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Pavel Etingof

Konstantin Styrkas

Algebraic integrability of Schrodinger operators and representations of Lie algebras

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PAVEL ETINGOF and KONSTANTIN STYRKAS
Department of Mathematics, Yale University, New Haven, CT 06520, USA

1. Introduction

A Schrödinger operator is a differential operator whose symbol is the Laplace's operator. A quantum integral of a Schrödinger operator is a differential operator that commutes with it.

A Schrödinger operator in m variables is called integrable if it has m algebraically independent quantum integrals in involution (i.e. commuting with each other). This notion is the quantum analogue of the notion of Liouville integrability of a classical Hamiltonian system.

A Schrödinger operator in m variables is called algebraically integrable if it is integrable but the algebra of its quantum integrals cannot be generated by m operators. In the one-variable case, algebraically integrable operators correspond to finite-gap potentials [Kr].

One of the most interesting examples of an integrable Schrödinger operator is the Calogero-Sutherland operator [C], [S], [OP]. This is the Hamiltonian of the quantum many-body problem with rational, trigonometric, or elliptic interaction potential. The Calogero-Sutherland operator depends on a parameter which is called the coupling constant.

It has been observed [CV1], [CV2], [VSC] that the Calogero-Sutherland operators become algebraically integrable when the coupling constant takes a discrete set of special values. This is proved for the rational and trigonometric case but still remains a conjecture in the elliptic case for two or more variables.

These results can be generalized to Calogero-Sutherland operators associated with root systems, which were defined in [OP].

In this paper we study integrability and algebraic integrability properties of certain matrix Schrödinger operators. More specifically, we associate such an operator (with rational, trigonometric, or elliptic coefficients) to every simple Lie algebra $\mathfrak{g}$ and every representation U of this algebra with a nonzero but finite dimensional zero weight subspace. (The Calogero-Sutherland operator is a special case of this construction). Such an operator is always integrable [E]. Our main result

is that it is also algebraically integrable in the rational and trigonometric case if the representation U is highest weight. This generalizes the corresponding result for Calogero-Sutherland operators ([CV1]). We also conjecture that this is true for the elliptic case as well, which is a generalization of the corresponding conjecture from [CV2].

The proof of the main result is based on the method of ψ-function – a joint eigenfunction of quantum integrals of the Schrödinger operator. This method was developed in [CV1]. The proof of existence and uniqueness of the ψ-function is based on an explicit construction of this function which uses representation theory of the Lie algebra $\mathfrak{g}$. To be more precise, the ψ-function is realized (up to a factor) as a weighted trace of an intertwining operator between a Verma module over $\mathfrak{g}$ and the tensor product of this module with U. Such realization goes back to [E], [EK1], where it is found that joint eigenfunctions of quantum integrals of a Calogero-Sutherland operator can be realized as traces. Using the theory of Shapovalov form for $\mathfrak{g}$ ([Sh], [KK]), we prove that the trace function satisfies the axioms for the ψ-function analogous to those formulated in [CV1], and is determined uniquely by them, and then establish algebraic integrability using the method of [CV1].

The paper is organized as follows. In Section 2 we make the necessary definitions, motivate them, and formulate the main result. In Section 3 we give information about Verma modules, Shapovalov form, and intertwining operators. In Section 4 we define the ψ-function as a normalized trace, and prove two properties of this function. In Section 5 we prove that these two properties uniquely determine the ψ-function. In Section 6 we prove algebraic integrability using the ψ-function. In the Appendix we describe how to get Weyl group invariant quantum integrals from central elements (Casimirs) of $U(\mathfrak{g})$.

2. Main definitions and results

Let V be a finite-dimensional complex vector space.

DEFINITION 2.1. A matrix differential operator is a differential operator whose coefficients are $\operatorname{End} V$-valued functions. A matrix Schrödinger operator is a differential operator of the form

$$L = \Delta - A(x), \ x \in \mathbb{C}^m, \tag{2-1}$$

where Δ is the Laplacian in $\mathbb{C}^m$, and A is a meromorphic function in $\mathbb{C}^m$ with values in $\operatorname{End} V$.

DEFINITION 2.2. A matrix Schrödinger operator L is called integrable if there exist pairwise commutative matrix differential operators $L_1 = L$, $L_2, \ldots, L_m$ such that the symbols of L_i have the form $p_i(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m})\mathrm{Id}$, where p_i are algebraically independent polynomials $(p_1(\mathbf{y}) = \mathbf{y}^2)$. The operators $L_1, \ldots, L_m$ are called the quantum integrals for L.

Let $R \subset \mathfrak{h}^* = \mathbb{C}^m$ be the root system of a simple Lie algebra $\mathfrak{g}$ of rank m, and let Δ^+ be the set of positive roots of R.

DEFINITION 2.3. The Calogero-Sutherland (CS) operator for R is the operator

$$L = \Delta - \sum_{\alpha \in \Delta^+} C_\alpha u(\langle x, \alpha \rangle) + K, \qquad (2\text{-}2)$$

where the scalar constant C_α may depend only on the length of the root α, and u is one of the following potential functions: (i) $u(x) = 2/x^2$ (rational potential), (ii) $u(x) = 2/\sinh^2 x$ (trigonometric potential), or (iii) $u(x) = 2\wp(x|\omega_1, \omega_2)$ (elliptic potential), where $\wp(x|\omega_1, \omega_2)$ is the Weierstrass elliptic function with periods ω_1, ω_2, and K is a constant (Cases (i) and (ii) and degenerations of case (iii)).

Such operators were introduced by Calogero [C] and Sutherland [S] for the root system A_m and by Olshanetsky and Perelomov [OP] in general.

THEOREM 2.1. *The operator L given by (2-2) is integrable. Furthermore, one can choose the integrals $\{L_i, i = 1, \dots, m\}$ in such a way that their symbols would generate the algebra of Weyl group invariant polynomials on $\mathfrak{h}$.*

For $R = A_m$ (and in some other special cases) this theorem was proved in [OP]. Cases (i) and (ii) for general root systems were settled by Heckman and Opdam [HO, H1, O1, O2]. Case (iii) for B_m, C_m and D_m is settled in [Osh]. The general proof for Case (iii) (and hence Cases (i) and (ii)) was given recently by I. Cherednik [Ch].

If $m = 1$ then any Schrödinger operator is integrable by the definition. In two or more variables integrability is a very rare property. This is illustrated by the following result.

THEOREM 2.2. [OOS, OS]. *Let $m \geqslant 2$. Let L be an integrable Schrödinger operator defined by (2-1) with $V = \mathbb{C}$. Assume that the quantum integrals L_i, $1 \leqslant i \leqslant m$, are invariant under the symmetric group S_{m+1} acting irreducibly in $\mathbb{C}^m$, and their symbols generate the ring of S_{m+1}-symmetric polynomials on $\mathbb{C}^m$ (the operator L_i is of order $i + 1$). Then L coincides with (2-2) for the root system A_m for some values of the parameters.*

Theorem 2.1 can be generalized to the matrix case, as follows.

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $R \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$, Δ^+ be the set of positive roots. $e_\alpha, f_\alpha \in \mathfrak{g}$ be the root elements corresponding to the positive root α. Let U be a representation of $\mathfrak{g}$ such that the space $V = U[0]$ of zero weight vectors in U is finite-dimensional. Define the matrix Schrödinger operator

$$H_{\mathfrak{g},U,u} = \Delta_\mathfrak{h} - \sum_{\alpha \in \Delta^+} u(\langle x, \alpha \rangle) f_\alpha e_\alpha, \quad x \in \mathfrak{h}, \qquad (2\text{-}3)$$

where u is of type (i), (ii), or (iii) from Definition 2.3. Such operators are considered in [E], [EK1].

