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Morse indices of Yang–Mills connections over the unit sphere

Dedicated to Professor Masaru Takeuchi on his sixtieth birthday

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Abstract. We give an optimal estimate for the (Morse) index $i(\nabla)$ of any nonflat Yang–Mills connection ∇ over the m -sphere S^m ($m \geq 5$) with the standard Riemannian metric as follows:

$$i(\nabla) \geq m + 1.$$

Notice the canonical connection on $S^m = \text{SO}(m + 1)/\text{SO}(m)$ ($m \geq 5$) achieves the equality.

1. Introduction

We consider a compact Riemannian manifold M , a principal G -bundle P over M , and the associated G -vector bundle E over M , where G is a compact Lie group. On the space $\mathcal{C}(E)$ of G -connections of E , we consider the *Yang–Mills functional*:

$$\mathcal{YM}(\nabla) = \frac{1}{2} \int_M \|R^\nabla\|^2 v_g, \quad \nabla \in \mathcal{C}(E),$$

where R^∇ is the curvature of the connection ∇ and the norm is defined in terms of the Riemannian metric g on M and a fixed $\text{Ad}(G)$ -invariant inner product on the Lie algebra \mathfrak{g} of G .

A critical point of \mathcal{YM} is called a *Yang–Mills connection*. Our interests are in the second variation of the functional at such a connection and instability of Yang–Mills connections. A Yang–Mills connection ∇ is said to be *weakly stable* if the second variation of \mathcal{YM} at ∇ is non-negative, i.e.,

$$\left. \frac{d^2}{dt^2} \right|_{t=0} \mathcal{YM}(\nabla^t) \geq 0,$$

for every smooth one-parameter family ∇^t , $|t| < \varepsilon$, with $\nabla^0 = \nabla$. Otherwise, we say that ∇ is *unstable*. Furthermore, Bourguignon and Lawson [B.L] gave

the second variation formula and the notions of (Morse) *index* $i(\nabla)$ and *nullity* $n(\nabla)$ of a Yang–Mills connection ∇ . Roughly speaking, the index is the dimension of the space of infinitesimal deformations decreasing \mathcal{YM} and the nullity is the dimension of the space of those preserving \mathcal{YM} (cf. Definition 2.5). Thus instability of a Yang–Mills connection ∇ means $i(\nabla) \geq 1$ (cf. Definition 2.6).

At the Tokyo Symposium on “Minimal Submanifolds and Geodesics” in September of 1977, J. Simons announced the following theorem in his talk:

THEOREM 1.3. *For $m \geq 5$, any nonflat Yang–Mills connection on any vector bundle E over the m -sphere S^m with the standard Riemannian metric is unstable.*

A proof of Theorem 1.3 can be found in the paper of Bourguignon and Lawson [B.L]. See also [K.O.T]. On the other hand, Laquel [L] showed a table of the indices and nullities of the canonical connections (which are typical examples of Yang–Mills connections) of all compact irreducible symmetric spaces and all compact simple Lie groups. On his table, the index $i(\nabla_0)$ of the canonical connection ∇_0 on the m -sphere S^m ($m \geq 5$) with the standard Riemannian metric is given by

$$i(\nabla_0) = m + 1. \tag{1.4}$$

Now our theorem is as follows:

THEOREM 1.5. (cf. Theorem 4.1) *For any nonflat Yang–Mills connection ∇ on any vector bundle E over the m -sphere S^m ($m \geq 5$) with the standard Riemannian metric,*

$$i(\nabla) \geq m + 1.$$

It might be of some interest to compare our result with instability results for harmonic maps.

THEOREM 1.6. (Xin [X]) *For $m \geq 3$ and for any Riemannian manifold N , there is no nonconstant weakly stable harmonic maps $f: S^m \rightarrow N$.*

Our proof of the above main theorem follows the method of El Soufi developed in the setting of minimal immersions (cf. [E1]) and harmonic maps (cf. [E1], [E2]) on the index $i(f)$ of a harmonic map $f: S^m \rightarrow N$:

THEOREM 1.7. (El Soufi [E2]) *For $m \geq 3$, any Riemannian manifold N and any nonconstant harmonic map $f: S^m \rightarrow N$,*

$$i(f) \geq m + 1. \tag{1.8}$$

The estimate (1.8) is also optimal, in fact, the identity map $\text{id}: S^m \rightarrow S^m$ ($m \geq 3$) and the Hopf fibering $\pi: S^3 \rightarrow S^2$ achieve the equality (cf. [S], [U]).

2. Preliminaries

2.1. YANG-MILLS CONNECTIONS

Let (M, g) be a compact Riemannian manifold, P a principal G -bundle over M , and E the associated G -vector bundle of rank r over M , with projections $\pi: P \rightarrow M$ and $\pi: E \rightarrow M$, where G is a compact Lie group. Recall for a given faithful representation $\rho: G \rightarrow O(r)$, E is given as

$$E = P \times_{\rho} \mathbf{R}^r = \{[u, y]; u \in P, y \in \mathbf{R}^r\},$$

where $[u, y]$ is an equivalence class containing $(u, y) \in P \times \mathbf{R}^r$, and the equivalence relation is $(u, y) \sim (ub, \rho(b)^{-1}y), b \in G$. Each C^∞ section $\sigma \in \Gamma(E)$ can be regarded as a C^∞ map $\tilde{\sigma}$ of P into \mathbf{R}^r by

$$\tilde{\sigma}(u) = u^{-1}(\sigma(x)), \quad u \in P, \tag{2.1}$$

where $\pi(u) = x$ and each $u \in P$ is regarded here as an onto isomorphism $u: \mathbf{R}^r \ni y \mapsto [u, y] \in E_x$, where E_x is the fiber of E over $x \in M$.

