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Étale fundamental groups of non-Archimedean analytic spaces

Dedicated to Frans Oort on the occasion of his 60th birthday

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1. Introduction

How to define the étale fundamental group of a non-Archimedean analytic space X ? As we know from the work of Grothendieck on the algebraic fundamental group for schemes, an étale fundamental group should be part of a dualism whose other constituent is the category of étale covering spaces. The definition of étale covering spaces was given by Berkovich, see [B2, Remark 6.3.4 (ii)]. This category includes the finite étale coverings of X and the topological covering spaces of X , of these two classes the first is dual to the algebraic fundamental group $\pi_1^{\text{alg}}(X)$ of X , the second gives rise to the topological fundamental group $\pi_1^{\text{top}}(X)$. A genuine example of an étale covering map is the logarithm

$$\log : \{z \in \mathbb{C}_p ; |z - 1| < 1\} \longrightarrow \mathbb{C}_p,$$

given by the usual power series. It is a Galois covering with group $\bigcup_n \mu_{p^n}(\mathbb{C}_p)$, and its existence evidences the nontriviality of the étale fundamental group of the affine line over \mathbb{C}_p , in contrast to the triviality of both the algebraic and topological fundamental group of $\mathbb{A}_{\mathbb{C}_p}^1$.

We may motivate the study of this étale fundamental group by analogy to the case of schemes or the case of complex analytic spaces. However, at least part of the theory in this work (especially in Sections 4, 6) was developed by the author in an attempt to understand the nature of the p -adic period maps introduced by Rapoport and Zink [RZ]. These maps are étale rigid analytic morphisms of the form

$$\pi : \mathcal{M}^{\text{rig}} \longrightarrow \mathcal{F}^{\text{wa}},$$

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and the logarithm is an example of such a morphism, see [RZ, Proposition 5.40]. In this formula, \mathcal{M} denotes a formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$ which represents a moduli problem for p -divisible groups quasi-isogenous to a fixed p -divisible group X_0 . The space $\mathcal{F}^{\mathrm{wa}}$ denotes the admissible open subset of weakly admissible filtrations in a flag variety of filtrations of the F -isocrystal associated to X_0 . Both the moduli problem and the flag variety involve polarizations and endomorphisms. We remark that in all cases π may be considered as an étale morphism of Berkovich spaces. Let U be the paracompact separated rigid variety whose associated analytic space (see [B2]) is the image of π , considered as a morphism of Berkovich spaces. The morphism $j : U \rightarrow \mathcal{F}^{\mathrm{wa}}$ is injective and identifies local rings, and if Fontaine's conjecture - weakly admissible \Leftrightarrow admissible - holds then j is actually a surjection (unless $\mathcal{M}^{\mathrm{rig}}$ is empty, see [RZ, Proposition 5.20]). However, even granting this conjecture, in general it cannot be expected that j is an isomorphism, see [RZ, 5.41].

By definition the morphism π factors as $j \circ \pi'$ for some

$$\pi' : \mathcal{M}^{\mathrm{rig}} \longrightarrow U.$$

Using the results of this article it follows that π' is an étale covering map. More precisely, it follows that there is a natural local system of \mathbb{Q}_p -vectorspaces \mathcal{V} on U , such that $\mathcal{M}^{\mathrm{rig}}$ is identified with a union of connected components in the space of lattices in \mathcal{V} . The arguments for this - see Section 7 - are given only in a special case, namely the case where \mathcal{M} is the Lubin-Tate moduli space, treated by Hopkins and Gross, [HG]. In this case, we have that $\mathcal{F}^{\mathrm{wa}} = \mathcal{F} = \mathbb{P}^{h-1}$, and, by the results of [HG], we know that $U \cong \mathbb{P}^{h-1}$. In particular, from the fact that π is not an isomorphism we deduce that $\pi_1(\mathbb{P}_{\mathbb{C}_p}^{h-1}) \neq \{1\}$.

Moreover, the local system \mathcal{V} corresponds to a continuous homomorphism

$$\pi_1(U) \rightarrow G(\mathbb{Q}_p),$$

where G is a certain algebraic group associated to the moduli problem. It is natural to ask what the image of this homomorphism is, or what the image of the geometric fundamental group $\pi_1(U_{\mathbb{C}_p})$ is. In the Lubin-Tate case we have $G = \mathrm{GL}_h$, where h is the height of the associated formal group and we can answer these questions. We prove in the final section that the map

$$\pi_1(\mathbb{P}_{\mathbb{C}_p}^{h-1}) \rightarrow \mathrm{SL}_h(\mathbb{Q}_p)$$

is a surjection.

The general theory of the étale fundamental group is exposed using the k -analytic spaces developed by Berkovich. In the proof of the duality theorem it is used in an essential way that such a space is locally path connected; I do not know how to formulate the corresponding fact for rigid spaces other than referring to these k -analytic spaces. However, in Section 5 we translate the results back to rigid analytic varieties. Indeed, when we speak in Section 6 about p -divisible

groups and filtered F -crystals over analytic spaces, it seems more natural to use the terminology of rigid analytic varieties. The result of this section is that the category of p -divisible groups up to isogeny over a rigid analytic space X is equivalent to a full subcategory of the category of filtered F -crystals, and in case X is reduced, equivalent to a full subcategory of the category of local systems of \mathbb{Q}_p -vector spaces. For example, the local system \mathcal{V} above is given by a p -divisible group up to isogeny over U .

The author thanks Prof. M. Rapoport and Prof. T. Zink for many discussions concerning the p -adic period maps and other subjects, for example the idea of having a local system of \mathbb{Q}_p -vector spaces on U is already contained in [RZ]. Further, he would like to thank Prof. D. Gross and Prof. J.-K. Yu for many discussions concerning the structure of the p -adic period map in the Lubin-Tate case, let us mention here that it was proved first by Prof. Yu that the period map is an étale covering map in this case. He also thanks Prof. M. Hopkins, Prof. W. Messing, and Prof. M. van der Put. Thanks are due to the referee, who corrected some mistakes, and gave a simpler proof of Proposition 7.5 which originally was proved only for special fields k . Finally, it is his pleasure to thank Prof. F. Oort, to whom this article is dedicated, for his constant help and advice during the last years.

2. Étale covering spaces of Berkovich spaces

Let k be a field complete with respect to a non-Archimedean valuation $|\cdot|$; we do not assume that $|\cdot|$ is nontrivial. In this section we work with k -analytic spaces as defined in [B2, page 17]. Let X be such a space. We remark that the underlying topological space $|X|$ is locally connected, see [B1, Corollary 2.2.8].

DEFINITION 2.1. (See [B2, Remark 6.3.4 (ii)].) Let $f : Y \rightarrow X$ be a morphism of k -An. We say that Y is an *étale covering space* of X or that f is an *étale covering map* if for all $x \in X$ there exists an open neighbourhood $\mathcal{U} \subset X$ of x such that $f^{-1}(\mathcal{U})$ is a disjoint union of spaces \mathcal{V}_i , each mapping finite étale to \mathcal{U} . We say that Y is a *topological covering space* of X if we can choose \mathcal{U} and the \mathcal{V}_i as above such that all the maps $\mathcal{V}_i \rightarrow \mathcal{U}$ are isomorphisms.

We note that finite étale morphisms $Y \rightarrow X$ are étale covering spaces; these are the coverings used in algebraic geometry to define the algebraic fundamental group. A morphism of étale covering spaces Y_1, Y_2 of X is a morphism $Y_1 \rightarrow Y_2$ compatible with the structural maps $Y_i \rightarrow X$. From the definition it follows that a fibre product $Y_1 \times_Y Y_2$ of étale covering spaces is an étale covering space. If $g : Y_1 \rightarrow Y_2$ is a morphism of such covering spaces then g is an étale covering map. However, it is in general not true that a composition of étale covering maps is an étale covering map. Finite disjoint unions of étale covering spaces are étale covering spaces; the empty space is an étale covering space of X . Infinite disjoint unions of étale covering spaces do not give étale covering spaces in general.

LEMMA 2.2. Let $f : Y \rightarrow X$ be an étale covering map.

- (i) The morphism f is étale and separated.
- (ii) For any morphism $Z \rightarrow X$ of An_k (see [B2, page 25]) the fibre product $Y \times_X Z$ is an étale covering space of Z .
- (iii) If X is paracompact, then so is Y . In this case, if Y is connected, the fibres of f are at most countable.
- (iv) The image of f is a union of connected components of X .

Proof. Statements (i), (ii) and (iv) follow immediately from the definitions. To prove (iii) we note that since $|X|$ is Hausdorff and $|Y| \rightarrow |X|$ separated, we get that $|Y|$ is Hausdorff. By definition there exists an open covering $X = \bigcup_i \mathcal{U}_i$ such that $f^{-1}(\mathcal{U}_i) = \coprod \mathcal{V}_{ij}$, with $\mathcal{V}_{ij} \rightarrow \mathcal{U}_i$ finite étale. Consider a closed locally finite refinement $X = \bigcup_s Z_s$ of the covering $X = \bigcup_i \mathcal{U}_i$ ([En, Theorem 5.1.11]). For each s we choose an i such that $Z_s \subset \mathcal{U}_i$ and put $W_{sj} = f^{-1}(Z_s) \cap \mathcal{V}_{ij}$. These are closed subsets of Y . The map $W_{sj} \rightarrow Z_s$ has finite fibres, Z_s is paracompact, hence W_{sj} is paracompact [En, Theorem 5.1.35].

