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Complex local systems and morphisms of varieties

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Abstract. Let X be a smooth projective algebraic variety over $\mathbb{C}$, and let Z be a scheme of finite type over $\mathbb{C}$, all of the connected components of which are complete normal varieties. Suppose further that $\phi: Z \to X$ is a morphism whose image is a connected closed subscheme Y of X, and that $\pi_1^{\rm alg}(Y)$ maps onto $\pi_1^{\rm alg}(X)$. Let N be the normal subgroup of $\pi_1(X)$ generated by the images of the fundamental groups of the connected components of Z. In this paper we prove results about the finite dimensional complex representations of $\pi_1(X)/N$ that suggest that it is small. In particular we prove that all the finite dimensional complex representations of $\pi_1(X)/N$ are unitary.

1. Introduction

M. V. Nori, in his paper [13], proves the following result:

THEOREM 1.1. *Let X be a smooth projective algebraic surface over the complex numbers, let C be an irreducible nodal curve on X with r nodes, let $f: \bar{C} \to C$ be the non-singular model for C, and choose base points d_0 for $\bar{C}$ and x_0 for C, with $f(d_0) = x_0$. Then if $C \cdot C > 2r$, $[\pi_1(X, x_0): f_*\pi_1(\bar{C}, d_0)] < \infty$. Here $C \cdot C$ denotes the self-intersection of C on X.*

This result is a consequence of Nori's "weak Lefschetz theorem," which he proves in [13]. The Lefschetz hyperplane theorem says that in the situation of Theorem 1.1, $\pi_1(C, x_0)$ maps onto $\pi_1(X, x_0)$. Elementary arguments show, however, that $\pi_1(C, x_0) \approx \pi_1(\bar{C}, d_0) \star F$, where F is the free group on r generators, so that the normal subgroup of $\pi_1(C, x_0)$ generated by $f_*(\pi_1(\bar{C}, d_0))$ is of infinite index. Theorem 1.1 says then that if C has sufficiently positive self-intersection, then only a finite shadow of the infinite contribution of the singularities of C to its fundamental group can be seen in the fundamental group of the smooth surface X.

More generally, suppose X is any smooth projective algebraic variety over the complex numbers, and Y is a closed connected subscheme of X such that $\pi_1^{\rm alg}(Y, y_0)$ maps onto $\pi_1^{\rm alg}(X, y_0)$ for some y_0 in Y. Suppose further that $f: Z \to Y$ is a morphism from a scheme Z of finite type over the complex numbers, all of the connected components of which are complete and normal varieties, onto Y. We denote by N the normal subgroup of $\pi_1(X, y_0)$ generated by images under f_* of the fundamental groups of the connected components of Z. The normal subgroup N is independent of the choice of base points for the connected components of Z.

Considering theorem 1.1, it is reasonable to ask whether $[\pi_1(X, y_0) : N] < \infty$. If $f : Z \to Y$ is a normalization of Y, this question is again whether the contribution of the non-normal singularities of Y to the fundamental group of Y can cast more than a finite shadow on the fundamental group of X. In this paper we prove that in general $\pi_1(X, y_0)/N$ has at most finitely many isomorphism classes of complex representations of any given finite dimension (Corollary 6.2.) In fact we prove a stronger technical result (Theorem 6.1), which also has as a corollary, as pointed out to me by Professor Nori, that the finite dimensional complex representations of $\pi_1(X, y_0)/N$ are unitary (Corollary 6.5.) These results provide support for the belief that $\pi_1(X, y_0)/N$ may be finite, inasmuch as they show that its finite dimensional complex representations theory shares some properties with the representation theory of finite groups. There are, however, examples of finitely presented infinite groups which have at most finitely many isomorphism classes of complex representations of any given dimension, such that all these representations are unitary. We give some of these examples in 7.

Note that there are theorems which give conditions for a closed subscheme Y of a complex projective variety X to be connected and for the map induced by the inclusion of Y in X to send $\pi_1(Y, y_0)$ onto $\pi_1(X, y_0)$ for any choice of base point y_0 for Y. See W. Fulton and B. Lazarsfeld's article [4]. For example, if X is a normal and irreducible closed subvariety of $\mathbb{P}^r(\mathbb{C})$ and W is closed subvariety of $\mathbb{P}^r(\mathbb{C})$ such that $\dim X + \dim W > r$, then $X \cap W$ is connected and $\pi_1(X \cap W)$ maps onto $\pi_1(X)$, for any choice of base points (p. 27 of [4].)

The proof of the main result (6.1) of this paper uses results from a number of different areas of research, and once we realize the possibility of bringing them together to cooperate, our theorem follows easily. Because of this, we devote a substantial portion of this paper to explaining how results we use can be extracted from the work of other authors, in the process repeating some of their work. In Section 2 we discuss schemes of representations of a finitely generated group, and give proofs of facts about the relation of the tangent spaces of these schemes to group cohomology. In this section we follow Weil's papers [18] and [19]. Section 3 is concerned with results of C. Simpson in [15] and [16] about those representations of the fundamental group of a smooth projective variety over the complex numbers which underlie variations of Hodge structure, considered as points on the moduli spaces of semi-simple complex local systems on the variety. We give definitions of variations of Hodge structure and of the moduli space of semi-simple complex local systems in that section. Section 4 is devoted to an application we make of ideas of P. Deligne and B. Saint-Donat in [14], previously applied in Deligne's papers [3]. Finally, in Section 5 we discuss P. Griffiths' classifying spaces for polarized real Hodge structures (see [5]), which we use to prove Corollary 6.5. Again, we give the relevant definitions in the section itself. By including these sections as the bulk of the paper, and only proceeding with the proof of Theorem 6.1 after the necessary aspects of previous work are isolated, I hope to clarify how research in diverse areas cooperates to give the main theorem. Also, since many readers are likely to be unfamiliar with some of the results which we use, I hope that these sections will help them to more completely understand the proof of the main result without an extensive search through the literature.

I'd like to thank Professor Nori, who first suggested to me the essential outline of the proof of Theorem 6.1 in the case where Y is a curve with only normal crossings for singularities and with positive self-intersection on a surface X, and Z is its normalization. The proof of Theorem 6.1 is based on this outline. I'd also like to thank Professor Alex Lubotzky for a number of helpful conversations.

2. Schemes of representations

In this section we recall work of Weil in [18] and [19] on schemes of representations of a finitely generated group. The main result of this section, which we employ in the proof of Theorem 6.1, is Proposition 2.6. In addition to the papers of Weil, the memoir [10] offers a good discussion of these results. This memoir is our primary source for this section, and any proofs we omit here may be found in the first two sections of [10].

DEFINITION 2.1. Let G be a finitely generated group and let n be a positive integer. Let $\{\gamma_1, \ldots, \gamma_r\}$ be a generating set for G, and let $\{r_q\}_{q\in Q}$ be a defining set of relations for G for some index set Q. Let $A = \mathbb{C}[x_{ij}, \det(x_{ij})^{-1}]$ be the coordinate ring of $\mathrm{GL}_n(\mathbb{C})$, where x_{ij} denotes the ij coordinate function on $\mathrm{GL}_n(\mathbb{C})$ for $1 \leqslant i, j \leqslant n$. Denote the coordinate functions corresponding to the kth factor in $A^{\otimes r}$ by $x_{ij}^{(k)}$, and let $X^{(k)}$ be the matrix $(x_{ij}^{(k)})$ in $M_n(A^{\otimes r})$. Then each relation r_q defines a matrix

$$R_q = r_q(X^{(1)}, \ldots, X^{(r)}) \in M_n(A^{\otimes r}).$$

If I_G is ideal of $A^{\otimes r}$ generated by

$$\{(R_q)_{ij} - \delta_{ij} \mid 1 \leqslant q \leqslant r, 1 \leqslant i, j \leqslant n\},$$

then we define $R_n(G) = \mathrm{Spec}(A^{\otimes r}/I_G)$.

Up to isomorphism this definition is independent of the choice of presentation for G.

Note that the complex points of this scheme correspond to representations of G in $\mathbb{C}^n$, since they are exactly those points $(X_1, \ldots, X_r)$ in $\mathrm{GL}_n(\mathbb{C}^n)^r$ which satisfy the equality

$$r_q(X_1, \ldots, X_r) = \text{identity}.$$

Given such a point, we can define a representation ρ of G by

$$\rho(\gamma_i) = X_i \quad \text{for} \quad 1 \leqslant i \leqslant r,$$

and conversely, given a representation $\rho\colon\ G \to \mathrm{GL}_n(\mathbb{C})$, we can define a point $(X_1,\dots,X_r)$ in $R_n(G)(\mathbb{C})$ by

$$X_i = \rho(\gamma_i) \quad \text{for} \quad 1 \leqslant i \leqslant r.$$

It is clear that this gives a bijective correspondence. The complex points of $R_n(G)$ do not, however, completely describe it, since in general it is not reduced.