Via (2.1), the connection form ω (cf. [K.N, p. 64]) corresponds to a unique G -connection ∇ on E by

$$(\nabla\sigma)^\sim = D\tilde{\sigma} = d\tilde{\sigma} + \rho(\omega)\tilde{\sigma}, \quad \sigma \in \Gamma(E), \tag{2.2}$$

where $D\tilde{\sigma}$ is the covariant exterior differentiation defined by

$$D\tilde{\sigma}(W) = d\tilde{\sigma}(W^H), \quad W \in T_uP,$$

and

$$(\nabla\sigma)^\sim(W) = u^{-1}(\nabla_{\pi_*W}\sigma), \quad W \in T_uP. \tag{2.2'}$$

Here π_* is the differential of the projection $\pi: P \rightarrow M$ and W^H is the horizontal component of W corresponding to the decomposition

$$T_uP = V_u \oplus H_u,$$

where $V_u = \{W \in T_uP; \pi_*(W) = 0\}$ (the *vertical space*) and $H_u = \{W \in T_uP; \omega(W) = 0\}$ (the *horizontal space*).

Let $\mathcal{C}(E)$ be the space of all C^∞ G -connections on E . The group of all automorphisms of E inducing the identity map of M is called the *gauge group*, denoted by $\mathcal{G}(E)$. The gauge group $\mathcal{G}(E)$ is identified with the group of all automorphisms φ of P satisfying $\varphi(ua) = \varphi(u)a$ for $u \in P$ and $a \in G$. The identification is given by $\mathcal{G}(E) \ni \varphi \mapsto \tilde{\varphi}$, where $\tilde{\varphi} := \varphi \circ u$ with $u \in P$ being considered as a linear isomorphism $u: \mathbf{R}^r \rightarrow E_x$. The group $\mathcal{G}(E)$ acts on $\mathcal{C}(E)$ by

$$\nabla^\varphi\sigma := \varphi^{-1}(\nabla(\varphi\sigma)),$$

for $\sigma \in \Gamma(E)$, $\varphi \in \mathcal{G}(E)$ and $\nabla \in \mathcal{C}(E)$. We can regard the space of C^∞ sections of the vector bundle $P \times_{\text{Ad}} \mathfrak{g}$, $\Gamma(P \times_{\text{Ad}} \mathfrak{g})$, as the Lie algebra of $\mathcal{G}(E)$. The bundle $P \times_{\text{Ad}} \mathfrak{g}$ is identified with a subbundle of $\text{End}(E)$ via ρ , denoted by \mathfrak{g}_E . The identification is given by

$$P \times_{\text{Ad}} \mathfrak{g} \ni [u, A] \mapsto u \circ \rho(A) \circ u^{-1} \in \text{End}(E).$$

Note that $\mathcal{C}(E)$ admits an affine structure, i.e., the difference of two connections $A = \nabla - \nabla'$ is in $\Omega^1(\mathfrak{g}_E)$ and $\mathcal{C}(E) = \{\nabla + A; A \in \Omega^1(\mathfrak{g}_E)\}$ for any fixed $\nabla \in \mathcal{C}(E)$. Equivalently, the difference of two connection forms $\alpha = \omega - \omega'$ is in $\Omega^1(P \times_{\text{Ad}} \mathfrak{g})$ and any connection form can be expressed as $\omega + \alpha$ with $\alpha \in \Omega^1(P \times_{\text{Ad}} \mathfrak{g})$ for a fixed connection form ω .

To each G -connection ∇ on E , the *curvature tensor* $R^\nabla \in \Omega^2(\mathfrak{g}_E)$ is by definition

$$R^\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$$

for C^∞ vector fields X and Y on M . It corresponds to a \mathfrak{g} -valued 2-form Ω on P , called the *curvature form*, defined by $\Omega := d\omega + \omega \wedge \omega$ with ω being the connection form of ∇ . It holds that

$$R^{\nabla^\varphi} = \varphi^{-1} \circ R^\nabla \circ \varphi,$$

equivalently, $\tilde{\Omega} = \tilde{\varphi}^* \Omega$ with $\tilde{\Omega}$ being the curvature form of the connection form $\tilde{\varphi}^* \omega$.

Let \langle, \rangle be an $\text{Ad}(G)$ -invariant inner product on \mathfrak{g} , which induces fiber metrics on $P \times_{\text{Ad}} \mathfrak{g}$ and \mathfrak{g}_E , respectively, denoted by the same symbol \langle, \rangle .

In general, for a connection ∇ on a vector bundle F over M , let $d^\nabla: \Omega^p(F) \rightarrow \Omega^{p+1}(F)$, $p \geq 0$ be the corresponding exterior differential operator (cf. [B.L]). A fiber metric \langle, \rangle of F induces an inner product on $\wedge^p T_x^* M \otimes F_x$ together with g , denoted by the same symbol \langle, \rangle . We denote the associated norm by $\| \cdot \|$. The global inner product $(,)$ on $\Omega^p(F)$ is defined by

$$(\psi, \varphi) = \int_M \langle \psi, \varphi \rangle v_g$$

for $\psi, \varphi \in \Omega^p(F)$. Let $\delta^\nabla: \Omega^{p+1}(F) \rightarrow \Omega^p(F)$, $p \geq 0$, be the formal adjoint of the operator d^∇ .

Note that for $\varphi \in \mathcal{G}(E)$

$$\|R^{\nabla^\varphi}\| = \|R^\nabla\|; \quad \text{or equivalently } \|\tilde{\varphi}^* \Omega\| = \|\Omega\|.$$

The tangent space to the orbit of the gauge group at ∇ , considered as a subspace of $\Omega^1(\mathfrak{g}_E) \cong T_\nabla \mathcal{C}(E)$, is $d^\nabla(\Omega^0(\mathfrak{g}_E))$. Its orthogonal complement to this subspace

in $\Omega^1(\mathfrak{g}_E)$ is $\text{Ker}(\delta^\nabla)$. The subspace $\text{Ker}(\delta^\nabla) \subset \Omega^1(\mathfrak{g}_E)$ is called the space of *infinitesimal deformations* of the connection ∇ .

