one considers non degenerate families of base point free pencils (not necessarily primitive), it is known that if $\mathrm{cliff}(C) = 1$, i.e. if C is trigonal or isomorphic to a smooth plane quintic, then C has one in degree $g - 1$ ([ACGH] p. 372). In any case, $\mathrm{Pic}^{g+1}(C)$ itself is a nondegenerate family of base point free pencils and this is optimal for hyperelliptic curves.

(C) THE MAPS $\gamma_{K_C,L}$

Let us now apply the previous results to the problem of finding explicit surjectivity statements for gaussian maps $\gamma_{L,M}$, where L and M are line bundle whose degrees satisfy certain lower bounds depending of the intrinsic geometry of the curve C. In order to apply Theorem 3.2 we need results on the surjectivity of multiplication maps $m_{E,F}$, where $E \neq F$ are line bundles on a curve C, as a function of the geometry of C (notably of the indexes $\mathrm{cliff}(C)$ and/or $\mathrm{gon}(C)$). This is done in the Appendix below, extending known results and methods, basically due to Green and Lazarsfeld, for multiplication maps $m_{E,E}$. Then the line of attack is clear:

(1) one looks for all nondegenerate families Y of base point free pencils;
(2) given L, M and Y one needs (at least) that $L \otimes A^{\vee}$ and $M \otimes A^{\vee}$ are base point free for the general A in Y;
(3) then one applies the results the Appendix to get the surjectivity of the multiplication maps $\gamma_{L\otimes A^{\vee}, M\otimes A^{\vee}}$.

We start by showing how this method works for gaussian maps of type $\gamma_{K_C,L}$. As mentioned in the introduction, such maps have a very interesting deformation-theoretic meaning, discovered by Wahl (see [W1], [W2], [W3] and references therein), yielding the striking fact that if L is normally generated and γ_{L,K_C} is surjective then $C \hookrightarrow \mathbf{P}(H^0(L))$ is not the hyperplane section of any normal surface other than a cone.

THEOREM 3.4. *Let C be a curve such that* $\mathrm{cliff}(C) > 2$ *and let L be a line bundle on C. The gaussian map $\gamma_{K_C,L}$ is surjective if one of the following conditions holds:*

(a) $\deg(L) \geqslant 4g + 5 + 2h(C) - 2\,\mathrm{gon}(C) - \mathrm{cliff}(C)$;
(b) $(3g+1)/7 \geqslant \mathrm{cliff}(C) \geqslant (g+4)/3$ *and* $\deg(L) \geqslant 5g + 13 + 4h(C) - 4\,\mathrm{gon}(C) - 2\,\mathrm{cliff}(C)$;
(c) $\mathrm{cliff}(C) \geqslant (3g+1)/7$ *and* $\deg(L) \geqslant 2g + 2 - h(C) + \mathrm{gon}(C)$.

Proof.

CLAIM 1. *Assume that there exists a non degenerate family $Y \in \mathrm{Pic}^{\mathrm{d}}(\mathrm{C})$ of primitive pencils and an open set $U \subset Y$ meeting every component of the locus Y_1^B such that $K_C \otimes A^{\vee}(B_A)$ is base point free for any A in U. If L is a line bundle such that $L \otimes A^{\vee}$ very ample for any A in U and*

$$\deg(L) \geqslant \max \begin{cases} g + 2d - 1 \\ 2g - 1 + 2d - \min \begin{cases} 2\,\mathrm{cliff}(C) \\ \max \begin{cases} \mathrm{cliff}(C) \\ \mathrm{cliff}(C) + (3g - 3 - \deg(L))/2 \end{cases} \end{cases} \end{cases}$$

then the gaussian map $\gamma_{K_C,L}$ is surjective.

Proof. We apply Theorem 3.2 taking $M = K_C$. If Y and L are as in the hypothesis of the present Claim, the surjectivity of the multiplication maps $m_{K_C\otimes L\otimes A^{\vee},A}$ is obvious. Therefore the Claim follows plugging Theorem 3 of the Appendix into Theorem 3.2 (the hypothesis cliff$(C) > 2$ yields $h^0(K_C \otimes A^{\vee}) \geqslant 3$). □

CLAIM 2. *If*

$$\deg(L) \geqslant \max \begin{cases} 2g + 2 - h(C) + \mathrm{gon}(C) \\ 4g + 5 + 2h(C) - 2\,\mathrm{gon}(C) - \mathrm{cliff}(C) - \max \begin{cases} 0 \\ (3g - 3 - \deg(L))/2 \end{cases} \end{cases}$$

the gaussian map $\gamma_{K_C,L}$ is surjective.

Proof. If

$$\deg(L) \geqslant 2g + 3 + d - \dim(Y) \qquad (1)$$

one has that for general A in Y the line bundle $L \otimes A^\vee$ is very ample. Indeed, as it follows from an easy dimension count, the dimension of the locus of line bundles of degree k which are not very ample is $\leqslant \min\{g, 2g+2-k\}$. Then we apply Claim 1 taking as Y the component of W_h^1 constructed in Lemma 3.3. One can check that it is possible to find U as in Claim 1 and that $L \otimes A^\vee$ is in fact very ample for any A in U. Then $\dim(Y) \geqslant \rho(d, g, 2) = 2d - g - 2$ and $d \leqslant g + 3 + h(C) - \mathrm{gon}(C)$. Plugging all that into (1) and plugging in turn the result into Claim 1 one gets that if

$$\deg(L) \geqslant \max \begin{cases} 2g - h(C) + \mathrm{gon}(C) + 2 \\ 3g + 2h(C) - 2\,\mathrm{gon}(C) + 5 \\ 4g + 2h(C) - 2\,\mathrm{gon}(C) + 5 - \min \begin{cases} 2\,\mathrm{cliff}(C) \\ \max \begin{cases} \mathrm{cliff}(C) \\ \mathrm{cliff}(C) + (3g - 3 - \deg(L))/2 \end{cases} \end{cases} \end{cases} \qquad (2)$$

then the map $\gamma_{K_C,L}$ is surjective. To see that (2) reduces to the inequality of Claim 2 is a rather tedious count. We sketch it for the benefit of the reader. First of all one checks that if the inequality $\deg(L) < 3g - 3$ is compatible with (2) then $\mathrm{cliff}(C) \geqslant [(g + 4)/3]$. Then one checks that in any case $\mathrm{cliff}(C) \geqslant (3g - 3 - \deg(L))/2$. Finally one checks that $3g + 2h(C) + 5 - 2\,\mathrm{gon}(C)$ never attains the (first) max in (2). □

End of the proof of Theorem 3.4. The statement follows at once from Claim 2. First of all one checks that in any case

$$2g + 2 - h(C) + \mathrm{gon}(C) \leqslant 4g + 2h(C) - 2\,\mathrm{gon}(C) + 5 - \mathrm{cliff}(C).$$

This proves (a). Next, one considers the case $\deg(L) < 3g - 3$. This is possible if and only if

$$\mathrm{cliff}(C) \geqslant (g + 4)/3. \qquad (3)$$

If (3) holds, the bound of Claim 2 becomes weaker:

$$\deg(L) \geqslant \max \begin{cases} 2g + 2 - h(C) + \mathrm{gon}(C) \\ 5g + 13 + 4h(C) - 4\,\mathrm{gon}(C) - 2\,\mathrm{cliff}(C). \end{cases}$$

Finally, one checks that the maximum is achieved by the function upstairs if and only if $\mathrm{cliff}(C) \geqslant (3g + 1)/7$. □

REMARK. Unless $h(C) = 1$ and $\mathrm{cliff}(C) = \mathrm{gon}(C) - 2$, the uniform bound

(a) of the previous theorem reads as $\deg(L) \geqslant 4g + 1 - 3\,\mathrm{cliff}(C)$, i.e. exactly the result of Bertran–Ein–Lazarsfeld ([BEL], Thm 2) which was proved with a different method. Unfortunately when $h(C) = 1$ and $\mathrm{cliff}(C) = \mathrm{gon}(C) - 2$ the bound (a) reads as $\deg(L) \geqslant 4g + 3 - 3\,\mathrm{cliff}(C)$ and which is worse of 2 than the bound of [BEL]. As already mentioned in Remark (c) after Lemma 3.3, the author suspects that this discrepancy is due to the fact that in this last case Lemma 3.3 should not be sharp. However, if $\mathrm{cliff}(C)$ satisfies the additional conditions (b) or (c) (yielding that the curve becomes increasingly general) then the corresponding bounds improve the quoted result of [BEL].

