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On the bifurcation set of complex polynomial with isolated singularities at infinity

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Abstract. Let f be a complex polynomial. In this paper we give complete descriptions of the bifurcation set of f, provided f has only isolated singularities at infinity. In particular, we generalize to such polynomials the Há-Lê Theorem and show that if the Euler characteristic of the fibres of f is constant over $U \subset \mathbf{C}$, where U contains only regular values of f, then f is actually locally C^∞-trivial over U. The proof is based on a criterion which allows us to show for some families of isolated hypersurface singularities that μ-constant implies topological triviality.

Let $f: \mathbf{C}^n \to \mathbf{C}$ be a polynomial function. It is well known (see e.g. [Ph2]) that there exists a finite set $\Delta \subset \mathbf{C}$ such that $f: \mathbf{C}^n \backslash f^{-1}(\Delta) \to \mathbf{C} \backslash \Delta$ is a locally trivial C^∞-fibration. We call the smallest such set *the bifurcation set of* f and denote by B_f. Besides the critical values of f, B_f may contain some other numbers – the values of so called "critical points at infinity". This may happen since f is not proper and we cannot apply Ehresmann's Fibration Theorem.

There are two approaches to study B_f. First, one may consider the family $\bar{f}: X \to \mathbf{C}$ of the projective closures of the fibres of f, that is $\bar{f}^{-1}(t)$ are the closures in $\mathbf{P}^n$ of $f^{-1}(t)$ (see Section 1.1 for the details). Now $\bar{f}$ is proper but the generic fibre of $\bar{f}$ may have singularities. So instead of Ehresmann's Fibration Theorem one may apply the theory of Whitney stratification and trivialize $\bar{f}$ using Thom–Mather Isotopy Lemma. For this approach see for instance [Ph2], [Hà-Lê] and [Di1, Ch. 1 Sect. 4]. The second approach is to work entirely in the affine space $\mathbf{C}^n$ and trivialize f using explicitly constructed trivializing vector field. This vector field is defined using the gradient of f so this approach requires some assumptions on the asymptotic behaviour of grad $f(x)$ as $\|x\| \to \infty$. For this approach see Section 1.2 and [Br1-2], [F], [Né] and [Né-Z].

The purpose of this paper is to study B_f under the assumption that f has only isolated singularities at infinity, that is the projective closure of a generic fibre of f has only isolated singularities. In this case we give complete descriptions of B_f in the spirit of the first and the second approach. Moreover, in this case, each "critical point at infinity" changes the affine fibre by "substracting" from a generic fibre a number of handles of index n. In particular, if t_0 is a regular value of f then f is trivial over a neighbourhood of t_0 if and only if the Euler characteristic of $f^{-1}(t_0)$ is the same as the one of a generic fibre. This generalizes the Hà-Lê Theorem [Hà-Lê]. Our results are stated in Theorem 1.4 which is then proven in Section 3.

A word about the method of proof. We may study the singularities of $\bar{f}^{-1}(t)$ as a finite number of families of isolated singularities. By [Hà-Lê], [Di1, Ch. 1 Sect. 4] these families are μ-constant if and only if the Euler characteristic of the fibres $f^{-1}(t)$ does not change (here we restrict ourselves to regular fibres of f only). So the main difficulty is to show that the μ-constancy implies topological triviality. In general, this is not known. In this paper we develop a method which allow us to show it in our case. The presentation of such a method (Section 3.1) is another purpose of this paper. The method is based on a study of the Thom condition $\mathbf{a}_F$ and the conormal space which we present in Section 2.

1. Main results

1.1. Family of projective closures of fibres of f

Let $f: \mathbf{C}^n \to \mathbf{C}$ be a polynomial function. To study the fibres of f we consider their projective closures. We follow a classical construction (see e.g. [Di1, Ch. 1 Sect. 4], [Br2]).

Let $d = \deg f$ and let $f = f_0 + f_1 + \cdots + f_d$ be the decomposition of f into homogeneous components, $f_d \not\equiv 0$. Consider the homogenization of f

$$\tilde{f}(x_0, x_1, \ldots, x_n) = x_0^d f(x_1/x_0, \ldots, x_n/x_0)$$

and the hypersurface in $\mathbf{P}^n \times \mathbf{C}$ defined by

$$X = \{(x, t) \in \mathbf{P}^n \times \mathbf{C} \mid F(x, t) = 0\}, \quad F(x, t) = \tilde{f}(x) - t x_0^d.$$

Let $H_\infty = \{x_0 = 0\} \subset \mathbf{P}^n$ be the hyperplane at infinity and let $X_\infty = X \cap (H_\infty \times \mathbf{C})$. The cone at infinity C_∞ of the fibres $X_t = f^{-1}(t)$ of f does not depend on t and is given by $C_\infty = \{x \in H_\infty \mid f_d(x) = 0\}$. Hence $X_\infty = C_\infty \times \mathbf{C}$. Let

$$\bar{f}: X \to \mathbf{C}$$

be the map induced by the projection $\mathbf{P}^n \times \mathbf{C}$ onto the second factor. The fibres of $\bar{f}$, which we denote by $\bar{X}_t$, are the projective closures of X_t's. The singular part of X is precisely $A \times \mathbf{C}$, where

$$A = \{x \in H_\infty \mid \partial f_d/\partial x_1 = \cdots = \partial f_d/\partial x_n = f_{d-1} = 0\}.$$

The singular part of $X_\infty = X \cap H_\infty$ can be bigger and equals precisely $B \times \mathbf{C}$, where

$$B = \{x \in H_\infty \mid \partial f_d/\partial x_1 = \cdots = \partial f_d/\partial x_n = 0\}.$$

DEFINITION 1.1. We say that f *has isolated singularities at infinity* if A is a finite set.