THEOREM 2.3. *The operator (2-3) is integrable. The symbols of its quantum integrals are generators of the algebra of Weyl group invariant polynomials on $\mathfrak{h}$.*

This result is proved in [E] for the special case $\mathfrak{g} = \mathfrak{sl}_{m+1}$, but the method used for the proof works for any Lie algebra. This method uses representation theory of the Lie algebra $\mathfrak{g}$ in the trigonometric case, and representation theory of the affine Lie algebra $\hat{\mathfrak{g}}$ in the elliptic case. The quantum integrals of $H_{\mathfrak{g},U,u}$ are constructed from central elements of the universal enveloping algebra. We discuss this method in the Appendix.

As a particular case, Theorem 2.3 includes Theorem 2.1 for the root system A_m. Indeed, let us take $\mathfrak{g} = \mathfrak{sl}_{m+1}$ and a special representation of $\mathfrak{g}$: $U_\mu = (z_1 \ldots z_{m+1})^\mu \mathbb{C}[\frac{z_1}{z_2}, \ldots \frac{z_m}{z_{m+1}}, \frac{z_{m+1}}{z_1}]$, $\mu \in \mathbb{C}$, with the action of $\mathfrak{g}$ by linear transformations of variables (this representation has no highest weight). All weight subspaces in U_μ are one-dimensional; in particular, $V = U_\mu[0] = \mathbb{C}$. It is easy to compute that $f_\alpha e_\alpha|_{U[0]} = \mu(\mu + 1)$. Therefore, if μ is chosen in such a way that $C_\alpha = \mu(\mu + 1)$, then operator (2-3) transforms into (2-2) for the root system A_m.

Krichever [Kr] introduced the notion of an algebraically integrable Schrödinger operator (see also [CV1]). Here we generalize this definition to the matrix case.

Let L be an integrable matrix Schrödinger operator, let $L_1 = L, \ldots, L_m$ be its quantum integrals, and let the symbols of L_i be $p_i(\partial/\partial x_1, \ldots, \partial/\partial x_m)\mathrm{Id}$, where p_i are algebraically independent polynomials.

DEFINITION 2.4. L is called algebraically integrable if there exists a matrix differential operator L_0 commuting with $L_1, \ldots, L_m$ with symbol $p_0(\partial/\partial x_1, \ldots, \partial/\partial x_m)\mathrm{Id}$, such that for generic $E_1, \ldots, E_m \in \mathbb{C}$ the polynomial p_0 takes distinct values at the roots of the system of equations $p_i(y_1, \ldots, y_m) = E_i$, $1 \leqslant i \leqslant m$.

For $V = \mathbb{C}$ this definition coincides with the one in [Kr], [CV1]. In the matrix case and $m = 1$ the property of algebraic integrability of differential operators was studied in [G].

It turns out that a Calogero-Sutherland operator is algebraically integrable for a discrete spectrum of values of the constants C_α.

THEOREM 2.4. [VSC]. *If $C_\alpha = \frac{1}{2}\mu_\alpha(\mu_\alpha + 1)\langle\alpha, \alpha\rangle$ for all roots $\alpha \in \Delta^+$, where μ_α is an integer depending only on the length of α, then the operator (2-2) is algebraically integrable for the rational and trigonometric potential.*

CONJECTURE 2.5. [CV2]. *Theorem 2.4 is true for the elliptic potential.*

Conjecture 2.5 is proved only for the case of the root system A_1. In this case, operator (2-2) is the Lamé operator $L = \partial^2 - C\wp$, and algebraic integrability of this operator is equivalent to the finite gap property, which takes place for

$C = \mu(\mu + 1)$, $\mu \in \mathbb{Z}$ [Kr]. In this case, there is a quantum integral L_0 of order $2\mu + 1$.

Now let us consider the case of the root system A_m. Looking at the interpretation of the Calogero-Sutherland operator via the representation U_μ, we see why the integer values of μ should be special: they are exactly those values for which the representation U_μ has a finite dimensional submodule or quotient module which is isomorphic to a symmetric power of $\mathbb{C}^{m+1}$ (or $(C^{m+1})^*$). Since the zero weight vector is contained in this finite-dimensional module, we can use it instead of U_μ. Thus we observe that algebraic integrability occurs at those values of μ where U_μ can be replaced by a highest weight module. This motivates the following general theorem which is the main result of this paper.

THEOREM 2.6. *If U is a highest weight $\mathfrak{g}$-module then $H_{\mathfrak{g},U,u}$ is algebraically integrable for the rational and trigonometric potential.*

In Sections 3–6 we prove this theorem for the trigonometric case. The rational case can be obtained in the limit, so we don't discuss it.

Note that Theorem 2.4 for the root system A_m is a special case of Theorem 2.6.

Finally, we would like to formulate a natural conjecture concerning the elliptic case.

CONJECTURE 2.7. *Theorem 2.6 is true for the elliptic potential.*

This conjecture contains Conjecture 2.5 for the root system A_m. We believe that it could be proved by applying the methods of this paper to the elliptic case and using the techniques of representation theory of affine Lie algebras and theory of vertex operators introduced in [E], [EK1].

3. Verma modules, Shapovalov form, intertwining operators

Let $\mathfrak{g}$ be a simple complex Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Fix an element $\lambda \in \mathfrak{h}^*$. Denote by M_λ the Verma module over $\mathfrak{g}$ with highest weight λ, i.e. the module with one generator v_λ and relations

$$\mathfrak{n}_+ v_\lambda = 0, \qquad h v_\lambda = \langle \lambda, h \rangle v_\lambda \quad \text{for} \quad h \in \mathfrak{h}.$$

We have the decomposition $M_\lambda = \oplus_{\mu \leqslant \lambda} M_\lambda[\mu]$ of M_λ into the direct sum of finite dimensional weight subspaces $M_\lambda[\mu]$. Denote also $M_\lambda^* = \oplus_{\mu \leqslant \lambda} M_\lambda[\mu]^*$ the restricted dual module to M_λ with the action of $\mathfrak{g}$ defined by duality. For generic λ, M_λ^* is a lowest weight module with the lowest weight vector $v_{-\lambda}^*$ of weight $-\lambda$.

We have a vector space decomposition $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$. Define the Harish-Chandra homomorphism $\phi : U(\mathfrak{g})[0] \to U(\mathfrak{h})$ by $\phi|_{U(\mathfrak{h})} = \text{Id}$, and $\phi(g) = 0$ if $g \in U(\mathfrak{g})[0]$ can be represented as $g = g_1 e$ for some $g_1 \in U(\mathfrak{g})$, $e \in \mathfrak{n}_+$.

This in turn gives rise to a contravariant bilinear $U(\mathfrak{h})$-valued from $\mathcal{F}$ on $U(\mathfrak{n}_-)$ defined by

$$\mathcal{F}(g_1, g_2) = \phi(\omega(g_1)g_2),$$

when g_1, g_2 belong to the same weight subspace of $U(\mathfrak{n}_-)$, and $\mathcal{F}(g_1, g_2) = 0$ otherwise. Here ω is the Cartan antiautomorphism of $\mathfrak{g}$ defined by

$$\omega(e_\alpha) = f_\alpha, \quad \omega(f_\alpha) = e_\alpha, \quad \omega(h_\alpha) = h_\alpha$$

It is easy to see that this form is symmetric.

As $U(\mathfrak{h})$ can be identified with the space of all polynomials on $\mathfrak{h}^*$, we can introduce a symmetric contravariant $\mathbb{C}$-valued form F on M_λ defined by

$$F(g_1 v_\lambda, g_2 v_\lambda) = \mathcal{F}(g_1, g_2)(\lambda).$$

Let $Q^+ = \sum_{\alpha \in \Delta^+} \mathbb{Z}_+ e_\alpha$.