Now let us recall the Yang–Mills functional.

DEFINITION 2.3. The function $\mathcal{YM}: \mathcal{C}(E) \rightarrow \mathbf{R}$ defined by

$$\mathcal{YM}(\nabla) = \frac{1}{2} \int_M \|R^\nabla\|^2 v_g$$

is called the *Yang–Mills functional*. A critical point $\nabla \in \mathcal{C}(E)$ of the Yang–Mills functional is called a *Yang–Mills connection* which is equivalent to $\delta^\nabla R^\nabla = 0$. The *Hodge-deRham Laplacian* for vector bundle valued exterior p -forms is by definition

$$\Delta^\nabla = d^\nabla \delta^\nabla + \delta^\nabla d^\nabla.$$

Define an endomorphism \mathfrak{R}^∇ of the space $\Omega^1(\mathfrak{g}_E)$ by

$$\mathfrak{R}^\nabla(\varphi)(X) = \sum_{j=1}^m [R^\nabla(e_j, X), \varphi(e_j)]$$

for $\varphi \in \Omega^1(\mathfrak{g}_E)$, where $\{e_1, \dots, e_m\}$ is any local orthonormal frame field on M with respect to g . Then the second variation formula is given by:

THEOREM 2.4. (cf. [B.L]) *Suppose $\nabla = \nabla^0$ is a Yang–Mills connection and $B = \frac{d}{dt}|_{t=0} \nabla^t \in \Omega^1(\mathfrak{g}_E)$. Then the second variation of the Yang–Mills functional is given by*

$$\begin{aligned} \left. \frac{d^2}{dt^2} \right|_{t=0} \mathcal{YM}(\nabla^t) &= \int_M \langle \delta^\nabla d^\nabla B + \mathfrak{R}^\nabla(B), B \rangle v_g \\ &= \int_M \langle \mathcal{S}^\nabla(B), B \rangle v_g, \end{aligned}$$

provided $\delta^\nabla B = 0$, where

$$\mathcal{S}^\nabla(B) = \Delta^\nabla B + \mathfrak{R}^\nabla(B).$$

The operator \mathcal{S}^∇ is elliptic and self-adjoint and the restriction to the space $\text{Ker}(\delta^\nabla) \subset \Omega^1(\mathfrak{g}_E)$ has eigenvalues $\lambda_1 < \lambda_2 < \dots \rightarrow \infty$, with associated finite dimensional eigenspaces $E_{\lambda_1}, E_{\lambda_2}, \dots$. Then we can define the (Morse) index and the (Morse) nullity as follows (cf. [B.L]).

DEFINITION 2.5. The *index* of a Yang–Mills connection ∇ is the dimension $i(\nabla) = \dim(\oplus_{\lambda < 0} E_\lambda)$. The *nullity* of ∇ is the dimension $n(\nabla) = \dim(E_0)$.

DEFINITION 2.6. A Yang–Mills connection ∇ is said to be *weakly stable* if $i(\nabla) = 0$, i.e., the second variation ≥ 0 . Otherwise, ∇ is said to be *unstable*.

2.2. CONFORMAL TRANSFORMATIONS

Let $S^m = \{x \in \mathbf{R}^{m+1}; \|x\| = 1\}$ be the unit sphere and for any $a \in \mathbf{R}^{m+1}$, define a C^∞ vector field \bar{a} on S^m by

$$\bar{a}(y) = a - \langle a, y \rangle y, \quad y \in S^m, \tag{2.7}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on \mathbf{R}^{m+1} and $\|x\|^2 = \langle x, x \rangle, x \in \mathbf{R}^{m+1}$. Let $\gamma_t = \gamma_t^a, t \in \mathbf{R}$, be a *flow* of \bar{a} , i.e., each γ_t is a C^∞ diffeomorphism of S^m onto itself satisfying

$$\gamma_s \circ \gamma_t = \gamma_{s+t}, \quad s, t \in \mathbf{R}, \quad \text{and} \tag{2.8}$$

$$\left. \frac{d}{dt} \right|_{t=0} \gamma_t(x) = \bar{a}(x), \quad x \in S^m. \tag{2.9}$$

It is wellknown (cf. [E1], [O]) that each γ_t is a conformal transformation of S^m , i.e.

$$\gamma_t^* g = \alpha_t g \tag{2.10}$$

where g is the standard Riemannian metric on S^m with constant curvature 1, and α_t is a C^∞ positive function on S^m which is given (cf. [E.1]) by

$$\alpha_t(x) = \|a\|^2 (\sinh t \langle x, a \rangle + \|a\| \cosh t)^{-2}, \quad x \in S^m. \tag{2.11}$$

Let D be the Levi–Civita connection of (S^m, g) and \tilde{D} that of $(S^m, \gamma_t^* g)$. Then

$$d\gamma_t(\tilde{D}_X Y) = D_{d\gamma_t X} d\gamma_t Y, \tag{2.12}$$

for all C^∞ vector fields X and Y on S^m . It is known (cf. [E1]) that

$$\begin{aligned} \tilde{D}_X Y - D_X Y &= \frac{1}{2} \alpha_t^{-1} \{g(X, \text{grad } \alpha_t) Y + g(Y, \text{grad } \alpha_t) X \\ &\quad - g(X, Y) \text{grad } \alpha_t\} \end{aligned} \tag{2.13}$$

for C^∞ vector fields X and Y on S^m , where $\text{grad } \alpha_t$ is the gradient vector field of α_t with respect to g which is given (cf. [E1]) by

$$\text{grad } \alpha_t = -\frac{2}{\|a\|} (\alpha_t)^{3/2} \sinh t \bar{a}. \tag{2.14}$$

3. Deformations of connections

In what follows, we always assume the base Riemannian manifold (M, g) is the unit sphere (S^m, g) with the standard Riemannian metric g of constant curvature 1 and preserve the situations in Section 2.