Take $y \in Y$ and let $x = f(y)$. By definition we may choose an open neighbourhood \mathcal{U} of x such that the set $\{s' \mid Z_{s'} \cap \mathcal{U} \neq \emptyset\}$ is finite, say it consists of s_1, \dots, s_r . We replace \mathcal{U} by $\mathcal{U} \setminus Z_{s_i}$ if $x \notin Z_{s_i}$. Hence we may assume $x \in Z_{s_i}$ for all i . Shrinking \mathcal{U} we may therefore assume that $\mathcal{U} \subset \mathcal{U}_i$, where \mathcal{U}_i is the chosen subset such that $Z_{s_i} \subset \mathcal{U}_i$. Finally, we may choose \mathcal{U} so small that for some connected open neighbourhood \mathcal{V} of y the map $\mathcal{V} \rightarrow \mathcal{U}$ is finite étale. It is clear that for each $i = 1, \dots, r$ there is a unique j_i such that $\mathcal{V} \subset \mathcal{V}_{ij_i}$. Thus among the W_{sj} only the sets $W_{s_i j_i}$ meet \mathcal{V} . We get that the covering $Y = \bigcup W_{sj}$ is locally finite and Y is paracompact [En, Theorem 5.1.34]. If Y is connected, then Y is Lindelöf [En, Theorem 5.1.27]. Since the fibre $f^{-1}(x) \subset Y$ is a discrete closed subspace of Y , it is Lindelöf and hence countable. \square

LEMMA 2.3. Let \mathcal{F} be a sheaf on the étale site $X_{\text{ét}}$ of X . Suppose that $\{g_i : U_i \rightarrow X\}$ is a covering for the étale topology of X . The sheaf \mathcal{F} is representable by an étale covering space of X if and only if \mathcal{F}_{U_i} is representable by an étale covering space of U_i for each i .

Proof. The only if statement follows from Lemma 2.2 and [B2, Corollary 4.1.4]. Assume that \mathcal{F}_{U_i} is representable by an étale covering space for each i . Take $x \in X$. By assumption there exists an open neighbourhood $\mathcal{U} \subset X$, a finite étale morphism $g : \mathcal{V} \rightarrow \mathcal{U}$ and a point $y \in \mathcal{V}$ mapping to x such that $\mathcal{F}_{\mathcal{V}}$ is representable by an étale covering space $Y_{\mathcal{V}}$ over \mathcal{V} . By shrinking \mathcal{U} and \mathcal{V} we may even assume that \mathcal{V} is Galois over \mathcal{U} , say with group G . The action of G on \mathcal{V} lifts to an action on $Y_{\mathcal{V}}$, since this action lifts to $\mathcal{F}_{\mathcal{V}}$. Using that $Y_{\mathcal{V}}$ is an étale covering map, we may shrink \mathcal{V} further to get the situation where $Y_{\mathcal{V}}$ is a disjoint union of spaces finite étale over \mathcal{V} . There is an étale covering space $Y_{\mathcal{U}} \rightarrow \mathcal{U}$ such that there is a G -equivariant isomorphism $Y_{\mathcal{U}} \times_{\mathcal{U}} \mathcal{V} \cong Y_{\mathcal{V}}$. To prove this, write $Y_{\mathcal{V}} = \coprod \mathcal{V}_j$ such that $\mathcal{V}_j \rightarrow \mathcal{V}$ is finite étale and $\mathcal{V}_j \subset Y_{\mathcal{V}}$ is G -stable. This is possible as G is finite. The spaces \mathcal{V}_j and \mathcal{V} may be viewed as (finite étale) schemes over (any affinoid subdomain of) \mathcal{U} , see [B2, Section 2.6]. In the category of schemes we know the result; this gives

the spaces \mathcal{V}_j/G over \mathcal{U} and the disjoint union of these gives $Y_{\mathcal{U}}$. Obviously, $Y_{\mathcal{U}}$ represents $\mathcal{F}_{\mathcal{U}}$ and is an étale covering space of \mathcal{U} . We leave it to the reader to glue the spaces $Y_{\mathcal{U}}$ to an étale covering space Y representing \mathcal{F} (use [B2, Proposition 1.3.3]). \square

LEMMA 2.4. Consider an étale covering space Y of X . Suppose $R \subset Y \times_X Y$ is a union of connected components, which is an equivalence relation on Y over X . The quotient sheaf Y/R on $X_{\text{ét}}$ is representable by an étale covering space of X .

Proof. By the previous lemma the question is local on X . By definition of étale covering spaces this reduces the question to the case where $Y \rightarrow X$ is finite étale. In case X is affinoid, the spaces Y and R are affinoid also and the result is known by algebraic geometry [SGA1, Exposé V]. As usual we leave the gluing process to the reader. \square

EXAMPLE 2.5. Suppose that $g : Y_1 \rightarrow Y_2$ is a morphism of étale covering spaces of X . The fibre product $R = Y_1 \times_{Y_2} Y_1$ is a union of connected components of $Y_1 \times_X Y_1$, since the map $R \rightarrow Y_1 \times_X Y_1$ is an injective étale covering map. Thus R is an equivalence relation on Y_1 over X . Of course the quotient Y_1/R is represented by $\text{Im}(g) \subset Y_2$, a union of connected components of Y_2 (Lemma 2,2). The result of Lemma 2.4 is that any equivalence relation as in the lemma is realized in this fashion. \square

LEMMA 2.6. The category of topological covering spaces of X is equivalent to the category of covering spaces of $|X|$.

Proof. Suppose $\mathcal{T} \rightarrow |X|$ is a covering space. We define a sheaf $\mathcal{F} = \mathcal{F}(\mathcal{T})$ on $X_{\text{ét}}$ as follows: The sections of \mathcal{F} over $g : U \rightarrow X$ étale are morphisms $|U| \rightarrow \mathcal{T}$ of topological spaces over $|X|$. Using Lemma 2.3 we see immediately that \mathcal{F} is representable by a topological covering space $T \rightarrow X$ with $T \cong |T|$. This defines a quasi-inverse to the functor $T \mapsto |T|$. \square

To define the fundamental group of X we need some notations. We use the symbole $\underline{\text{Cov}}_X$ (resp. $\underline{\text{Cov}}_X^{\text{top}}$, $\underline{\text{Cov}}_X^{\text{alg}}$) to denote the category of étale covering spaces of X (resp. topological covering spaces of X , finite étale covering spaces of X). A geometric point \bar{x} of X is a morphism $\bar{x} : \mathcal{M}(K) \rightarrow X$ of An_k where K is an algebraically closed complete valued extension of k . As in [SGA1] we consider the fibre functor

$$F_{\bar{x}} : \underline{\text{Cov}}_X \longrightarrow \underline{\text{Sets}}$$

defined by the formula

$$F_{\bar{x}}(Y) = \{ \bar{y} : \mathcal{M}(K) \rightarrow Y \mid f(\bar{y}) = \bar{x} \}.$$

Note that $F_{\bar{x}}$ is compatible with fibre products and disjoint unions. We denote $F_{\bar{x}}^{\text{top}}$, resp. $F_{\bar{x}}^{\text{alg}}$ the restriction of $F_{\bar{x}}$ to the subcategory $\underline{\text{Cov}}_X^{\text{top}}$, resp. $\underline{\text{Cov}}_X^{\text{alg}}$.

The *fundamental group of X with base point \bar{x}* is a topological group. The underlying abstract group is

$$\pi_1(X, \bar{x}) = \text{Aut}(F_{\bar{x}}),$$

i.e., it is the group of isomorphisms of functors $F_{\bar{x}} \rightarrow F_{\bar{x}}$. We define a topology on $\pi_1(X, \bar{x})$ as follows. For any pair (Y, \bar{y}) , $Y \in \underline{\text{Cov}}_X$ and $\bar{y} \in F_{\bar{x}}(Y)$ we put

$$H(Y, \bar{y}) \subset \pi_1(X, \bar{x})$$

equal to the stabilizer of \bar{y} for the natural action of $\pi_1(X, \bar{x})$ on $F_{\bar{x}}(Y)$. The family \mathcal{F} of subgroups so obtained satisfies i) $H, H' \in \mathcal{F} \Rightarrow H \cap H' \in \mathcal{F}$ and ii) $H \in \mathcal{F}$, $\gamma \in \pi_1(X, \bar{x}) \Rightarrow \gamma H \gamma^{-1} \in \mathcal{F}$. The first property follows from the existence of fibre products in $\underline{\text{Cov}}_X$; the second since $\gamma H(Y, \bar{y}) \gamma^{-1} = H(Y, \gamma \cdot \bar{y})$. Consequently, there is a unique topology on $\pi_1(X, \bar{x})$ making it into a topological group such that the family of $H \in \mathcal{F}$ forms a cofinal system of open neighbourhoods of the identity element (see [Bou, General topology III §2]).

The same definition, but using the functor $F_{\bar{x}}^{\text{top}}$ (resp. $F_{\bar{x}}^{\text{alg}}$), gives a topological group $\pi_1^{\text{top}}(X, \bar{x})$ (resp. $\pi_1^{\text{alg}}(X, \bar{x})$). These are called the topological (resp. algebraic) fundamental group of X . By Lemma 2.6 the topological group $\pi_1^{\text{top}}(X, \bar{x})$ depends only on the pair $(|X|, x)$, where $x \in X$ denotes the unique point in the image of \bar{x} . There are restriction maps $\text{Aut}(F_{\bar{x}}) \rightarrow \text{Aut}(F_{\bar{x}}^{\text{top}})$ and $\text{Aut}(F_{\bar{x}}) \rightarrow \text{Aut}(F_{\bar{x}}^{\text{alg}})$; these give rise to continuous homomorphisms $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\text{top}}(X, \bar{x})$ and $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\text{alg}}(X, \bar{x})$. Finally, there is an obvious functoriality of these fundamental groups with respect to morphisms of pairs $(Y, \bar{y}) \rightarrow (X, \bar{x})$.

LEMMA 2.7. The topological group $\pi_1(X, \bar{x})$ is Hausdorff and prodiscrete. More precisely, the map

$$\pi_1(X, \bar{x}) \longrightarrow \varprojlim_{H \in \mathcal{F}} \pi_1(X, \bar{x})/H$$

is an isomorphism of topological spaces. Similar for the topological resp. algebraic fundamental group of X .