Let U be any $\mathfrak{g}$-module with highest weight $\mu \in Q^+$. The completed tensor product $M_\lambda \hat{\otimes} U = \mathrm{Hom}_\mathbb{C}(M_\lambda^*, U)$ has a natural $\mathfrak{g}$-module structure. We say that an element $v \otimes u$ has order η if $v \in M_\lambda[\lambda - \eta]$. Clearly, only elements of order $\eta \in Q^+$ may occur. We say that $\nu < \eta$ if $\nu \neq \eta$ and $\eta - \nu \in Q^+$.

Let $u \in U$. Let $\Phi_\lambda^u : M_\lambda \to M_\lambda \hat{\otimes} U$ be an intertwining operator such that $v_\lambda \mapsto v_\lambda \otimes u + \{higher\ order\ terms\}$.

It is clear from the intertwining property of Φ_λ^u that u has to be a zero weight vector.

PROPOSITION 3.1. *If M_λ is irreducible then Φ_λ^u exists and is unique for any $u \in U[0]$.*

Proof. Because M_λ is freely generated by v_λ over $U(\mathfrak{n}_-)$, we only need to prove that the module $M_\lambda \hat{\otimes} U$ contains a unique singular vector of the form $v_\lambda \otimes u + \{higher\ order\ terms\}$. This is the same as to construct a map $\Theta : M_\lambda^* \to U$ such that $\Theta(v_{-\lambda}^*) = u$ and Θ is a $\mathfrak{n}_+$-intertwiner. But M_λ^* is a free $U(\mathfrak{n}_+)$-module generated by $v_{-\lambda}^*$, so Θ can be uniquely extended from $v_{-\lambda}^*$ to the whole M_λ^*. $\square$

It is known that M_λ is irreducible for generic λ. For special λ's M_λ may be reducible, and it happens when the contravariant bilinear form on M_λ is degenerate. Shapovalov [Sh] obtained an explicit formula for the determinant of this form:

$$\det F_\mu(\lambda) = \mathrm{const} \prod_{\alpha \in \Delta^+} \prod_{n \in \mathbb{N}} \left(\langle \alpha, \lambda + \rho \rangle - \frac{n}{2} \langle \alpha, \alpha \rangle \right)^{K(\mu - n\alpha)}, \tag{3-1}$$

where $F_\mu = (F_\mu)_{i,j}$, $i, j = 1, 2, \ldots, \dim M_\lambda[\lambda - \mu]$ is the matrix of the restriction of the form to $M_\lambda[\lambda - \mu]$, $K(\mu)$ – the Kostant partition function, and the nonzero constant depends on the choice of basis in $M_\lambda[\mu]$.

Let

$$\chi_n^\alpha(\lambda) = \langle \alpha, \lambda + \rho \rangle - \frac{n}{2} \langle \alpha, \alpha \rangle. \tag{3-2}$$

The conditions for reducibility of M_λ can then be rewritten as

$$\chi_n^\alpha(\lambda) = 0 \text{ for some } \alpha \in \Delta^+, \ n \in \mathbb{N}$$

Now we fix weight μ and let $n_\mu^\alpha = \max\{n \in \mathbb{N} \mid U[-n\alpha] \neq 0\}$, where $U[\beta]$ is the subspace of vectors of weight β in U. Denote

$$\chi_\mu(\lambda) = \prod_{\alpha \in \Delta^+} \prod_{n=1}^{n_\mu^\alpha} \chi_n^\alpha(\lambda). \tag{3-3}$$

We need the following

LEMMA 3.2. *Matrix elements* $(F_\mu^{-1})_{i,j}$ *of the inverse matrix* F_μ^{-1} *can be written in the form*

$$(F_\mu^{-1})_{i,j} = \frac{P_{ij}^\mu(\lambda)}{\chi_\mu(\lambda)}$$

for some suitable polynomials $P_{ij}^\mu(\lambda)$.

Proof. Shapovalov formula implies that matrix elements are rational functions in λ with only possible poles in hyperplanes defined by $\chi_n^\alpha(\lambda) = 0$. Our goal is to show that only simple poles may occur.

Fix $\alpha \in \Delta^+$, $n \leqslant n_\mu^\alpha$. Take λ such that $\chi_m^\beta(\lambda) = 0$ iff $m = n$, $\beta = \alpha$. Then M_λ is reducible and contains a unique maximal submodule of M_λ^1, generated by a singular vector $v_{\lambda - n\alpha}$.

Fix $z \in \mathfrak{h}^*$ such that $\langle \alpha, z \rangle \neq 0$ for any $\alpha \in \Delta^+$, and let t be an independent variable. Using the $U(\mathfrak{h})$-valued bilinear form $\mathcal{F}$ we can introduce a new $\mathbb{C}[t]$-valued bilinear form F^t and M_λ defined by

$$F^t(g_1 v_\lambda, g_2 v_\lambda) = \mathcal{F}(g_1, g_2)(\lambda + tz), \quad g_1, g_2 \in U(\mathfrak{n}_-).$$

Clearly, specialization $t \mapsto 0$ gives the usual Shapovalov form.

Denote $N = \dim M_\lambda[\lambda - \mu] = K(\mu)$, $M = \dim M_\lambda^1[\lambda - \mu] = K(\mu - n\alpha)$. Choose a basis v_k in $M_\lambda[\lambda - \mu]$ so that $\{v_i\}$, $i = 1, 2, \ldots, M$, would form a basis for $M_\lambda^1[\lambda - \mu]$. Then the matrix elements $(F_\mu^t)_{i,j}$ will be divisible by t if $i \leqslant M$ or $j \leqslant M$:

$$
F_\mu^t =
\begin{pmatrix}
t \cdot f_{1,1} & \cdots & t \cdot f_{1,M} & t \cdot f_{1,M+1} & \cdots & t \cdot f_{1,N} \\
\vdots & & \vdots & \vdots & & \vdots \\
t \cdot f_{M,1} & \cdots & t \cdot f_{M,M} & t \cdot f_{M,M+1} & \cdots & t \cdot f_{M,N} \\
t \cdot f_{M+1,1} & \cdots & t \cdot f_{M+1,M} & f_{M+1,M+1} & \cdots & f_{M+1,N} \\
\vdots & & \vdots & \vdots & & \vdots \\
t \cdot f_{N,1} & \cdots & t \cdot f_{N,M} & f_{N,M+1} & \cdots & f_{N,N}
\end{pmatrix}.
$$

where $f_{i,j}$ are some polynomials in λ, t. It is clear now that the determinant of any $(N-1) \times (N-1)$ submatrix of F_μ^t is divisible by t^{M-1}. Shapovalov formula implies that $\det F_\mu^t$ is divisible by exactly Mth power of t, which means that when we compute the matrix elements of $(F_\mu^t)^{-1}$, only simple poles will be allowed when $t = 0$, or, equivalently, $(F_\mu^{-1})_{i,j}$, will have at most simple poles on the hyperplanes $\chi_n^\alpha(\lambda) = 0$. Repeating this argument for all μ, n, α we prove the lemma. $\square$

We can apply this result to get more information about the intertwining operator Φ_λ^u. In the proof of Proposition we defined $\Phi_\lambda^u v_\lambda$ as a map $\Psi \colon M_\lambda^* \to U$. We would like to obtain a more explicit formula for $\Phi_\lambda^u v_\lambda$ as an element of $M_\lambda \hat\otimes U$.

For any basis $v_k = g_k^\mu v_\lambda$, $g_k^\mu \in U(\mathfrak{n}_-)$ of $M_\lambda[\lambda - \mu]$ we have the basis of $M_\lambda^*[-\lambda + \mu]$ given by $v_k^* = (\omega g_k^\mu) v_{-\lambda}^*$, where ω is the Cartan involution. It is clear that $\langle v_i^*, v_k \rangle = F(v_i, v_k)$.