We fix a G -connection ∇ on E with a connection form ω . For $0 \neq a \in \mathbf{R}^{m+1}$, let γ_t be a flow of \bar{a} on S^m . We can define a one-parameter family $\tilde{\gamma}_t, t \in \mathbf{R}$ of bundle maps of E lifting γ_t as the flow of the horizontal lift \bar{a}^* of the vector field \bar{a} to E . That is, for each $w \in E$ with $\pi(w) = x$, the curve $t \mapsto \tilde{\gamma}_t(w)$ is a horizontal lift of the curve $t \mapsto \gamma_t(x)$. We also define a one-parameter family $\varphi_t, t \in \mathbf{R}$, of bundle maps of P so that each $u \in P$ with $\pi(u) = x$, a curve $t \mapsto \varphi_t(u)$ is the parallel transport of u with respect to ω along the curve $t \mapsto \gamma_t(x)$. Then it holds that

$$\varphi_t(u) = \tilde{\gamma}_t \circ u, \quad u \in P, \tag{3.1}$$

where each $u \in P$ is regarded as an onto linear isomorphism $u: \mathbf{R}^r \rightarrow E_x$ and $\tilde{\gamma}_t \circ u$ is its composition with $\tilde{\gamma}_t: E_x \rightarrow E_{\gamma_t(x)}$.

For each C^∞ section $\sigma \in \Gamma(E)$, define a new C^∞ section $\tilde{\gamma}_t \cdot \sigma \in \Gamma(E)$ by

$$(\tilde{\gamma}_t \cdot \sigma)(x) = \tilde{\gamma}_t(\sigma(\gamma_t^{-1}(x))), \quad x \in S^m. \tag{3.2}$$

It is known (cf. [K.N, p. 114]) that

$$\left. \frac{d}{dt} \right|_{t=0} \tilde{\gamma}_t \cdot \sigma = -\nabla_{\bar{a}} \sigma. \tag{3.3}$$

Moreover, the C^∞ map of P into \mathbf{R}^r corresponding to the section $\tilde{\gamma}_t \cdot \sigma \in \Gamma(E)$ as in (2.1) is given by

$$(\tilde{\gamma}_t \cdot \sigma)^\sim = \tilde{\sigma} \circ \varphi_t^{-1}, \tag{3.4}$$

that is, $(\tilde{\gamma}_t \cdot \sigma)^\sim(u) = \tilde{\sigma}(\varphi_t^{-1}(u)), u \in P$.

DEFINITION 3.5. For a G -connection ∇ of E with a connection form ω , define a one-parameter family of connections $\nabla^t, t \in \mathbf{R}$ by

$$\nabla_X^t \sigma = \tilde{\gamma}_t^{-1} \cdot (\nabla_{d\gamma_t X}(\tilde{\gamma}_t \cdot \sigma)), \quad \sigma \in \Gamma(E) \tag{3.6}$$

for a vector field X on S^m .

LEMMA 3.7. (i) ∇^t is a G -connection with the connection form $\varphi_t^* \omega$.

(ii) Its infinitesimal deformation satisfies that

$$\left. \frac{d}{dt} \right|_{t=0} \nabla^t = i(\bar{a})R^\nabla. \tag{3.8}$$

That is, for a vector field X on S^m ,

$$\left. \frac{d}{dt} \right|_{t=0} \nabla_X^t \sigma = R^\nabla(\bar{a}, X)\sigma, \quad \sigma \in \Gamma(E).$$

(iii) The curvature form $\Omega^{\varphi_t^* \omega}$ is equal to $\varphi_t^* \Omega^\omega$, and for C^∞ vector fields X and Y on S^m ,

$$R^{\nabla^t}(X, Y)\sigma = \tilde{\gamma}_t^{-1} \cdot R^\nabla(d\gamma_t X, d\gamma_t Y)(\tilde{\gamma}_t \cdot \sigma), \quad \sigma \in \Gamma(E). \quad (3.9)$$

Proof. For (i), in fact, it is easy to see ∇^t is a connection on E . For instance, for $f \in C^\infty(S^m)$, $\sigma \in \Gamma(E)$ and a C^∞ vector field X on S^m ,

$$\begin{aligned} \nabla_X^t(f\sigma) &= \tilde{\gamma}_t^{-1} \cdot (\nabla_{d\gamma_t X}(\tilde{\gamma}_t \cdot (f\sigma))) \\ &= \tilde{\gamma}_t^{-1} \cdot \{d\gamma_t(X)(f \circ \gamma_t^{-1})\tilde{\gamma}_t \cdot \sigma + f \circ \gamma_t^{-1} \nabla_{d\gamma_t X}(\tilde{\gamma}_t \cdot \sigma)\} \\ &= (Xf)\sigma + f\tilde{\gamma}_t^{-1} \cdot (\nabla_{d\gamma_t X}(\tilde{\gamma}_t \cdot \sigma)) \\ &= (Xf)\sigma + f\nabla_X^t \sigma \end{aligned}$$

since $d\gamma_t(X)(f \circ \gamma_t^{-1}) = Xf \circ \gamma_t^{-1}$, and the other conditions to be a connection are verified in a similar way. On the other hand, since $\varphi_t \circ R_b = R_b \circ \varphi_t$, where for $b \in G$, $R_b: P \ni u \mapsto u \cdot b \in P$, $\varphi_t^* \omega$ satisfies two conditions to be a connection form, in fact. To see that ∇^t is a G -connection, we only have to see the connection form of ∇^t coincides with $\varphi_t^* \omega$. To see this, let $\bar{\nabla}^t$ be the G -connection corresponding to $\varphi_t^* \omega$. By (2.2), for $\sigma \in \Gamma(E)$,

$$(\bar{\nabla}^t \sigma)^\sim = d\tilde{\sigma} + \rho(\varphi_t^* \omega)\tilde{\sigma}. \quad (3.10)$$