1.2. AFFINE TRIVIALIZATIONS

By grad f we denote the vector $\mathrm{grad}\, f = \overline{(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)}$, so the chain rule may be expressed by the inner product $\partial f/\partial \mathbf{v} = \langle \mathbf{v}, \mathrm{grad}\, f\rangle$. Let t_0 be a regular value of f. There are several conditions restricting the asymptotic growth of grad $f(x)$ as $\|x\| \to \infty$, $f(x) \to t_0$, so that they imply the topological triviality of f over a small neighbourhood of t_0 in $\mathbf{C}$. For instance Fedoryuk's condition [F] (see also [Br1], [Br2]) says

$$\exists_{\delta>0} \quad \|\mathrm{grad}\, f(x)\| \geqslant \delta. \tag{f}$$

If one looks for a weaker condition then it is natural to take Malgrange's condition (quoted in [Ph1])

$$\exists_{\delta>0} \quad \|x\|\, \|\mathrm{grad}\, f(x)\| \geqslant \delta. \tag{m}$$

There are also other interesting conditions given in [Né], [Né-Z]. Let us introduce the following condition, which is intermediate between the Fedoryuk's and Malgrange's ones,

$$\exists_{N\geqslant 1}\exists_{\delta>0} \quad \|x\|^{(N-1)/N}\|\mathrm{grad}\, f(x)\| \geqslant \delta. \tag{1.1}$$

We shall show below that, if f has only isolated singularities at infinity, then (1.1) is equivalent to the μ-constant condition and also to the topological triviality. First, (1.1) implies topological triviality by the following standard argument (see e.g. [Né-Z]) which works without any assumption on the singularities of f at infinity.

LEMMA 1.2. *Let t_0 be a regular value of f such that (1.1) holds for $\|x\| \to \infty$, $f(x) \to t_0$. Then there is a neighbourhood U of t_0 in $\mathbf{C}$ such that f is topologically trivial over U. Moreover, one may find a trivialization which fixes all the points at infinity.*

Proof. We construct a vector field $\mathbf{w}$ by taking first the projection of grad f onto spheres and then renormalizing the projected vector. That is we define

$$\mathbf{v}(x) = \mathrm{grad}\, f(x) - \frac{\overline{\langle x, \mathrm{grad}\, f(x)\rangle}}{\|x\|^2}\, x, \quad \mathbf{w}(x) = \frac{\mathbf{v}(x)}{\langle \mathbf{v}(x), \mathrm{grad}\, f(x)\rangle}.$$

If (1.1) holds then $\mathbf{w}(x) \neq 0$ provided $\|x\|$ is sufficiently large and $f(x)$ is sufficiently close to t_0. Indeed, it suffices to check it on real analytic curves. Let

$$x(s) = s^{\alpha}(\mathbf{a}_0 + \mathbf{a}_1 s + \cdots), \quad \mathbf{a}_0 \neq 0,$$

be such a curve parametrized by $s \in [0, \varepsilon)$. Since $\|x(s)\| \to \infty,\ s \to 0$ we have $\alpha < 0$. We expand also

$$\operatorname{grad} f(x(s)) = s^{\beta}(\mathbf{b}_0 + \mathbf{b}_1 s + \cdots), \quad \mathbf{b}_0 \neq 0.$$

Condition (1.1) or even weaker condition (m) imply $\alpha + \beta \leqslant 0$. Since $f(x(s))$ is bounded, $f(x(s))$ is an analytic function of s at 0. So is $(d/\mathrm{d}s)f(x(s))$. But

$$\frac{d}{\mathrm{d}s} f(x(s)) = \left\langle \frac{d}{\mathrm{d}s} x(s),\ \operatorname{grad} f(x(s)) \right\rangle = s^{\alpha+\beta-1}(\langle \mathbf{a}_0, \mathbf{b}_0\rangle + \cdots),$$

and $\alpha + \beta - 1 < 0$. Hence

$$\langle \mathbf{a}_0, \mathbf{b}_0 \rangle = 0,$$

which implies not only $\mathbf{v}(x(s)) \neq 0$ but also

$$\operatorname{grad} f(x) \sim \mathbf{v}(x), \quad \text{as } \|x\| \to \infty. \tag{1.2}$$

Therefore $\mathbf{w}(x) \neq 0$, $\langle x, \mathbf{w}(x)\rangle = 0$, $\partial f/\partial \mathbf{w} = 1$. Thus, integrating $\mathbf{w}$ we get the desired trivialization of f outside a big ball $\mathbf{B}_R = \{\|x\| \geqslant R\}$. Inside $\mathbf{B}_R$ we may trivialize f integrating $\operatorname{grad} f/\|\operatorname{grad} f\|^2$, since, by the assumptions, grad f is nonzero. Glueing these two vector fields we obtain a global trivialization.

Note that (1.2) and (1.1) also give

$$\|\mathbf{w}(x)\| \sim \frac{1}{\|\operatorname{grad} f\|} \leqslant C\|x\|^{(N-1)/N}, \tag{1.3}$$

which implies the last statement of the lemma.