Introduce another basis w_k which is dual to v_k^* in the usual sense, i.e. $\langle v_i^*, w_j \rangle = \delta_{ij}$. These two bases v_k and w_k are related via the F_μ matrix:

$$
v_k = \sum_i (F_\mu)_{ki} w_i, \quad \text{or}
$$

$$
w_k = \sum_i (F_\mu^{-1})_{ki} v_i.
$$

It is clear that in this notation

$$
\Phi_\lambda^u v_\lambda = v_\lambda \otimes u + \cdots + \sum_k w_k \otimes (\omega g_k^\mu) u + \cdots
$$

$$
= v_\lambda \otimes u + \cdots + \sum_{k,l} (F_\mu^{-1})_{kl} \cdot g_k^\mu v_\lambda \otimes (\omega g_l^\mu) u + \cdots
$$

COROLLARY 3.3. *Suppose we have an intertwiner* $\Phi_\lambda^u \colon M_\lambda \to M_\lambda \hat\otimes U$, *where* U *is a highest weight $\mathfrak{g}$-module with the highest weight* $\theta \in Q^+$.

(1) *There are no order μ terms in the expression for* $\Phi_\lambda^u v_\lambda$, *unless* $\mu \leqslant \theta$.

(2) *If* $\mu \leqslant \theta$ *then the order μ part of* $\Phi_\lambda^u v_\lambda$ *can be written as*

$$\sum_{k,l} (F_\mu^{-1})_{kl} \cdot g_k^\mu v_\lambda \otimes (\omega g_l^\mu)u, \tag{3-4}$$

where

$$(F_\mu^{-1})_{kl} = \frac{P_{kl}^\mu(\lambda)}{\chi_\mu(\lambda)}$$

for some polynomials $P_{kl}^\mu(\lambda)$.

(3) *A sufficient condition of existence of Φ_λ^u can be written as $\chi_\theta(\lambda) \neq 0$.*

(4) *For any basis g_k^μ of $U(\mathfrak{n}_-)$ we can choose polynomials $S_{kl}^\mu(\lambda)$ so that*

$$\Phi_\lambda^u v_\lambda = \sum_{\mu \in Q^+} \sum_{k,l} \frac{S_{kl}^\mu(\lambda)}{\chi_\theta(\lambda)} g_k^\mu v_\lambda \otimes (\omega g_l^\mu)u. \tag{3-5}$$

For a rational function R, represented as a ratio of two polynomials $R = \frac{P}{Q}$ we set $\deg R = \deg P - \deg Q$.

Note that all coefficients $(F_\mu^{-1})_{kl}$ are of negative degree in λ. Later we will work with λ in the hyperplanes $\langle \alpha, \lambda \rangle = \text{const.}$, so we introduce notation

$$\lambda_\alpha = \frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad \lambda^\perp = \lambda - \lambda_\alpha, \tag{3-6}$$

so that $\lambda^\perp$ is a $(\dim \mathfrak{h} - 1)$-dimensional vector and $\langle \alpha, \lambda^\perp \rangle = 0$.

We will use the following

PROPOSITION 3.4. *When restricted to the hyperplane $\langle \alpha, \lambda \rangle = C$, matrix elements $(F_\mu^{-1})_{kl}$ are rational functions in $\lambda^\perp$ of nonpositive degree, and only constants may occur as terms of degree 0.*

Proof. We choose a special basis in $U(\mathfrak{n}_-)[\mu]$. For any sequence ω of positive roots $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_r$, where $\geqslant$ denotes now the lexicographical order, such that $\Sigma \beta_i = \mu$, set $X_\omega = f_{\beta_1} \ldots f_{\beta_r}$. Set $\deg X_\omega = \text{Card}(\{k \mid \beta_k \neq \alpha\})$.

The set of X_ω's is a basis in $U(\mathfrak{n}_-)[\mu]$. We also have

$$\deg \mathcal{F}(X_{\omega_1}, X_{\omega_2})(\lambda) \leqslant \frac{\deg X_{\omega_1} + \deg X_{\omega_2}}{2}. \tag{3-7}$$

Indeed, we can only raise the degree by commuting some e_β and f_β for $\beta \neq \alpha$, which results in the term $\langle \beta, \lambda \rangle + \text{const.}$, which is linear in $\lambda^\perp$. Note also that commuting with e_α or f_α will not increase the total number of terms e_β and f_β for all $\beta \neq \alpha$. Therefore, the maximal degree cannot be greater than half the original number of terms e_β and f_β, $\beta \neq \alpha$. This proves formula (3-7).

The determinant of the form in the hyperplane $\langle \lambda, \alpha \rangle = C$ is equal to

$$\det F_\mu(\lambda) = \text{const.} \prod_{\beta \neq \alpha} \prod_{n \in \mathbb{N}} \left(\langle \beta, \lambda + \rho \rangle - \frac{n}{2} \langle \beta, \beta \rangle \right)^{K(\mu - n\beta)},$$

where the constant depends on C. Then

$$N = \sum_{\beta \neq \alpha} \sum_{n \in \mathbb{N}} K(\mu - n\beta)$$

is the degree of the determinant as polynomial in $\lambda^{\perp}$. By the same argument as in [Sh], from (3-7) it follows that the $\lambda^{\perp}$-degree of any minor of the Shapovalov matrix cannot exceed N. Moreover, commuting with e_α or f_α does not change the set of $\beta \bmod \alpha$, and therefore any term of degree exactly N has highest term proportional to that of the determinant, which proves the Proposition.   □

4. Matrix Trace, ψ-function and its properties

Fix a highest weight $\mathfrak{g}$-module U with highest weight θ and finite dimensional zero weight space $U[0]$. Consider a new operator

$$\tilde{\Phi}_\lambda^u = \chi_\theta(\lambda)\Phi_\lambda^u. \tag{4-1}$$

From Corollary 3.3 it follows that

$$\tilde{\Phi}_\lambda^u v_\lambda = \sum_\mu \sum_{k,l} S_{kl}^\mu(\lambda) g_k^\mu v_\lambda \otimes g_l^\mu u.$$

This expression allows us to define $\tilde{\Phi}_\lambda^u$ even for λ where Φ_λ^u itself was not defined. It is clear that $\tilde{\Phi}_\lambda^u$ is an intertwining operator for all λ, and it has the property

$$\tilde{\Phi}_\lambda^u v_\lambda = v_\lambda \otimes \chi_\theta(\lambda)u + \{higher\ order\ terms\}. \tag{4-2}$$

Let $\Psi(\lambda, x)$ be a $\mathrm{End}(U[0])$-valued function on $\mathfrak{h}^* \times \mathfrak{h}$ defined by

$$\Psi(\lambda, x)u = \frac{\mathrm{Tr}|_{M_\lambda}(\tilde{\Phi}_\lambda^u e^x)}{\mathrm{Tr}|_{M_{-\rho}}(e^x)}. \tag{4-3}$$

PROPOSITION 4.1. *The Ψ-function defined above has the following properties:*

(1)

$$\Psi(\lambda, x) = e^{\langle \lambda+\rho, x \rangle} \tilde{P}(\lambda, x), \tag{4-4}$$

where $\tilde{P}(\lambda, x)$ is a $\mathrm{End}(U[0])$-valued polynomial in λ with the highest term

$$\prod_{\alpha \in \Delta^+} \langle \alpha, \lambda \rangle^{n_\alpha} \cdot \mathrm{Id},$$

where we put for brevity $n_\alpha = n_\theta^\alpha$.