Anyway, going back to the three main steps of the argument as outlined at the beginning of the present section, it should be noted that steps (3) and/or (2) can be improved in most (probably all) cases: in the first place if the multiplication maps $m_{K_C \otimes A^\vee, L \otimes A^\vee}$ are not surjective then L has to satisfy constraints following from the proofs of the Appendix. E.g. if $\deg(L \otimes K_C^\vee) \geqslant g + 1$ it follows from Lemma 1 of the Appendix that if $m_{K_C \otimes A^\vee, L \otimes A^\vee}$ is not onto then $L \otimes K_C^\vee$ contributes to the Clifford index and $\mathrm{cliff}(L \otimes K_C^\vee)$ has to be close to $\mathrm{cliff}(C)$. This allows one to exclude many possibilities. Furthermore, step (2) of the argument should be subject to improvements in many cases as well: here the problem is that, to make sure that $L \otimes A^\vee$ is very ample for general $A \in Y$, we used a very rough dimensional count, leading to the bound $\deg(L) \geqslant 2g + 2 - h(C) + \mathrm{gon}(C)$. Analizing carefully the geometry of the subvariety Y one should do much better, at least when the curve is close to be Brill–Noether general. This appears also from (c) of the previous Theorem, which is surely non sharp, since one expects as a bound a decreasing function of $\mathrm{gon}(C)$ (or $\mathrm{cliff}(C)$). These considerations also show that, looking for optimal results along these lines, such results should depend on a rather complicated interaction of many factors. In absence of a unified statement, or even conjecture, we leave a direct inspection of these phenomena to the interested reader.

As an important example, we show how the present techniques can be pushed further to get a result for Brill–Noether–Petri general curves considerably stronger than Theorem 3.4 (actually the Petri generality is not strictly necessary and we assume it only to avoid technical complications). We recall that a curve C of genus g is said *to satisfy the Brill–Noether condition* if the varieties $W_d^r(C)$ are empty when the Brill–Noether number $\rho(d, g, r)$ is negative, and $\dim(W_d^r(C)) = \rho(d, g, r)$ otherwise. Moreover C is said to *satisfy Petri's condition* (*or to be Brill–Noether–Petri general*) if $W_d^r(C)$ is smooth away of W_d^{r+1} for any r and d. If this is the case the variety $G_d^r(C)$ parametrizing g_d^r's on C is a (canonical) resolution of singularities of W_d^r (we refer to [ACGH] for all this material). For every d such that $\rho(d, g, 1) \geqslant 0$ the closure of the locus of pencils with a base point is $W_{d-1}^1 + C$. In particular the general pencil of W_d^1 is base point free. Moreover there is a pencil A in W_d^1 such that $K_C \otimes A^\vee$ has a base point if and only if there is a point p such that $A(p) \in W_{d+1}^2$ (therefore $\rho(d+1, g, 2) \geqslant 0$). Finally, let us recall that:

(a) $W_d^r(C)$ is CM and irreducible unless $\rho(d, g, r) = 0$ ([FL]),

(b) if $\rho(d,g,r) > 0$ $W_d^r(C)$ is nondegenerate ([FL], Remark 1.9). Therefore in this case, if $d \leqslant g-1$, $W_d^1(C)$ is a nondegenerate family of primitive pencils.
(c) If $\rho(d,g,r) > 1$ (e.g. $r=1$ and $d \geqslant [(g+5)/2]$) then $h^{1,0}(G_d^r(C)) = g$ and the map $G_d^r(C) \to W \hookrightarrow \mathrm{Pic}^d\mathrm{C}$ is an Albanese map for $G_d^r(C)$. (loc cit., see also e.g. [C]).

LEMMA 3.5. *Let C be a Brill–Noether–Petri general curve of genus g and let L be a line bundle on C such that* $\deg(L) \geqslant 2g+4$. *Then*

(a) *for some integer d, with* $[(g+7)/2] \leqslant d \leqslant [(g+7)/2]$, *there is a nondegenerate CM family of primitive pencils $Y \subset \mathrm{Pic}^d(\mathrm{C})$ and an open set $U \subset Y$ meeting every component of Y_1^B such that for any A in U the line bundle $L \otimes A^\vee$ is base point free,*
(b) *for some integer d, with* $[(g+3)/2] \leqslant d \leqslant [(g+9)/2]$, *there is a nondegenerate CM family of primitive pencils $Y \subset \mathrm{Pic}^d(\mathrm{C})$ and an open set $U \subset Y$ as in (a) such that for A in U the line bundle $L \otimes A^\vee$ is very ample.*

Proof. (a) Let us denote $W = W^1_{[(g+5)/2]}$. To start with, we claim that if for A general in W the line bundle $L \otimes A^\vee$ is base point free then the Lemma follows. To see this, let us observe that, since for A in W we have that $\deg(K_C \otimes L^\vee \otimes A) < [(g+3)/2] - 1$, the fact that C is B-N general implies that $h^1(L \otimes A^\vee) \leqslant 1$ and $h^1(L \otimes A^\vee(-p)) \leqslant 1$ for any $p \in C$. Therefore if $h^0(L \otimes A^\vee) = 1$ for any A in Y then $L \otimes A^\vee$ is base point free for any A in Y so that the Lemma is proved taking $Y = U = W$. If $L \otimes A^\vee$ is base point free and $h^1(L \otimes A^\vee) = 0$ for A general in W then for p general in C $L \otimes A^\vee(-p)$ is base point free and $h^1(L \otimes A^\vee(-p)) = 0$. Then, since the locus of line bundles with base points of $W^1_{[(g+7)/2]}$ is $W^1[(g+5)/2] + C$ the Lemma is proved taking $Y = W^1_{[(g+7)/2]}$ and $U \subset Y$ a suitable open subset. This proves what claimed.

Therefore we can suppose that $L \otimes A^\vee$ has a base point p for A general in W. Arguing as above if A has a base point then $h^1(L \otimes A^\vee)$ must vanish and the base point p has to be unique. Therefore, given a line bundle A in W, $h^1(L \otimes A^\vee) = 0$ if and only if A has a (unique) base point p_A. By Riemann–Roch this means that $K_C \otimes L^\vee \otimes A$ can be written in a unique way as $\mathcal{O}_C(D_A - p_A)$ where $h^0(\mathcal{O}_C(D_A)) = 1$ and $p_A \notin \mathrm{supp}(D_A)$. Let denote $d := 2g-2-\deg(L)+\deg(A)$ and $C^{(d+1)}$ the $(d+1)$-fold symmetric product of C. Let us consider the difference map $\pi\colon C^{(d+1)} \times C \to \mathrm{Pic}^d\mathrm{C}, (D,p) \mapsto \mathcal{O}_c(D-p)$. By abuse of language we will call W also the translate of W in $\mathrm{Pic}^d\mathrm{C}$ via $A \mapsto K_C \otimes L^\vee \otimes A$. Then W is contained in the image of π. If $h^1(L \otimes A^\vee) = 0$ then $\pi^{-1}(A) = (D_A, p_A)$ while if $h^1(L \otimes A^\vee) = 1$ and $K_C \otimes L^\vee \otimes A = \mathcal{O}_C(D'_A)$ then $\pi^{-1}(A) = \{(D'_A, p) \mid p \in C\}$. Let $\hat{W}$ be the component of $\pi^{-1}(W')$ surjecting onto W. Clearly $\hat{W}$ is birational onto W. Let p_1 and p_2 the two projections of $C^{(d+1)} \times C$.