The construction in Definition 2.1 is functorial. If H is another finitely generated group and $f\colon\ G \to H$ is a group homomorphism, then there is a natural morphism of schemes over $\mathbb{C}$:

$$f^*\colon\ R_n(H) \to R_n(G).$$

On the level of complex points, f^* takes a representation ρ of H in $\mathbb{C}^n$ to the representation $\rho \circ f$ of G in $\mathbb{C}^n$. It is not hard to write down an explicit formula for the map on coordinate rings associated to f^*.

While the complex points of $R_n(G)$ parameterize all representations of G in $\mathbb{C}^n$, we are interested only in isomorphism classes of n-dimensional representations of G. If $(X_1,\dots,X_r)$ is in $R_n(G)(\mathbb{C})$, then the set of points of $R_n(G)(\mathbb{C})$ which correspond to isomorphic representations is exactly

$$\{(AX_1A^{-1},\dots,AX_rA^{-1}) \mid A \in \mathrm{GL}_n(\mathbb{C})\}.$$

So isomorphism classes of n-dimensional complex representations of G are parameterized by points in the topological space $\mathrm{GL}_n(\mathbb{C})\backslash R_n(G)(\mathbb{C})$, where $\mathrm{GL}_n(\mathbb{C})$ acts as above the conjugation. This action in fact comes from an action of the reductive algebraic group GL_n on the affine scheme $R_n(G)$. According to Theorem 1.1 on page 27 in [12] we can form an affine scheme of finite type over $\mathbb{C}$ which is a universal categorical quotient (Definition .7 on page 4 of [12]) of $R_n(G)$ by the action of GL_n. We denote this scheme by $SS_n(G)$. It is not in general a geometric quotient; that is to say that if $\pi_G\colon\ R_n(G) \to SS_n(G)$ is the quotient morphism, then in general some of the fibers of the induced map on complex points contain more than one $\mathrm{GL}_n(\mathbb{C})$-orbit. We have, however, the following result, which is Proposition 1.12 in [10].

PROPOSITION 2.1. *Let $R_n^s(G)$ denote the open subscheme of $R_n(G)$ the complex points of which correspond to irreducible representations of G in $\mathbb{C}^n$ and let $S_n(G)$ be its categorical quotient by the action of* GL_n. *(See pp. 11–14 in [10] for precise definitions of these schemes.) Then $S_n(G)$ is an open subscheme of $SS_n(G)$, and π_G is a geometric quotient when restricted to $R_n^s(G)(\mathbb{C})$.*

This proposition says that complex points of $S_n(G)$ correspond exactly to isomorphism classes of irreducible n-dimensional complex representations of G.

As for the complex points of $SS_n(G)$, we have the following, which is Theorem 1.28 in [10].

THEOREM 2.2. *Every fiber of the map*

$$\pi_G\colon R_n(G)(\mathbb{C}) \to SS_n(G)(\mathbb{C})$$

contains a unique $\mathrm{GL}_n(\mathbb{C})$ *orbit* $O(\rho)$ *of a point corresponding to a semi-simple representation* ρ *of* G *in* $\mathbb{C}^n$, *and this orbit is closed. Furthermore, if* σ *is a representation of* G *on* $\mathbb{C}^n$ *corresponding to a complex point in this fiber, then the closure of the orbit* $O(\sigma)$ *meets* $O(\rho)$, *and* ρ *is isomorphic to a semi-simplification of* σ. *(For any* G*-representation* $\tau\colon G \to \mathrm{GL}(V)$, *where* V *is a finite dimensional complex vector space, if*

$$V = V_0 \supset V_1 \supset \cdots \supset V_m = 0$$

is a filtration of V *by* G*-invariant subspaces* V_i *such that* V_{i-1}/V_i *is irreducible for* $1 \leqslant i \leqslant m$, *then the "semi-simplification" of* τ *is*

$$\bigoplus_{i=1}^{m} V_{i-1}/V_i.$$

This is well-defined up to isomorphism.)

The complex points of $SS_n(G)$ thus correspond to isomorphism classes of semisimple representations of G.

If $f : G \to H$ is a homomorphism of finitely generated groups, then $f^* : R_n(H) \to R_n(G)$ is GL_n-equavariant, so that f^* induces a morphism of universal categorical quotients from $\pi_H : R_n(H) \to SS_n(H)$ to $\pi_G\colon R_n(G) \to SS_n(G)$. In particular there is a morphism $\bar{f}^* : SS_n(H) \to SS_n(G)$, which sends a point on $SS_n(H)$ corresponding to a semi-simple representation $\sigma : H \to \mathrm{GL}_n(\mathbb{C})$ to the point on $SS_n(G)$ corresponding to a semi-simplification of $\sigma \circ f$.

The final basic results about these schemes which are important to us are those proved by Weil in [19], which describe tangent spaces. Recall that for a not necessarily reduced scheme X over $\mathbb{C}$, the tangent space $T_x(X)$ to X at a complex point x of X is the space of morphisms of $\mathrm{Spec}(\mathbb{C}[T]/(T^2))$ into X over the point x. If $U = \mathrm{Spec}(A)$ is an affine open neighborhood of x in X, then x corresponds to a $\mathbb{C}$-linear homomorphism $i_x\colon A \to \mathbb{C}$, and $T_x(X)$ is the space of all $\mathbb{C}$-linear homomorphisms $j\colon A \to \mathbb{C}[T]/(T^2)$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[rr, "j"] \arrow[dr, "i_x"'] & & \mathbb{C}[T]/(T^2) \arrow[dl, "p"] \\
& \mathbb{C} &
\end{tikzcd}$$

Here $p\colon\ \mathbb{C}[T]/(T^2) \to \mathbb{C}$ is the $\mathbb{C}$-algebra homomorphism which sends T to 0. If X is reduced and of finite type over $\mathbb{C}$, then this is the usual tangent space, and in general $T_x(X)$ contains $T_z(X^{\mathrm{red}})$. A morphism of schemes $\phi\colon\ X \to Y$ induces in the obvious way a morphism $D\phi_x\colon\ T_x(X) \to T_{\phi(x)}(Y)$ for any complex point x in X.

For schemes of representations, we have the following two results due to Weil, which are Propositions 2.2 and 2.3 in [10]. In stating these results, we identify a representation $\rho\colon\ G \to \mathrm{GL}_n(\mathbb{C})$ with a complex point on $R_n(G)$.

PROPOSITION 2.3. *Let ρ be in $R_n(G)(\mathbb{C})$. Then there is a natural isomorphism*

$$Z^1(G, \mathrm{Ad} \circ \rho) \to T_\rho(R_n(G)),$$

where $\mathrm{Ad} \circ \rho$ *is the representation of G on $M_n(\mathbb{C})$ given by:*

$$\mathrm{Ad} \circ \rho(g)(M) = \rho(g)M\rho(g)^{-1}$$

for any g in G and any M in $M_n(\mathbb{C})$, and $Z^1(G, \mathrm{Ad} \circ \rho)$ is the group of 1-cocyles for this representation.

PROPOSITION 2.4. *Let ρ be in $R_n(G)(\mathbb{C})$ and let $\psi_\rho\colon\ \mathrm{GL}_n \to R_n(G)$ be the morphism of schemes defined by the action of* GL_n *on the orbit of ρ. (For any A in* $\mathrm{GL}_n(\mathbb{C})$, $\psi_\rho(A)\colon\ G \to \mathrm{GL}_n(\mathbb{C})$ *is given by:*

$$\psi_\rho(A)(g) = A\rho(g)A^{-1}$$

for any g in G.) Then the image of the composite

$$D\psi_{\rho,id}\colon\ T_{id}(\mathrm{GL}_n) \to T_\rho(R_n(G)) \approx Z^1(G, \mathrm{Ad} \circ \rho),$$

where the isomorphism is that in Proposition 2.3, is $B^1(G, \mathrm{Ad} \circ \rho)$, the space of 1-coboundaries for the representation $\mathrm{Ad} \circ \rho$.