(2) *If* $\langle \alpha, \lambda + \rho \rangle - \frac{n}{2} \langle \alpha, \alpha \rangle = 0$ *for some* $\alpha \in \Delta^+$, $n = 1, 2, \ldots, n_\alpha$ *then*

$$\Psi(\lambda, x) = \Psi(\lambda - n\alpha) \tilde{B}_n^\alpha(\lambda), \tag{4-5}$$

for some $\tilde{B}_n^\alpha(\lambda) \in \mathrm{End}(U[0])$, *which is rational in* λ *of nonpositive degree, and only constant operators may appear in it as degree zero terms. (That is, the highest term of the numerator coincides with the highest term of the denominator up to a constant factor).*

Proof. The first part is clear from the formula

$$\mathrm{Tr}|_{M_\lambda}(\tilde{\Phi}_\lambda^u e^x) = \sum_\mu e^{\langle \mu, x \rangle} \mathrm{Tr}|_{M_\lambda[\mu]}(\tilde{\Phi}_\lambda^u)$$

$$= e^{\langle \lambda + \rho, x \rangle} \sum_{\beta \in Q^+} e^{\langle -\rho - \beta, x \rangle} \mathrm{Tr}|_{M_\lambda[\lambda - \beta]}(\tilde{\Phi}_\lambda^u)$$

and the fact that all the $\mathrm{Tr}|_{M_\lambda[\lambda - \beta]}(\tilde{\Phi}_\lambda^u)$ are some combinations of S_{kl}^μ's, and therefore polynomials in λ. Their highest terms are obviously all equal to $\prod_{\alpha \in \Delta^+} \langle \alpha, \lambda \rangle^{n_\alpha}$, so the highest term of $P(\lambda, x)$ is equal to

$$\frac{\Sigma_{\beta \in Q^+} e^{\langle -\rho - \beta, x \rangle} K(\beta)}{\mathrm{Tr}|_{M_{-\rho}}(e^x)} \prod_{\alpha \in \Delta^+} \langle \alpha, \lambda \rangle^{n_\alpha} \cdot \mathrm{Id} = \prod_{\alpha \in \Delta^+} \langle \alpha, \lambda \rangle^{n_\alpha} \cdot \mathrm{Id}.$$

We now prove the second property of the Ψ-function. Let $\langle \alpha, \lambda + \rho \rangle - \frac{n}{2} \langle \alpha, \alpha \rangle = 0$ for some $\alpha \in \Delta^+$, $1 \leqslant n \leqslant n_\alpha$, but $\langle \beta, \lambda + \rho \rangle - \frac{m}{2} \langle \beta, \beta \rangle \neq 0$ unless $\beta = \alpha$, $m = n$.

From Corollary 3.3 it follows that $\tilde{\Phi}_\lambda^u v_\lambda$ has no order ν terms unless $\nu \geqslant n\alpha$. In particular, there are no order zero terms. On the other hand, $\tilde{\Phi}_\lambda^u v_\lambda$ has to be a singular vector. Therefore we must have

$$\tilde{\Phi}_\lambda^u v_\lambda = v_{\lambda - n\alpha} \otimes \tilde{u} + \{higher\ order\ terms\}, \tag{4-6}$$

where $v_{\lambda - n\alpha}$ is the unique singular vector generating the submodule $M_\lambda^1 \cong M_{\lambda - n\alpha}$. This implies that $\tilde{\Phi}_\lambda^u$ is a triangular operator: $\tilde{\Phi}_\lambda^u M_\lambda \subset M_\lambda^1 \otimes U$, so

$$\mathrm{Tr}|_{M_\lambda}(\tilde{\Phi}_\lambda^u e^x) = \mathrm{Tr}|_{M_\lambda^1}(\tilde{\Phi}_\lambda^u e^x).$$

Let $\tilde{\Phi}_\lambda^u v_{\lambda - n\alpha} = v_{\lambda - n\alpha} \otimes w + \{higher\ order\ terms\}$. Using the fact that $M_\lambda^1 \cong M_{\lambda - n\alpha}$ we see that

$$\mathrm{Tr}|_{M_\lambda}(\tilde{\Phi}_\lambda^u e^x) = \mathrm{Tr}|_{M_{\lambda - n\alpha}}(\Phi_{\lambda - n\alpha}^w e^x) = \frac{\mathrm{Tr}|_{M_{\lambda - n\alpha}}(\tilde{\Phi}_{\lambda - n\alpha}^w e^x)}{\chi_\theta(\lambda - n\alpha)}. \tag{4-7}$$

Set $\mu = n\alpha$.

It is an easy calculation to show that

$$w = \sum_{k,l} S^{\mu}_{kl}(\lambda) g^{\mu}_{k} \omega(g^{\mu}_{l}) u = \sum_{k,l} (\chi_{\theta}(\lambda)(F^{-1}_{\mu})_{kl}) \cdot g^{\mu}_{k} \omega(g^{\mu}_{l}) u.$$

Introduce a linear operator

$$\tilde{B}^{\alpha}_{n}(\lambda) = \sum_{k,l} \frac{S^{\mu}_{kl}(\lambda)}{\chi_{\theta}(\lambda - n\alpha)} \cdot g^{\mu}_{k} \omega(g^{\mu}_{l}) \in \mathrm{End}(U[0]). \qquad (4\text{-}8)$$

We can rewrite (4-7) as

$$\Psi(\lambda, x) = \Psi(\lambda - n\alpha, x) \tilde{B}^{\alpha}_{n}(\lambda),$$

To complete the proof we only need to show that $\tilde{B}^{\alpha}_{n}(\lambda)$ satisfies the required condition. It is clear that $\tilde{B}^{\alpha}_{n}(\lambda)$ is rational in λ and is not singular in the hyperplane $\langle \alpha, \lambda + \rho \rangle - \frac{n}{2} \langle \alpha, \alpha \rangle = 0$. As we can rewrite

$$\tilde{B}^{\alpha}_{n}(\lambda) = \sum_{k,l} \frac{\chi_{\theta}(\lambda)(F^{-1}_{\mu})_{kl}(\lambda)}{\chi_{\theta}(\lambda - n\alpha)} \cdot g^{\mu}_{k} \omega(g^{\mu}_{l}) \in \mathrm{End}(U[0]),$$

the rest follows from the Proposition 3.4.   □

Now we can introduce our main object of study. Set $\kappa = \Sigma_{\alpha \in \Delta +} n_{\alpha}$. Put

$$\psi(\lambda, x) = 2^{\kappa} \Psi\left(\frac{\lambda}{2} - \rho, 2x\right). \qquad (4\text{-}9)$$

The properties of Ψ-function can now be rewritten in the following form:

COROLLARY 4.2.

(1) ψ-function can be represented as

$$\psi(\lambda, x) = e^{\langle \lambda, x \rangle} P(\lambda, x), \qquad (4\text{-}10)$$

where $P(\lambda, x)$ is a polynomial in λ of the form

$$P(\lambda, x) = \prod_{\alpha \in \Delta^{+}} \langle \alpha, \lambda \rangle^{n_{\alpha}} + \{lower\ degree\ terms\}.$$

(2) If $\langle \alpha, \lambda \rangle = 0$ for some $\alpha \in \Delta^{+}$, then for $s = 1, 2, \ldots, n_{\alpha}$

$$\psi(\lambda + s\alpha, x) = \psi(\lambda - s\alpha, x) \cdot B^{\alpha}_{s}(\lambda), \qquad (4\text{-}11)$$

where $B_s^\alpha(\lambda) = \tilde{B}_s^\alpha(\frac{\lambda}{2} - \rho)$ is a rational $\mathrm{End}(U[0])$-valued function of λ and can be represented as

$$B_s^\alpha(\lambda) = b_s^\alpha + \{\text{lower degree terms}\}$$

for some constant $b_s^\alpha \in \mathrm{End}(U[0])$.