Proof. Suppose that $(\gamma_H)_{H \in \mathcal{F}}$ is an element of the inverse limit. For each pair (Y, \bar{y}) , $Y \in \underline{\text{Cov}}_X$, $\bar{y} \in F_{\bar{x}}(Y)$ we can define $\gamma \cdot \bar{y}$ as $\tilde{\gamma}_H \cdot \bar{y}$ where $H = H(Y, \bar{y})$ and $\tilde{\gamma}_H \in \pi_1(X, \bar{x})$ is a lift of the element $\gamma_H \in \pi_1(X, \bar{x})/H$. It is clear that this is well defined. We leave it to the reader to see that this defines an element $\gamma \in \text{Aut}(F_{\bar{x}}) = \pi_1(X, \bar{x})$. This is the inverse to the natural map of the lemma; the statement on topologies is clear. \square

Let us introduce some notation. If G is a topological group we write $G\text{-Sets}$ to denote the category of pairs (V, ρ) , where V is a set and $\rho : G \times V \rightarrow V$ is an action of G on V which is continuous for the discrete topology on V . This means that the stabilizer of any element of V is open in G . Let us write *finite- G -Sets* for the full subcategory consisting of pairs (V, ρ) with V finite.

It follows immediately from the definitions that $F_{\bar{x}}$ may be considered as a functor

$$F_{\bar{x}} : \underline{\text{Cov}}_X \longrightarrow \pi_1(X, \bar{x})\text{-}\underline{\text{Sets}}.$$

The same holds for $F_{\bar{x}}^{\text{top}}$; the functor $F_{\bar{x}}^{\text{alg}}$ is seen as a functor

$$F_{\bar{x}}^{\text{alg}} : \underline{\text{Cov}}_X^{\text{alg}} \longrightarrow \text{finite-}\pi_1^{\text{alg}}(X, \bar{x})\text{-}\underline{\text{Sets}}.$$

EXAMPLE 2.8. Suppose that $X = \{x\}$ consists of one point and that $\bar{x} : \mathcal{M}(K) \rightarrow X$ is given by the embedding $\mathcal{H}(x) \rightarrow K$. We write $\mathcal{H}(x)^{\text{sep}}$ for the separable closure of $\mathcal{H}(x)$ in K . In this case, étale covering spaces $Y \rightarrow X$ with Y connected correspond to finite separable field extensions $\mathcal{H}(x) \subset L$. The set $F_{\bar{x}}(Y)$ is equal to the set of $\mathcal{H}(x)$ -embeddings $L \rightarrow K$.

From this discussion it follows that $\underline{\text{Cov}}_X = \text{Ind}(\underline{\text{Cov}}_X^{\text{alg}})$. By Grothendieck's theory of the fundamental group [SGA1, Exposé V] we get the equalities $\pi_1(X, \bar{x}) = \pi_1^{\text{alg}}(X, \bar{x}) = \text{Gal}(\mathcal{H}(x)^{\text{sep}}/\mathcal{H}(x))$ and the duality statements that $\underline{\text{Cov}}_X^{\text{alg}} \cong \text{finite-}\pi_1^{\text{alg}}(X, \bar{x})\text{-}\underline{\text{Sets}}$ and $\underline{\text{Cov}}_X \cong \pi_1(X, \bar{x})\text{-}\underline{\text{Sets}}$. Finally, we note the trivial fact that $\pi_1^{\text{top}}(X, \bar{x}) = \{1\}$. \square

THEOREM 2.9. Suppose X is connected. For any two geometric points \bar{x}, \bar{x}' there exists an isomorphism of functors $F_{\bar{x}} \cong F_{\bar{x}'}$.

This will be proved later on; first we deduce the duality theorem from it.

THEOREM 2.10. Here \bar{x} is a geometric point of the connected k -analytic space X .

(i) The functor $F_{\bar{x}}$

$$\underline{\text{Cov}}_X \longrightarrow \pi_1(X, \bar{x})\text{-}\underline{\text{Sets}}$$

is fully faithful; any $\pi_1(X, \bar{x})$ -set which consists of a single orbit is in the essential image of $F_{\bar{x}}$. Thus the category $\pi_1(X, \bar{x})\text{-}\underline{\text{Sets}}$ is naturally equivalent to the category of disjoint unions of étale covering spaces of X . An analogous result holds for the functor $F_{\bar{x}}^{\text{top}}$. The functor $F_{\bar{x}}^{\text{alg}}$

$$\underline{\text{Cov}}_X^{\text{alg}} \longrightarrow \text{finite-}\pi_1^{\text{alg}}(X, \bar{x})\text{-}\underline{\text{Sets}}$$

is an equivalence.

(ii) The topology on $\pi_1^{\text{top}}(X, \bar{x})$ is defined by a fundamental system of open normal subgroups. In the category $\underline{\text{Cov}}_X^{\text{top}}$ any connected object is covered by a Galois covering of X .

(iii) The topological group $\pi_1^{\text{alg}}(X, \bar{x})$ is profinite. Any continuous map of $\pi_1(X, \bar{x})$ to a pro-finite group factors through $\pi_1^{\text{alg}}(X, \bar{x})$.

(iv) The maps $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\text{top}}(X, \bar{x})$ and $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\text{alg}}(X, \bar{x})$ have dense image.

Proof. For any $Y \in An_k$ and geometric points \bar{y}, \bar{y}' of Y we call a path connecting \bar{y} to \bar{y}' an isomorphism $\alpha : F_{\bar{y}} \rightarrow F_{\bar{y}'}$. This is functorial: If $f : Y \rightarrow Z$ is a morphism in An_k we get a path $f(\alpha)$ on Z connecting $f \circ \bar{y}$ to $f \circ \bar{y}'$. Use Lemma 2.2. If $f \circ \bar{y} = f \circ \bar{y}'$ then we get a loop $f(\alpha) \in \pi_1(Z, \bar{z})$ with $\bar{z} = f \circ \bar{y}$. If we are in the special situation that $f : Y \rightarrow X$ is an étale covering map and $\bar{y}, \bar{y}' \in F_{\bar{x}}(Y)$ then $\gamma = f(\alpha) \in \pi_1(X, \bar{x})$ will be an element such that $\gamma \cdot \bar{y} = \bar{y}'$. Therefore, Theorem 2.9 implies that the orbits of $\pi_1(X, \bar{x})$ on $F_{\bar{x}}(Y)$ correspond bijectively to the connected components of Y .

This implies that the functor $F_{\bar{x}}$ is fully faithful: morphisms $Y_1 \rightarrow Y_2$ in Cov_X correspond 1-1 with connected components Γ of $Y_1 \times_X Y_2$ such that $F_{\bar{x}}(\Gamma) \rightarrow F_{\bar{x}}(Y_1)$ is bijective. Consider the $\pi_1(X, \bar{x})$ -set $\pi_1(X, \bar{x})/H$, where $H \subset \pi_1(X, \bar{x})$ is an open subgroup. By definition this means that $H(Y, \bar{y}) \subset H$ for some $Y \in \text{Cov}_X$ and $\bar{y} \in F_{\bar{x}}(Y)$. We have to show that $H = H(Y', \bar{y}')$ for some pair (Y', \bar{y}') . The connected components of $Y \times_X Y$ correspond 1-1 to orbits in $F_{\bar{x}}(Y \times_X Y) = F_{\bar{x}}(Y) \times F_{\bar{x}}(Y)$. Let $R \subset Y \times_X Y$ correspond to the union of the orbits of the points $(\bar{y}, h\bar{y})$ where $h \in H$. It is easily seen that R is an equivalence relation on Y over X . The result follows by taking $Y' = Y/R$ (see Lemma 2.4).

The same arguments work for the case of topological covering spaces. Regarding finite étale coverings and the algebraic fundamental group of X we may refer to [SGA1, Exposé V]. Further, statement (iii) is also clear.

To see that any connected topological covering space $Y \rightarrow X$ is dominated by a Galois covering is a topological question (see Lemma 2.6). For any open covering $\mathcal{U} : |X| = \bigcup \mathcal{U}_i$ there exists a universal covering space $\tilde{X}_{\mathcal{U}} \rightarrow |X|$ dominating any covering split over the members of \mathcal{U} . The construction of $\tilde{X}_{\mathcal{U}}$ works for arbitrary locally connected spaces X . Let $\mathcal{U} : X = \bigcup \mathcal{U}_i$ be an open covering with connected \mathcal{U}_i . Let $N(\mathcal{U})$ be the nerve of the covering \mathcal{U} . The category of topological covering spaces of X split over \mathcal{U} is equivalent to the category of coverings of $N(\mathcal{U})$. The result follows as there is a universal covering \tilde{N} of the simplicial set $N(\mathcal{U})$.

The final statement is a formal consequence of the above and the fact that the functors $\text{Cov}_X^{\text{top}} \rightarrow \text{Cov}_X$ and $\text{Cov}_X^{\text{alg}} \rightarrow \text{Cov}_X$ are fully faithful. □

Remark 2.11. (i) The map $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\text{alg}}(X, \bar{x})$ is not surjective in general. To see this take X to be two copies of the projective line (over k algebraically closed) glued together in 0 and ∞ . In this case $\pi_1^{\text{alg}}(X, \bar{x}) = \hat{\mathbb{Z}}$ and $\pi_1^{\text{top}}(X, \bar{x}) = \mathbb{Z}$ (since $\pi_1^{\text{top}}(|\mathbb{P}^1|, \bar{x}) = \{1\}$, see [B1, 4.2.2]). The map $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\text{alg}}(X, \bar{x}) = \hat{\mathbb{Z}}$ factors through $\mathbb{Z} \subset \hat{\mathbb{Z}}$ in this case. We are able to show that it is surjective in certain cases, see Proposition 7.5.

(ii) We do not know whether $\pi_1^{\text{top}}(X, \bar{x})$ is in general a discrete topological group. This would follow if the topological space $|X|$ were semi-locally simply connected (in the sense of covering spaces). For the case of curves, see 3.9.