(i) *Assume that ${p_2}_{\hat{W}}$ is not constant.* Let $q\colon \tilde{W} \to \hat{W}$ be a desingularization of $\hat{W}$. Then, since $\tilde{W}$ is birational to $G^1_{[(g+5)/2]}$ we have that $h^{0,1}(\tilde{W}) = g$. Moreover, from the birational morphism $\tilde{W} \to W \hookrightarrow \mathrm{Pic^d(C)}$ it follows that the image of the Albanese map of $\tilde{W}$ has maximal dimension (here as usual we assign to $\mathrm{Pic^d(C)}$ a structure of abelian variety via a translation $\mathrm{Pic^0(C)} \to \mathrm{Pic^d(C)}$). But this leads to a contradiction: we would have a commutative diagram

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{a_{\tilde{W}}} & \mathrm{Alb}(\tilde{W}) \\ \Big\downarrow{\scriptstyle p_2} & & \Big\downarrow \\ C & \xrightarrow{a_C} & \mathrm{Pic^0(C)} \end{array}$$

where the $a_{\tilde{W}}$ is an Albanese map for $\tilde{W}$, a_C is an Abel–Jacobi map and the right vertical arrow, induced by the universal property of the Albanese variety, should be an isogeny since $\dim(\mathrm{Alb}(\tilde{W})) = g$. But this is impossible since $\dim \mathrm{Im}(a_{\tilde{W}}) = \dim \tilde{W} > 1$.

(ii) *Assume that the map* ${p_2}_{\hat{W}} \to C$ *is constant.* In other words for any $A \in W$ such $h^1(L \otimes A^\vee) = 0$ the base point p_A is constant, say p. Therefore in this case $\hat{W} = \{(D_A, p)\}_{A \in W}$ and $h^1(L \otimes A^\vee) = 1 \Leftrightarrow p \in \mathrm{supp}(D_A)$. Then the subvariety $Y := \{A \in W \mid h^1(L \otimes A^\vee)\}$ is an ample divisor of W, since it corresponds to the intersection of W with the ample divisor $\{D \in C^{d+1} \mid p \in \mathrm{supp}(D)\}$. Hence $Y \subset \mathrm{Pic^d(C)}$ is nondegenerate by Lefshetz's hyperplane theorem. Then in this case Y will be a family as in the statement as soon as the general pencil parametrized by Y is primitive. Within the present hypotheses it is sufficient to check that the general pencil A in Y is base point free i.e. that Y is not (an irreducible component of) $W^1_{[(g+3)/2]} + C$ inside W. In fact if g is even, this is impossible since no component of $W^1_{[(g+2)/2]} + C$ is ample in W (in this case W is a (smooth) surface, $W^1_{[(g+2)/2]}$ is made of many isolated points and if A', A'' are two of them, it is easy to check that $\{A'(q)\}_{q \in C}$ and $\{A''(q)\}_{q \in C}$ don't meet). On the other hand if g is odd and $Y = W^1_{[(g+3)/2]} + C$ then for A' general in $W^1_{[(g+3)/2]}$ and q general in C we have that $L \otimes A'^\vee$ is base point free and consequently $L \otimes A'^\vee$ itself is base point free. Then we can take $Y = W^1_{[(g+3)/2]}$ in this case. This concludes the proof of (a).

(b) The proof of the second assertion is similar to the previous one and therefore we will only outline it. If for A general in $W := W^1_{[(g+7)/2]}$ the line bundle $L \otimes A^\vee$ is very ample then the Lemma is proved taking $Y = W^1_{[(g+9)/2]}$. Otherwise the general line bundle A of W can be written in a unique way as $\mathcal{O}_C(D_A - p_A - q_A)$, with $h^0(\mathcal{O}_C(D_A)) = 1$ and $p_A, q_A \notin \mathrm{supp}(D_A)$. Therefore (a translate of) W is contained in the image of the difference map $\pi \colon C^{(d+2)} \times C^{(2)} \to \mathrm{Pic^d(C)}$ (notation as above) and the surjective component $\hat{W}$ of $\pi^{-1}(W)$ is birational onto W. The fact that $\hat{W}$ surjects, via p_2, onto $C^{(2)}$ leads to a contradiction as in the previous proof. The analysis of the case when $p_{2_{\hat{W}}} \to C^{(2)}$ is not surjective is omitted. □

THEOREM 3.6. *Let C be a Brill–Noether–Petri general curve of odd* (resp. *even) genus $g = 20$ or $g \geqslant 22$. Let L be a line bundle of degree* $\deg(L) \geqslant 2g + 9$ (resp. $\deg(L) \geqslant 2g + 8$). *Then the gaussian map $\gamma_{K_C,L}$ is surjective.*

Proof. We take as a nondegenerate family of primitive pencils the variety Y of the previous Lemma. If L is as in the statement then, by Lemma 3.5(i), $L \otimes A^\vee$ is base point free for general A in Y and, by Corollary 7 of the Appendix, the multiplication map $m_{K_C \otimes A^\vee, L \otimes A^\vee}$ is surjective. Thus the statement follows at once from Theorem 3.2. □

REMARK. Theorem 3.6, besides improving the theorem of [BEL] in case of B–N–P general curves, improves substantially also the result of Lopez ([Lo], Cor. 1.7), which is obtained arguing by specialization, while here we show that the result is implied by the Brill–Noether–Petri condition. Theorem 3.6 should be compared with a question of Wahl's ([W3], Q. 2.5), asking if for a general (not necessarily B–N general) curve C of genus $\geqslant 12$ the gaussian map $\gamma_{K_C,L}$ is surjective as soon as L is very ample and $\deg(L) \geqslant 2g - 2$.

In analogy with the case of multiplication maps ([GL]), Theorems 3.4 and 3.6 suggest that the stratification on $\mathcal{M}_g$ given by the invariant $d(C)$, where

$$d(C) := \min\{d \mid \gamma_{K_C,L} \text{ is onto for } \deg(L) \geqslant d, L \text{ very ample}\}$$

should be compatible with the usual stratifications of Brill–Noether theory, as the ones given by $\mathrm{gon}(C)$ or $\mathrm{cliff}(C)$. This is suggested also by the results of the next section. All the above should be contrasted by the fact that the surjectivity of the Wahl map γ_{K_C,K_C} does not seem to have much to do with Brill–Noether theory: on the one hand the general curve of genus $g = 10$ or $g \geqslant 12$ has surjective Wahl map, as well as many curves which are very special from the B–N point of view (as complete intersections, cf. [W3]). On the other hand for any genus, there are Brill–Noether general curves lying on a K3 surfaces ([L3]) and, by Wahl's theorem ([W1]), the Wahl map of such curves is not surjective.

(D) OTHER GAUSSIAN MAPS

To start with, we argue as in the previous section to prove surjectivity results for gaussian maps $\gamma_{L,L}$ and $\gamma_{L,M}$, with L and M line bundles, as a function of the intrinsic geometry of C.

THEOREM 3.7. *Let C be a curve of genus g and L a line bundle on C. The gaussian map $\gamma_{L,L}$ is surjective if one of the following conditions holds:*

(a) $\deg(L) \geqslant \max \begin{cases} 3g + 4 + h(C) - \mathrm{cliff}(C) - \mathrm{gon}(C) \\ 2g + 2 + h(C) + \mathrm{gon}(C) \end{cases}$

(b) *C is a Brill–Noether–Petri general curve of genus $g \geqslant 11$ and* $\deg(L) \geqslant 2g+7$.