REMARK. Consider the case $\mathfrak{g} = sl(n)$, $U = S^{kn}\mathbb{C}^n$. In this case $n_\alpha = k$ for all α, and it is easy to show that $B_s^\alpha(\lambda)$ is not identically zero for any $\alpha, s \leqslant k$. Indeed, if α is a simple root then it is easy to get from (4-8) that $B_s^\alpha(\lambda) = 1$. If $\alpha = \sum_{l=i}^j \alpha_l$ $(i < j)$ then choose λ in such a way that $\langle \alpha_l, \lambda - 2s \sum_{m=i}^{l-1} \alpha_m \rangle = 2s, l = i, \ldots, j$. Then (4-11) implies: $\psi(\lambda, x) = \psi(\lambda - 2s\alpha, x)$, i.e. $B_s^\alpha(\lambda) = 1$ for this particular λ. This means, B_s^α is not identically 0. In fact, a more careful analysis shows that it is identically 1, i.e. can be removed from (4-11). This result agrees with [CV1].

The fact that B_λ^α is not identically zero implies that in this case the polynomial $\chi_\theta(\lambda)$ is exactly the common denominator of the components of the operator Φ_λ^u. Indeed, if some function $f = \langle \alpha, \lambda + \rho \rangle - s, 1 \leqslant s \leqslant k$, does not occur in such common denominator, then by (4-5) we would have $\tilde{B}_s^\alpha = 0$ on the hyperplane $f = 0$, which is impossible.

5. Uniqueness of the ψ-function

In this section we prove the uniqueness property of the function $\psi(\lambda, x)$, satisfying (4-10) and (4-11).

PROPOSITION 5.1. *Suppose we have an $\mathrm{End}(U[0])$-valued function*

$$\phi(\lambda, x) = e^{\langle \lambda, x \rangle} Q(\lambda, x),$$

where $Q(\lambda, x)$ is a polynomial in λ, satisfying (4-11). Then the highest term of $Q(\lambda, x)$ is divisible by

$$\prod_{\alpha \in \Delta^+} \langle \alpha, \lambda \rangle^{n_\alpha}.$$

Proof. Consider the highest term of $Q(\lambda, x)$. We need to show that it is divisible by $\langle \alpha, \lambda \rangle^{n_\alpha}$ for any $\alpha \in \Delta^+$.

Fix an $\alpha \in \Delta^+$. We can uniquely represent $Q(\lambda, x)$ as

$$Q(\lambda, x) = \sum_{l=0}^L \sum_{k=0}^{K_l} \lambda_\alpha^k Q_{kl}(\lambda^\perp, x), \tag{5-1}$$

where $Q_{kl}(\lambda^\perp, x)$ are homogeneous $\mathrm{End}(U[0])$-valued polynomials in $\lambda^\perp$ of degree l.

The highest term of $Q(\lambda, x)$ will be some combination of the terms of $\lambda_\alpha^{K_l} Q_{K_l l}(\lambda, x)$. We claim that it is enough to show that $K_L \geqslant n_\alpha$. Indeed, it will follow then that the highest term will have degree at least $L + n_\alpha$, and therefore all terms of the form $\lambda_\alpha^{K_l} Q_{K_l l}(\lambda, x)$ contributing to the highest term must have $K_l \geqslant L + n_\alpha - l \geqslant n_\alpha$, which proves the statement.

By our assumption $\psi(\lambda, x)$ satisfies (4-11), so we can write

$$e^{s\langle \alpha, x \rangle} \left(\sum_{l=0}^{L} \sum_{k=0}^{K_l} s^k Q_{kl}(\lambda^\perp, x) \right) =$$

$$e^{-s\langle \alpha, x \rangle} \left(\sum_{l=0}^{L} \sum_{k=0}^{K_l} (-s)^k Q_{kl}(\lambda^\perp, x) \right) (b_s^\alpha + \{lower\ degree\ terms\}), \quad (5\text{-}2)$$

where {*lower degree terms*} are understood with respect to $\lambda^\perp$.

We can consider homogeneous parts of (5-2) of degree L in $\lambda^\perp$. Formally, given a function $f(\lambda^\perp)$, we consider

$$\lim_{t \to \infty} \frac{f(t\lambda^\perp)}{t^L}.$$

This gives us

$$e^{s\langle \alpha, x \rangle} \left(\sum_{k=0}^{K_L} s^k Q_{kL}(\lambda^\perp, x) \right) = e^{-s\langle \alpha, x \rangle} \left(\sum_{k=0}^{K_L} (-s)^k Q_{kL}(\lambda^\perp, x) \right) b_s^\alpha \quad (5\text{-}3)$$

for $s = 1, \ldots, K_L$.

The rest is based on the following.

LEMMA 5.2. *Consider a homogeneous system of N linear equations on K vector variables $A_k(z) \in \mathbb{C}^M$, which are meromorphic in some additional parameter z:*

$$\sum_{k=1}^{K} s^k (e^{sz} + C_s) A_k = 0, \quad (5\text{-}4)$$

$s = 1, \ldots, N$, $C_s \in \mathrm{Mat}_M(\mathbb{C})$.

If $K \leqslant N$ then this system has only trivial solution $A_k(z) = 0$.

Proof of lemma. We can think of this system as a system of linear equations on KM variables $(A_k)_m$ and rewrite (5-4) in the block-matrix form

$$\begin{pmatrix} (e^z \cdot \mathrm{Id} + C_1) & (e^z \cdot \mathrm{Id} + C_1) & \cdots & (e^z \cdot \mathrm{Id} + C_1) \\ 2(e^{2z} \cdot \mathrm{Id} + C_2) & 2^2(e^{2z} \cdot \mathrm{Id} + C_2) & \cdots & 2^K(e^{2z} \cdot \mathrm{Id} + C_2) \\ \vdots & \vdots & & \vdots \\ N(e^{Nz} \cdot \mathrm{Id} + C_N) & N^2(e^{Nz} \cdot \mathrm{Id} + C_N) & \cdots & N^K(e^{Nz} \cdot \mathrm{Id} + C_N) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_K \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Suppose $K \leqslant N$. Then the determinant of the submatrix, consisting of first K blocks (or, equivalently, first KM equations) is an entire $\mathrm{Mat}_M(\mathbb{C})$-valued function of z with the asymptotics as $z \to +\infty$

$$\det \begin{pmatrix} (e^z \cdot \mathrm{Id}) & (e^z \cdot \mathrm{Id}) & \cdots & (e^z \cdot \mathrm{Id}) \\ 2(e^{2z} \cdot \mathrm{Id}) & 2^2(e^{2z} \cdot \mathrm{Id}) & \cdots & 2^K(e^{2z} \cdot \mathrm{Id}) \\ \vdots & \vdots & & \vdots \\ K(e^{Kz} \cdot \mathrm{Id}) & K^2(e^{Kz} \cdot \mathrm{Id}) & \cdots & K^K(e^{Kz} \cdot \mathrm{Id}) \end{pmatrix},$$

which is equal to

$$(D \cdot e^{\frac{K(K+1)}{2}z})^M,$$

where the nonzero constant

$$D = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^K \\ \vdots & \vdots & \ddots & \vdots \\ K & K^2 & \cdots & K^K \end{pmatrix}$$

is the Vandermonde determinant.

Therefore, this determinant is a nonzero entire function, which implies that for generic z it is not zero, so the system has only trivial solution. The meromorphic functions $A_k(z)$ are equal to zero for generic z, and therefore must be identically equal to zero.

The lemma is proved. $\qquad\qquad\square$

We now apply Lemma 5.2 to the system given by (5-3), for $N = n_\alpha$, $K = K_L + 1$, $M = \dim U[0]$ and setting $C_s = (-1)^k(b_s^\alpha)^t$, $z = x_\alpha = 2\langle \alpha, x \rangle$, where A^t is the transposed matrix A.