COROLLARY 2.5. *If for every semi-simple complex representation ρ of G of dimension n, $H^1(G, \mathrm{Ad} \circ \rho) = 0$, then there are only finitely many isomorphism classes of n-dimensional complex representations of G, and all of them are semi-simple.*

Proof. By the previous two propositions, all the orbits of $\mathrm{GL}_n(\mathbb{C})$ on $R_n(G)(\mathbb{C})$ corresponding to semi-simple representations are open, while by Theorem 2.2 these orbits are closed, and the closure of any orbit of $\mathrm{GL}_n(\mathbb{C})$ meets one of these orbits. Therefore $R_n(G)(\mathbb{C})$ is a disjoint union of a finite number of orbits of $\mathrm{GL}_n(\mathbb{C})$ corresponding to semi-simple representations of G. □

These results combine to give us the result which we apply in the proof of Theorem 6.1. Before stating the result, we introduce some notation.

Let $f: G \to H$ be a homomorphism of finitely generated groups, and let $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ be a semi-simple representation. We denote by $\bar{\rho}$ the corresponding complex point of $SS_n(G)(\mathbb{C})$. The fiber of $\bar{f}^*: SS_n(H) \to SS_n(G)$ over $\bar{\rho}$ is a subscheme the complex points of which correspond to isomorphism classes of semi-simple representations $\sigma: H \to \mathrm{GL}_n(\mathbb{C})$ such that all semi-simplifications of $\sigma \circ f$ are isomorphic to ρ. (see Lemma 1.26 in [10].) For the proof of Theorem 6.1, however, we are interested in isomorphism classes of representations σ such that $\sigma \circ f$ is *itself* isomorphic to ρ. These correspond to complex points in a closed subscheme W_ρ of $\bar{f}^{*-1}(\bar{\rho})$, which we describe as follows. If $O(\rho)$ denotes the orbit of ρ in $R_n(G)$ under GL_n, then by Theorem 2.2 $O(\rho)$ is a closed subscheme of $R_n(G)$, and so $f^{*-1}(O(\rho))$ is a closed GL_n-invariant subscheme of $R_n(H)$. Therefore there is a closed subscheme of $SS_n(H)$ which is the categorical quotient of $f^{*-1}(O(\rho))$ by GL_n. (This follows from arguments used in the proof of Theorem 1.1 on pages 25–27 of [12].) The complex points of this scheme are those corresponding to semi-simplifications σ_{ss} of representations $\sigma: H \to \mathrm{GL}_n(\mathbb{C})$ such that $\sigma \circ f$ is isomorphic to ρ. Then $\sigma_{ss} \circ \rho$ is also isomorphic to ρ, so that this scheme is the W_ρ we want.

Note that by construction and Theorem 2.2, the map on complex points induced by the morphism

$$\pi_H|_{f^{*-1}(O(\rho))}\colon f^{*-1}(O(\rho)) \to W_\rho$$

has connected fibers. For if $\tau: H \to \mathrm{GL}_n(\mathbb{C})$ is semi-simple and $\bar{\tau}$ is in W_ρ, then $O(\tau)$ is connected and contained in $(\pi_H|_{f^{*-1}(O(\rho))})^{-1}(\bar{\tau})$, and if τ' is also in $(\pi_H|_{f^{*-1}(O(\rho))})^{-1}(\bar{\tau})$, then the closure of $O(\tau')$, which is also connected, meets $O(\tau)$.

PROPOSITION 2.6. *In the situation described immediately above, if every connected component of the topological space associated to the complex points of* W_ρ *contains a point corresponding to a semi-simple representation* $\sigma\colon H \to \mathrm{GL}_n(\mathbb{C})$ *such that* f *induces an injection*

$$f^*\colon H^1(H, \mathrm{Ad} \circ \sigma) \hookrightarrow H^1(G, \mathrm{Ad} \circ \rho),$$

then W_ρ *has only finitely many complex points.*

Proof. Suppose the theorem is false. Then there is a complex point $\bar{\sigma}$ in W_ρ corresponding to a semi-simple representation $\sigma\colon H \to \mathrm{GL}_n(\mathbb{C})$ with

$$H^1(H, \mathrm{Ad} \circ \sigma) \hookrightarrow H^1(G, \mathrm{Ad} \circ \rho),$$

such that $\bar{\sigma}$ is contained in an irreducible component V of W_ρ of positive dimension. Let

$$U = \pi_H^{-1}(V) \cap f^{*-1}(O(\rho)).$$

The Hausdorff space formed by the complex points of U is connected, since the fibers of $\pi_H|_{f^{*-1}(O(\rho))}$ are connected. Because V is of positive dimension, and $\pi_H(O(\sigma))$ is the point $\bar{\sigma}$, we have $T_\sigma(O(\sigma)) \subsetneqq T_\sigma(U)$. However $f^*(U) \subseteq O(\rho)$, so that

$$Df^*_\sigma(T_\sigma(U)) \subseteq T_\rho(O(\rho)).$$

Then the kernel of the composite:

$$H^1(H, \mathrm{Ad} \circ \sigma) \approx \frac{T_\sigma(R_n(H))}{T_\sigma(O(\sigma))} \xrightarrow{Df^*_\sigma} \frac{T_\rho(R_n(G))}{T_\rho(O(\rho))} \approx H^1(G, \mathrm{Ad} \circ \rho)$$

contains the non-zero subgroup $\frac{T_\sigma(U)}{T_\sigma(O(\sigma))}$. But by the naturality of the morphisms in Propositions 2.3 and 2.4, this composite is the natural map:

$$H^1(H, \mathrm{Ad} \circ \sigma) \hookrightarrow H^1(G, \mathrm{Ad} \circ \rho),$$

which is injective by assumption. This completes the proof of the proposition. □

3. Local systems underlying variations of Hodge structure

For any analytic space X over the complex numbers, and any positive integer n, if $\{X_i\}_{1 \leqslant i \leqslant r}$ is the set of connected components of X, we denote by $M_{B,n}(X)$ the Hausdorff topological space consisting of the complex points of the scheme

$$\prod_{1 \leqslant i \leqslant r} SS_n(\pi_1(X_i, x_i))$$

for some choice of base points x_i for each of the X_i. $M_{B,n}(X)$ is then well-defined up to homeomorphism, and by the correspondence between representations of fundamental groups and local systems, along with Theorem 2.2, the points of $M_{B,n}(X)$ correspond naturally to isomorphism classes of semi-simple n-dimensional complex local systems on X. In the proof of Theorem 6.1, we apply Proposition 2.6 to schemes $SS_n(\pi_1(X, x_0))$ for smooth projective algebraic varieties X, viewed as analytic spaces. To verify the hypotheses of Proposition 2.6, we use certain results from the work of C. Simpson in [15] and [16] about the spaces $M_{B,n}(X)$ when X is a smooth projective algebraic variety. This section is devoted to a discussion of these results.

First we give some definitions from [5] and [15].

DEFINITION 3.1. A *complex variation of Hodge structure* on a complex manifold M is a C^∞ complex vector bundle V on M with the following data.

(1) A C^∞ decomposition $V = \bigoplus_{r,s \in \mathbb{Z}} V^{r,s}$.

(2) A flat connection D_V satisfying

$$D\colon V^{r,s} \to \mathcal{A}^{0,1}(V^{r+1,s-1}) \oplus \mathcal{A}^{1,0}(V^{r,s}) \\ \oplus \mathcal{A}^{0,1}(V^{r,s}) \oplus \mathcal{A}^{1,0}(V^{r-1,s+1}).$$

(3) A Hermitian form Q_V on V which is flat with respect to D, positive definite on $V^{r,s}$ is r is even, negative definite on $V^{r,s}$ if r is odd, and such that $Q_V(V^{r_1,s_1}, V^{r_2,s_2}) = 0$ unless $r_1 = r_2$ and $s_1 = s_2$.

Here $\mathcal{A}^{p,q}(V^{r,s})$ denotes the bundle of C^∞ 1-forms on M with coefficients in $V^{r,s}$. Note that the (0,1)-component of the connection D_V defines a holomorphic structure on V such that for any integer the subbundle

$$\mathcal{F}^p = \bigoplus_{r \geqslant p} V^{r,s}$$

is holomorphic.

DEFINITION 3.2. Let V be a complex variation of Hodge structure on a complex manifold M, and suppose we are given a real structure on the underlying C^∞ vector bundle. We define the *conjugate* variation of Hodge structure $\bar{V}$ to V to be the complex variation of Hodge structure with the same underlying C^∞ vector bundle given by the following data.