Proof. (a) We apply Theorem 3.2 taking as Y the variety $Y \subset \mathrm{Pic}^h(C)$, with $h \leqslant g+3+h(C)-\mathrm{gon}(C)$ constructed in Lemma 3.3 (with the necessary adjustments if $\mathrm{cliff}(C) \leqslant 2$, cf. Remark (c) after Lemma 3.3). If $\deg(L) \geqslant 2g+2-h(C)+\mathrm{gon}(C)$, arguing as in the proof of Theorem 3.4, we get that $L \otimes A^\vee$ is very ample, as well as $L \otimes A^\vee(B_A)$, for A in U. If $L \otimes A^\vee$ is very ample and $\deg(L \otimes A^\vee) \geqslant 2g+1-\mathrm{cliff}(C)$ the multiplication map $m_{L\otimes A^\vee, L\otimes A^\vee}$, as well as $m_{L\otimes A^\vee(B_A), L\otimes A^\vee}$, is surjective by Green–Lazarsfeld's result ([GL] Thm 1). Therefore (a) follows.

(b) If C is Brill–Noether–Petri general we apply Theorem 3.2 taking as Y the variety provided by Lemma 3.5(ii). As usual, the maps $m_{L^{\otimes 2}\otimes A^\vee, A}$ are easily seen to be surjective. By Lemma 3.5(ii), $L \otimes A^\vee$ is very ample for A general in Y. Finally, by Green–Lazarsfeld's theorem (loc cit.), if $L \otimes A^\vee$ is very ample and $\deg(L\otimes A^\vee) \geqslant 2g+1-[(g-1)/2]$ then the map $m_{L\otimes A^\vee, L\otimes A^\vee}$ is surjective. □

THEOREM 3.8. *Let N and L be line bundles on C. The gaussian map $\gamma_{L,M}$ is surjective if one of the following conditions holds:*

(a) $\deg(N), \deg(L) \geqslant 2g+2, \deg(N)+\deg(L) \geqslant 6g+3-\mathrm{cliff}(C)$;

(b) $\deg(N), \deg(N) \geqslant 2g+1-h(C)+\mathrm{gon}(C)$ *and* $\deg(N)+\deg(L) \geqslant 6g+7+2h(C)-\mathrm{cliff}(C)-2\,\mathrm{gon}(C)$;

(c) $g-1 \geqslant \deg(L)-\deg(N) \geqslant g-1-2\,\mathrm{cliff}(C), \deg(N) \geqslant 2g+1-h(C)+\mathrm{gon}(C)$ *and* $\deg(N)+3\deg(L) \geqslant 13g+13+4h(C)-4\,\mathrm{gon}(C)-2\,\mathrm{cliff}(C)$;

(d) $0 \leqslant \deg(L)-\deg(N) \leqslant g-1-2\,\mathrm{cliff}(C), \deg(N) \geqslant 2g+1-h(C)+\mathrm{gon}(C)$ *and* $\deg(N)+\deg(L) \geqslant 6g+7+2h(C)-2\,\mathrm{gon}(C)-2\,\mathrm{cliff}(C)$;

(e) *C is Brill–Noether–Petri general,* $0 \leqslant \deg(L)-\deg(N) \leqslant g-1, \deg(N) \geqslant 2g+2$ *and* $\deg(N)+3\deg(L) \geqslant 10g+12$.

The proof is along the lines of the proof of Theorems 3.4 and 3.7. Because of the cumbersome numerology we omit it. □

Concerning Theorems 3.7(a) and 3.8(a), the absolute bounds, obtained plugging $\mathrm{cliff}(C)=0$, were established in [BEL] (Thm 1) with a different method. They also show that they are optimal for hyperelliptic curves. Moreover Theorems 3.7(b) and 3.8(e) improve and precise results in [Lo].

It is suggested by the constructions above that the invariant $\nu(C)$, defined as *the minimal degree of a nondegenerate family Y of base point free pencils*, is relevant to the problem of the surjectivity of gaussian maps. In fact one has

PROPOSITION 3.9 *If L and M are line bundles on C such that* $\deg(L) \geqslant 2g+1+\nu(C)$ *and* $\deg(M) \geqslant 2g+\nu(C)$ *then the gaussian map $\gamma_{L,M}$ is surjective.*

Proof. Let us consider our family $W^1_{\nu(C)} = Y$ and let us apply Theorem 3.2. As usual the condition on the surjectivity of $M_{K_C, L\otimes M}$ is easily checked. Therefore the first part of the statement follows from Mumford's theorem (cf. [M1] and also the Appendix below).

We leave to the interested reader the statements about the surjectivity of maps $\gamma_{L,M}$, with $L \neq M$, as a function of the invariants $\nu(C)$ and cliff(C).

Let us define the index $pl(C)$ as *the minimal degree of a plane model of the curve* C. Obviously we have that $\nu(C) \leqslant pl(C) - 1$. In specific cases $pl(C)$ is computable and the above estimate of $\nu(C)$, plugged into Proposition 3.9, gives more than Theorems 3.7 and 3.8.

EXAMPLES. (a) *Smooth plane curves.* Let C be a smooth plane curve of degree d. They by Proposition 5.9 we get that if $\deg(L) \geqslant 2g + d$ then $\gamma_{L,L}$ is surjective. This is almost sharp since, arguing as in Beauville–Merindol's paper [BM], one can prove that $\gamma_{K_C\otimes\mathcal{O}(1), K_C\otimes\mathcal{O}(1)}$ is not surjective. Moreover it turns out that if L is a line bundle of degree $2g + d - 1$ such that $\gamma_{L,L}$ is not surjective then L is of the form $K_C \otimes \mathcal{O}(1)(p)$. It might be interesting to know whether or not $\gamma_{L,L}$ is surjective in this case.

(b) *Curves of Clifford dimension* $\geqslant 3$. As in the previous case, curves of Clifford dimension $\geqslant 3$ are such that $\nu(C) = \text{gon}(C)$. Then, by Proposition 3.9 one has that if $\deg(L) \geqslant 2g + 1 + \text{gon}(C)$ then $\gamma_{L,L}$ is surjective.

(c) *Trigonal curves and Maroni invariant.* Let C be a trigonal curve and Maroni invariant $e(C)$ (cf. Maroni's paper [Ma] and also [MS], Sect. 1). We recall that the integer $e(C)$ has the same parity as $g(C)$ and $0 \leqslant e(C) \leqslant (g+2)/3$. Moreover all the possible values are attained. If C is a general trigonal curve of odd (resp. even) genus then $e(C) = 1$ (resp. $e(C) = 0$). Furthermore we have that $pl(C) = (g + e(C))/2 + 2$ unless g is even and $e(C) = 0$. In this last case $pl(C) = g/2 + 3$. Therefore, by Proposition 3.9, one has that $\gamma_{L,L}$ is surjective as soon as $\deg(L) \geqslant (5g + e(C))/2 + 3$ (if $e(C) = 0$, as soon as $\deg(L) \geqslant 5g/2 + 3$). Besides $pl(C)$, also the exact configuration of the singularities of the plane model (cf. loc cit.) is known. E.g. if C is a general trigonal curve of odd genus, i.e. $e(C) = 1$, it turns out that the minimal plane model of C has a unique ordinary singularity of multiplicity $pl(C) - 3$. Viewing C as a smooth curve in $\mathbf{P}^2$ blown up at a point and arguing as in [BM] one gets again that $\gamma_{K_C\otimes\mathcal{O}(1), K_C\otimes\mathcal{O}(1)}$ is not surjective. Therefore when $e(C) = 1$ the bound given by Proposition 3.9 is almost sharp (as in (a)). It seems an interesting question is to know whether or not the minimal degree such that $\gamma_{L,L}$ is surjective depends on $e(C)$ or not. When $e(C) \neq 1$ there are more than one singular points (usually infinitely near). In analogy with the case $e(C) = 1$ one can see the curve C as a smooth curve in a suitable blowing up of $\mathbf{P}^2$ but in this case the argument of [BM] applied $K_C \otimes \mathcal{O}(1)$ does not work.