(iii) We do not know whether the map $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\text{top}}(X, \bar{x})$ is surjective in general. This is true if $\pi_1^{\text{top}}(X, \bar{x})$ is a discrete topological group: if in this case

(Y, \bar{y}) is the universal topological covering space, $\gamma \in \pi_1^{\text{top}}(X, \bar{x})$ and α is a path connecting \bar{y} to $\gamma\bar{y}$ then $\tilde{\gamma} = f(\alpha) \in \pi_1(X, \bar{x})$ maps to γ .

PROOF OF THEOREM 2.9. It suffices to prove the theorem for connected affinoid spaces X . Thus we assume X is affinoid; using the functoriality of paths the reader may even suppose X is strictly affinoid.

For any finite open covering $\mathcal{U} : X = \bigcup_{i=1}^n \mathcal{U}_i$ of X we denote $\underline{\text{Cov}}_{X, \mathcal{U}} \subset \underline{\text{Cov}}_X$ the full subcategory of étale covering maps $f : Y \rightarrow X$ that are split over the members of \mathcal{U} : the spaces $f^{-1}(\mathcal{U}_i)$ are disjoint unions of spaces lying finite étale over \mathcal{U}_i . By our definitions we have

$$\underline{\text{Cov}}_X = \varinjlim_{\mathcal{U}} \underline{\text{Cov}}_{X, \mathcal{U}}.$$

Let us denote by $F_{\bar{x}, \mathcal{U}}$ (resp. $F_{\bar{x}', \mathcal{U}}$) the fibre functor $F_{\bar{x}}$ (resp. $F_{\bar{x}'}$) restricted to $\underline{\text{Cov}}_{X, \mathcal{U}}$. It is clear that

$$\text{Isom}(F_{\bar{x}}, F_{\bar{x}'}) = \varinjlim_{\mathcal{U}} \text{Isom}(F_{\bar{x}, \mathcal{U}}, F_{\bar{x}', \mathcal{U}}).$$

By [B1, 3.2.1] the space $|X|$ is arcwise connected. Let us choose a closed subset $\ell \subset |X|$ and an isomorphism $[0, 1] \rightarrow \ell$ such that 0 corresponds to x and 1 corresponds to x' (see [En, 6.3.11 & 6.3.12]). Let us consider coverings $\mathcal{U} : X = \bigcup_{i=1}^n \mathcal{U}_i$ such that there exists an m , $1 \leq m \leq n$ with

$$\begin{aligned} \mathcal{U}_1 \cap \ell &= [0, t_1), \\ \mathcal{U}_i \cap \ell &= (s_i, t_i), \quad i = 2, \dots, m-1 \\ \mathcal{U}_m \cap \ell &= (s_m, 1] \text{ and} \\ \mathcal{U}_i \cap \ell &= \emptyset, \quad i \leq m+1 \end{aligned}$$

where

$$0 < s_2 < t_1 < \dots < s_m < t_{m-1} < 1.$$

We note that these coverings are cofinal in the system of all finite open coverings of X . For any such \mathcal{U} we choose

$$x_i \in \mathcal{U}_i \cap \mathcal{U}_{i+1} \cap \ell, \quad i = 1, \dots, m-1$$

and a geometric point \bar{x}_i lying over x_i . For convenience we put $x_0 = x$, $\bar{x}_0 = \bar{x}$, $x_m = x'$ and $\bar{x}_m = \bar{x}'$.

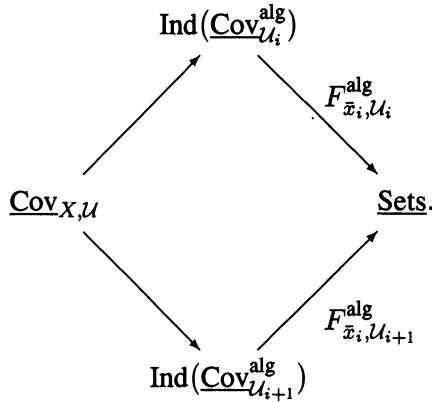
By [SGA1, Exposé V] we can find isomorphisms

$$\alpha_i : F_{\bar{x}_i, \mathcal{U}_{i+1}}^{\text{alg}} \longrightarrow F_{\bar{x}_{i+1}, \mathcal{U}_{i+1}}^{\text{alg}}$$

as functors on $\underline{\text{Cov}}_{\mathcal{U}_{i+1}}^{\text{alg}}$. We remark that this will also induce an isomorphism

$$\alpha_i : F_{\bar{x}_i, \mathcal{U}} \longrightarrow F_{\bar{x}_{i+1}, \mathcal{U}},$$

since each $F_{\bar{x}_i, \mathcal{U}}$ factors in two ways:



We put

$$K_{\mathcal{U}} \subset \text{Isom}(F_{\bar{x}, \mathcal{U}}, F_{\bar{x}', \mathcal{U}})$$

equal to all possible compositions $\alpha_{m-1} \circ \dots \circ \alpha_0$ of choices α_i as described above.

Note that the set $\text{Isom}(F_{\bar{x}, \mathcal{U}}, F_{\bar{x}', \mathcal{U}})$ has a natural Hausdorff topology which makes it into a principal homogeneous space under the topological group $\text{Aut}(F_{\bar{x}, \mathcal{U}})$. (Topologies defined as before.) The essential remark is that the subspaces $K_{\mathcal{U}}$ are (nonempty) compact. This follows from the fact that what we just defined was a continuous map

$$\begin{aligned}
 & \text{Isom}(F_{\bar{x}_0, \mathcal{U}_1}^{\text{alg}}, F_{\bar{x}_1, \mathcal{U}_1}^{\text{alg}}) \times \dots \times \text{Isom}(F_{\bar{x}_{m-1}, \mathcal{U}_m}^{\text{alg}}, F_{\bar{x}_m, \mathcal{U}_m}^{\text{alg}}) \\
 & \longrightarrow \text{Isom}(F_{\bar{x}, \mathcal{U}}, F_{\bar{x}', \mathcal{U}})
 \end{aligned}$$

of a compact topological space onto $K_{\mathcal{U}}$. (Each of the spaces occurring on the left is homogeneous under a profinite group.) Finally, if \mathcal{U}' refines \mathcal{U} the $K_{\mathcal{U}'}$ maps into $K_{\mathcal{U}}$ under the natural map

$$\text{Isom}(F_{\bar{x}, \mathcal{U}'}, F_{\bar{x}', \mathcal{U}'}) \longrightarrow \text{Isom}(F_{\bar{x}, \mathcal{U}}, F_{\bar{x}', \mathcal{U}}).$$

Modulo the usual arguments, this boils down to the fact that the subspace $K_{\mathcal{U}}$ is independent of the choice of the points x_i in the intersections $\mathcal{U}_i \cap \mathcal{U}_{i+1} \cap \ell$. To see this, for another set of choices x'_i , choose paths

$$\gamma_i : F_{\bar{x}_i, \mathcal{U}_i \cap \mathcal{U}_{i+1}}^{\text{alg}} \longrightarrow F_{\bar{x}'_i, \mathcal{U}_i \cap \mathcal{U}_{i+1}}^{\text{alg}}.$$

(This is possible, x_i and x'_i are connected by a segment of ℓ in $\mathcal{U}_i \cap \mathcal{U}_{i+1}$, hence lie in the same connected component.) We get $\alpha_{m-1} \circ \dots \circ \alpha_0 = \alpha'_{m-1} \circ \dots \circ \alpha'_0$ with $\alpha'_i = \gamma_{i+1} \circ \alpha_i \circ \gamma_i^{-1}$ (use that γ_i can be viewed both as a path in \mathcal{U}_i and in \mathcal{U}_{i+1}).

The theorem follows:

$$\emptyset \neq \varprojlim_{\mathcal{U}} K_{\mathcal{U}} \subset \varprojlim_{\mathcal{U}} \text{Isom}(F_{\bar{x}, \mathcal{U}}, F_{\bar{x}', \mathcal{U}})$$

since the limit of nonempty compact spaces is nonempty. □

Let us prove some general properties of these fundamental groups. Given a connected k -analytic space and a geometric point \bar{x} , the structure map $X \rightarrow \mathcal{M}(k)$ gives a homomorphism of fundamental groups.

PROPOSITION 2.12. If X is geometrically connected then the map

$$\pi_1(X, \bar{x}) \longrightarrow \pi_1(\mathcal{M}(k), \bar{x}) \cong \text{Gal}(k^{sep}/k)$$

is surjective; for general X the image of this map is open.

Proof. The fact that X is geometrically connected means that $\text{Gal}(k^{sep}/k)$ is a quotient of $\pi_1^{\text{alg}}(X, \bar{x})$. Hence Theorem 2.10 implies that the map has dense image in this case. Thus it suffices to prove the image is open in the general case. This assertion does not depend on the choice of the base point x of X . Hence, by [B2, Lemma 7.3.3], we may choose x such that the image of $\pi_1(x, \bar{x}) \rightarrow \text{Gal}(k^{sep}/k)$ has finite index. This map factors through $\pi_1(X, \bar{x})$, thus the result is clear. \square

We write \mathbb{C} for a completion of a separable closure of k ; note that \mathbb{C} is algebraically closed. Further, we choose a geometric point \bar{x} of the \mathbb{C} -analytic space $X_{\mathbb{C}}$. Note that \bar{x} can also be considered as a geometric point of X using the morphism $X_{\mathbb{C}} \rightarrow X$ of An_k .

PROPOSITION 2.13. If X is geometrically connected the sequence

$$\pi_1^{\text{alg}}(X_{\mathbb{C}}, \bar{x}) \longrightarrow \pi_1^{\text{alg}}(X, \bar{x}) \longrightarrow \text{Gal}(k^{sep}/k) \longrightarrow 1$$

is exact.