REMARK 1.3. The proof of Lemma 1.2 also shows that (m) implies topological triviality. Then, instead of (1.3) we get

$$\|\mathbf{w}(x)\| \sim \frac{1}{\|\operatorname{grad} f\|} \leqslant C\|x\|.$$

Therefore, in this case, it is not clear whether our trivialization extends to infinity i.e. gives a topological trivialization of $\bar{f}$. Nevertheless, this is the case if f has only isolated singularities at infinity. Indeed, we show in this paper that in our case the topological triviality of f implies (1.1).

1.3. MAIN RESULT

The main result of this paper is the following theorem.

THEOREM 1.4. *Let $f: \mathbf{C}^n \to \mathbf{C}$ be a polynomial function with isolated singularities $A = \{a_1, \dots, a_p\}$ at infinity. Let t_0 be a regular value of f. Then the following conditions are equivalent*

(i) *f is C^∞ trivial over a neighbourhood of t_0, i.e. $t_0 \notin B_f$;*
(ii) *the condition (1.1) holds for x such that $\|x\|$ is big and $f(x)$ is close to t_0;*
(iii) *the families of isolated singularities $(\bar{X}_t, a_i \times t)$ are μ-constant for t close to t_0;*
(iv) *The Euler characteristic $\chi(X_t)$ of the fibres of f is consant for t close to t_0.*

Moreover, if a regular value t_0 is a bifurcation point of f, that is $t_0 \in B_f$, then a generic fibre X_t of f may be obtained from X_{t_0}, up to homotopy, by adding a finite number of n-handles.

(i) $\Rightarrow$ (iv) trivially, (ii) $\Rightarrow$ (i) by Lemma 1.1. We shall show (iii) $\Rightarrow$ (ii) in Section 3.1. (iii) $\Longleftrightarrow$ (iv) is proven in [Di1, Ch. 1 (4.6)]. We show the last statement of the theorem in Section 3.2. Our argument gives also an alternative proof of (iii) $\Longleftrightarrow$ (iv).

2. Polar varieties and a_F stratifications

Throughout this section we work in a more general analytic setup. Consider first the following classical situation. Let $F(t, x_0, \dots, x_n)$ be an analytic function which for each fixed t has an isolated singularity at the origin. Then we may consider F as a family of isolated singularities along a line $S = \{x_0 = \cdots = x_n = 0\}$. By a theorem of Lê and Saito [L-S] this family is μ-constant if and only if the Thom condition $\mathbf{a}_F$ is satisfied along S. In this section we generalize this characterization of $\mathbf{a}_F$ condition to singularities of any codimension. The condition replacing the μ-constant condition is the emptiness of a relative polar variety. In the case of isolated singularities this variety is a polar curve $\Gamma = \{(x, t) \,|\, \partial F/\partial t \neq 0,\, \partial F/\partial x_0 = \cdots = \partial F/\partial x_n\}$ and it is easy to see that Γ is empty if and only if the family is μ-constant. We start with a quick reminder of conormal spaces.

Let X be an analytic subset of an open $U \subset \mathbf{C}^{n+1}$ and let $\operatorname{Reg} X$, Σ_X denote the sets of regular and singular points of X respectively. By the (projectivized) *conormal space of X* we mean an analytic set

$$C_X = \text{Closure}\{(x, H) \in \operatorname{Reg} X \times \check{\mathbf{P}}^n;\ H \supset T_x \operatorname{Reg} X\} \subset X \times \check{\mathbf{P}}^n$$

together with projections

$$\begin{array}{ccc} C_X & \xrightarrow{\gamma_X} & \check{\mathbf{P}}^n \\ {\scriptstyle \tau_X}\downarrow & & \\ X & & \end{array} .$$

Similarly, let $F(x_0, \ldots, x_n)$ be a holomorphic function in U and let $\Sigma_F = \{x \in U \mid \operatorname{grad} F(x) = 0\}$ be the set of critical points of F. For $x \notin \Sigma_F$ we denote by $T_x F$ the relative tangent space to F that is $T_x F = T_x F^{-1}(F(x))$. Then by the (projectivized) *relative conormal space to F* we mean

$$C_F = \text{Closure}\{(x, H) \in (U \backslash \Sigma_F) \times \check{\mathbf{P}}^n;\ H = T_x F\} \subset U \times \check{\mathbf{P}}^n,$$

together with the induced projections

$$\begin{array}{ccc} C_F & \xrightarrow{\gamma_F} & \check{\mathbf{P}}^n \\ {\scriptstyle \tau_F}\downarrow & & \\ U & & \end{array} .$$

In this case $\tau_F \colon C_F \to U$ coincides with the Jacobian blowing-up of F.

Let $\mathcal{S} = \{S_i\}$ be an analytic stratification of $X \subset U$. Let $p \in S_1 \subset \bar{S}_2$. Then Whitney's **a**-condition for (S_2, S_1) at p is equivalent to

$$(C_{\bar{S}_2})_p \subset (C_{\bar{S}_1})_p,$$

where customarily by $(C)_p$ we denote the fibre of C over p.

Similarly, if $X = F^{-1}(0)$ we say that the Thom condition $\mathbf{a}_F$ holds along S at $p \in S$ if

$$(C_F)_p \subset (C_{\bar{S}})_p.$$

A stratification of X satisfying $\mathbf{a}_F$ condition is customarily called "good" ("bonne" in French).

Fix a stratum S_0 of $\mathcal{S}$. The failure of $\mathbf{a}_F$ condition at $p \in S_0$ can be detected with help of relative polar varieties at p. The following proposition is well-known. Since we could not have found the exact statement in literature, we present it with a proof.