(d) *Castelnuovo's curves.* Usually $pl(C)$ can be computed also for Castelnuovo's curves. For example, as shown in [D] (see also [A]), given a curve C

of genus $g \equiv 0 \pmod 3$, admitting a birational $g^{g/3}_{g-1}$, one can attach to C an integer $e(C)$, analogous to the Maroni invariant of a trigonal curve. It turns out that $pl(C) = g/3 + e(C) + 2$ unless $m = 0$ where $pl(C) = g/3 + 4$ (cf. also [A]). This case seems totally analogous to the previous one since one has also a similar description of the singularities and for $e(C) = 1$ it turns out that C has only one ordinary singularity. As above, in this case it turns out that the bound of Proposition 3.9 is almost sharp since $\gamma_{K_C\otimes\mathcal{O}(1),K_C\otimes\mathcal{O}(1)}$ is not surjective. Note that these curves are tetragonal ([A]). Again, these results and the ones of (c) seem to suggest a certain compatibility between Brill–Noether theory and the stratification given by the minimal degree d such that $\gamma_{L,L}$ is surjective for any line bundle L of degree d.

(E) GAUSSIAN MAPS ON VECTOR BUNDLES

Finally, plugging into our construction Butler's theorems on the surjectivity of multiplication maps of global sections of vector bundles on a curve ([Bu]), one obtains a surjectivity result for gaussian maps. To this purpose, let us recall that, given a vector bundle E on a curve C the *slope* of E is the integer $\mu(E) := \deg(E)/\mathrm{rk}(E)$. Furthermore, one can associate to E its *Harder–Narashiman* filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_s = E,$$

defined by the property that the E_i/E_{i-1}'s are semistable and $\mu(E_i/E_{i-1})$ is strictly decreasing in i. Then one defines the *minimal slope* of E as the integer $\mu^-(E) := \mu(E_s/E_{s-1})$. Clearly $\mu^-(E) = \mu(E)$ if E is semistable.

THEOREM 3.10. *Let C be a curve of genus g and let E and F be vector bundles such that $\mu^-(E) \geqslant 2g + \nu(C)$ and $\mu^-(F) \geqslant 2g + 1 + \nu(C)$ (e.g., as a special case, if E and F are semistable, $\mu(E) \geqslant 2g + \nu(C)$ and $\mu(F) \geqslant 2g + 1 + \nu(C)$). Then the gaussian map $\gamma_{E,F}$ is surjective.*

Since, as we have seen, in any case $\nu(C) \leqslant g + 3 + h(C) - \mathrm{gon}(C)$ (Lemma 3.3 and Remark (c) following it) one obtains

COROLLARY 3.11. *If $\mu^-(E) \geqslant 3g + 3 + h(C) - \mathrm{gon}(C)$ and $\mu^-(F) \geqslant 3g + 4 + h(C) - \mathrm{gon}(C)$ (e.g., as a special case, if E and F are semistable, $\mu(E) \geqslant 3g + 3 + h(C) - \mathrm{gon}(C)$ and $\mu(F) \geqslant 3g + 4 + h(C) - \mathrm{gon}(C)$) then the gaussian map $\gamma_{E,F}$ is surjective.*

As in the examples above, in many cases one can largely improve the bound on $\nu(C)$ and consequently also the result of the previous corollary.

Proof of Theorem 3.10. We apply Theorem 3.1 taking as Y a nondegenerate family of base point free pencils of degree $\nu(C)$. Note that (loc cit., Lemma 2.6) $\mu^-(E \otimes F) = \mu^-(E) + \mu^-(F)$. Then, since $\mu^-(K_C \otimes E \otimes F) > 2g - 2$, we have that $H^1(K_C \otimes E \otimes F) = 0$ (loc cit. 1.12(2)). Moreover, since Y is nondegenerate, condition (a) of Theorem 1.1 reduces to the surjectivity of the multiplication map $m_{K_C,E\otimes F}$, which holds by loc cit., Proposition 2.2, since $\mu^-(E\otimes F) > 2g$. Next, we turn to condition (b) of Theorem 1.1. Let us consider a base point free pencil A parametrized by Y. Since $\mu^-(E\otimes F\otimes A^{\otimes -2}) = \mu^-(E)+\mu^-(F) - 2\nu(C)$ is largely $> 2g-2$, again by loc cit., 1.12(2), we have that $H^1(E\otimes F\otimes A^{\otimes -2}) = 0$. Therefore the map $m_{E\otimes F\otimes A^\vee,A}$ is surjective by the base point free pencil trick. Finally, condition (c) of Theorem 3.1 is satisfied because of loc cit., Theorem 2.1. □

When E is a line bundle, one can improve the results using Proposition 2.2 of loc cit. If e.g. $E = K_C$, using our Lemma 3.3 we get

PROPOSITION 3.12. *Let C be a curve of Clifford index $\geqslant 2$ and let F be a vector bundle such that $\mu^-(F) > 4g+8+2h(C) - 2\,\mathrm{gon}(C)$. Then the gaussian map $\gamma_{K_C,F}$ is surjective.* □

Appendix. Surjectivity of multiplication maps of line bundles on curves

(A) INTRODUCTION

In this appendix we will adapt methods from [G], [L2] and [GL] to prove the announced results (cf. Section 3 above) about the surjectivity of multiplication maps of global sections of line bundles on curves. Although the main reason why the author turned his attention to this problem is the application to the surjectivity of gaussian maps, the question might have some independent interest.

Let us briefly introduce the problem. A classical theorem of Castelnuovo can be restated by saying that: *if* $\deg(L) \geqslant 2g+1$ *then the multiplication map*

$$m_{L,L}\colon H^0(L)\otimes H^0(L) \to h^0(L^{\otimes 2})$$

is surjective. This result has been considerably sharpened by Green–Lazarsfeld ([GL]) taking into account the geometry of C via the Clifford index (see Section 5(b) above). Specifically, they proved that: *if L is very ample and* $\deg(L) \geqslant 2g+1-2h^1(L) - \mathrm{cliff}(C)$ *then $m_{L,L}$ is surjective*, and that this is sharp, at least when $\mathrm{cliff}(C)$ is "small". On the other hand the result of Castelnuovo has been generalized by Mumford ([M1]), Green ([G], 4.e.4), Eisenbud–Koh–Stillman ([EKS]) to multiplication maps

$$m_{N,L}\colon H^0(N)\otimes H^0(L) \to H^0(N\otimes L)$$

with $N \not\cong L$ line bundles on C. Specifically, they proved that *if N is base point free,* $\deg(L) \geqslant \deg(N)$ *and* $\deg(N)+\deg(L) \geqslant 4g+1$ *then the multiplication map $m_{N,L}$ is surjective. Moreover if* $\deg(N)+\deg(L) \geqslant 4g$ *then $m_{N,L}$ is surjective unless C is hyperelliptic and $L\otimes N^\vee$ is a (possibly trivial) multiple of the g^1_2.*

Here we prove a strengthening of the results of Green, Mumford and Eisenbud *et al.* along the lines of [GL]. In order to do that, it is convenient to separate somehow the ranges

$$A: \quad \deg(L) - \deg(N) \geqslant g + 1,$$

$$B: \quad \deg(L) - \deg(N) \leqslant g.$$

Our first result concerns range A:

LEMMA 1. *Let N and L two line bundles on a curve C of genus g such that:* (a) $\deg(L) \geqslant \deg(N)$; (b) *$N$ base pointfree;* (c) $h^0(N)+2h^1(L) \geqslant 3$; (d) $h^1(N \otimes L) = 0$; (e) $\deg(L) \geqslant \deg(N) + g + 1$, *and*

(f) $\quad \deg(N) + \deg(L) > 4g - 2h^1(N) - 4h^1(L) - \text{cliff}(C).$

Then the map $m_{N,L}$ is surjective. Moreover if (a), (b), (c), (d), (e) *hold and $m_{N,L}$ is not surjective then $L \otimes N^\vee$ contributes to the Clifford index of C and*

$$\text{cliff}(L \otimes N^\vee) \leqslant 4g - 2h^1(N) - 4h^1(L) - \deg(N) - \deg(L).$$

As a particular case, if there is equality in (f) *and $m_{N,L}$ is not surjective then $L \otimes N^\vee$ computes the Clifford index of C.*

The next result holds, in principle, for both ranges, but it applies in a relevant way to range B only:

LEMMA 2. *Let N and L be base point free line bundles on C such that:* (a) $\deg(L) \geqslant \deg(N)$; (b) *at least one of them is very ample;* (c) $h^0(N) \geqslant 3, h^0(L) \geqslant 3$ *and* (d)

$$\deg(N)+\deg(L) \geqslant \max \begin{cases} 3g-3 \\ 4g+1-2h^1(N)-2h^1(L)-\min \begin{cases} 2\,\text{cliff}(C) \\ \text{cliff}(C) + [(\deg(N \otimes L^\vee) + g - 1) \end{cases} \end{cases}$$

Then the multiplication map $m_{N,L}$: $H^0(N) \otimes H^0(L) \to H^0(N \otimes L)$ is surjective.

Note that, since $\text{cliff}(C) \leqslant [(g-1)/2]$, when $N = L$ one recovers Green–Lazarsfeld's result. Putting together the two previous lemmas one easily gets

THEOREM 3. *Let N and L be two base point free line bundles on C such that:* (a) $\deg(L) \geqslant \deg(M)$; (b) *at least one of them is very ample;* (c) $h^0(N), h^0(L) \geqslant 3$; *and*

$$\text{(d)}\quad \deg(N)+\deg(L) \geqslant \max\begin{cases} 3g-3, \\ 4g+1-2h^1(N)-2h^1(L)-\min\begin{cases} 2\,\mathrm{cliff}(C), \\ \max\begin{cases} \mathrm{cliff}(C), \\ \mathrm{cliff}(C)+[(\deg(N\otimes L^\vee)+g-1)/2] \end{cases} \end{cases} \end{cases}$$

Then the multiplication map $m_{N,L}$ is surjective.

It is probably worth to record also the following less precise, but more expressive, version of Theorem 3:

COROLLARY 4. *Let N and L be two base point free line bundles on C such that:* (a) *at least one of them is very ample;* (b) $h^0(N), h^0(L) \geqslant 3$ *and*

$$\text{(c)}\qquad \deg(N)+\deg(L) \geqslant \max\begin{cases} 3g-3 \\ 4g+1-2h^1(N)-2h^1(L)-\mathrm{cliff}(C). \end{cases}$$

Then the multiplication map $m_{N,L}$ is surjective.

In this appendix we do not tackle the question of the sharpness of such theorems, since this would lead to complications which are beyond the scope of the present paper. However, if one wants to consider the possible pairs of line bundles (N, L) subject to conditions close to the ones of the previous results, and such that the map $m_{N,L}$ is not surjective, if we are in Range A a strong constraint appears in Lemma 1. Concerning Range B, as the careful reader will notice, other constraints follow from the proof of Lemma 2.

(B) Proofs

First of all let us recall that, given a line bundle N on a curve C, the kernel of the evaluation map $H^0(N)\otimes\mathcal{O}_C \to N$ is usually denoted M_N (note, by the way, that, given another line (or vector) bundle L, the vector bundle $M_N \otimes L$ is nothing else than the vector bundle $R^1_{L,N}$ of Section 1(a)). When N is a base point free pencil we have evidently that $M_N = N^\vee$. The main tool in proving Lemma 1 will be the following

LEMMA 5. *Let N and L line bundles on C such that:* (a) *N is base point free; and* (b) $h^1(N\otimes L) = 0$. *If the multiplication map $m_{N,L}$: $H^0(N)\otimes H^0(L) \to H^0(N\otimes L)$ is not surjective then*

$$h^0(K_C\otimes N\otimes L^\vee) \geqslant h^0(N)+2h^1(L)-1.$$

Proof. Since N is base point free, M_N is a vector bundle sitting in the exact sequence

$$0 \to M_N \to H^0(N) \otimes \mathcal{O}_C \to N \to 0.$$

Tensoring with L one obtains

$$0 \to M_N \otimes L \to H^0(N) \otimes L \to N \otimes L \to 0$$

where the third arrow, at the global sections level, is the map $m_{N,L}$. As we are assuming $h^1(N \otimes L) = 0$, $m_{N,L}$ is not surjective if and only if

$$h^1(M_N \otimes L) > h^0(N)h^1(L). \tag{1}$$

Now let $p_1, \ldots, p_n$ be points on C imposing independent conditions to $H^0(N)$ and such that $N(-p_1 \cdots - p_n)$ is base point free (if $n \leqslant h^0(N) - 2$ then n *general* points on C satisfy this condition). Then ([L2], Lemma 1.4.1) there is an exact sequence

$$0 \to M_{N(-p_1 \cdots - p_n)} \to M_N \to \bigoplus_{i=1}^{n} \mathcal{O}_C(-p_i) \to 0.$$

In particular, when $n = h^0(N) - 2$ one gets

$$0 \to N^\vee(p_1 + \cdots + p_n) \to M_N \to \bigoplus_{i=1}^{n} \mathcal{O}_C(-p_i) \to 0$$

and finally, tensoring with L

$$0 \to N^\vee \otimes L(p_1 + \cdots + p_n) \to M_N \otimes L \to \bigoplus_{i=1}^{n} L(-p_i) \to 0.$$

If the p_i's are chosen generally then $h^1(L(-p_i)) = h^1(L)$ and therefore

$$h^1(M_N \otimes L) \leqslant (h^0(N) - 2)h^1(L) + h^1(N^\vee \otimes L(p_1 + \cdots + p_n)).$$

This is compatible with (1) if and only if

$$h^1(N^\vee \otimes L(p_1 + \cdots p_n)) > 2h^1(L)$$

i.e., by Serre duality, if and only if

$$h^0(K_C \otimes N \otimes L^\vee(-p_1 - \cdots - p_n)) > 2h^1(L).$$

Since $n = h^0(N) - 2$ and $p_1, \dots, p_n$ are *general* points on C the last inequality is equivalent to

$$h^0(K_C \otimes N \otimes L^\vee) > h^0(N) + 2h^1(L) - 2.$$

Proof of Lemma 1. If $\deg(L) \geqslant \deg(N) + 2g - 2$, then, by Lemma 5, $m_{N,L}$ is clearly surjective. Let us assume now $\deg(L) \leqslant \deg(N) + 2g - 2$ and that the map $m_{N,L}$ is not surjective.