Proof. It is clear that the composition is trivial and that the right arrow is surjective. Let $Y \rightarrow X$ be finite étale and suppose there is a connected component $Y' \subset Y_{\mathbb{C}}$ mapping isomorphically to $X_{\mathbb{C}}$. We have to construct a map $X_{k'} \rightarrow Y$ for some finite separable extension $k \subset k'$. See [SGA 1, page 139]. Since by the lemma below the connected component Y' already exists over such a k' we are done. \square

LEMMA 2.14. Let $X \in k\text{-An}$.

- (i) The map $\varphi : |X_{\mathbb{C}}| \rightarrow |X|$ is open and compact.
- (ii) Let X be connected. The space $X_{\mathbb{C}}$ has finitely many connected components; there exists a finite separable field extension $k \subset k'$ such that the connected components of $X_{k'}$ correspond to those of $X_{\mathbb{C}}$.

Proof. Compactness of the map $X_{\mathbb{C}} \rightarrow X$ is in [B2, page 30]. If $x \in X_{\mathbb{C}}$ and $x \in \mathcal{U} \subset X_{\mathbb{C}}$ is an open neighbourhood, then by the argument of [B2, page 103] there exists a finite separable extension $k \subset k' \subset \mathbb{C}$ and an open $\mathcal{U}' \subset X_{k'}$ such that $x \in \mathcal{U}'_{\mathbb{C}} \subset \mathcal{U}$. Thus openness follows as the finite étale morphism $X_{k'} \rightarrow X$ is open.

To prove (ii) we take $x \in X$ as in [B2, Lemma 7.3.3]. Since φ is open any connected component of $X_{\mathbb{C}}$ has a nontrivial intersection with the set $\varphi^{-1}(x)$. This

set is equal to $\mathcal{M}(\mathcal{H}(x) \hat{\otimes} \mathbb{C})$, which is finite by our choice of x . Thus we see that the action of $\text{Gal}(k^{sep}/k)$ on the set of connected components of $X_{\mathbb{C}}$ is continuous. We leave it to the reader to see that these components actually exist over the field k' whose Galois group acts trivially on the set of components. \square

Remark 2.15. The author doesn't know whether the sequence $\pi_1(X_{\mathbb{C}}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow \text{Gal}(k^{sep}/k)$ is exact. However, an argument similar to the above shows that the closure of the image of the first arrow is the kernel of the second arrow.

3. The case of curves

In this section we consider a k -analytic space X , which is pure of dimension 1 and separated. Let us call such a space a curve. We put together some results on curves proved in the literature. An important result, proved by Berkovich, is that $|X|$ is locally simply connected.

LEMMA 3.1. An affinoid curve X has a finite boundary ∂X . In general the boundary ∂X of a curve X is a discrete subset of $|X|$.

Proof. By [B1, 2.5.12] we may assume X strictly affinoid. Since X allows a finite morphism to a closed disc $E = E(0, 1)$, it suffices to do the case $X = E$, use [B1, 2.5.8 & 2.5.13]. The boundary ∂E of E consists of the single point corresponding to the supremum norm; this can be seen by embedding E into the projective line and using [B1, 2.5.8 & 2.5.13]. The second assertion follows since the boundary of X is closed and its intersection with any affinoid is at most finite. \square

PROPOSITION 3.2. [FM] If $|k^*| \neq \{1\}$, then any irreducible compact curve is either affinoid or projective.

Proof. This result in case X is a strict k -analytic space is in [FM]. In the general case, take a finite affinoid G -covering $X = \bigcup U_i$. We can choose a field extension $k \subset K_r$ such that all the affinoid spaces $U_i \hat{\otimes} K_r$ and $(U_i \cap U_j) \hat{\otimes} K_r$ are strictly affinoid. The strictly K_r -analytic space $X \hat{\otimes} K_r$ is still irreducible. Suppose it is affinoid. In this case it follows that the banach k -algebra $A = \Gamma(X, \mathcal{O}_X)$ is such that $A \hat{\otimes} K_r$ is affinoid. Consequently, we get that A is affinoid and that $X \cong \mathcal{M}(A)$.

If $X \hat{\otimes} K_r$ is not affinoid then it is projective. It follows that X is closed, see [B2, 1.5.5]. By Lemma 3.5 we see that X is a strict k -analytic space. Hence we may apply [FM]. \square

Remark 3.3. If $|k^*| = 1$ the first part of the proof given above still works. To deal with the case $X \hat{\otimes} K_r$ projective, we reduce X and normalize. (We may even reduce to the case k algebraically closed.) Take $\mathcal{L} = \Omega_X^1$ in case the genus g of $X \hat{\otimes} K_r$ is at least 2 or put $\mathcal{L} = (\Omega_X^1)^{-1}$ for $g = 0$. The graded ring $R = \bigoplus \Gamma(X, \mathcal{L}^{\otimes n})$ is

such that $\text{Proj}(R \otimes K_r) \cong X \hat{\otimes} K_r$, since we know that $X \hat{\otimes} K_r$ is projective and $\mathcal{L} \hat{\otimes} K$ is ample. We deduce that $X \cong \text{Proj}(R)$. For the case $g = 1$, the difficulty remains to find a point $x \in X$ whose residue field is a finite extension of k .

COROLLARY 3.4. Any curve is a good k -analytic space.

Proof. Any point has a compact analytic domain as neighbourhood. This domain has finitely many components, each of which may be assumed to be affinoid by the proposition or the remark in case $|k^*| = 1$. In this case the domain is affinoid, since its normalisation is affinoid. \square

LEMMA 3.5. Suppose Y is a closed k -analytic space. If the valuation of k is nontrivial then Y has a unique structure of a strictly k -analytic space.

Proof. A closed k -analytic space is good, see [B2, Section 1.5]. Unicity by [B2, 1.2.17]. Since $Y \rightarrow \mathcal{M}(k)$ is good and closed, we may see Y as an analytic space in the sense of [B1] (use [B2, Section 1.5]) and use [B1, Proposition 3.1.2 (ii)] to conclude. \square

Remark 3.6. This implies, if $\sqrt{|k^*|} \neq \{1\}$, that a curve X has a big open part $\mathcal{U} = X \setminus \partial X$ which is a strict k -analytic space. This is true as $\partial \mathcal{U} = \emptyset$. In particular, any smooth curve, in the sense of [B2], is strict.

PROPOSITION 3.7. [B1, LP] Any curve is paracompact.

Proof. This may be deduced from the results of [B1, Section 4]; but here is another proof. It suffices to prove the result when the valuation on k is nontrivial: consider the surjective compact map $|X \hat{\otimes} K| \rightarrow |X|$ and use [En, 5.1.33]. We may even assume that $|k^*| = \mathbb{R}_+^*$ by taking K to be a complete valued field over k with $|K^*| = \mathbb{R}_+^*$. Clearly, this implies that X is strictly k -analytic, so that we may apply [LP]. \square

PROPOSITION 3.8. [B1] The space $|X|$ is locally simply connected in the sense of coverings.

Proof. This means that any point $x \in X$ has a neighbourhood \mathcal{U} such that $\pi_1^{\text{top}}(\mathcal{U}, x) = \{1\}$. By Lemma 3.4 we may assume X affinoid. The result follows from [B1, 4.3.3] if the valuation of k is nontrivial, from [B1, 4.3.5] if the valuation is trivial. \square

COROLLARY 3.9. (i) The topological group $\pi_1^{\text{top}}(X, \bar{x})$ is a discrete group; the homomorphism $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\text{top}}(X, \bar{x})$ is surjective.

(ii) The kernel of $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\text{top}}(X, \bar{x})$ is topologically generated by profinite subgroups.

Proof. For the first statement, see Remark 2.11. For the second, we consider for all geometric points \bar{x}' and paths $\alpha : F_{\bar{x}} \rightarrow F_{\bar{x}'}$ the map $\varphi_{\bar{x}', \alpha} : \pi_1(x', \bar{x}') \rightarrow$

$\pi_1(X, \bar{x})$ given by $\gamma \mapsto \alpha^{-1} \circ \gamma \circ \alpha$. The closure N of the subgroup generated by the images of the maps $\varphi_{\bar{x}', \alpha}$ is contained in the kernel of $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\text{top}}(X, \bar{x})$. To prove these subgroups are equal it suffices to show that any étale covering map $f : Y \rightarrow X$ where N acts trivially on $F_{\bar{x}}(Y)$ is a topological covering space. It is easily seen that N acts trivially if and only if for all $y' \in Y$ the field extension $\mathcal{H}(f(y')) \subset \mathcal{H}(y')$ is trivial.

Take a point $x' \in X$. There exists an open neighbourhood \mathcal{U} of x' such that $f^{-1}(\mathcal{U}) = \coprod \mathcal{V}_i$, each $\mathcal{V}_i \rightarrow \mathcal{U}$ finite étale. By the lemma below the maps $\mathcal{V}_i \rightarrow \mathcal{U}$ are topological covering maps. By the proposition above we could have chosen \mathcal{U} to be simply connected. In this case we see that f splits over \mathcal{U} , i.e., we see that f is a topological covering map. \square

LEMMA 3.10. If $f : Z \rightarrow Y$ is a finite étale morphism and all the field extensions $\mathcal{H}(f(z)) \subset \mathcal{H}(z)$ are trivial, then f is a topological covering map.

Proof. This is easy, compare [B2, Section 6.3]. \square

Remark 3.11. Another proof of the result of the proposition can be deduced from Proposition 5.3 below. As in [B1] this relies on the stable reduction theorem for algebraic curves.

4. Berkovich spaces and local systems

Let k be a non-Archimedean valued field and let X be an analytic space over k . We fix any prime number ℓ .