(1) For any pair of integers (r, s), $\overline{V^{r,s}} = V^{s,r}$.
(2) The connection $D_{\bar{V}}$ on $\bar{V}$ is the complex conjugate of the connection D_V on V; that is to say that for any C^∞ section f of V, $D_{\bar{V}}(f)$ is the complex conjugate of $D_V(\bar{f})$.
(3) For any two C^∞ sections f and g of $\overline{V^{r,s}} = V^{s,r}$, the Hermitian form $Q_{\bar{V}}$ on $\bar{V}$ is such that $Q_{\bar{V}}(f, g)$ is the complex conjugate of $(-1)^{r+s}Q(\bar{f}, \bar{g})$.

DEFINITION 3.3. We say that a complex variation of Hodge structure on a complex manifold M is a *real variation of Hodge structure* if the underlying C^∞ vector bundle is given a real structure so that $V = \bar{V}$.

If V is a real variation of Hodge structure, then there is a local system of $\mathbb{R}$-vector spaces $V_{\mathbb{R}}$ such that $V_{\mathbb{R}} \otimes \mathbb{C}$ is the flat bundle defined by D_V, and the complex conjugate of $V^{r,s}$ with respect to this structure is $V^{s,r}$ for any pair of integers (r, s). Furthermore, there is a flat bilinear form S on V, defined over $V_{\mathbb{R}}$, such that S restricted to $\bigoplus_{r+s=m} V^{r,s}$ is symmetric if m is even and skew is m is odd, $S(V^{r_1,s_1}, V^{r_2,s_2}) = 0$ unless $r_1 = s_2$ and $s_1 = r_2$, and $i^{-r-s}S(f, \bar{g}) = Q(f, g)$ for any pair (f, g) of C^∞ sections of $V^{r,s}$.

Let V and W be any two complex variations of Hodge structure on a complex manifold M.

DEFINITION 3.4. The complex variation of Hodge structure $\mathrm{Hom}(V, W)$ is the variation of Hodge structure with underlying C^∞ vector bundle $\mathrm{Hom}(V, W)$ given by the following data.

(1) $\mathrm{Hom}(V,W)^{r,s} = \{\lambda \in C^\infty(M, \mathrm{Hom}(V,W)) \mid \lambda(V^{p,q}) \subseteq W^{p+r,q+s}$ for all pairs of integers p and $q\}$.
(2) $D_{\mathrm{Hom}(V,W)}(\lambda) = D_W\lambda - \lambda D_V$.
(3) $Q_{\mathrm{Hom}(V,W)}$ is on each fiber the natural Hermitian form induced by Q_V and Q_W.

With Definition 3.4 we can define the dual variation of Hodge structure $V^* = \mathrm{Hom}(V, \mathbb{C})$, where $\mathbb{C}$ denotes the trivial variation of Hodge structure on M with $\mathbb{C}^{0,0} = \mathbb{C}$, and thus we can also define the tensor product $V \otimes W$ of V and W. Since the complex conjugate of $\mathrm{Hom}(V,W)$ is $\mathrm{Hom}(\bar{V}, \bar{W})$, all these constructions apply to real variations of Hodge structure as well.

DEFINITION 3.5. The *direct sum* $V \oplus W$ of the complex variations of Hodge structure V and W is the complex variation of Hodge structure with underlying C^∞ vector bundle structure $V \oplus W$ given by the following data.

(1) $$(V \oplus W)^{r,s} = \bigoplus_{\substack{r_1,s_1,r_2,s_2 \in \mathbb{R} \\ r_1+r_2=r, s_1+s_2=s}} (V^{r_1,s_1} \oplus W^{r_2,s_2}).$$
(2) $D_{V\oplus W} = D_V \oplus D_W$.
(3) $Q_{V\oplus W} = Q_V \oplus Q_W$.

Note that for any complex variation of Hodge structure V on a complex manifold M, and any choice of real structure on the C^∞ vector bundle underlying V, $V \oplus \bar{V}$ is a real variation of Hodge structure.

For the remainder of this section we denote by X any smooth projective algebraic variety over the complex numbers, which we consider as a complex manifold.

Now we can state what for our purposes are the main results in [15] and [16].

THEOREM 3.1. *There is a continuous action of $\mathbb{C}^*$ on $M_{B,n}(X)$ the fixed points of which are those points on $M_{B,n}(X)$ corresponding to semi-simple complex local systems on X which underlie variations of Hodge structure.*

THEOREM 3.2. *For any point $\bar{\rho}$ in $M_{B,n}(X)$ corresponding to a semi-simple complex local system on X,*

$$\lim_{\substack{t\to 0 \\ t \in \mathbb{C}^*}} t\bar{\rho}$$

exists.

The limit point which exists by Theorem 3.2 is $\mathbb{C}^*$-invariant, so by Theorem 3.1 the corresponding semi-simple local system underlies a variation of Hodge structure.

The complete proofs of Theorems 3.1 and 3.2 are well beyond the scope of this paper. For our applications, though, we need to know more precisely how the $\mathbb{C}^*$-action is defined. For this we need first more definitions given in [15].

DEFINITION 3.6. A *Higgs bundle* on X is a pair (E,θ), where E is a holomorphic vector bundle and $\theta\colon\ E \to E \otimes \Omega_X^1$ is a map of holomorphic vector bundles such that $\theta \wedge \theta = 0$. Here Ω_X^1 is the bundle of holomorphic differentials on X.

DEFINITION 3.7. A Higgs bundle (E,θ) is *stable* if for any non-zero proper holomorphic subbundle F of E such that $\theta(F) \subseteq F \otimes \Omega_X^1$, we have

$$\frac{\deg(F)}{\operatorname{rank}(F)} < \frac{\deg(E)}{\operatorname{rank}(E)},$$

where the degree of a vector bundle on X is the product of its first Chern class with an appropriate power of the class of a hyperplane section of X, viewed as an integer via the canonical isomorphism

$$H^{2\dim X}(X;\mathbb{Z}) \approx \mathbb{Z}.$$

In [16], for any positive integer, n, Simpson constructs a complex algebraic variety with points corresponding to the isomorphism classes of direct sums of stable Higgs bundles (E,θ) such that all the Chern classes of E are trivial. We denote the Hausdorff topological space associated to this variety by $M_{\mathrm{Dol},n}(X)$. There is a natural action of $\mathbb{C}^*$ on $M_{\mathrm{Dol},n}(X)$, which sends the point corresponding to a Higgs bundle (E,θ) to the point corresponding to $(E,t\theta)$. This action turns out to be continuous, and we get the $\mathbb{C}^*$-action on $M_{B,n}(X)$ via the following theorem from [16].

THEOREM 3.3. *There is a functorial bijective correspondence between isomorphism classes of semi-simple complex local systems of dimension n on X and isomorphism classes of direct sums of stable Higgs bundles of dimension n on X, and this bijection induces a homeomorphism between $M_{\mathrm{Dol},n}(X)$ and $M_{B,n}(X)$.*

Explicitly, we get the semi-simple local system corresponding via Theorem 3.3 to a Higgs bundle (E,θ) of dimension n which is a direct sum of stable Higgs bundles with trivial Chern classes as follows. First, for any Hermitian metric K on E, let $\bar\partial + \partial_K$ be the connection on E which is compatible with both the complex structure and the metric K on E, and let

$$\bar\theta_K\colon\ E \to E \otimes \bar\Omega_X^1$$

be the unique map of vector bundles such that

$$(e,\bar\theta_K f)_K = (\theta e, f)_K$$

for any two C^∞ sections e and f of E. Then we set $D'_K = \partial_K + \bar\theta_K$, $D'' = \bar\partial + \theta$, and $D_K = D'_K + D''$. The metric K is called *harmonic* if $D_K^2 = 0$. If K is harmonic, then the flat connection D_K defines a local system which has C^∞ bundle E. By a theorem of K. Corlette in [2], this local system is semi-simple. A result in nonlinear analysis due to several authors shows that an arbitrary Higgs bundle has a harmonic metric if and only if it is a direct sum of stable Higgs bundles with trivial Chern classes. For references see [15]. Therefore our Higgs bundle (E, θ) has a harmonic metric. Finally, the semi-simple local system so obtained is well-defined up to isomorphism, independent of the choice of harmonic metric. In this way we get the isomorphism class of semi-simple local systems corresponding to the isomorphism class of the Higgs bundle (E, θ).