CLAIM. *If hypotheses* (c) *and* (e) *hold then* $L \otimes N^\vee$ *contributes to the Clifford index.*

First let us show that the Claim implies the Lemma. Because of the Claim we have that

$$\operatorname{cliff}(K_C \otimes N \otimes L^\vee) \geqslant \operatorname{cliff}(C). \tag{2}$$

On the other hand, working out with Riemann–Roch the right-hand side of the inequality of Lemma 5 we get

$$h^0(K_C \otimes N \otimes L^\vee) \geqslant \deg(N) + h^1(N) + 2h^1(L) - g, \tag{3}$$

or, equivalently

$$\operatorname{cliff}(K_C \otimes N \otimes L^\vee) \leqslant 4g - \deg(N) - \deg(L) - 2h^1(N) - 4h^1(L). \tag{4}$$

Putting together (2) and (4) one gets that: if (a), (b), (c), (d), (e) hold and $m_{N,L}$ is not surjective, then

$$\begin{aligned} 4g - \deg(N) - \deg(L) - 2h^1(N) - 4h^1(L) &\geqslant \operatorname{cliff}(K_C \otimes N \otimes L^\vee) \\ &\geqslant \operatorname{cliff}(C). \end{aligned}$$

This proves Lemma 1. □

Proof of the Claim. From Lemma 5 and hypothesis (c) it follows that $h^1(L \otimes N^\vee) \geqslant 2$. That $h^0(L \otimes N^\vee) \geqslant 2$ follows simply from (e) and Riemann–Roch. □

Let us record, by the way, the following slight improvement of the aforementioned result of Green:

COROLLARY 6. *Assume that:* (a) N *base point free,* (b) $\deg(L) \geqslant \deg(N)$ *and*

(c) $\quad \deg(N) + \deg(L) \geqslant \max\{2g - 1, 4g + 1 - 2h^1(N) - 4h^1(L)\}$.

Then the map $m_{N,L}$ *is surjective.*

Proof. Since $\deg(N) + \deg(L) \geqslant 2g - 1$ we have that $h^1(N \otimes L) = 0$, so that we can apply Lemma B.5. Then the proof is like the previous one noting that if $\deg(N) \leqslant \deg(L) \leqslant \deg(N) + 2g - 2$ then by Clifford's theorem $\mathrm{cliff}(K_C \otimes N \otimes L^\vee) \geqslant 0$. □

Now let us turn to the proof of Theorem 2, which is a plain extension of the proof of the aforementioned result of [GL].

Proof of Lemma 2. Let us recall that dualizing the map $m_{N,L}$ one gets, via Serre duality, a map

$$m_{N,L}^\vee : \mathrm{Ext}^1(L, K_C \otimes N^\vee) \to \mathrm{Hom}(H^0(L), H^1(K_C \otimes N^\vee))$$

sending, as it is well known, the class of an extension

$$0 \to K_C \otimes N^\vee \to E \to L \to 0 \tag{5}$$

to its coboundary map $\delta: H^0(L) \to H^1(K_C \otimes N^\vee)$.

We all argue by contradiction. Thus assume that hypotheses (a), (b), (c), (d) of the statement of the present Lemma hold and that the map $m_{N,L}^\vee$ is not injective. Then there is a non split extension like (5) which is exact at the global section level. By a theorem of Segre ([Gh], [L2], see also [N]), E has a line subbundle A such that

$$\deg(A) \geqslant (\deg(E) - g + 1)/2 = (\deg(L) - \deg(N) + g - 1)/2. \tag{6}$$

One has an exact diagram

$$\begin{array}{c} 0 \\ \downarrow \\ A \\ \downarrow \\ 0 \to K_C \otimes N^\vee \to E \to L \to 0 \\ \downarrow \\ K_C \otimes N^\vee \otimes L \otimes A^\vee \\ \downarrow \\ 0 \end{array} \tag{7}$$

The hypothesis $\deg(N)+\deg(L) \geqslant 3g-3$ is equivalent to $(\deg(L)-\deg(N)+g-1)/2 > \deg(K_C \otimes N^\vee)$. Therefore from (6) we get that any map from A to $K_C \otimes N^\vee$ has to be zero. Consequently, from diagram (7) it follows that $A = L(-D)$, with D effective (nonzero, since otherwise the extension would be split) divisor and, (6) becomes

$$\deg(D) \leqslant (\deg(N) + \deg(L) - g + 1)/2. \tag{8}$$

Since sequence (5) is supposed exact at the global sections level one has

$$h^0(K_C \otimes N^\vee) + h^0(L) \leqslant h^0(L(-D)) + h^0(K_C \otimes N^\vee(D))$$

and this, by Riemann–Roch, is equivalent to

$$\begin{aligned} &\deg(L(-D)) - (h^0(L(-D)) + h^0(N(-D))) \\ &\quad \leqslant \deg(L) - (h^0(L) + h^0(N)) \end{aligned} \tag{9}$$

and, as it is easily seen, to

$$\mathrm{cliff}(N(-D)) + \mathrm{cliff}(L(-D)) \leqslant \mathrm{cliff}(N) + \mathrm{cliff}(L). \tag{10}$$

CLAIM. *At least one of $N(-D)$ and $L(-D)$ contributes to the Clifford index of C.*

Let us first show how the Claim implies Lemma 2. We consider two cases:

(i) *$L(-D)$ contributes to the Clifford index.* Then

$$\mathrm{cliff}(L(-D)) \geqslant \mathrm{cliff}(C) \tag{11}$$

while obviously

$$\begin{aligned} \mathrm{cliff}(N(-D)) &\geqslant \min\{\mathrm{cliff}(C), \deg(N) - \deg(D)\} \\ &\overset{(8)}{\geqslant} \min\{\mathrm{cliff}(C), (\deg(N) - \deg(L) + g - 1)/2\}. \end{aligned} \tag{12}$$

Therefore if $m_{N,L}$ is not surjective then

$$\begin{aligned} &\mathrm{cliff}(C) + \min\{\mathrm{cliff}(C), (\deg(N) - \deg(L) + g - 1)/2\} \leqslant \\ &\quad \overset{(11)(12)}{\leqslant} \mathrm{cliff}(L(-D)) + \mathrm{cliff}(N(-D)) \\ &\quad \overset{(8)}{\leqslant} \mathrm{cliff}(L) + \mathrm{cliff}(N), \end{aligned} \tag{13}$$

and this, via the formula

$$\mathrm{cliff}(\mathcal{L}) = 2g - \deg(\mathcal{L}) - 2h^1(\mathcal{L}) \tag{$*$}$$

is in contradiction with hypothesis (d).

(ii) *$N(-D)$ contributes to the Clifford index and $L(-D)$ does not.* Then we have

$$\mathrm{cliff}(N(-D)) \geqslant \mathrm{cliff}(C) \tag{11$'$}$$

and, since $\deg(L) \geqslant \deg(N)$,

$$\begin{aligned}\mathrm{cliff}(L(-D)) &\geqslant \deg(L) - \deg(D)\\ &\geqslant \deg(N) - \deg(D)\\ &\overset{(8)}{\geqslant} (\deg(N) - \deg(L) + g - 1)/2.\end{aligned} \tag{12'}$$

Consequently (13) is still true and we get a contradiction in the same way.