PROPOSITION 2.1. *Let X be the zero set of a holomorphic function F defined in open $U \subset \mathbf{C}^{n+1}$ and such that the set of singular points $\Sigma_F = \Sigma_X$ of F is nowhere dense in X. Let $\mathcal{S}$ be a stratification of Σ_X and let S_0 be a stratum of $\mathcal{S}$ such that:*

(a) *for all S in $\mathcal{S}$ such that $S_0 \subset \bar{S}$ the pair (S, S_0) satisfies* **a**-*condition of Whitney;*
(b) *condition $\mathbf{a}_F$ is satisfied along all strata of $\mathcal{S}$ but maybe S_0.*
Then, for $p \in S_0$, the following conditions are equivalent:
 (i) *$\mathbf{a}_F$ holds at $p \in S_0$;*
 (ii) $\dim(C_F)_p \leqslant n - \dim S_0$*;*
 (iii) *For some local coordinate system $x_0, x_1, \dots, x_n$ at p such that $S_0 = \{x_0 = x_1 = \dots = x_s = 0\}$, the relative polar variety*

$$P_{s+1}(F) = \overline{\{x \notin \Sigma_F | \partial F/\partial x_0 = \dots = \partial F/\partial x_s = 0\}},$$

is empty near p.
 (iv) *The relative polar variety $P_{s+1}(F)$ is empty near p for any such system of coordinates.*

Proof. Clearly (iv) $\Rightarrow$ (iii), and (i) $\Rightarrow$ (iv) is trivial (see Remark 2.2 (a) below). Also the implication (i) $\Rightarrow$ (ii) is easy. Indeed, by $\mathbf{a}_F$ we get $(C_F)_p \subset (C_{\bar{S}_0})_p$ so $\dim(C_F)_p \leqslant \dim(C_{\bar{S}})_{p_0} = n - \dim S_0$.

We shall prove (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (i). Fix a coordinate system $x_0, x_1, \dots, x_n$ at p such that $S_0 = \{x_0 = x_1 = \dots = x_s = 0\}$.

First note that by the assumption (b) for any stratum $S \neq S_0$ and any $q \in S$ $(C_F)_q \subset (C_{\bar{S}})_q$. Therefore

$$\overline{\tau_F^{-1}(\Sigma_F \backslash S_0)} \subset \bigcup_{S \neq S_0} (C_{\bar{S}}).$$

Hence by (a)

$$(\overline{\tau_F^{-1}(\Sigma_F \backslash S_0)})_p \subset \bigcup_{S \neq S_0} (C_{\bar{S}})_p \subset (C_{S_0})_p.$$

Moreover, let Y be the set of those points of S_0 at which $\mathbf{a}_F$ along S_0 fails. Then Y is analytic and nowhere dense in S_0 and by the same argument as above

$$(\overline{\tau_F^{-1}(\Sigma_F \backslash Y)})_p \subset (C_{S_0})_p. \tag{2.1}$$

(ii) $\Rightarrow$ (i). Since $\tau_F \colon C_F \to U$ is a blowing-up its exceptional divisor $\tau_F^{-1}(\Sigma_F)$ is of pure dimension n. If $\dim(C_F)_p \leqslant n - \dim S_0$ at p then $\dim \tau_F^{-1}(Y) \leqslant n - \dim S_0 + \dim Y < n$. Hence $\tau_F^{-1}(Y)$ is nowhere dense in $\tau_F^{-1}(\Sigma_F)$ and

$$\overline{\tau_F^{-1}(\Sigma_F \backslash Y)} \supset \tau_F^{-1}(Y),$$

which by (2.1) gives $(C_F)_p \subset \tau_F^{-1}(Y) \subset (C_{S_0})_p$ as required.

(iii) $\Rightarrow$ (i). Assume, by contradiction, that (iii) holds at $p \in Y$. Then, since (iii) is an "open" condition, (iii) holds in a neighbourhood of p. Thus, to get a contradiction, it suffices to show (iii) $\Rightarrow$ (i) at a generic point of Y.

Therefore, we may assume that Y is nonsingular and $\mathbf{a}_F$ holds for Y as a new stratum. Then in particular

$$\tau_F^{-1}(Y) \subset C_Y. \tag{2.2}$$

Let $L \subset \mathbf{P}^n$ be the projective linear subspace defined by $\{x_{s+1} = \cdots = x_n = 0\}$ and let $\check{L} \subset \check{\mathbf{P}}^n$ be its dual, $\dim \check{L} = n - s - 1$. Since $\check{L} \cap (C_{S_0})_p = \emptyset$, by (2.1)

$$\gamma_F^{-1}(\check{L}) \cap \tau_F^{-1}(\Sigma_F \backslash Y) = \emptyset. \tag{2.3}$$

The polar variety $P_{s+1}(F)$ equals (as a set) $\tau_F(\tilde{\Gamma})$, where

$$\tilde{\Gamma} = \overline{\gamma_F^{-1}(\check{L}) \backslash \tau_F^{-1}(\Sigma_F)}.$$

Therefore, if by (iii) $\tilde{\Gamma}$ is empty, then $\gamma_F^{-1}(\check{L}) \subset \tau_F^{-1}(\Sigma_F)$. Hence, by (2.3), $\gamma_F^{-1}(\check{L}) \subset \tau_F^{-1}(Y)$ or equivalently

$$\gamma_F^{-1}(\check{L}) \subset \gamma_F^{-1}(\check{L}) \cap \tau_F^{-1}(Y). \tag{2.4}$$

Since $Y \subset S_0$ and L is transverse to S_0 we have $\dim(C_Y \cap \gamma_F^{-1}(\check{L})) = \dim C_Y - \operatorname{codim} \check{L} = n - s - 1$ and hence by (2.2) and (2.3)

$$\dim \gamma_F^{-1}(\check{L}) \cap \tau_F^{-1}(Y) \leqslant n - s - 1.$$

On the other hand, if nonempty, $\dim \gamma_F^{-1}(\check{L}) \geqslant (n+1) - \operatorname{codim} \check{L} = n - s$. So we see, calculating the dimensions of both sides, that (2.4) is impossible. This ends the proof.