Using this explicit form of the homeomorphism $M_{\mathrm{Dol},n}(X) \to M_{B,n}(X)$ we can prove a result which applies to the situation we're interested in. Let $f: (Z, z_0) \to (X, x_0)$ be a morphism of smooth projective algebraic varieties over the complex numbers with base points. Then the map

$$f_*\colon \pi_1(Z, z_0) \to \pi_1(X, x_0)$$

induces a morphism of schemes

$$f^*\colon SS_n(\pi_1(X, x_0)) \to SS_n(\pi_1(Z, z_0)),$$

and so also a continuous map

$$f^*\colon M_{B,n}(X) \to M_{B,n}(Z).$$

If Z isn't a variety, but is simply a smooth projective scheme, all the connected components of which are varieties, then we let $\{Z_i\}_{1 \leqslant i \leqslant r}$ be the set of components of Z. By considering each component of Z separately, we still have a continuous map $f^*: M_{B,n}(X) \to M_{B,n}(Z)$, a $\mathbb{C}^*$-action on $M_{B,n}(Z)$ with fixed points corresponding to local systems on Z underlying complex variations of Hodge structure, and, as in section 2, for any semi-simple complex local system V of dimension n on Z we have a closed subscheme W_V of $f^{*-1}(V)$ with points corresponding to those local systems U on X for which $f^*(U) \approx V$. Here we identify such a local system V with its corresponding point on $M_{B,n}(Z)$.

PROPOSITION 3.4. *$W_V = f^{*-1}(V)$, and f^* is $\mathbb{C}^*$-equivariant.*

Proof. It clearly suffices to prove the proposition when Z is a variety.

Let V be the semi-simple local system on Z corresponding to a representation $\rho\colon \pi_1(Z, z_0) \to \mathrm{GL}_n(\mathbb{C})$. Let U be semi-simple local system on X, associated to a Higgs bundle (E, θ) with a harmonic metric K, such that the point in $M_{B,n}(X)$ corresponding to U is in $f^{*-1}(V)$. We define the pullback to (E, θ) to be the Higgs bundle $(f^*E, f^*\theta)$, where f^*E is the C^∞ pullback bundle of E and $f^*\theta$ is the global holomorphic section of $\mathrm{End}(f^*E) \otimes \Omega^1_Z$ obtained by pulling back θ, a global holomorphic section of $\mathrm{End}(E) \otimes \Omega^1_X$. In the notation use in describing the construction of U from (E, θ) and K, we have $f^*(\partial_K) = \partial_{f^*(K)}$ and $f^*\bar{\theta}_K =$

$\overline{(f^*\theta)}_{f^*(K)}$, so that $f^*(D_K) = D_{f^*(K)}$. It follows that f^*K is a harmonic metric for $(f^*E, f^*\theta)$, and the local system we construct with f^*K is f^*U. By the theorem in non-linear analysis cited above, since it has a harmonic metric, $f^*(E,\theta)$ is a direct sum of stable Higgs bundles with trivial Chern classes, and by the theorem of Corlette in [2] referred to above, f^*U is semi-simple. Since we assume that the point on $M_{B,n}(X)$ corresponding to U is in $f^{*-1}(V)$, this shows that f^*U is isomorphic to V, and so U is in W_V. This completes the proof of the first part of the proposition.

From the above we also see that the correspondence between Higgs bundles and local systems commutes with pullbacks, and because $f^*(E,t\theta) = tf^*(E,\theta)$ for any Higgs bundle (E,θ) on X and any t in $\mathbb{C}^*$, it follows that f^* is $\mathbb{C}^*$-equivariant. □

COROLLARY 3.5. *If V is a local system on Z which underlies a variation of Hodge structure, then every connected component of $W_V = f^{*-1}(V)$ contains a point corresponding to a local system underlying a variation of Hodge structure on X.*

Proof. By Proposition 3.4, $f^{*-1}(V)$ is $\mathbb{C}^*$-invariant. Therefore if U is any semisimple local system on X corresponding to a point in $M_{B,n}(X)$, the point

$$\lim_{\substack{t\to 0\\ t\in\mathbb{C}^*}} tU$$

given by Theorem 3.2 is contained in the same connected component of $f^{*-1}(V)$ as U, and corresponds to a local system underlying a variation of Hodge structure on Z by Theorem 3.1. □

4. The necessary mixed Hodge theory

Recall the general situation of the introduction. We have the following data:

(1) A smooth projective variety over the complex numbers X.
(2) A connected closed subscheme Y of X such that $\pi_1(Y,y_0) \twoheadrightarrow \pi_1(Y,y_0)$ for any choice of base point y_0 for Y.
(3) A scheme Z of finite type over $\mathbb{C}$, all the connected components of which are complete normal varieties.
(4) A morphism ϕ from Z onto Y.

We want to apply Proposition 2.6 in an appropriate way to this situation. To do this we will need to show that for certain local systems V on X, the natural map

$$H^1(X,V) \to H^1(Z,V)$$

is injective. Since X is smooth and the singularities of Z are normal, we can hope to apply differential geometric results from Hodge theory fairly easily to these varieties, as indeed we do in Section 3 when Z is smooth. To use the assumption Y, however, presents greater difficulties, since its singularities are arbitrary. To overcome these difficulties, we follow Deligne in considering mixed Hodge structures on cohomologies. The result we need is Theorem 4.1. A detailed proof may be found in [9]. In this section we merely sketch the relevant arguments, which follow arguments in [14] and [3], leaving out the plentiful technical detail.

First we recall the relevant definitions.

DEFINITION 4.1. Let m be an integer. Then a *real Hodge structure* of weight m is a finite dimensional real vector space H with a decomposition

$$H_{\mathbb{C}} = H \otimes_{\mathbb{R}} \mathbb{C} \approx \bigoplus_{p+q=m} H^{p,q}$$

such that $\overline{H^{p,q}} = H^{q,p}$.

DEFINITION 4.2. A *real mixed Hodge structure* is a finite dimensional real vector space with the following data:

(1) An increasing filtration (the "weight filtration") W of H such that $W_k = 0$ for some k and $W_l = H$ for some l; and
(2) A decreasing filtration (the "Hodge filtration") F of $H_{\mathbb{C}} = H \otimes_{\mathbb{R}} \mathbb{C}$;

such that for any j, if we define

$$(Gr_j^W(H_{\mathbb{C}}))^{p,q} = F^p Gr_j^W(H_{\mathbb{C}}) \cap \overline{F^{j+1-q} Gr_j^W(H_{\mathbb{C}})}$$

then $Gr_j^W(H)$ is a real Hodge structure of weight j with decomposition

$$Gr_j^W(H_{\mathbb{C}}) = \bigoplus_{p+q=j} Gr_j^W(H_{\mathbb{C}})^{p,q}.$$

In [3], Deligne proves that there are contravariant functors that assign to every separated scheme X of finite type over the complex numbers a real mixed Hodge structure of weight m on $H^m(X(\mathbb{C}), \mathbb{R})$. These functors are defined in [3] using so-called "simplicial hypercoverrings" of schemes. A smooth simplicial hypercovering of a scheme X is a certain kind of augmented simplicial object in the category of schemes.

In general, an (augmented) simplicial object in a category is a contravariant functor from the (augmented) standard simplicial category to that category. The (augmented) standard simplicial category is the category whose objects are the standard simplexes of all positive dimensions (resp. all dimensions greater than or equal to -1, the standard -1-simplex being the empty set), and whose morphisms are all face and degeneracy maps and their composites. Thus a simplicial set (that is a simplicial object in the category of sets) is just a choice of sets of n-simplexes for every n, along with rules for how they should be pasted together to form a topological space. A simplicial set, together with the associated topological space, forms an augmented simplicial topological space: the space of -1-simplexes is defined to be the associated topological space, while for $n \geqslant 0$ the space of n-simplexes is the product of the given set of n-simplexes with the standard n-simplex. The cohomology of a topological space associated to a given simplicial set is of course easy to compute in terms of the cohomologies of each space of n-simplexes and the maps between them given by the simplicial set.

Deligne, in [3], generalizes this to augmented simplicial schemes, showing that under certain conditions the cohomology of a sheaf on the "scheme of -1-simplexes" may be computed in terms of the cohomologies of the pullback of that sheaf to each scheme of n-simplexes. When these conditions are satisfied, the simplicial scheme is called a simplicial hypercoverring of its scheme of -1-simplexes. The homological algebra in the more general situation is, however, much more complicated than in the situation of simplicial sets, since the higher cohomologies of the sheaves on schemes of n-simplexes are not necessarily zero. The cohomology of simplicial schemes is discussed in detail in [14].