Since, for a vector field W on P ,

$$\begin{aligned} d\tilde{\sigma}(W) &= (d(\tilde{\sigma} \circ \varphi_t^{-1})(d\varphi_t W)) \circ \varphi_t \\ &= (d(\tilde{\gamma}_t \circ \sigma)^\sim(d\varphi_t W)) \circ \varphi_t \end{aligned}$$

and $(\varphi_t^* \omega)(W) = \omega(d\varphi_t W) \circ \varphi_t$,

$$\begin{aligned} d\tilde{\sigma}(W) + \rho((\varphi_t^* \omega)(W))\tilde{\sigma} &= \{d(\tilde{\gamma}_t \cdot \sigma)(d\varphi_t W) \\ &\quad + \rho(\omega(d\varphi_t W))(\tilde{\gamma}_t \cdot \sigma)\} \circ \varphi_t \\ &= (\nabla_{d\gamma_t(\bullet)}(\tilde{\gamma}_t \cdot \sigma))^\sim(W) \circ \varphi_t \\ &= (\tilde{\gamma}_t^{-1} \cdot \nabla_{d\gamma_t(\bullet)}(\tilde{\gamma}_t \cdot \sigma))^\sim(W) = (\nabla^t \sigma)^\sim(W), \end{aligned}$$

by (2.2'). Thus we get $\bar{\nabla}^t = \nabla^t$, i.e., (i).

(ii) For a vector field X on S^m and $\sigma \in \Gamma(E)$,

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \nabla_X^t \sigma &= \frac{d}{dt} \Big|_{t=0} \tilde{\gamma}_t^{-1} \cdot \nabla_{d\gamma_t X} (\tilde{\gamma}_t \cdot \sigma) \\ &= \frac{d}{dt} \Big|_{t=0} \tilde{\gamma}_t^{-1} \cdot \nabla_X \sigma + \nabla_X \left(\frac{d}{dt} \Big|_{t=0} \tilde{\gamma}_t \cdot \sigma \right) \\ &\quad + \frac{d}{dt} \Big|_{t=0} \nabla_{d\gamma_t X} \sigma. \end{aligned}$$

By (3.3) and $\frac{d}{dt} \Big|_{t=0} d\gamma_t X = -[\bar{a}, X]$, the right hand side coincides with $R^\nabla(\bar{a}, X)\sigma$.

The assertion (iii) follows from definitions of R^{∇^t} , ∇^t and the curvature form. □

LEMMA 3.11. *If $i(\bar{a})R^\nabla = 0$, then*

$$\varphi_t^* \omega = \omega, \quad t \in \mathbf{R}.$$

Proof. By Lemma 3.7 (ii) and (2.2), for $\sigma \in \Gamma(E)$,

$$\begin{aligned} (i(\bar{a})R^\nabla \sigma)^\sim &= \frac{d}{dt} \Big|_{t=0} (\nabla^t \sigma)^\sim \\ &= \frac{d}{dt} \Big|_{t=0} \{d\tilde{\sigma} + \rho(\varphi_t^* \omega) \tilde{\sigma}\} \\ &= \rho \left(\frac{d}{dt} \Big|_{t=0} \varphi_t^* \omega \right) \tilde{\sigma}. \end{aligned}$$

Thus if $i(\bar{a})R^\nabla \sigma = 0$ for any $\sigma \in \Gamma(E)$, then $\frac{d}{dt} \Big|_{t=0} \varphi_t^* \omega = 0$ because of the faithfulness of ρ . This implies $\varphi_t^* \omega = \omega, t \in \mathbf{R}$. □

LEMMA 3.12. (cf. [B.L, p. 215]). *If ∇ is a Yang–Mills connection, then $B := i(\bar{a})R^\nabla, a \in \mathbf{R}^{m+1}$, belongs to the subspace $\text{Ker}(\delta^\nabla)$ of $\Omega^1(\mathfrak{g}_E)$.*

Proof. This lemma is an immediate consequence of Lemma 7.3 in [B.L, p. 215] since $\delta^\nabla R^\nabla = 0$ and \bar{a} is a vector field of gradient type in the sense of [B.L]. □

4. Main theorem

We shall prove:

THEOREM 4.1. *Let $(S^m, g)(m \geq 5)$ be the m -sphere with the standard Riemannian metric g of constant curvature 1. Let E be any G -vector bundle over S^m with G a compact Lie group and ∇ any nonflat Yang–Mills connection on E . Then*

$$i(\nabla) \geq m + 1.$$

Let $0 \neq a \in \mathbf{R}^{m+1}$ be arbitrarily fixed. Let $\nabla^t, t \in \mathbf{R}$, be the one-parameter family of G -connections in Definition 3.5. We shall show the following two propositions in the following sections.