Consider the rows of the matrix, corresponding to $\psi(\lambda^{\perp}, x)$, and transpose them so that they become columns. By (5-3) all these columns satisfy the system of equations (5-4), and as ψ-function is not identically equal to zero, it implies that the system (5-4) has a nontrivial solution. By Lemma 5.2, we have $K_L + 1 > n_\alpha$, or, equivalently, $K_L \geqslant n_\alpha$.

The proposition is proved.   □

COROLLARY 5.3. *Any* End$(U[0])$-*valued function*

$$\phi(\lambda, x) = e^{\langle \lambda, x \rangle} Q(\lambda, x),$$

satisfying (4-11), can be represented as

$$\phi(\lambda, x) = q(\lambda)\psi(\lambda, x)$$

for some End$(U[0])$-*valued polynomial* $q(\lambda)$.

Proof. The proof is similar to the proof of the Lemma in Section 1 of [CV1]. We use induction on the degree of $Q(\lambda, x)$.

If $\deg Q(\lambda, x) < \deg P(\lambda, x)$, where

$$\psi(\lambda, x) = e^{\langle \lambda, x \rangle} P(\lambda, x),$$

then by proposition we have $Q(\lambda, x) \equiv 0$, so we can take $q(\lambda) \equiv 0$.

Suppose we have proved the statement for all polynomials of degree less than the degree of $Q(\lambda, x)$. By Proposition we can find a End$(U[0])$-valued polynomial $q_1(\lambda)$ such that

$$\text{highest term of } Q(\lambda) = q_1(\lambda) \prod_\alpha \langle \alpha, \lambda \rangle^{n_\alpha}.$$

Consider the function

$$\phi_1(\lambda, x) = \phi(\lambda, x) - q_1(\lambda)\psi(\lambda, x).$$

Obviously, it satisfies (4-11). Moreover, it can be represented as

$$\phi_1(\lambda, x) = e^{\langle \lambda, x \rangle} \tilde{Q}(\lambda, x),$$

where polynomial $\tilde{Q}(\lambda, x)$ has degree smaller than that of $Q(\lambda, x)$.

By induction hypothesis we can introduce a End$(U[0])$-valued polynomial $q_2(\lambda)$ such that

$$\phi_1(\lambda, x) = q_2(\lambda)\psi(\lambda, x).$$

The polynomial $q(\lambda) = q_1(\lambda) + q_2(\lambda)$ satisfies the required property.   □

COROLLARY 5.4. *The function $\psi(\lambda, x)$, satisfying both (4-10) and (4-11), exists and is unique.*

   Proof. It is a direct consequence of (4-10) and Corollary 5.3.   □

6. Existence of differential operators

The properties of the ψ-function obtained in Chapters 4, 5 are very close to the axioms in [CV1]. The function satisfying these axioms was used to construct a ring of differential operators that contained dim $\mathfrak{h}$ algebraically independent operators, corresponding to the generators of the ring of W-invariant polynomials, but was bigger than the ring generated by those operators.

   Here we apply these ideas to construct a similar ring of matrix differential operators and thus prove Theorem 2.6.

THEOREM 6.1. *For any* $\mathrm{End}(U[0])$-*valued polynomial* $Q(\lambda)$ *satisfying the property*

$$Q(\lambda + s\alpha) = Q(\lambda - s\alpha), \quad s = 1, 2, \ldots, n_\alpha \tag{6-1}$$

whenever $\langle \alpha, \lambda \rangle = 0$, *there exists a differential operator* D_Q *with coefficients in* $\mathrm{End}(U[0])$, *such that*

$$D_Q \psi(\lambda, x) = Q(\lambda)\psi(\lambda, x). \tag{6-2}$$

The correspondence $Q(\lambda) \mapsto D_Q$ *is a homomorphism of rings. In particular, all* D_Q *commute with each other.*

   Proof. We use induction on the degree of $Q(\lambda)$. If $\deg Q(\lambda) = 0$, then $Q(\lambda) =$ const., so the operator D_Q will be just the operator of multiplication by this constant.

   Suppose we have proved the theorem for all polynomials of degree less than that of $Q(\lambda)$. Let the highest term of $Q(\lambda)$ be equal to

$$\text{highest term of } (Q(\lambda)) = \sum_{(\mathbf{n})} a_{(\mathbf{n})} \lambda^{(\mathbf{n})},$$

where $(\mathbf{n})$ is a multiindex, $a_{(\mathbf{n})} \in \mathrm{End}(U[0])$. Consider the operator $\tilde{D}_Q$ defined by

$$\tilde{D}_Q = \sum_{(\mathbf{n})} a_{(\mathbf{n})} \frac{\partial^{|\mathbf{n}|}}{\partial x^{(\mathbf{n})}}.$$

It has the property that

$$\tilde{D}_Q \psi(\lambda, x) = Q(\lambda)\psi(\lambda, x) + \{\textit{lower degree terms}\},$$

and it also satisfies (6-1). Consider the difference

$$\phi(\lambda, x) = \tilde{D}_Q \psi(\lambda, x) - Q(\lambda)\psi(\lambda, x).$$

It satisfies (4-11) and therefore can be represented as

$$\phi(\lambda, x) = \tilde{Q}(\lambda)\psi(\lambda, x)$$

for some $\mathrm{End}(U[0])$-valued polynomial $\tilde{Q}(\lambda)$ such that $\deg \tilde{Q}(\lambda) < \deg Q(\lambda)$. By induction hypothesis we can introduce an operator $D_{\tilde{Q}}$ such that

$$\phi(\lambda, x) = D_{\tilde{Q}} \psi(\lambda, x),$$

and the operator

$$D_Q = \tilde{D}_Q + D_{\tilde{Q}}$$

has the required property, which completes the proof of the induction step.

The assertion that the constructed correspondence is a homomorphism of rings follows from the fact that the operator $D_{Q_1 Q_2} - D_{Q_1} D_{Q_2}$ annihilates the ψ-function for any λ, and therefore has to be identically zero. □

Among the polynomials $Q(\lambda)$, satisfying (6-1), are all the generators of the ring of W-invariant polynomials $p_1(\lambda), \ldots, p_r(\lambda)$. It is known that they are algebraically independent, and the ring generated by corresponding differential operators is a ring of polynomials in generators $D_{p_1}, \ldots, D_{p_r}$.

There are also other polynomials, satisfying (6-1), which are not W-invariant. They give rise to differential operators which are not W-invariant and therefore do not belong to the ring generated by $D_{p_1}, \ldots, D_{p_r}$.

In particular, all polynomials contained in the ideal generated by

$$Q_0(\lambda) = \prod_{\alpha \in \Delta^+} \prod_{n=1}^{n_\alpha} (\langle \alpha, \lambda \rangle^2 - n^2 \langle \alpha, \alpha \rangle^2)$$

satisfy the (6-1).

PROPOSITION 6.2. *The differential operator corresponding to the invariant polynomial* $p_1(\lambda) = \langle \lambda, \lambda \rangle$, *is equal to*

$$D_{p_1} = \Delta_{\mathfrak{h}} - \sum_{\alpha \in \Delta^+} \frac{2 f_\alpha e_\alpha}{\sinh^2 \langle \alpha, x \rangle}. \tag{6-3}$$

This fact can be proved by a direct computation. Another proof of it using the relationship between the center of $U(\mathfrak{g})$ and commuting differential operators is sketched in the Appendix (see also [E]).

The operator (6-3) coincides with the generalized Calogero-Sutherland operator (2-3) for the trigonometric case. We have shown that this operator is algebraically integrable. Theorem 2.6 is proved.

Appendix

In conclusion we briefly describe how to construct quantum integrals of $H_{\mathfrak{g},U,u}$ with trigonometric potential from central elements of $U(\mathfrak{g})$. This construction works for arbitrary module U, not necessarily highest weight.