For the purposes of defining mixed Hodge structures on cohomologies of a proper scheme X over the complex numbers, Deligne uses the fact, proved in [14], that there is a simplicial hypercoverring of X such that for $n \geqslant 0$, the scheme of n-simplexes is a smooth projective (possibly not connected) variety. Then the cohomology of each of the schemes of n-simplexes with coefficients in $\mathbb{R}$ has a natural real Hodge structure. When the cohomology of X with coefficients in $\mathbb{R}$ is expressed in terms of that of its simplicial hypercoverring, using a spectral sequence, the Hodge structures on the cohomologies of the schemes of n-simplexes define a *mixed* Hodge structure on the cohomology of X. The $(m-j)$th weight graded piece of the mth real cohomology of X essentially is that coming from the scheme of j-simplexes. See [3] for details.

The existence of a smooth projective simplicial hypercoverring of a proper scheme is established in [14] inductively. the scheme of 0-simplexes may be defined to be any smooth projective variety Y which surjects onto X. Such a variety exists by Hironaka's resolution of singularities. (See [8].) If the fibre product $Y \times_Z Y$ is smooth and projective, then the scheme of 1-simplexes may be defined to be $Y \times_X Y$, with face maps given by the two natural projections

$$Y \times_X Y \to Y$$

and degeneracy map given by the diagonal map

$$Y \to Y \times_X Y.$$

In general, let

$$W \to Y \times_X Y$$

be a resolution of singularities of $Y \times_X Y$. Then the scheme of 1-simplexes may be defined to be $Y \amalg W$. This process may be continued indefinitely, at the nth stage defining the scheme of n-simplexes as a disjoint union of:

(1) a resolution of singularities of a certain projective limit of schemes of k-simplexes for $k \leqslant n-1$; and
(2) a disjoint union of certain copies of k-simplexes for $k \leqslant n-1$.

In this way one defines a smooth projective simplicial hypercoverring of X. We refer an interested reader to Saint-Donat's article [14] or the author's detailed treatment in section four of [9].

In our situation, we are concerned with cohomologies of proper schemes over the complex numbers not just with coefficients in the real numbers, but with coefficients in some real variation of Hodge structure V of weight k. Deligne's approach applies to this situation as well, giving functorial mixed Hodge structures on the mth cohomologies of proper schemes over $\mathbb{C}$ with coefficients in a real variation of Hodge structure. The idea is to replace the proper scheme X with a smooth projective simplicial hypercoverring and compute the cohomology of X with coefficients in V by pulling V back to this hypercoverring. Then for any $j \geqslant 0$, the mth cohomology of the scheme of j-simplexes with coefficients in the pullback of V has a natural Hodge structure of weight $m+k$ (see [20]), and as for the case with real coefficients, a spectral sequence gives a mixed Hodge structure on the mth cohomology of X with coefficients in V. The $(m+k-j)$ weight graded piece of this cohomology essentially is that coming from the scheme of j-simplexes. These mixed Hodge structures are discussed in detail in section five of [9].

The following is result using mixed Hodge theory that we need for this paper.

THEOREM 4.1. *Under assumptions (1)–(4) at the beginning of this section, if V is a complex variation of Hodge structure on X, then the map*

$$H^1(X,V) \to H^1(Z,V)$$

is injective.

This is an immediate consequence of Corollary 5.7 in [9], and a detailed proof may be found there. Here we will simply indicate why Theorem 4.1 is true.

First, we may easily reduce to showing that if V is a real variation of Hodge structure of weight k on X, then the map

$$H^1(X,V) \to H^1(Z,V)$$

is injective. This map is in fact a morphism of real mixed Hodge structures. As indicated above, we may compute the mixed Hodge structure on $H^1(Y,V)$ by using a simplicial hypercoverring of Y whose scheme of 0-simplexes is any smooth projective variety mapping onto Y. In particular we may use a simplicial hypercoverring whose scheme of 0-simplexes is Z. Since the scheme of 0-simplexes of a simplicial hypercoverring of Y essentially gives the weight $k+1$ graded part of $H^1(Y,V)$, and all other weights are at most k, the kernel of the map

$$H^1(Y,V) \to H^1(Z,V)$$

has weights at most k. By assumption, the fundamental group of Y surjects onto that of X, so that the map

$$H^1(X,V) \to H^1(Y,V)$$

is injective. (We use here the identification of the first sheaf cohomology of a local system with the first group cohomology of the corresponding representation of the fundamental group.) As $H^1(X,V)$ is pure of weight $k+1$ (that is, $W_k H^1(X,V) = 0$ and $W_{k+1}H^1(X,V) = H^1(X,V)$), the image of $H^1(X,V)$ in $H^1(Y,V)$ does not meet the kernel of the map

$$H^1(Y,V) \to H^1(Z,V),$$

and so the composite

$$H^1(X,V) \to H^1(Z,V)$$

is injective.

5. Classifying spaces for polarized real Hodge structures

In this section we give the definitions from [5] of P. Griffiths' classifying spaces for polarized real Hodge structures and the so-called period maps defined by a real variation of Hodge structure, and state a result of Griffiths and W. Schmid from [6] about these objects, which we employ in the proof of Corollary 6.5.

DEFINITION 5.1. Let m be a positive integer, let $H_{\mathbb{R}}$ be a real vector space of dimension h with a bilinear form S which is symmetric if m is even and skew if m is odd. Let $\{h^{r,s}\}_{r+s=m}$ be a set of non-negative integers, all but a finite number of which are zero, such that $\Sigma_{r+s=m} h^{r,s} = h$ and $h^{r,s} = h^{s,r}$ for all pairs of integers (r,s) with $r+s=m$. We denote by $\check{D}$ the closed subvariety of a partial flag variety consisting of filtrations

$$H_{\mathbb{C}} \supseteq \cdots \supseteq F^{p-1} \supseteq F^p \supseteq F^{p+1} \cdots \supseteq 0$$

such that

(1) $\dim F^p = \Sigma_{r \geqslant p} h^{r,m-r}$, and

(2) $S(F^p, F^{m-p+1}) = 0$.

The orthogonal group of S operates transitively on $\check{D}$, so that $\check{D}$ is smooth.

DEFINITION 5.2. Under the same assumptions as in Definition 5.1, we denote by D the open submanifold of $\check{D}$ consisting of filtrations

$$H_{\mathbb{C}} \supseteq \cdots \supseteq F^{p-1} \supseteq F^p \supseteq F^{p+1} \cdots \supseteq 0$$

which satisfy conditions (1) and (2), above, and for which $i^{2p-m}S(v, \bar{v}) \geqslant 0$ for any non-zero v in $F^p \cap \overline{F^{m-p}}$.

The points of D correspond bijectively to polarized real Hodge structures of weight m on $H_{\mathbb{R}}$ with $\dim H_{\mathbb{C}}^{r,s} = h^{r,s}$ and with polarization S. A point on D corresponding to a filtration

$$H_{\mathbb{C}} \supseteq \cdots \supseteq F^{p-1} \supseteq F^p \supseteq F^{p+1} \cdots \supseteq 0$$

defines a polarized real Hodge structure on $H_{\mathbb{R}}$ with $H_{\mathbb{C}}^{r,s} = F^r \cap \overline{F^{m-s}}$ for any pair of integers (r, s) with $r + s = m$.

DEFINITION 5.3. For any integer p, let $\mathcal{F}^p$ be the universal bundle on $\check{D}$ corresponding to the subspace F^p of H at a point in $\check{D}$ given by a filtration

$$H_{\mathbb{C}} \supseteq \cdots \supseteq F^{p-1} \supseteq F^p \supseteq F^{p+1} \cdots \supseteq 0.$$

The bundle $\mathcal{F}^p$ is a holomorphic subbundle of the trivial bundle $\check{D} \times H_{\mathbb{C}}$. We define the *horizontal tangent bundle* $T_h(\check{D})$ of $\check{D}$ to be the subbundle consisting at a point x of $\check{D}$ of those holomorphic tangent vectors X such that

$$\nabla_X \mathcal{F}^p \subseteq \mathcal{F}^{p-1}$$

for all integers p, where ∇ is the trivial flat connection on $\check{D} \times H_{\mathbb{C}}$. This is in fact a holomorphic subbundle of $T(\check{D})$, since the notion of a horizontal tangent vector is invariant with respect to the action of the orthogonal group of S on $\check{D} \times H_{\mathbb{C}}$.