PROPOSITION 4.2. *Put*

$$f(t) = \mathcal{YM}(\nabla^t) = \frac{1}{2} \int_{S^m} \|R^{\nabla^t}\|^2 v_g.$$

Then we obtain

$$(i) \quad f'(t) = \frac{4 - m}{\|a\|} \sinh t \int_{S^m} \alpha_t^{1/2} \|i(\bar{a})R^{\nabla^t}\|^2 v_g.$$

$$(ii) \quad f''(0) = \frac{4 - m}{\|a\|} \int_{S^m} \|i(\bar{a})R^\nabla\|^2 v_g.$$

PROPOSITION 4.3. *If $i(\bar{a})R^\nabla = 0$ for some $0 \neq a \in \mathbf{R}^{m+1}$, then ∇ is flat.*

Proof of Theorem 4.1. By these two propositions, we obtain immediately Theorem 4.1. In fact, assume that ∇ is a nonflat Yang–Mills connection on E and $m > 4$. Then by Lemma 3.12 and Proposition 4.3, $V := \{i(\bar{a})R^\nabla \in \Omega^1(\mathfrak{g}_E); a \in \mathbf{R}^{m+1}\}$ is an $(m + 1)$ -dimensional subspace of $\text{Ker}(\delta^\nabla)$, and by Propositions 4.2, the second variation of \mathcal{YM} at ∇ is negative definite on V . Thus, $i(\nabla) \geq m + 1$. □

5. Proof of Proposition 4.2

We only have to show (i) since (ii) follows immediately from (i).

Let $B^t = \frac{d}{ds}|_{s=0} \nabla^{t+s}$. By (3.8), we have

$$B^t = i(\bar{a})R^{\nabla^t} = R^{\nabla^t}(\bar{a}, \bullet). \tag{5.1}$$

We first show

$$\begin{aligned} \delta^{\nabla^t} R^{\nabla^t}(X) &= - \sum_{j=1}^m \{R^{\nabla^t}(\tilde{D}_{e_j} e_j - D_{e_j} e_j, X) \\ &\quad + R^{\nabla^t}(e_j, \tilde{D}_{e_j} X - D_{e_j} X)\}, \end{aligned} \tag{5.2}$$

where $\{e_j\}_{j=1}^m$ is a local orthonormal frame field on (S^m, g) , and D and \tilde{D} are the Levi–Civita connections of (S^m, g) and (S^m, γ_t^*g) , respectively. In fact, for $\sigma \in \Gamma(E)$,

$$\begin{aligned}
 (\delta^{\nabla^t} R^{\nabla^t})(X)\sigma &= -\sum_{j=1}^m (\nabla_{e_j}^t R^{\nabla^t})(e_j, X)\sigma \\
 &= -\sum_{j=1}^m \{ \nabla_{e_j}^t (R^{\nabla^t}(e_j, X)\sigma) - R^{\nabla^t}(D_{e_j}e_j, X)\sigma \\
 &\quad - R^{\nabla^t}(e_j, D_{e_j}X)\sigma - R^{\nabla^t}(e_j, X)\nabla_{e_j}^t\sigma \} \\
 &= -\sum_{j=1}^m \tilde{\gamma}_t^{-1} \cdot \{ \nabla_{d\gamma_t e_j} (R^\nabla(d\gamma_t e_j, d\gamma_t X)\tilde{\gamma}_t \cdot \sigma) \\
 &\quad - R^\nabla(d\gamma_t(D_{e_j}e_j), d\gamma_t X)\tilde{\gamma}_t \cdot \sigma \\
 &\quad - R^\nabla(d\gamma_t e_j, d\gamma_t(D_{e_j}X))\tilde{\gamma}_t \cdot \sigma \\
 &\quad - R^\nabla(d\gamma_t e_j, d\gamma_t X)\nabla_{d\gamma_t e_j}\tilde{\gamma}_t \cdot \sigma \}
 \end{aligned}$$

because of the definition of $\nabla^t R^{\nabla^t}$ and (3.9). Here $\nabla_{d\gamma_t e_j} (R^\nabla(d\gamma_t e_j, d\gamma_t X)\tilde{\gamma}_t \cdot \sigma)$ coincides with

$$\begin{aligned}
 &(\nabla_{d\gamma_t e_j} R^\nabla)(d\gamma_t e_j, d\gamma_t X)\tilde{\gamma}_t \cdot \sigma + R^\nabla(D_{d\gamma_t e_j} d\gamma_t e_j, d\gamma_t X)\tilde{\gamma}_t \cdot \sigma \\
 &+ R^\nabla(d\gamma_t e_j, D_{d\gamma_t e_j} d\gamma_t X)\tilde{\gamma}_t \cdot \sigma + R^\nabla(d\gamma_t e_j, d\gamma_t X)\nabla_{d\gamma_t e_j}\tilde{\gamma}_t \cdot \sigma. \quad (5.3)
 \end{aligned}$$

Since $e'_j(\gamma_t x) := \alpha_t(x)^{-1/2}(d\gamma_t e_j)_{\gamma_t x}$, $j = 1, \dots, m$, is a local orthonormal frame field with respect to g and the Levi–Civita connection \tilde{D} of $\gamma_t^*g = \alpha_t g$ satisfies (2.12), (5.3) coincides with

$$\begin{aligned}
 &\alpha_t(\nabla_{e'_j} R^\nabla)(e'_j, d\gamma_t X)\tilde{\gamma}_t \cdot \sigma + R^\nabla(d\gamma_t(\tilde{D}_{e_j}e_j), d\gamma_t X)\tilde{\gamma}_t \cdot \sigma \\
 &+ R^\nabla(d\gamma_t e_j, d\gamma_t(\tilde{D}_{e_j}X))\tilde{\gamma}_t \cdot \sigma + R^\nabla(d\gamma_t e_j, d\gamma_t X)\nabla_{d\gamma_t e_j}\tilde{\gamma}_t \cdot \sigma.
 \end{aligned}$$