REMARK 2.2.

(a) Let $x_0, x_1, \ldots, x_n$ be a local coordinate system at p such that $S_0 = \{x_0 = x_1 = \cdots = x_s = 0\}$. Then $\mathbf{a}_F$ at p is equivalent to

$$\frac{\|(\partial F/\partial x_{s+1}, \ldots, \partial F/\partial x_n)(x)\|}{\|(\partial F/\partial x_0, \ldots, \partial F/\partial x_s)(x)\|} \to 0 \tag{2.5}$$

as $U \backslash \Sigma_F \ni x \to p$, and (iii) of Proposition 2.1 is equivalent to saying that there is no sequence of points $x \to p$ such that

$$(\partial F/\partial x_{s+1}, \ldots, \partial F/\partial x_n)(x) \neq 0 \quad \text{and}$$

$$(\partial F/\partial x_0, \ldots, \partial F/\partial x_s)(x) = 0. \tag{2.6}$$

So Proposition 2.1 says that (2.5), (2.6) as well as the following intermediate condition are equivalent:

$$\frac{\|(\partial F/\partial x_{s+1}, \ldots, \partial F/\partial x_n)\|}{\|(\partial F/\partial x_0, \ldots, \partial F/\partial x_s)\|} \text{ is bounded near } p.$$

(b) The subscript $s+1$ in $P_{s+1}(F)$ means that the expected codimension of $P_{s+1}(F)$ is $s+1$. But if the system of coordinates is not generic, as in our case, it may happen that $\operatorname{codim} P_{s+1} \leq s$. If the assumptions of Proposition 2.1 are satisfied then at a generic point of Y (in the notation of the proof of Proposition 2.1) $\operatorname{codim} P_{s+1}(F) = s + 1 - \dim Y$. In particular, for a one parameter family of isolated singularities $\Gamma = P_n(F)$ is always a curve (if non-empty).

(c) For a hypersurface X of a smooth variety M the condition $\mathbf{a}_F$ along $S \in X$ does not depend on the choice of a local equation $X = \{x \in M | F(x) = 0\}$. Here we understand X as a subvariety of M and F has to generate the ideal of X. We use this observation in the next section to study $\mathbf{a}_F$ stratification of $X = F^{-1}(0) \subset \mathbf{P}^n \times \mathbf{C}$, where $F(x_0, x_1, \ldots, x_n, t)$ is a polynomial homogeneous in $(x_0, x_1, \ldots, x_n)$.

3. The proofs

3.1. $\mathbf{a}_F$-STRATIFICATION OF X

Let f be a polynomial function as in Section 1. We assume that f has only isolated singularities at infinity. Then $(X \setminus A \times \mathbf{C}, A \times \mathbf{C})$ is a stratification of X. Let t_0 be a regular value of f and assume that (iii) of Theorem 1.4 holds. By a theorem of Lê and Saito [L-S] this is equivalent to saying that our stratification satisfies the Thom condition $\mathbf{a}_F$, where, recall, $F(x, t) = \tilde{f}(x) - t x_0^d = 0$ is the equation of defining X (this makes sense by Remark 2.2 (c)).

If we suppose, as in [Di1, Ch. 1 Sect. 4], [Hà-Lê], that the pair $(X \setminus A \times \mathbf{C}, A \times \mathbf{C})$ satisfies Whitney's Conditions **a** and **b**, then the Thom-Mather Isotopy Lemma gives a trivialization with required properties (see [Di1, Ch. 1] for the details). Note that in our case, that is in the case of family of isolated singularities, Whitney's conditions are equivalent to μ^*-constant condition. But it is known ([B-S]) that in general μ^*-constant is a condition much stronger than μ-constant. Also, whether, in general, μ-constant implies topological triviality is not known. To overcome these difficulties, we use some particular properties of our stratification and Proposition 2.1. Note that even if our stratification can be considered as a family of isolated singularities, we shall use the results of Section 2 for more general singularities. This is due to the fact that in our construction we will have to 'enlarge' the singularities of X.

Fix $p_0 \in A \times \{t_0\}$. We may assume, and we always do in the sequel, that $p_0 = (0 : 0 : \cdots : 0 : 1), t_0)$, so that $x_0, x_1, \ldots, x_{n-1}, t$ form a system of coordinates near p_0. Then, $\mathbf{a}_F$ along $\{0\} \times \mathbf{C}$ means

$$|x_0^d| = |\partial F/\partial t(x, t)| \ll \|(\partial F/\partial x_0, \ldots, \partial F/\partial x_{n-1})(x, t)\|,$$

as (x, t) approaches $(0, t_0)$.

LEMMA 3.1. *Let $p = (p', t) \in X_\infty$ be close to p_0. If either $p \notin A \times \mathbf{C}$ or if $p \in A \times \mathbf{C}$ and $\mathbf{a}_F$ holds along $\{p'\} \times \mathbf{C}$ at p, then for every positive integer N*

$$|x_0^d| = |\partial F/\partial t| \ll \|(x_0^{(N-1)/N} \partial F/\partial x_0, \partial F/\partial x_1, \ldots, \partial F/\partial x_{n-1})(x, t)\|. \quad (3.1)$$

as $B \times \mathbf{C} \not\ni (x, t)$ approaches p.