The most convenient way to define a *local system of \mathbb{Z}_ℓ -lattices* is the following: It is an abelian sheaf \mathcal{F} on $\text{Ét}(X)$, which is ℓ -divisible, ℓ -power torsion and such that $\mathcal{F}[\ell]$ is locally constant with finite fibres. This means that $\mathcal{F} = \bigcup \mathcal{F}_n$, with $\mathcal{F}_n = \mathcal{F}[\ell^n]$, each \mathcal{F}_n is a finite locally free sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$ -modules on $\text{Ét}(X)$. It is clear what morphisms of such local systems are. Let us denote $\mathbb{Z}_\ell\text{-LOC}_X$ the \mathbb{Z}_ℓ -linear category so obtained. Note that this category has a tensor product and internal Hom satisfying the usual axioms. For any morphism $Y \rightarrow X$ in An_k there are pullback functors $\mathbb{Z}_\ell\text{-LOC}_X \rightarrow \mathbb{Z}_\ell\text{-LOC}_Y$.

Suppose \bar{x} is a geometric point of X . We define the *stalk* $\mathcal{F}_{\bar{x}}$ of \mathcal{F} at \bar{x} to be

$$\mathcal{F}_{\bar{x}} = \varprojlim_n \mathcal{F}_{n, \bar{x}},$$

where the transition maps are given by multiplication by ℓ . This is a finite free \mathbb{Z}_ℓ -module. The rank of this module is a locally constant function on $|X|$. Further, it is clear that the \mathbb{Z}_ℓ -module $\mathcal{F}_{\bar{x}}$ has a continuous $\pi_1^{\text{alg}}(X, \bar{x})$ -action. In fact if X is connected this gives an equivalence of $\mathbb{Z}_\ell\text{-LOC}_X$ with the category of such $\pi_1^{\text{alg}}(X, \bar{x})$ -modules. This equivalence is compatible with tensor product and internal Hom. By Theorem 2.10 we may also replace $\pi_1^{\text{alg}}(X, \bar{x})$ by the topological group $\pi_1(X, \bar{x})$.

Suppose that we have a site C and a category $p : \mathcal{S} \rightarrow C$ fibred in \mathbb{Z}_ℓ -linear categories over C . There is an obvious category $\mathcal{S} \otimes_{\mathbb{Q}_\ell}$ fibred in \mathbb{Q}_ℓ -linear categories over C : The objects of the categories are the same, for objects A, B in the fibre category $\mathcal{S}_x = p^{-1}(x)$, $x \in C$ we have

$$\text{Hom}_{(\mathcal{S} \otimes_{\mathbb{Q}_\ell})_x}(A, B) = \text{Hom}_{\mathcal{S}_x}(A, B) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Even if \mathcal{S} is a stack, the category $\mathcal{S} \otimes_{\mathbb{Q}_\ell}$ is not a stack in general. The stack associated to $\mathcal{S} \otimes_{\mathbb{Q}_\ell}$ is denoted $\mathcal{S}_{\mathbb{Q}_\ell}$. Note that if \mathcal{S} has a tensor product and internal Hom then so does $\mathcal{S}_{\mathbb{Q}_\ell}$.

We note that $\mathbb{Z}_\ell\text{-Loc}$ is a stack over the category of k -analytic spaces with the étale topology. In this case we write $\mathbb{Q}_\ell\text{-Loc}$ for the associated stack in \mathbb{Q}_ℓ -linear categories.

DEFINITION 4.1. A *local system of \mathbb{Q}_ℓ -vectorspaces* \mathcal{V} over X is an object of $\mathbb{Q}_\ell\text{-Loc}_X$, where $\mathbb{Q}_\ell\text{-Loc}$ denotes the stack associated to the fibred category $\mathbb{Z}_\ell\text{-Loc} \otimes \mathbb{Q}_\ell$.

By going through the definitions, we see that such a local system \mathcal{V} is given by the following data

$$\mathcal{V} = (\{U_i \rightarrow X\}, \mathcal{F}_i, \phi_{ij})$$

where

- $\{U_i \rightarrow X\}$ is an étale covering of X ,
- for each i there is given an object $\mathcal{F}_i \in \mathbb{Z}_\ell\text{-Loc}_{U_i}$,
- for each pair i, j there is given an isomorphism $\phi_{ij} : \mathcal{F}_i|_{U_i \times_X U_j} \rightarrow \mathcal{F}_j|_{U_i \times_X U_j}$ in the fibre category of $\mathbb{Z}_\ell\text{-Loc} \otimes \mathbb{Q}_\ell$ over $U_i \times_X U_j$.

These data are subject to the cocycle condition $pr_{ij}^*(\phi_{ij}) \circ pr_{jk}^*(\phi_{jk}) = pr_{ik}^*(\phi_{ik})$ on the triple product $U_i \times_X U_j \times_X U_k$. A refinement of the covering $\{U_i \rightarrow X\}$ gives by definition an isomorphic object. Thus morphisms $\mathcal{V} \rightarrow \mathcal{V}'$ need only be defined for systems given over the same covering $\{U_i \rightarrow X\}$. In this case (after possibly refining the covering $\{U_i \rightarrow X\}$ further) such a morphism is defined by a collection of morphisms $\phi_i : \mathcal{F}_i \rightarrow \mathcal{F}'_i$ from $\mathbb{Z}_\ell\text{-Loc} \otimes \mathbb{Q}_\ell$ over U_i satisfying $\phi'_{ij} \circ pr_i^*(\phi_i) = pr_j^*(\phi_j) \circ \phi_{ij}$.

This presentation of objects also indicates how to define tensor products and internal Hom. It is clear that $\mathbb{Q}_\ell\text{-Loc}_X$ is a \mathbb{Q}_ℓ -linear abelian tensor category with internal Hom. If X is connected then $\mathbb{Q}_\ell\text{-Loc}_X$ is also rigid. The functor $\mathbb{Z}_\ell\text{-Loc}_X \rightarrow \mathbb{Q}_\ell\text{-Loc}_X$ is denoted $\mathcal{F} \mapsto \mathcal{F} \otimes \mathbb{Q}_\ell$; it is compatible with tensor product and internal Hom. Remark that a morphism $\phi : \mathcal{F}_1 \otimes \mathbb{Q}_\ell \rightarrow \mathcal{F}_2 \otimes \mathbb{Q}_\ell$ induces a morphism on stalks $\phi_{\bar{x}} : \mathcal{F}_{1, \bar{x}} \otimes \mathbb{Q}_\ell \rightarrow \mathcal{F}_{2, \bar{x}} \otimes \mathbb{Q}_\ell$ (just choose any geometric point lying over \bar{x} on one of the U_i and consider the action of ϕ_i on the stalks at this point). By definition a quasi-isogeny of \mathcal{F}_1 to \mathcal{F}_2 is an isomorphism $\mathcal{F}_1 \otimes \mathbb{Q}_\ell \rightarrow \mathcal{F}_2 \otimes \mathbb{Q}_\ell$.

Let G be a topological group. We write $\text{Rep}_{\mathbb{Q}_\ell}(G)$ for the category of continuous representations of G in finite dimensional \mathbb{Q}_ℓ -vectorspaces. This is a \mathbb{Q}_ℓ -linear tensor category. There is an obvious faithful exact functor $\omega : \text{Rep}_{\mathbb{Q}_\ell}(G) \rightarrow \text{Vect}_{\mathbb{Q}_\ell}$.

THEOREM 4.2. For any geometric point \bar{x} of X there is a natural \mathbb{Q}_ℓ -linear tensor functor

$$\omega_{\bar{x}} : \mathbb{Q}_\ell\text{-}\underline{\text{Loc}}_X \longrightarrow \underline{\text{Rep}}_{\mathbb{Q}_\ell}(\pi_1(X, \bar{x})).$$

If X is connected then $\omega_{\bar{x}}$ is an equivalence.

Proof. We consider the sheaf \mathcal{G} of lattices on $\mathcal{V} \in \mathbb{Q}_\ell\text{-}\underline{\text{Loc}}_X$. It is defined as follows: a section of \mathcal{G} over $U \rightarrow X$ étale is given by a pair (\mathcal{F}, φ) up to isomorphism, where (i) \mathcal{F} is an object of $\mathbb{Z}_\ell\text{-}\underline{\text{Loc}}_U$ and (ii) φ is an isomorphism $\mathcal{F} \otimes \mathbb{Q}_\ell \rightarrow \mathcal{V}|_U$. We leave it to the reader to see that \mathcal{G} is indeed a sheaf (use that $\mathbb{Z}_\ell\text{-}\underline{\text{Loc}}$ is a stack). If \mathcal{V} is given by the triple $\mathcal{V} = (\{U_i \rightarrow X\}, \mathcal{F}_i, \phi_{ij})$ then it follows immediately that $\mathcal{G}|_{U_i}$ is representable by a disjoint union of spaces finite étale over U_i . Hence Lemma 2.3 implies that \mathcal{G} is representable by an étale covering space Y over X .

Consider the universal object (\mathcal{F}, φ) over Y . Since \mathcal{V} lives over X we get a quasi-isogeny

$$\phi : pr_1^*(\mathcal{F}) \otimes \mathbb{Q}_\ell \longrightarrow pr_2^*(\mathcal{F}) \otimes \mathbb{Q}_\ell$$

over $Y \times_X Y$ satisfying the cocycle relation over $Y \times_X Y \times_X Y$. The morphisms $\mathcal{F}_n \rightarrow Y$ are finite étale, hence \mathcal{F}_n may be considered as an étale covering space of X also. Consider the set

$$W = \varprojlim_n F_{\bar{x}}(\mathcal{F}_n).$$

It has a prodiscrete topology and a continuous $\pi_1(X, \bar{x})$ -action. There is a map $W \rightarrow F_{\bar{x}}(Y)$ compatible with the $\pi_1(X, \bar{x})$ -action. Further, there is a natural group structure on W over $F_{\bar{x}}(Y)$, given by the fact that the fibre of $W \rightarrow F_{\bar{x}}(Y)$ at \bar{y} is the stalk $\mathcal{F}_{\bar{y}}$ of \mathcal{F} at \bar{y} . Let us write $w_1 \sim w_2$ for $w_i \in W$ if and only if $\phi_{y_1 y_2}(w_1) = w_2$, where $y_i \in F_{\bar{x}}(Y)$ is the image of w_i and $\phi_{y_1 y_2} : \mathcal{F}_{y_1} \otimes \mathbb{Q}_\ell \rightarrow \mathcal{F}_{y_2} \otimes \mathbb{Q}_\ell$ is the action of ϕ on the stalks at the point (y_1, y_2) of $Y \times_X Y$. This is an equivalence relation as follows immediately from the cocycle condition. We note that \sim is compatible with the $\pi_1(X, \bar{x})$ -action and hence that the topological space

$$\omega_{\bar{x}}(\mathcal{V}) = W / \sim$$

is a continuous $\pi_1(X, \bar{x})$ -set.