Suppose that M is a complex manifold and V is a real variation of Hodge structure of weight m on M, with holomorphic subbundles $\mathcal{F}^p$ as in Section 3. If $\tilde{M} \xrightarrow{p} M$ is the universal cover for M, then p^*V is a real variation of Hodge structure of weight m on $\tilde{M}$. Since $\tilde{M}$ is simply connected, $p^*V_{\mathbb{R}}$ is canonically trivial on $\tilde{M}$. Let $H_{\mathbb{R}}$ by the space of global sections of $p^*V_{\mathbb{R}}$, let S be the bilinear form on $H_{\mathbb{R}}$ defined by that of p^*V, and let $h^{r,s} = \dim(V^{r,s})$ for all pairs of integers (r, s) with $r + s = m$. We let D be the classifying space for polarized real Hodge structures corresponding to this data, as in Definition 5.2. The holomorphic subbundles $p^*\mathcal{F}^p$ of the trivial bundle $p^*(V_{\mathbb{R}})$ then define a holomorphic map.

$$f\colon \tilde{M} \to D.$$

The transversality condition ((2) in Definition 3.1), that

$$D_V(\mathcal{F}^p) \subseteq \Omega^1_M(\mathcal{F}^{p-1})$$

for the flat connection D_V on V, means that the image of the map on tangent spaces

$$f_*\colon T(\tilde{M}) \to T(D)$$

is contained in $T_h(D)$. The map f is Griffiths' "period mapping" for the variation of Hodge structure V.

Finally, we need to record for future reference a result, which is Corollary 8.3 in [6].

PROPOSITION 5.1. *A holomorphic map $f\colon M \to D$ from a connected compact complex manifold M to a classifying space for polarized real Hodge structures D, as in Definition 5.2, such that the image of $T(M)$ under f_* is contained in $T_h(D)$ is constant.*

COROLLARY 5.2. *If V is a real variation of Hodge structure on a connected compact complex manifold M such that the local system $V_{\mathbb{R}}$ is trivial, then the period mapping $f\colon \tilde{M} \to D$, is constant.*

Proof. Since $V_{\mathbb{R}}$ is trivial on M, the period mapping factors through a horizontal holomorphic map from M to D, which by Proposition 5.1 must be constant. □

6. Proofs of our main results

We return in this section to the situation outlined in the introduction and apply the results of the previous sections to prove Theorem 6.1, which is the focus of this paper, and its corollaries.

Let X be a smooth projective algebraic variety over the complex numbers, let Y be a connected closed subscheme of X with a base point y_0, such that $\pi_1(Y, y_0) \twoheadrightarrow \pi_1(X, y_0)$, and let Z be a scheme of finite type over the complex numbers, all the connected components of which are smooth projective varieties, with a surjective morphism $\phi\colon Z \to Y$.

As in Section 3, for any positive integer n, we let $M_{B,n}(X)$ and $M_{B,n}(Z)$ be the Hausdorff topological spaces with points corresponding to n-dimensional semisimple complex local systems on X and Z, respectively, where n is a positive integer. Let $\{Z_i\}$ be the set of connected components of Z, so that

$$M_{B,n}(Z) = \prod M_{B,n}(Z_i).$$

THEOREM 6.1. *The fibers of the natural map*

$$\phi^*\colon\ M_{B,n}(X) \to M_{B,n}(Z)$$

are finite.

Proof. In the language of Corollary 3.5, we show first that if U is any complex variation of Hodge structure on Z of dimension n,

$$W_{U\mathbb{C}} = \phi^{*-1}(U_{\mathbb{C}})$$

is finite. By Corollary 3.5, every connected component of $W_{U\mathbb{C}}$ contains a point $V_{\mathbb{C}}$ corresponding to a complex variation of Hodge structure V on X. According to Proposition 2.6, applied to each connected component Z_i of Z and using the standard identification of the first cohomology of a representation of a fundamental group and the first sheaf cohomology of the associated local system, if the map

$$\phi^*\colon\ H^1(X, \mathrm{End}(V_{\mathbb{C}})) \to H^1(Z, \phi^*\mathrm{End}(V_{\mathbb{C}}))$$

is an injection, then the finiteness of the fiber of ϕ^* and $U_{\mathbb{C}}$ follows. But this map is an injection by Theorem 4.1.

According to Theorem 3.3, it follows from the above that the natural morphism of schemes:

$$\phi^*\colon\ M_{\mathrm{Dol},n}(X) \to M_{\mathrm{Dol},n}(Z)$$

has finite fibers over points corresponding to Higgs bundles which come from variations of Hodge structure. But since there are points in the closures of each $\mathbb{C}^*$-orbit corresponding on these moduli spaces corresponding to variations of Hodge structure, and the action of $\mathbb{C}^*$ is algebraic, the morphism ϕ^* is necessarily finite on every irreducible component of $M_{\mathrm{Dol},n}(X)$. Again, by Theorem 3.3 it follows that the map

$$\phi^*\colon\ M_{B,n}(X) \to M_{B,n}(Z)$$

has finite fibers. □

COROLLARY 6.2. *Let n be a positive integer, let X and Y be as above, let Z be a scheme of finite type over the complex numbers, all the connected components of which are complete normal varieties, and let $\phi\colon\ Z \to Y$ be a surjective morphism. If N is the normal subgroup of $\pi_1(X, y_0)$ generated by the images under ϕ of the fundamental groups of the connected components of Z, then $\pi_1(X, y_0)/N$ has only finitely many isomorphism classes of n-dimensional complex representations, all of which are semi-simple.*

Proof. By Corollary 2.5, this corollary would follow if we could show that

$$H^1(\pi_1(X, y_0)/N, \mathrm{Ad} \circ \rho) = 0$$

for any finite dimensional semi-simple complex representation ρ of $\pi_1(X, y_0)$. This in turn would follow if we could show that any complex local system V on X the map

$$H^1(X, V) \to \bigoplus_i H^1(Z_i, \phi^* V)$$

is injective. If Z is smooth and projective, then since a trivial local system on Z underlies a variation of Hodge structure, the corollary follows immediately from the proof of Theorem 6.1.

For a general Z, let $\{Z_i\}$ be the set of connected components of Z, and let $\rho_i : \overline{Z_i} \to Z_i$ be a projective resolution of singularities for Z_i, which exists according to [8]. If U_i is a Zarski-open subset of Z_i such that ρ_i restricted to $\rho_i^{-1}(U_i)$ is an isomorphism, then since both $\bar{Z}_i$ and Z_i are normal, the homomorphisms

$$\pi_1(\rho_i^{-1}(U_i), \overline{u_i}) \to \pi_1(\overline{Z_i}, \overline{u_i}), \quad \text{and} \quad \pi_1(U_i, u_i) \to \pi_1(Z_i, u_i)$$

are surjective for any choice of base points u_i of U_i and $\overline{u_i}$ of $\rho_i^{-1}(U_i)$ with $\rho_i(\overline{u_i}) = u_i$. Therefore the induced map

$$\rho_{i*} \colon \pi_1(\overline{Z_i}, \overline{u_i}) \to \pi_1(Z_i, u_i)$$

is surjective. Let

$$\bar{Z} = \coprod_i \overline{Z_i}$$

and let

$$\bar{\rho} \colon \bar{Z} \to Z$$

be the map given by ρ_i on $\overline{Z_i}$ for each i. Then the normal subgroup generated by the images of the fundamental groups of the connected components of $\bar{Z}$ under $\phi \circ \rho$ is N, and we can proceed as before to prove the corollary. □

We don't in fact need to assume that the connected components Z_i of Z are normal to conclude that the homomorphisms

$$\pi_1(U_i, u_i) \to \pi_1(Z_i, u_i)$$

are surjective for any Zariski open subset U_i of Z_i and any choice of base point u_i. We can assume simply that the universal cover of each Z_i is an irreducible analytic space. See [4].

There is a version of Corollary 6.2 using algebraic fundamental groups. Before giving this version, we recall some definitions and a general theorem of A. Grothendieck in [7].

DEFINITION 6.1. Let G be any group. Then we can topologize G by taking the set of all finite index subgroups of G as a fundamental system of open neighborhoods of the identity, and we define $\hat{G}$, the *profinite completion of* G, to be the completion of G with respect to this topology.

The topological group $\hat{G}$ is totally disconnected and compact, and there is a natural homomorphism from G to $\hat{G}$ with dense image. We denote by $\bar{G}$ the quotient of G by the kernel of this homomorphism. Clearly the natural homomorphism $q\colon G \to \bar{G}$ induces an isomorphism $\hat{q}\colon \hat{G} \to \hat{\bar{G}}$.

For any commutative ring A we denote by $\mathrm{Rep}_A(G)$ the category of finitely generated A-modules with a G-action.