Notice that

$$\sum_{j=1}^m (\nabla_{e'_j} R^\nabla)(e'_j, d\gamma_t X)\tilde{\gamma}_t \cdot \sigma = 0$$

since $\delta^\nabla R^\nabla = 0$ (∇ is a Yang–Mills connection). Therefore we get

$$\begin{aligned}
 (\delta^{\nabla^t} R^{\nabla^t})(X)\sigma &= -\sum_{j=1}^m \tilde{\gamma}_t^{-1} \cdot \{R^\nabla(d\gamma_t(\tilde{D}_{e_j}e_j), d\gamma_t X)\tilde{\gamma}_t \cdot \sigma \\
 &\quad - R^\nabla(d\gamma_t(D_{e_j}e_j), d\gamma_t X)\tilde{\gamma}_t \cdot \sigma \\
 &\quad + R^\nabla(d\gamma_t e_j, d\gamma_t(\tilde{D}_{e_j}X))\tilde{\gamma}_t \cdot \sigma \\
 &\quad - R^\nabla(d\gamma_t e_j, d\gamma_t(D_{e_j}X))\tilde{\gamma}_t \cdot \sigma\} \\
 &= -\sum_{j=1}^m \{R^{\nabla^t}(\tilde{D}_{e_j}e_j - D_{e_j}e_j, X)\sigma \\
 &\quad + R^{\nabla^t}(e_j, \tilde{D}_{e_j}X - D_{e_j}X)\sigma\},
 \end{aligned}$$

which implies (5.2).

By (2.13), we have for any C^∞ vector field X on S^m ,

$$\begin{aligned}
 \tilde{D}_{e_j}X - D_{e_j}X &= \frac{1}{2}\alpha_t^{-1}\{g(e_j, \mathbf{grad}\alpha_t)X + g(X, \mathbf{grad}\alpha_t)e_j \\
 &\quad - g(e_j, X)\mathbf{grad}\alpha_t\},
 \end{aligned} \tag{5.4}$$

$$\sum_{j=1}^m (\tilde{D}_{e_j}e_j - D_{e_j}e_j) = \frac{1}{2}(2-m)\alpha_t^{-1}\mathbf{grad}\alpha_t. \tag{5.5}$$

Therefore, substituting (5.4) and (5.5) into (5.2),

$$\begin{aligned}
 (\delta^{\nabla^t} R^{\nabla^t})(X)\sigma &= -\frac{1}{2}(2-m)\alpha_t^{-1}R^{\nabla^t}(\mathbf{grad}\alpha_t, X)\sigma \\
 &\quad - \frac{1}{2}\sum_{j=1}^m \alpha_t^{-1}R^{\nabla^t}(e_j, g(e_j, \mathbf{grad}\alpha_t)X \\
 &\quad + g(X, \mathbf{grad}\alpha_t)e_j - g(e_j, X)\mathbf{grad}\alpha_t)\sigma \\
 &= -\{\frac{1}{2}(2-m)\alpha_t^{-1}R^{\nabla^t}(\mathbf{grad}\alpha_t, X)\sigma \\
 &\quad + \frac{1}{2}\alpha_t^{-1}R^{\nabla^t}(\mathbf{grad}\alpha_t, X)\sigma \\
 &\quad - \frac{1}{2}\alpha_t^{-1}R^{\nabla^t}(X, \mathbf{grad}\alpha_t)\sigma\} \\
 &= -\frac{1}{2}(4-m)\alpha_t^{-1}R^{\nabla^t}(\mathbf{grad}\alpha_t, X)\sigma.
 \end{aligned}$$

Substituting (2.14) into this, we obtain

$$(\delta^{\nabla^t} R^{\nabla^t})(X)\sigma = \frac{4-m}{\|a\|}\alpha_t^{1/2}\sinh t R^{\nabla^t}(\bar{a}, X)\sigma,$$

that is,

$$\delta^{\nabla^t} R^{\nabla^t} = \frac{4-m}{\|a\|}\alpha_t^{1/2}\sinh t i(\bar{a})R^{\nabla^t}.$$

Therefore together with (5.1), we finally obtain

$$\begin{aligned} f'(t) &= \int_{S^m} \langle \delta^{\nabla^t} R^{\nabla^t}, B^t \rangle v_g \\ &= \frac{4 - m}{\|a\|} \sinh t \int_{S^m} \alpha_t^{1/2} \|i(\bar{a})R^{\nabla^t}\|^2 v_g. \end{aligned}$$

□

6. Proof of Proposition 4.3.

Let ∇ be a G -connection on E with a connection form ω . Assume that

$$i(\bar{a})R^\nabla = 0$$

for some $0 \neq a \in \mathbf{R}^{m+1}$. Then by Lemma 3.11,

$$\varphi_t^* \omega = \omega, \quad t \in \mathbf{R}, \tag{6.1}$$

where $\varphi_t, t \in \mathbf{R}$ is a one-parameter family of C^∞ bundle maps of P corresponding to $a \in \mathbf{R}^{m+1}$ defined by: for each $u \in P$ with $\pi(u) = x$, the curve $t \mapsto \varphi_t(u)$ is the parallel transport of ∇ along the curve $t \mapsto \gamma_t(x)$ and γ_t is a flow on S^m of \bar{a} .

Then we have

LEMMA 6.2. (i) *We have*

$$H_{\varphi_t(u)} = \varphi_{t*} H_u, \quad u \in P, \tag{6.3}$$

where $H_u = \{X \in T_u P; \omega(X) = 0\}$ is the horizontal subspace of $T_u P$ and φ_{t*} is the differential of φ at u . (6.3) means that φ_t sends a horizontal curve $s \mapsto u(s)$ in P to another horizontal one $s \mapsto \varphi_t(u(s))$ in P .