Let us show that for any pair $w_1, w_2 \in W$ there are equivalent w'_1, w'_2 such that w'_1 and w'_2 are in the same fibre $\mathcal{F}_{\bar{y}}$. To do this, note that for some $n \gg 0$ we have $\ell^n \phi_{y_1 y_2}(w_1) \in \mathcal{F}_{y_2}$ (a priori this lies in $\mathcal{F}_{y_2} \otimes \mathbb{Q}_\ell$). We take $\bar{y} \in F_{\bar{x}}(Y)$ corresponding to the lattice $\ell^{-n} \mathcal{F}_{y_2}$. Clearly, $w'_1 = \phi_{y_1 \bar{y}}(w_1)$ and $w'_2 = \phi_{y_2 \bar{y}}(w_2)$ work. This means that we can define an addition on $\omega_{\bar{x}}(\mathcal{V})$, putting $[w_1] + [w_2] = [w'_1 + w'_2]$ and a scalar multiplication, putting $\lambda \cdot [w_1] = [\lambda w_1]$, for $\lambda \in \mathbb{Z}_\ell$. We leave it to the reader to check this is well defined and that the \mathbb{Z}_ℓ -module so defined is in fact ℓ -divisible, hence a \mathbb{Q}_ℓ -vectorspace. Finally, the action of $\pi_1(X, \bar{x})$ preserves this vectorspace

structure and is continuous. This defines our functor $\omega_{\bar{x}}$. We leave it to the reader to check it is compatible with tensor products and internal Hom.

Remark that for any $\bar{y} \in F_{\bar{x}}(Y)$ there is a natural isomorphism $\mathcal{F}_{\bar{y}} \otimes_{\mathbb{Q}_\ell} \cong \omega_{\bar{y}}(\mathcal{V})$. Using this and internal Hom one may check that $\omega_{\bar{x}}$ is fully faithful, if X is connected.

Let us show that $\omega_{\bar{x}}$ is essentially surjective. Suppose that

$$\rho : \pi_1(X, \bar{x}) \longrightarrow \mathrm{GL}_n(\mathbb{Q}_\ell)$$

is a continuous homomorphism. This gives a continuous action on the (discrete) set of lattices $\mathrm{GL}_n(\mathbb{Q}_\ell)/\mathrm{GL}_n(\mathbb{Z}_\ell)$. An orbit O of this action corresponds to an étale covering map $f : Y \rightarrow X$ with $F_{\bar{x}}(Y) = O$, see Theorem 2.10. Take $\bar{y} \in F_{\bar{x}}(Y)$ corresponding to the lattice $\Lambda(\bar{y}) \in \mathbb{Q}_\ell^n$. The action of $\pi_1(Y, \bar{y})$ on \mathbb{Q}_ℓ^n stabilizes $\Lambda(\bar{y})$, hence we get a local system of \mathbb{Z}_ℓ -lattices $\mathcal{F}(\bar{y})$ over Y associated to this action. First let us show that $\mathcal{F}(\bar{y})$ is independent of the choice of \bar{y} . For another point \bar{y}' lying over \bar{x} we choose a path $\alpha : F_{\bar{y}} \rightarrow F_{\bar{y}'}$, this determines an element $\gamma = f(\alpha)$ in $\pi_1(X, \bar{x})$ such that $\gamma\bar{y} = \bar{y}'$, i.e., $\rho(\gamma)\Lambda(\bar{y}) = \Lambda(\bar{y}')$. Thus we get $\mathcal{F}(\bar{y})_{\bar{y}} \cong \mathcal{F}(\bar{y})_{\bar{y}'}$ via α and $\Lambda(\bar{y}) \cong \Lambda(\bar{y}')$ via $\rho(\gamma)$. Together these define an isomorphism $\mathcal{F}(\bar{y})_{\bar{y}'} \cong \mathcal{F}(\bar{y}')_{\bar{y}'}$ compatible with $\pi_1(Y, \bar{y}')$ -actions. This gives the desired isomorphism $\mathcal{F}(\bar{y}) \cong \mathcal{F}(\bar{y}')$.

This means that $\mathcal{F} = \mathcal{F}(\bar{y})$ is a local system of \mathbb{Z}_ℓ -lattices such that at each point \bar{y}' the fibre $\mathcal{F}_{\bar{y}'} \otimes \mathbb{Q}_\ell$ is identified with \mathbb{Q}_ℓ^n with $\pi_1(Y, \bar{y}')$ action given by the composite of ρ and $\pi_1(Y, \bar{y}') \rightarrow \pi_1(X, \bar{x})$. The existence of a quasi-isogeny over $Y \times_X Y$ satisfying the cocycle condition follows formally from this. Thus we get $\mathcal{V} \in \mathbb{Q}_\ell\text{-Loc}_X$ as desired. \square

Remark 4.3. Suppose that \bar{x}' is a second geometric point of the connected analytic space X . The equivalence of categories $\mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(X, \bar{x})) \cong \mathrm{Rep}_{\mathbb{Q}_\ell}(\pi_1(X, \bar{x}'))$ gotten from applying the theorem twice is isomorphic to the equivalence one gets from the isomorphism $\pi_1(X, \bar{x}) \cong \pi_1(X, \bar{x}')$ one gets by choosing a path from \bar{x} to \bar{x}' .

COROLLARY 4.4. A local system of \mathbb{Q}_ℓ -vectorspaces \mathcal{V} can always be given as $\mathcal{V} = (\{U_i \rightarrow X\}, \mathcal{F}_i, \phi_{ij})$, where the maps $U_i \rightarrow X$ are open immersions.

Proof. This means that for any point x of X we can find a neighbourhood $x \in U \subset X$ and a lattice of \mathcal{V} over U . Let Y denote the space of lattices in \mathcal{V} as in the proof of the theorem above. We have to prove that there exists a point $y \in Y$ with $\mathcal{H}(x) = \mathcal{H}(y)$. This means exactly that there exists a lattice $\Lambda \subset \omega_{\bar{x}}(\mathcal{V})$ stable for the action of $\pi_1(x, \bar{x})$. Such a lattice exists since $\pi_1(x, \bar{x})$ is a compact group. \square

5. Étale coverings of rigid analytic spaces

Here we just remark how to transpose the definitions and results above to the category of rigid analytic spaces. In this section the field k has a nontrivial valu-

ation. For simplicity we only work with quasi-separated rigid spaces. We use the terminology developed in [JP].

Let us define when a morphism of rigid analytic spaces $f : Y \rightarrow X$ is an étale covering space. If X is affinoid, we demand that Y be paracompact and that the associated morphism of Berkovich spaces (see [B2, 1.6.1]) is an étale covering space. This can be spelled out as follows: for any analytic point a of our affinoid X there exists a wide affinoid neighbourhood U of a such that $Y \times_X U$ is a disjoint union of affinoids finite étale over U . If X is arbitrary, we demand that any base change $Y \times_X X' \rightarrow X'$ with X' affinoid be an étale covering space. Similar definition for topological covering spaces.

We note that if the space X comes from a Hausdorff strictly k -analytic space X_B as in [B2, Section] then the categories of étale covering spaces of X and X_B coincide. (Use that if $Y_B \rightarrow X_B$ is an étale covering space of X_B in k - An then Y_B has a unique structure of a strictly k -analytic space such that $Y_B \rightarrow X_B$ in an arrow of st - k - An .)

For any geometric point (called étale point in [JP]) \bar{x} of the rigid space X there is a fibre functor and fundamental groups $\pi_1(X, \bar{x})$, $\pi_1^{\text{top}}(X, \bar{x})$ and $\pi_1^{\text{alg}}(X, \bar{x})$. The results of Theorems 2.9 and 2.10 hold in this situation also.

Instead of the definition of étale covering map given above we could have considered the following (seemingly more natural) condition on the morphism $f : Y \rightarrow X$: there exists an admissible affinoid covering $X = \bigcup U_i$ such that each $f^{-1}(U_i)$ is a disjoint union of affinoids finite étale over U_i . Let us call these maps almost étale covering maps for the moment. The author has not been able to show that any almost étale covering space is an étale covering space (nor does he believe strongly that this should be the case). Going through the usual arguments (compare [SGA4, Exposé IV Exercice 2.7.5]), one sees that the fundamental group associated to this category is a strict pro-system of topological groups, but it is not clear that this system is "realized" by a topological group.

Let us define $\mathbb{Z}_\ell\text{-Loc}_X$ as in the case of Berkovich spaces, but using the étale topology on X as defined in [JP] (or [SS]). Let us define $\mathbb{Q}_\ell\text{-Loc}$ as the stack associated to $\mathbb{Z}_\ell\text{-Loc} \otimes \mathbb{Q}_\ell$ using this same topology. In particular, objects of $\mathbb{Q}_\ell\text{-Loc}_X$ are given by triples $\mathcal{V} = (\{U_i \rightarrow X\}, \mathcal{F}_i, \phi_{ij})$ exactly as before, but here $\{U_i \rightarrow X\}$ is a étale covering of X in the rigid étale topology. In particular any admissible affinoid covering of X will do. In view of the above it is interesting to note that this does give the right concept.