THEOREM 6.3. ([7]). *Let A be commutative ring and let $u : G' \to G$ be a morphism of discrete groups.*

(1) *If $\hat{u}\colon \hat{G}' \to \hat{G}$ is surjective and G is finitely generated, then the functor*

$$u^*\colon \mathrm{Rep}_A(G) \to \mathrm{Rep}_A(G')$$

is fully faithful.

(2) *If $\hat{u}\colon \hat{G}' \to \hat{G}$ is an isomorphism, then the functor*

$$u^*\colon \mathrm{Rep}_A(G) \to \mathrm{Rep}_A(G')$$

is an equivalence of categories.

Theorem 6.3 asserts that for the group homomorphism $q\colon G \to \bar{G}$, the functor

$$q^*\colon \mathrm{Rep}_A(\bar{G}) \to \mathrm{Rep}_A(G)$$

is an equivalence of categories. In particular, for any finitely generated A-module M with a G-action, q^* induces a natural isomorphism

$$H^{\cdot}(\bar{G}, M) \to H^{\cdot}(G, M).$$

If $G = \pi_1(X, y_0)$ is the fundamental group of a connected scheme X over $\mathbb{C}$ of finite type, given its Hausdorff topology, then $\hat{G}$ is the algebraic fundamental group $\pi_1^{\mathrm{alg}}(X, x_0)$ of X with base point x_0.

COROLLARY 6.4. *Let X be a smooth projective complex algebraic variety, let Y be a subvariety of X such that for some choice of base point y_0 of Y the homomorphism*

$$\pi_1^{\mathrm{alg}}(Y, y_0) \to \pi_1^{\mathrm{alg}}(X, y_0)$$

is surjective, let Z be a scheme of finite type over $\mathbb{C}$, all the connected components of which are complete normal varieties, and let ϕ be a surjective morphism from Z to Y. Then the conclusions of Theorem 6.1 and Corollary 6.2 still hold.

Proof. If we denote $\pi_1(X, y_0)$ by G and $\pi_1(Y, y_0)$ by H, then the assumption that

$$\hat{i}_*: \hat{H} \to \hat{G}$$

is surjective implies that the homomorphism

$$\bar{i}_*: \bar{H} \to \bar{G}$$

is surjective. If V is an n-dimensional complex vector space with a G-action, then this action factors through $\bar{G}$ by Theorem 6.3, so that V also has $\bar{G}$-, $\bar{H}$-, and H-actions. For any integer k the following diagram is commutative:

$$\begin{tikzcd} H^k(\bar{G}, V) \arrow[r, "\bar{i}^*"'] \arrow[d, "q^*"'] & H^k(\bar{H}, V) \arrow[d, "q^*"'] \\ H^k(G, V) \arrow[r, "i^*"'] & H^k(H, V) \end{tikzcd}$$

The vertical arrows in this diagram are isomorphisms by Theorem 6.3.

In the proof of Theorem 6.1 and Corollary 6.2, the only point at which we use the assumption that the homomorphism i_* from H to G is surjective is when we show, in the proof of Theorem 4.1, that the homomorphism

$$i^*: H^1(G, W) \to H^1(H, W)$$

is injective for a finite dimensional complex G-representation W. But by the above diagram this follows from the surjectivity of the homomorphism $\bar{i}_*$ from $\bar{H}$ to $\bar{G}$. Therefore, under the assumptions of Corollary 6.4, we still have the conclusions of Theorem 6.1 and Corollary 6.2. □

COROLLARY 6.5. *With the same assumptions as in Corollary 6.4, all the finite dimensional complex representations of $\pi_1(X, y_0)/N$ are unitary.*

Proof. As in the proof of Corollary 6.2, we may assume that all the connected components of Z are smooth and projective. By Corollary 6.4, all finite dimensional complex representations of $\pi_1(X, y_0)/N$ are semi-simple. By Theorem 6.1, any semi-simple finite dimensional complex representation σ of $\pi_1(X, y_0)/N$ comes from a complex variation of Hodge structure V on X. If we can show that the representation of $\pi_1(X, y_0)$ coming from $V \oplus \bar{V}$ is unitary, then the representation σ corresponding to V is clearly unitary, so we may assume that σ comes from a real variation of Hodge structure V on X. Since $V \approx \oplus_l V_l$, where V_l is a real variation of Hodge structure of weight l, we may assume that V is of weight m for some integer m.

Denote by $\tilde{Z}, \tilde{Y}$, and $\tilde{X}$ the covering spaces of Z, Y, and X, respectively, corresponding to the images of the ordinary fundamental groups of the connected components of Z, of Y, and of X, respectively, in the corresponding algebraic fundamental groups. Then the maps $\phi: Z \to Y$ and $i: Y \to X$ induce maps $\tilde{\phi}: \tilde{Z} \to \tilde{Y}$ and $\tilde{i}: \tilde{Y} \to \tilde{X}$, and $\tilde{\phi}$ is surjective. The real variation of Hodge structure V gives a period map

$$f: \tilde{X} \to D$$

for an appropriate classifying space for polarized real Hodge structures D (see Section 5) such that the composite

$$f \circ \tilde{i} \circ \tilde{\phi}: \tilde{Z} \to D$$

is constant. This follows from Theorem 6.3, which requires that all finite dimensional representations of the fundamental group of a variety must factor through its image in the algebraic fundamental group of that variety. Since $\tilde{\phi}$ is surjective,

$$f \circ \tilde{i}: \tilde{Y} \to D$$

is also constant. This means that the subbundles $i^*V^{r,s}$ of i^*V are flat for all pairs of integers (r, s) such that $r + s = m$. Therefore the positive definite Hermitian form $\langle\, ,\, \rangle$ such that

$$\langle v, w \rangle = (-1)^r Q_V(v, w)$$

for any v and w in $V^{r,s}$ is also flat, and so the representation of $\pi_1(Y, y_0)$ corresponding to $i^*V_{\mathbb{C}}$ is unitary. This representation is the pullback by the surjective homomorphism

$$i_*: \pi_1(Y, y_0) \to \pi_1(X, y_0)$$

of σ, and it follows that σ is also unitary. This completes the proof of the corollary. □

The conclusion of Corollary 6.5 in fact implies the conclusion of Corollary 6.4. Any finitely generated group G such that all the finite dimensional complex representations of G are semi-simple must have only finitely many isomorphism classes of complex representations in any given dimension. See proposition 2.9 in [10].

7. An example

In the situation of the Corollaries 6.2 and 6.5 of Theorem 6.1, the group $\pi_1(X, y_0)/N$ has the following properties.

(1) It is finitely presented.
(2) All of its finite dimensional complex representations are unitary.

Given these properties, it is reasonable to hope that $\pi_1(X, y_0)/N$, or at least $\pi_1^{\mathrm{alg}}(X, y_0)/\hat{N}$ is finite, but it is not a consequence of these properties alone. We give in this section an example of a group G which has properties (1) and (2), but for which $\hat{G}$ is infinite.

Let K be a global field of characteristic p for some prime p; i.e., K is a finite extension of a rational function field $\mathbb{F}_p(T)$ over the field $\mathbb{F}_p$ with p elements. Let S be a finite set of primes of K. We denote by $K(S)$ the subring of elements x of K such that $|x|_v \leqslant 1$ for every v not in S. Then the following results give us a group with properties (1), (2) and (3), above.

THEOREM 7.1. (S. Splitthoff, [17]). *If S contains at least two primes and $n \geqslant 3$, then $\mathrm{SL}_n(K(S))$ is finitely presented.*

THEOREM 7.2. (A special case of Theorem 3.8(c) in Chapter 8 of [11]). *Any finite dimensional representation of $\mathrm{SL}_n(K(S))$ in characteristic zero has finite image if $n \geqslant 3$ and S is non-empty.*

For $G = \mathrm{SL}_n(K(S))$ with $n \geqslant 3$, Theorem 7.1 shows that G is finitely presented, while Theorem 7.2 shows that all finite dimensional complex representations of G have finite image, and thus are unitary. But it is easy to see that $\hat{G}$ is infinite. For example, for a given prime ideal $\wp$ in $K(S)$, we have a filtration

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

of G by the congruence subgroups

$$G_r = \{A \in \mathrm{SL}_n(K(S)) | A \equiv \text{identity} \mod \wp^r\}.$$

Since $K(S)/\wp^r$ is finite for any $r \geqslant 0$, G_r is a finite index subgroup of G for any r, so that

$$\ker(G \to \hat{G}) \subseteq \bigcap_r G_r.$$

But clearly

$$\bigcap_r G_r = 0,$$

so that the natural homomorphism from the infinite group G into $\hat{G}$ is injective.

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