(ii) *Let $u \in P \setminus \pi^{-1}(-\frac{a}{\|a\|})$. Then $u_\infty = \lim_{t \rightarrow \infty} \varphi_t(u)$ exists and lies in $\pi^{-1}(-\frac{a}{\|a\|}) \subset P$, and the correspondence $u \mapsto u_\infty$ is smooth.*

Proof. The assertion (i) is an immediate consequence of (6.1). We prove (ii). For each $x \in S^m \setminus \{-\frac{a}{\|a\|}\}$, $\lim_{t \rightarrow \infty} \gamma_t(x) = \frac{a}{\|a\|}$, and the curve $t \mapsto \gamma_t(x)$ is a smooth curve with finite length. Thus we get a smooth curve $s \mapsto c(s)$ connecting x and $\frac{a}{\|a\|}$, reparametrizing by the arc length $s = s(t)$. Then for any $u \in P$ with $\pi(u) = x$, $\varphi_t(u)$ is the parallel transport of u also along $c(s)$ with $s = s(t)$. The assertion (ii) is now straightforward. □

Fix any point $u_0 \in \pi^{-1}(\frac{a}{\|a\|})$. By Lemma 6.2, for each $u \in P \setminus \pi^{-1}(-\frac{a}{\|a\|})$, there exists a unique $b(u) \in G$ such that

$$\lim_{t \rightarrow \infty} \varphi_t(u) = u_0 b(u). \tag{6.4}$$

Therefore we can define a smooth mapping Φ of $P \setminus \pi^{-1}(-\frac{a}{\|a\|})$ into the product bundle $(S^m \setminus \{-\frac{a}{\|a\|}\}) \times G$ by

$$\Phi(u) = (\pi(u), b(u)). \quad (6.5)$$

Clearly Φ has a smooth inverse and satisfies

$$\Phi(ub) = \Phi(u)b, \quad b \in G$$

for all $u \in P \setminus \pi^{-1}(-\frac{a}{\|a\|})$. Thus Φ gives an isomorphism between the principal G -bundles $P \setminus \pi^{-1}(-\frac{a}{\|a\|})$ and $(S^m \setminus \{-\frac{a}{\|a\|}\}) \times G$.

Moreover, we have

LEMMA 6.6. *The differential Φ_* of Φ at $u \in P \setminus \pi^{-1}(-\frac{a}{\|a\|})$ is given by*

$$\Phi_*(X) = (\pi_*X, A_{b(u)}), \quad X \in T_uP,$$

where A is a unique element of \mathfrak{g} such that the vertical component of X, X^V , equals A_u^* .

Proof. For a smooth function f on $S^m \times G$, we shall calculate

$$\Phi_*(X)f = \Phi_*(X^H)f + \Phi_*(X^V)f = X^H(f \circ \Phi) + X^V(f \circ \Phi).$$

As for Φ_*X^V ,

$$\begin{aligned} \Phi_*X^Vf &= X^V(f \circ \Phi) = A_u^*(f \circ \Phi) \\ &= \left. \frac{d}{ds} \right|_{s=0} f(\pi(u \exp(sA)), b(u \exp(sA))) \\ &= \left. \frac{d}{ds} \right|_{s=0} f(\pi(u), b(u) \exp(sA)) \\ &= (0, A_{b(u)})f \end{aligned}$$

since $b(uc) = b(u)c$ for $c \in G$ by (6.4) and the definition of φ_t . Thus we get $\Phi_*X^V = (0, A_{b(u)})$.

As for Φ_*X^H , take any horizontal curve $I \ni s \mapsto u(s) \in P$ with $u(0) = u$ and $\dot{u}(0) = X^H$, where I is an open interval containing 0. By Lemma 6.2 (i), for each t , the curve $s \mapsto \varphi_t(u(s))$ is horizontal, i.e.

$$\omega((\varphi_t \circ u)^\bullet(s)) = 0$$

for all $s \in I$. By Lemma 6.2 (ii), letting $t \rightarrow \infty$, we get

$$\omega(\dot{u}_\infty(s)) = 0$$

for all $s \in I$. That is, the curve $s \mapsto u_\infty(s)$ is also a horizontal curve. But

$$\pi(u_\infty(s)) = \frac{a}{\|a\|}.$$

Therefore $u_\infty(s), s \in I$, is a single point. Thus $b(u(s)) = b(u)$ for all $s \in I$. Hence we get

$$\begin{aligned} \Phi_* X^H f &= \left. \frac{d}{ds} \right|_{s=0} f(\Phi(u(s))) \\ &= \left. \frac{d}{ds} \right|_{s=0} f(\pi(u(s)), b(u(s))) \\ &= (\pi_* X, 0)f. \end{aligned}$$

We obtain Lemma 6.6. □

Due to Lemma 6.6, for all $X = X^H + X^V$ where $X^H \in H_u$ and $X^V = A_u^* \in V_u$ with $A \in \mathfrak{g}$, we get

$$\begin{aligned} (\Phi^{-1*} \omega)(\pi_* X, A_{b(u)}) &= \omega(\Phi_*^{-1}(\pi_* X, A_{b(u)})) \\ &= \omega(X) \\ &= A. \end{aligned}$$

This means that $\Phi^{-1*} \omega$ coincides with the canonical flat connection form on the product bundle $(S^m \setminus \{ -\frac{a}{\|a\|} \}) \times G$ (cf. [K.N, p. 92]). Therefore the connection form ω corresponding to ∇ gives a flat connection on the G -bundle $P \setminus \pi^{-1}(-\frac{a}{\|a\|})$ (cf. [K.N, p. 92]). Since $P \setminus \pi^{-1}(-\frac{a}{\|a\|})$ is an open and dense subset of P and the curvature form Ω^ω is continuous, ω is a flat connection on P . We obtain Proposition 4.3. □

REMARK 6.7. In the case $m = 4$, for any Yang–Mills connection ∇ of any G -vector bundle over S^m , the nullity is estimated as

$$n(\nabla) \geq m + 1 = 5,$$

by a similar argument.

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