Proof. We leave the case $p \notin A \times \mathbf{C}$ to the reader and consider only the more difficult case $p \in A \times \mathbf{C}$.

Consider the singularities of X_∞ that is $B \times \mathbf{C}$. If A is finite then B is of dimension at most 1. Choose a stratification of B such that A is a union of strata. Taking products with $\mathbf{C}$ we get a stratification $\mathcal{S}$ of $B \times \mathbf{C}$ which clearly is Whitney $\mathbf{a}$ regular and $\mathbf{a}_F$ holds along each stratum contained in $(B \backslash A) \times \mathbf{C}$. Indeed, $\mathbf{a}$ regularity follows trivially from $\dim B \leqslant 1$, and $\mathbf{a}_F$ is always satisfied along strata contained in the regular part of X. We may also require that our stratification satisfies the following extra property:

(ex) *For F restricted to $H_\infty \times \mathbf{C}$, that is for $f_d(x, t) = f_d(x)$, the Thom condition $\mathbf{a}_{f_d}$ holds along each stratum of $\mathcal{S}$.*

Fix a positive integer $N > 1$ and consider a function

$$F_N(y_0, x_1, \ldots, x_{n-1}, t) = F(y_0^N, x_1, \ldots, x_{n-1}, t).$$

Then,

$$\frac{\partial F_N}{\partial t} = \frac{\partial F}{\partial t}, \quad \frac{\partial F_N}{\partial y_0} = N y_0^{N-1} \frac{\partial F}{\partial x_0},$$

$$\frac{\partial F_N}{\partial x_i} = \frac{\partial F}{\partial x_i} \quad \text{for } i = 1, 2, \ldots, n-1. \quad (3.2)$$

Therefore, $B \times \mathbf{C}$ is the set of singular points of F_N. We claim that for the function F_N the Thom condition $\mathbf{a}_{F_N}$ holds along each stratum of $\mathcal{S}$. We show it by descending induction on the dimension of strata. Therefore, we may assume that the assumptions (a) and (b) of Proposition 2.1 are satisfied and it is enough to show the emptiness of the polar variety associated to each stratum. Note that since $B \subset H_\infty$ we make take $y_0 = x_0 = 0$ as one of the equations of our stratum S_0.

So, if $S_0 \subset (B\backslash A) \times \mathbf{C}$ the polar variety is contained in the zero set of $\partial F_N/\partial y_0$. But

$$\partial F_N/\partial y_0 = N y_0^{N-1} f_{d-1}(x) + y_0^N g(x,t)$$

and f_{d-1} does not vanish on $B\backslash A$. Thus, the polar variety has to be contained in $\{y_0 = 0\}$ which is not possible by the extra condition (ex).

Now consider a stratum $S_0 \subset A \times \mathbf{C}$ that is $S_0 = \{p'\} \times \mathbf{C}$. Then again by (ex) the polar variety Γ_N, in fact in this case the polar curve, cannot be contained in $\{y_0 = 0\}$. But

$$\Gamma_N = \text{Closure}\left(\left\{(x,t) \notin B \times \mathbf{C} \;\middle|\; \frac{\partial F_N}{\partial y_0} = \frac{\partial F_N}{\partial x_1} = \cdots = \frac{\partial F_N}{\partial x_{n-1}} = 0\right\}\right)\Bigg|$$

so, by (3.2), Γ_N does not depend on N and, since we have assumed $\mathbf{a}_F$, is empty. This shows the Thom condition for $\mathbf{a}_{F_N}$ along the strata.

Now, by (3.2), $\mathbf{a}_{F_N}$ along $\{p'\} \times \mathbf{C}$ at p means exactly (3.1). The proof of lemma is complete.

By the curve selection lemma, (3.1) for all N implies

$$|x_0^d| = |\partial F/\partial t| \leqslant C\|(x_0\, \partial F/\partial x_0,\, \partial F/\partial x_1, \ldots, \partial F/\partial x_{n-1})\|. \tag{3.3}$$

Indeed, if (3.3) fails, then

$$\frac{|x_0^d|}{\|(x_0\, \partial F/\partial x_0,\, \partial F/\partial x_1, \ldots, \partial F/\partial x_{n-1})(x,t)\|} \to \infty$$

along an analytic curve, which contradicts (3.1) for N sufficiently big.

On $X = \{F = 0\}$ we get even stronger properties.

LEMMA 3.2. *Let $p = (p', t') \in X_\infty$ be close to p_0. If either $p \notin A \times \mathbf{C}$ or if $p \in A \times \mathbf{C}$ and $\mathbf{a}_F$ holds along $\{p'\} \times \mathbf{C}$ at p, then for $X\backslash(B \times \mathbf{C}) \ni (x,t) \to p$*

$$|x_0\, \partial F/\partial x_0| \ll \|(\partial F/\partial x_1, \ldots, \partial F/\partial x_{n-1})\|. \tag{3.4}$$

$$\exists_{N>0}\, |x_0^d| = |\partial F/\partial t| \leqslant |x_0|^{1/N}\|(\partial F/\partial x_1, \ldots, \partial F/\partial x_{n-1})\|. \tag{3.5}$$