Proof of the Claim. Let us rewrite (9) as

$$(h^0(N) - h^0(N(-D)) + (h^0(L) - h^0(L(-D)) \leqslant \deg(D). \tag{14}$$

We have that

$$h^0(N(-D)) + h^0(L(-D)) \geqslant 3 \tag{15}$$

because

$$\begin{aligned}h^0(N(-D)) + h^0(L(-D)) &\geqslant\\ &\overset{(14)}{\geqslant} h^0(N) + h^0(L) - \deg(D)\\ &\overset{(8)}{\geqslant} h^0(N) + h^0(L)\\ &\quad - \frac{\deg(L) + \deg(N) - g + 1}{2}\\ &= -\frac{\mathrm{cliff}(N) + \mathrm{cliff}(L) - g + 1}{2} + 2\\ &\geqslant \frac{g - 1 - 2\,\mathrm{cliff}(C)}{2} + 3\\ &\geqslant 3,\end{aligned}$$

where the fifth inequality holds by hypothesis (d) (via formula $(*)$, as usual), and the last inequality follows from $\mathrm{cliff}(C) \leqslant [(g-1)/2]$. Thus from (15) it follows that at least one of $h^0(N(-D))$ and $h^0(L(-D))$ is $\geqslant 2$. On the other hand from (14) and the fact that both N and L are base point free and at least one of them is very ample, it follows easily that both $h^0(N)-h^0(N(-D))$ and $h^0(L)-h^0(L(-D))$ are strictly positive and $\deg(D) \geqslant 3$. This, together with (14) again, yields that at least one of the differences $h^0(N) - h^0(N(-D))$ and $h^0(L) - h^0(L(-D))$ is $\leqslant \deg(D) - 2$. If $h^0(L) - h^0(L(-D)) \leqslant \deg(D) - 2$ (resp. $h^0(N) - h^0(N(-D)) \leqslant \deg(D) - 2$)

then by Riemann–Roch $h^1(L(-D)) \geqslant 2$ (resp. $h^1(N(-D)) \geqslant 2$). Hence at least one of $h^1(N(-D))$ and $h^1((L(-D))$ is $\geqslant 2$. Thus to prove the claim it is sufficient to exclude the combinations

$$h^0(N(-D)) \geqslant 2, \quad h^0(L(-D)) \leqslant 1,$$
$$h^1(N(-D)) = 1, \quad h^1(L(-D)) \geqslant 2, \tag{$\bullet$}$$

and

$$h^0(N(-D)) \leqslant 1, \quad h^0(L(-D)) \geqslant 2,$$
$$h^1(N(-D)) \geqslant 2, \quad h^1(L(-D)) \geqslant 2. \tag{$\bullet\bullet$}$$

But assume e.g. that ($\bullet$) holds. The fact that $h^1(N(-D)) = 1$ yields $h^0(N) - h^0(N(-D)) = \deg(D) - 1$. Then (14) yields $h^0(L) - h^0(L(-D)) = 1$. Then $h^0(L(-D) \leqslant 1$ implies that $h^0(L) \leqslant 2$ contrary to the hypothesis (c). One can exclude ($\bullet\bullet$) in the same way. □

The above proof suggests that one could obtain more precise results by means of a precise knowledge of line bundles L contributing to the Clifford index and such that cliff(L) is close to cliff(C).

EXAMPLE. As an example of this last remark, let us assume that C is Brill–Noether general and let us consider any base point free pencil A of degree $[(g+7)/2]$. By easy Brill–Noether theory, if $g = 20$ or $g \geqslant 22$ then $K_C \otimes A^\vee$ is always very ample. In connection with Theorem 3.6 above, we want an integer d such that the multiplication map $m_{K_C \otimes A^\vee, L}$ is surjective for any base point free line bundle L such that $\deg(L) \geqslant d$. Since $K_C \otimes A^\vee$ is very ample, from Theorem 3 it follows easily that

$$d \geqslant g + 1 + 3\deg(A) - 2\,\text{cliff}(C) = g + \deg(A) + 7.$$

In fact one can do slightly better:

COROLLARY 7. *Let C be a Brill–Noether general curve of genus $g = 20$ or $g \geqslant 22$ and let A be a base point free line bundle on C such that $h^0(A) = 2$ and* $\deg(A) = [(g+7)/2]$. *Then for any base point free line bundle L on C such that* $\deg(L) \geqslant g + \deg(A) + 2$ *the multiplication map $m_{K_C \otimes A^\vee, L}$ is surjective.*

Proof. First of all, an easy – and standard – argument shows that it is sufficient to prove the statement when $\deg(L) = g + \deg(A) + 2$. We leave this to the reader.

We will argue by contradiction. Thus assume that the map $m_{K_C \otimes A^\vee, L}$ is not surjective. Since we have that $\deg(K_C \otimes A^\vee) + \deg(L) \geqslant 3g - 3$, we can apply the proof of Lemma 2 and conclude that there is an effective divisor D such that:

(a) $3 \leqslant \deg(D) \leqslant g$ (cf. (8)); (b) at least one of $K_C \otimes A^\vee(-D)$ and $L(-D)$ contributes to the Clifford index; and

$$\text{(c)} \qquad \operatorname{cliff}(K_C \otimes A^\vee(-D)) + \operatorname{cliff}(L(-D)) \leqslant \operatorname{cliff}(K_C \otimes A^\vee) + \operatorname{cliff}(L).$$

Therefore

$$\begin{aligned} \operatorname{cliff}(K_C \otimes A^\vee(-D)) + \operatorname{cliff}(L(-D)) &\overset{(c)}{\leqslant} \operatorname{cliff}(K_C \otimes A^\vee) + \operatorname{cliff}(L) \\ &= \deg(A) - 2 + \operatorname{cliff}(L) \\ &\overset{(*)}{\leqslant} \deg(A) - 2 + 2g - \deg(L) \\ &= g - 4. \end{aligned} \tag{16}$$

We consider three possibilities:

(i) *both $K_C \otimes A^\vee(-D)$ and $L(-D)$ contribute to the Clifford index.* Then we have

$$\begin{aligned} 2\operatorname{cliff}(C) &\leqslant \operatorname{cliff}(K_C \otimes A^\vee(-D)) + \operatorname{cliff}(L(-D)) \\ &\overset{(16)}{\leqslant} g - 4. \end{aligned}$$

This is impossible since $\operatorname{cliff}(C) = [(g-1)/2]$.

(ii) *$K_C \otimes A^\vee(-D)$ contributes to the Clifford index and $L(-D)$ does not.* Since the minimal degree of a line bundle contributing to the Clifford index is $[(g+3)/2]$, from the fact that $K_C \otimes A^\vee(-D)$ contributes to $\operatorname{cliff}(C)$ we have

$$\deg(D) \leqslant g - 5. \tag{17}$$

Then we have

$$\operatorname{cliff}(K_C \otimes A^\vee(-D)) \geqslant \operatorname{cliff}(C) \tag{18}$$

and

$$\begin{aligned} \operatorname{cliff}(L(-D)) &\geqslant \deg(L) - \deg(D) \\ &\overset{(17)}{\geqslant} \deg(A) + 7. \end{aligned} \tag{18$'$}$$

Therefore

$$\begin{aligned} g - 4 &\overset{(16)}{\geqslant} \operatorname{cliff}(K_C \otimes A^\vee(-D)) + \operatorname{cliff}(L(-D)) \\ &\overset{(18,18')}{\geqslant} [(g-1)/2] + [(g+5)/2] + 5, \end{aligned}$$

a contradiction.

(iii) *$L(-D)$ contributes to the Clifford index and $K_C \otimes A^\vee(-D)$ does not.* We have

$$\deg(L(-D)) \overset{(a)}{\geqslant} \deg(A) + 2.$$

Since $L(-D)$ contributes to the Clifford index and C is B–N general it is easy to deduce that then

$$\text{cliff}(L(-D)) \geqslant \deg(A). \tag{19}$$

Moreover, since $K_C \otimes A^\vee(-D)$ does not contribute to cliff(C) we have that

$$\begin{aligned} \text{cliff}(K_C \otimes A^\vee(-D)) &\geqslant 2g - 2 - \deg(A) - \deg(D) \\ &\overset{(a)}{\geqslant} g - \deg(A) - 2. \end{aligned} \tag{20}$$

Therefore

$$\begin{aligned} g - 2 &= g - \deg(A) - 2 + \deg(A) \\ &\overset{(19,20)}{\leqslant} \text{cliff}(K_C \otimes A^\vee(-D)) + \text{cliff}(L(-D)), \end{aligned}$$

in contradiction with (16). □

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