PROPOSITION A1. [E] *Let $X \in U(\mathfrak{g})$ be an element of degree 0, i.e. $[h, X] = 0$, $h \in \mathfrak{h}$. Then there exists a unique matrix differential operator $\mathcal{D}(X)$ with $\text{End}(U[0])$-valued coefficients such that for any $\lambda \in \mathfrak{h}^*$ and any intertwining operator $\Phi \colon M_\lambda \to M_\lambda \otimes U$*

$$\text{Tr}|_{M_\lambda}(\Phi X e^x) = \mathcal{D}(X)\text{Tr}|_{M_\lambda}(\Phi e^x).$$

The proof of this theorem and a recursive construction of $\mathcal{D}(X)$ is given in [E]. We illustrate the idea of this construction by computing $D(EF)$ for $\mathfrak{g} = \mathfrak{sl}_2$ (E, F, H are standard generators of $\mathfrak{g}$). The main trick is to carry E around the trace, using the intertwining property of Φ and the cyclic property of the trace:

$$\text{Tr}|_{M_\lambda}(\Phi E F e^x) = \text{Tr}|_{M_\lambda}((E \otimes 1)\Phi F e^x) + E\,\text{Tr}|_{M_\lambda}(\Phi F e^x) =$$

$$\text{Tr}|_{M_\lambda}(\Phi F e^x E) + E\,\text{Tr}|_{M_\lambda}(\Phi F e^x) =$$

$$e^{\langle a,x \rangle}\,\text{Tr}|_{M_\lambda}(\Phi F E e^x) + E\,\text{Tr}|_{M_\lambda}(\Phi F e^x) =$$

$$e^{\langle \alpha,x \rangle}\,\text{Tr}|_{M_\lambda}(\Phi(EF + H)e^x) + E\,\text{Tr}|_{M_\lambda}(\Phi F e^x) =$$

$$e^{\langle \alpha,x \rangle}\,\text{Tr}|_{M_\lambda}(\Phi E F e^x) + e^{\langle \alpha,x \rangle}\frac{\partial}{\partial \alpha}\text{Tr}|_{M_\lambda}(\Phi e^x) + E\,\text{Tr}|_{M_\lambda}(\Phi F e^x), \quad (\text{A1})$$

(where α is the positive root of $\mathfrak{g}$) which implies

$$\text{Tr}|_{M_\lambda}(\Phi E F e^x) = \frac{1}{1 - e^{\langle \alpha,x \rangle}}$$

$$\left(e^{\langle \alpha,x \rangle}\frac{\partial}{\partial \alpha}\text{Tr}|_{M_\lambda}(\Phi e^x) + E\,\text{Tr}|_{M_\lambda}(\Phi F e^x)\right). \quad (\text{A2})$$

Further, we have

$$\text{Tr}|_{M_\lambda}(\Phi F e^x) = \text{Tr}|_{M_\lambda}((F \otimes 1)\Phi e^x) + F\,\text{Tr}|_{M_\lambda}(\Phi e^x)$$

$$= e^{-\langle \alpha,x \rangle}\,\text{Tr}|_{M_\lambda}(\Phi F e^x) + F\,\text{Tr}|_{M_\lambda}(\Phi e^x), \quad (\text{A3})$$

which implies

$$\mathrm{Tr}|_{M_\lambda}(\Phi F e^x) = \frac{1}{1 - e^{-\langle\alpha,x\rangle}} F \, \mathrm{Tr}|_{M_\lambda}(\Phi e^x). \tag{A4}$$

Combining (A2) and (A4), we find

$$\mathrm{Tr}|_{M_\lambda}(\Phi E F e^x) =$$

$$\left(\frac{e^{\langle\alpha,x\rangle}}{1 - e^{\langle\alpha,x\rangle}} \frac{\partial}{\partial\alpha} + \frac{1}{(1 - e^{\langle\alpha,x\rangle})(1 - e^{-\langle\alpha,x\rangle})} E F \right) \mathrm{Tr}|_{M_\lambda}(\Phi e^x). \tag{A5}$$

Thus

$$\mathcal{D}(EF) = \frac{e^{\langle\alpha,x\rangle}}{1 - e^{\langle\alpha,x\rangle}} \frac{\partial}{\partial\alpha} + \frac{1}{(1 - e^{\langle\alpha,x\rangle})(1 - e^{-\langle\alpha,x\rangle})} E F. \tag{A6}$$

In general, it is not true that $\mathcal{D}(X_1 X_2)$ equals either $\mathcal{D}(X_1)\mathcal{D}(X_2)$ or $\mathcal{D}(X_2)\mathcal{D}(X_1)$. However:

PROPOSITION A2. [E]. *If X_1 belongs to the center of $U(\mathfrak{g})$, then for any X_2 one has $\mathcal{D}(X_2)\mathcal{D}(X_1) = \mathcal{D}(X_1 X_2)$.*

This proposition follows from the fact that if Φ is an intertwining operator then ΦX_1 is also an intertwining operator.

COROLLARY. *If X_1, X_2 are both in the center of $U(\mathfrak{g})$, then $\mathcal{D}(X_1)$ and $\mathcal{D}(X_2)$ commute with each other.*

Let $D(X)$ be the differential operator obtained from $\mathcal{D}(X)$ by conjugation by the function $\mathrm{Tr}|_{M_{-\rho}}(e^x)$, i.e. defined by $D(X)\xi(x) = \mathrm{Tr}|_{M_{-\rho}}(e^x)\mathcal{D}(X)(\frac{\xi(x)}{\mathrm{Tr}|_{M_{-\rho}}(e^x)})$. The following statement is checked by a direct computation:

PROPOSITION A3. [E]. *Let B be an orthonormal basis of $\mathfrak{g}$, and $C_1 = \Sigma_{a\in B} a^2$ be the Casimir element. Then*

$$D(C_1) = \Delta_{\mathfrak{h}} - \sum_{\alpha\in\Delta+} \frac{f_\alpha e_\alpha}{2\sinh^2(\langle\alpha,\, x\rangle/2)} + \mathrm{const.} \tag{A7}$$

PROPOSITION A4. *Let $C_1, \dots, C_m$ be algebraically independent generators of the center of $U(\mathfrak{g})$. Let $\hat{L}_i = D(C_i)$. Then $\hat{L}_1, \dots, \hat{L}_m$ are algebraically independent quantum integrals of the Schrödinger operator $\hat{L}_1$ given by (A7). The symbols of $\hat{L}_1$ generate the algebra of Weyl group invariant polynomials on $\mathfrak{h}$. The function $\Psi(\lambda, x)$ is a joint eigenfunction for $\hat{L}_1, \dots, \hat{L}_m$ with eigenvalues $\phi(C_i)(\lambda)$, where ϕ is the Harish-Chandra homomorphism defined in Section 3.*

Observe that operator (A7) transforms into $\frac{1}{4} H_{\mathfrak{g},U,u} + \mathrm{const}$ when one makes a change of variables $x \to 2x$. Therefore, we have

PROPOSITION A5. *Let L_i be the operators obtained from $\hat{L}_i$ by replacing x with $2x$. Then $L_1 = \frac{1}{4}H_{\mathfrak{g},U,u} + $ const, and $L_1,\dots,L_m$ are algebraically independent quantum integrals of $H_{\mathfrak{g},U,u}$ for trigonometric u. The function $\psi(\lambda, x)$ is a joint eigenfunction of $L_1,\dots,L_m$.*

This implies Theorem 2.3 in the trigonometric case.

Finally, we observe that if C is a central element of $U(\mathfrak{g})$ then by Propositions A4, A5 one has $D(C) = D_p$, where D_p is defined in Section 6, and p is the Weyl group invariant polynomial given by $p(\lambda) = \phi(C)(2(\lambda + \rho))$. This proves Proposition 6.2.

Acknowledgements

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