Proof. Again, by the curve selection lemma, it suffices to show (3.4) on each real analytic curve. Let $(x(s), t(s))$ be such curve, $s \in [0, \varepsilon)$, $(x(s), t(s)) \in X\backslash(B \times \mathbf{C})$ for $s \neq 0$. Then

$$0 = \frac{d}{ds} F(x(s), t(s)) = \frac{dt}{ds}\frac{\partial F}{\partial t} + \frac{dx_0}{ds}\frac{\partial F}{\partial x_0} + \sum_{i=1}^{n-1} \frac{dx_i}{ds}\frac{\partial F}{\partial x_i},$$

which gives

$$\left|x_0 \frac{\partial F}{\partial x_0}\right| \sim s \left|\frac{\mathrm{d}x_0}{\mathrm{d}s} \frac{\partial F}{\partial x_0}\right| \leqslant C\|(x,\, t) - p\| \, \|(\partial F/\partial t,\, \partial F/\partial x_1, \ldots, \partial F/\partial x_{n-1})\| \ll \|(\partial F/\partial t,\, \partial F/\partial x_1, \ldots, \partial F/\partial x_{n-1})\|.$$

This together with (3.3) gives (3.4). By (3.1) and (3.4)

$$|\partial F/\partial t| \ll \|(\partial F/\partial x_1, \ldots, \partial F/\partial x_{n-1})\|.$$

Therefore the following function

$$\varphi(x,\, t) = \frac{|\partial F/\partial t|}{\|\partial F/\partial x_1, \ldots, \partial F/\partial x_{n-1})\|}$$

extends continuously to X_∞ by setting $\varphi(p) = 0$ for $p \in X_\infty$. Since this extension is clearly semi-algebraic, by Łojasiewicz's Inequality [Ło, Sect. 18], there is a positive integer N such that $\varphi(x,\, t) \leqslant |x_0|^{1/N}$. This shows (3.5) and ends the proof of lemma.

REMARK 3.3. Note that (3.5) can be understood as a "Verdier-like" condition. In fact, we may take N an integer. Then for F_N from the proof of Lemma 3.1 we get

$$|\partial F_N/\partial t| \leqslant |y_0| \, \|\partial F_N/\partial x_1, \ldots, \partial F_N/\partial x_{n-1})\|,$$

which implies the condition **w** of Verdier [V] (in the complex analytic geometry **w** is equivalent to Whitney's **a** and **b** conditions).

One may follow this idea and show the topological triviality along $A \times \mathbf{C}$ using (3.5). Instead we choose a shorter argument.

PROPOSITION 3.4. *Let a polynomial function f have only isolated singularities at infinity which form μ-constant families over a neighbourhood of a regular value t_0 of f. Then the condition (1.1) holds for $\|x\| \to \infty$ and $f(x,\, t) \to t_0$. In particular, $\bar{f}\colon X \to \mathbf{C}$ is trivial over a neighbourhood of t_0.*

Proof. This is just translation of (3.5) to the old affine coordinates. We divide both sides of (3.5) by x_0^d and replace $x_0 \to x_n^{-1}$ and for $i = 1, \ldots, n-1$

$$\partial F/\partial x_i = \partial \tilde{f}/\partial x_i(x_0,\, x_1, \ldots, x_{n-1},\, 1) = x_n^{-(d-1)} \partial f/\partial x_i.$$

Thus (3.5) gives

$$1 \leqslant |x_n|^{(N-1)/N} \|\partial f/\partial x_1, \ldots, \partial f/\partial x_{n-1}\|.$$

This implies (1.1). (The reader may check that in fact (3.5) is equivalent to (1.1).) The proof of proposition is complete.

3.2. FAILURE OF μ-CONSTANT CONDITION

If A is finite and t is a regular value of f then, by [Di2] or [Pa], the Euler characteristic of $X_t = f^{-1}(t)$ is given by

$$\chi(X_t) = \chi(V^d_{\text{smooth}}) + (-1)^n \sum_{x\in A} \mu(\bar{X}_t, x) - \chi(C_\infty),$$

where V^d_{smooth} is a nonsingular hypersurface in $\mathbf{P}^n$ of degree d.

Therefore, if t_0 is another regular value then

$$\chi(X_{t_0}) - \chi(X_t) = (-1)^n \sum_{x\in A} (\mu(\bar{X}_{t_0}, x) - \mu(\bar{X}_t, x)).$$

So if $\mathbf{a}_F$, or equivalently μ-constant condition, fails at some points of $A \times \{t_0\}$, then the Euler characteristic of fibre changes at t_0. Indeed, the change of the Milnor number at p_0

$$\mu(\bar{X}_{t_0}, x) - \mu(\bar{X}_t, x) = (\Gamma \cdot \{t - t_0 = 0\})_{p_0} > 0,$$

where Γ is a polar curve and $(\,\cdot\,)_{p_0}$ denotes the intersection index at p_0. In local coordinates at p_0

$$\begin{aligned}\Gamma = \text{Closure}(\{(x, t) \notin A \times \mathbf{C} | \partial F/\partial x_0 = \partial F/\partial x_1 \\ = \cdots = \partial F/\partial x_{n-1} = 0\}).\end{aligned}$$

To prove the last statement of Theorem 1.4 we consider on $\mathbf{C}^n$ the following function

$$\varphi(x_1, \dots, x_n) = |x_1|^2 + \cdots + |x_n|^2.$$

Since φ is semi-algebraic, for each regular value t of f, the set of critical values of $\varphi_t = \varphi|_{X_t}$ is finite. Hence, we may assume that outside a ball big enough $B(0, R_t) = \{||x|| \geqslant R_t||\}$, φ_t is regular (this also follows from the curve selection lemma). Nevertheless, the radius R_t may depend on t, that is there could exist a sequence of points such that

$$||x|| \to \infty, \quad f(x) \to t, \quad \text{grad}\, \varphi_{f(x)}(x) = 0. \tag{3.6}$$

Assume that such sequence tends to $p' \in C_\infty$, $p = (p', t')$ is close to p_0. Then, in the local coordinates at p_0,

$$\varphi(x_0, \dots, x_{n-1}, t) = |x_0|^{-2}(|x_1|^2 + \cdots + |x_{n-1}|^2 + 1).$$

Note that

$$|x_0\,\partial\varphi/\partial x_0| \geqslant \|\partial\varphi/\partial x_i\|_{i=1,\dots,n-1}. \tag{3.7}$$

The critical points of φ_t are exactly those for which $\overline{\operatorname{grad}\varphi_t}$ and $\overline{\operatorname{grad} F|_{X_t}} = (\partial F/\partial x_0, \dots, \partial F/\partial x_{n-1})$ are parallel. By (3.7) this cannot happen if (3.4) holds at p. Therefore the only possible limits of points satisfying (3.6) are such $p \in A \times \mathbf{C}$ at which $\mathbf{a}_F$ fails. Near such point, say again $p_0 = ((0 : \cdots : 0 : 1), t_0)$, we replace φ by

$$\psi(x_0, \dots, x_{n-1}, t) = |x_0|^{-2}.$$

(which corresponds to $|x_n|^2$ in the affine piece $\mathbf{C}^n$). Note that ψ satisfies also (3.7). Therefore, if (3.4) holds near p, then we may "move" φ to ψ without producing new critical points. Since the points where $\mathbf{a}_F$ and so (3.4) fails are isolated we may use this construction to the boundary of a small neighbourhood U_{p_0} of p_0.

Thus, near ∂U_{p_0} we "move" φ to ψ and all the critical points (near H_∞) of this new function are those of ψ and they lie on the (absolute) polar curve

$$\begin{aligned}\Gamma_{ab} &= \text{Closure}(\{(x, t) \notin B \times \mathbf{C} | \partial F/\partial x_1 \\ &= \cdots = \partial F/\partial x_{n-1} = F = 0\}).\end{aligned} \tag{3.8}$$

Carrying out this procedure at each point of $A \times \{t_0\}$ at which $\mathbf{a}_F$ fails we construct a smooth functin $\tilde\varphi$ such that

(1) There is R such that $\tilde\varphi|_{X_{t_0}}$ has all its critical points in $B(0, R)$.
(2) For $t \neq t_0$ but close to t_0 all the critical points of $\tilde\varphi_t = \tilde\varphi|_{X_t}$ lie on the associated polar curves (3.8).

Hence, the following spaces are homotopically equivalent for t close to t_0

$$X_{t_0} \sim X_{t_0} \cap B(0, R) \sim X_t \cap B(0, R).$$

Now the critical points of $\tilde\varphi_t$ near the infinity are the critical points $\psi = |x_0|^{-2}$ which are the critical points of $x_0|_{X_t}$. Each critical point of $x_0|_{X_t}$ is isolated and by [M] it can by morsified (in complex sense) with a number of critical Morse points equal to the local Milnor number of $x_0|_{X_t}$ at this point. Therefore each such point contributes to the homotopy type of X_t by adding a number of n-handles. The total contribution of $p_0 \in A \times \{t_0\}$ equals the intersection index

$$(\Gamma_{ab} \cdot \{t - t_0 = 0\})_{p_0}.$$

To complete the proof it suffices to show

LEMMA 3.5.

$$(\Gamma_{ab}\,.\,\{t-t_0=0\})_{p_0}=(\Gamma\,.\,\{t-t_0=0\})_{p_0}.$$

Proof. Let

$$Y=\mathrm{Closure}(\{(x,t)\notin B\times\mathbf{C}|\partial F/\partial x_1=\cdots=\partial F/\partial x_{n-1}=0\}).$$

Then $Y=Y'\times\mathbf{C}$, where Y' is a curve, and $\Gamma_{ab}=Y\cap\{F=0\}$, $\Gamma=Y\cap\{\partial F/\partial x_0=0\}$. We assume for simplicity that Y' is irreducible and let $x(s)$ be a parametrization of Y'. Then, up to a factor which depends only on Y'

$$(\Gamma_{ab}\,.\,\{t-t_0=0\})_{p_0}=\mathrm{ord}_0F(x(s),t_0)-\mathrm{ord}_0F(x(s),t), \tag{3.9}$$

where $t\neq t_0$ and $\mathrm{ord}_0\alpha(s)$ denotes the order of vanishing of $\alpha(s)$ at 0. Analogously, up to the same factor,

$$(\Gamma\,.\,\{t-t_0=0\})_{p_0}=\mathrm{ord}_0\,\partial F/\partial x_0(x(s),t_0)$$
$$-\,\mathrm{ord}_0\,\partial F/\partial x_0(x(s),t). \tag{3.10}$$

Note that for any t near t_0 (including t_0) along the curve $(x(s),t)$

$$s\frac{\mathrm{d}F}{\mathrm{d}s}=s\frac{\mathrm{d}x_0}{\mathrm{d}s}\frac{\partial F}{\partial x_0},$$

which gives $\mathrm{ord}_0F(x(s),t)=\mathrm{ord}_0\,\partial F/\partial x_0(x(s),t)+\mathrm{ord}_0x_0(s)$, and since $\mathrm{ord}_0x_0(s)$ does not depend on t, (3.9) and (3.10) give the result.

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