PROPOSITION 5.1. If the rigid analytic variety X comes from a paracompact strictly k -analytic space X_B , then there is a natural equivalence of categories $\mathbb{Q}_\ell\text{-Loc}_{X_B} \rightarrow \mathbb{Q}_\ell\text{-Loc}_X$.

Proof. It is clear that there is a natural functor $\mathbb{Q}_\ell\text{-Loc}_{X_B} \rightarrow \mathbb{Q}_\ell\text{-Loc}_X$ which is fully faithful, by [B2, 1.6.1]. To show that an object $\mathcal{V} \in \mathbb{Q}_\ell\text{-Loc}_X$ is in the essential image of this functor we have to show that for any $x \in X_B$ there is an

open neighbourhood $\mathcal{U} \subset X_B$ such that \mathcal{V} has a lattice over the admissible open subset \mathcal{U}_0 of $X = (X_B)_0$.

We define the sheaf of lattices in \mathcal{V} exactly as in the proof of 4.2. The same proof works to show that this is representable by an almost étale covering map $f : Y \rightarrow X$. The same argument as in the proof of Corollary 4.4 gives an analytic point y of Y such that $\mathcal{H}(x) = \mathcal{H}(y)$. By definition of almost étale covering spaces we can find finitely many affinoids $U_1, \dots, U_n \subset X_B$ with $x \in U_i$ such that $\bigcup U_i$ is a neighbourhood of x and such that $f^{-1}(U_{i,0})$ is a disjoint union of affinoids finite étale over $U_{i,0}$. By shrinking U_i we may assume that the connected component $V_{i,0}$ of $f^{-1}(U_{i,0})$ containing y maps isomorphically onto $U_{i,0}$ (compare [JP]). At this point it is clear that $\bigcup V_{i,0} \subset Y$ maps isomorphically to $\bigcup U_{i,0}$ (use that f is separated). Thus we may take \mathcal{U} to be any open neighbourhood of x contained in $\bigcup U_i$. \square

COROLLARY 5.2. If X is a connected rigid analytic space then $\mathbb{Q}_\ell\text{-Loc}_X$ is equivalent to the category $\text{Rep}_{\mathbb{Q}_\ell}(\pi_1(X, \bar{x}))$.

Proof. In the case that X is paracompact, this is clear from the above. In the general case the result follows by gluing. \square

Let us consider a connected rigid analytic curve X , i.e., X is a separated purely one dimensional rigid analytic space. By [LP] this space has a paracompact formal model \mathfrak{X} , i.e., a separated admissible formal scheme \mathfrak{X} over $\text{Spf}(k^\circ)$ which has a locally finite covering by affinoids. If the base field k is algebraically closed and X is smooth we may even assume that \mathfrak{X} is semi-stable, see [LP]. This means that \mathfrak{X} locally looks like an open subset of the completion of a semi-stable curve $C \rightarrow \text{Spec}(k^\circ)$. In particular, the special fibre \mathfrak{X}_s is a semi-stable curve over $\text{Spec}(k^\circ/k^\circ)$, which may have infinitely many components. Let Γ denote the graph associated to this semi-stable curve \mathfrak{X}_s . Here is a result which is contained in [B1, Section 4.3]; we give a ‘classical’ proof to indicate how to prove results like Proposition 3.9 using the methods of formal and rigid geometry.

PROPOSITION 5.3. In the situation described above there is a natural isomorphism $\pi_1^{\text{top}}(X, \bar{x}) \cong \pi_1(\Gamma, sp(x))$.

Proof. The category of topological covering spaces of X is equal to the direct limit of the categories of topological covering spaces of \mathfrak{X}'_s where \mathfrak{X}' runs through the semi-stable models of X . This follows as any admissible affinoid covering of X may be refined by a locally finite one and subsequently refined by a pure covering coming from a semi-stable blowing up \mathfrak{X}' of \mathfrak{X} (this may involve blowing in an ideal with infinite support). If a topological covering space Y is split by a pure covering coming from the model \mathfrak{X}' then Y has a formal model \mathcal{Y}' mapping to \mathfrak{X}' such that $\mathcal{Y}'_s \rightarrow \mathfrak{X}'_s$ is a topological covering map. Of course these correspond bijectively to coverings of the graph.

However, the blowing up $\mathfrak{X}' \rightarrow \mathfrak{X}$ does not change the homotopy type of the graphs associated to the special fibre. Indeed, the blowing up $\mathfrak{X}' \rightarrow \mathfrak{X}$ can be

factored into a sequence of simple blowing ups of points. Thus, the graph Γ' can be obtained from Γ by subsequently adding a vertex and connecting this by one edge to Γ (blowing up of a smooth point of \mathfrak{X}_s) or replacing one edge by two edges joined in one point (blowing up of one of the double points of \mathfrak{X}_s). \square

6. Filtered F -crystals, p -divisible groups and local systems

In this section we work with a complete discrete valuation ring \mathcal{O} of mixed characteristic $(0, p)$. We assume the residue field $k = \mathcal{O}/\pi\mathcal{O}$ is perfect. The quotient field of \mathcal{O} is denoted K . We introduce three categories:

- Sch_k is the category of schemes of finite type over k ,
- $\text{FS}_{\mathcal{O}}$ is the category of formal schemes of finite type over $\text{Spf}(\mathcal{O})$,
- Rig_K denotes the category of quasi-separated rigid analytic spaces over K .

Each of these we consider as a site with the rigid topology on Rig_K and the Zariski topology on both $\text{FS}_{\mathcal{O}}$ and Sch_k .

We are going to discuss a construction which associates to a stack over Sch_k (resp. $\text{FS}_{\mathcal{O}}$) a stack over $\text{FS}_{\mathcal{O}}$ (resp. Rig_K). For example suppose $p : \mathcal{S} \rightarrow \text{Sch}_k$ is a stack. We define $p^{\text{for}} : \mathcal{S}^{\text{for}} \rightarrow \text{FS}_{\mathcal{O}}$ by putting

$$(p^{\text{for}})^{-1}(\mathfrak{X}) = p^{-1}(\mathfrak{X}_{\text{red}}).$$

The reader checks immediately that \mathcal{S}^{for} is a stack; actually \mathcal{S}^{for} is the pushforward (see [Gi]) of \mathcal{S} under the morphism of sites $\text{Sch}_k \rightarrow \text{FS}_{\mathcal{O}}$ given by the functor $\mathfrak{X} \mapsto \mathfrak{X}_{\text{red}}$. (Another possibility is to consider the functor $\mathfrak{X} \mapsto \text{Spec}(\mathcal{O}_{\mathfrak{X}}/p\mathcal{O}_{\mathfrak{X}})$.)

The second construction. Take a stack $p : \mathcal{S} \rightarrow \text{FS}_{\mathcal{O}}$. We want to construct a stack $p^{\text{rig}} : \mathcal{S}^{\text{rig}} \rightarrow \text{Rig}_K$. Suppose $X \in \text{Rig}_K$ is quasi-compact. In this case we define

$$(p^{\text{rig}})^{-1}(X) = \varinjlim_{\mathfrak{X}} p^{-1}(\mathfrak{X}).$$

The limit is taken over the category of models of X , see [BL]. It suffices to consider \mathcal{O} -flat formal schemes. It follows from the result [BL, 4.1] that this defines a stack over QCRig_K , the full subcategory of Rig_K of quasi-compact rigid spaces. Any stack over QCRig_K extends canonically to a stack over Rig_K , since we consider only quasi-separated rigid varieties. Thus we get

$$p^{\text{rig}} : \mathcal{S}^{\text{rig}} \longrightarrow \text{Rig}_K$$

associated to \mathcal{S} . We remark that both constructions are functorial with respect to morphisms of stacks.

Notation: if $\mathcal{S} \rightarrow \text{Sch}_k$ is a stack, we write \mathcal{S}^{rig} in stead of $(\mathcal{S}^{\text{for}})^{\text{rig}}$.

EXAMPLE 6.1. (i) Take $\mathcal{S} = \text{Isoc}$ to be the stack of convergent isocrystals over Sch_k . This means that \mathcal{S}_T is the category of convergent isocrystals over T . See [B].

Per convention the associated stack over Rig_K is denoted Isoc^{rig} . The same can be done with F -isoc, the stack of convergent F -isocrystals over Sch_k .

(ii) Take $\mathcal{S} = \text{BT}$ to be the stack over $\text{FS}_{\mathcal{O}}$ whose category of sections over $\mathfrak{X} \in \text{FS}_{\mathcal{O}}$ is the category of p -divisible groups over \mathfrak{X} (see [dJ]). There results the stack of p -divisible groups BT^{rig} over Rig_K . \square

Note that there is an obvious morphism of stacks $\text{BT}^{\text{rig}} \rightarrow \mathbb{Z}_{\ell}\text{-Loc}$. This associates to a p -divisible group its Tate-module. Notation $G \mapsto T_p(G)$. Recall that $\text{BT}_{\mathbb{Q}}$ is the \mathbb{Q}_p -linear stack associated to the \mathbb{Z}_p -linear stack BT . See Section 4 for notations. The functor T_p transforms quasi-isogenies into quasi-isogenies, hence it extends to a functor $T_p : (\text{BT}_{\mathbb{Q}})^{\text{rig}} \rightarrow \mathbb{Q}_{\ell}\text{-Loc}$.

PROPOSITION 6.2. Let X be an object of Rig_K .

(i) There is a natural equivalence $(\text{BT}_{\mathbb{Q}})^{\text{rig}} = (\text{BT}^{\text{rig}})_{\mathbb{Q}}$. The common value is denoted $\text{BT}_{\mathbb{Q}}^{\text{rig}}$.

(ii) The functor $T_p : \text{BT}_X^{\text{rig}} \rightarrow \mathbb{Z}_{\ell}\text{-Loc}_X$ is faithful. It is fully faithful if X is reduced. The same for $T_p : \text{BT}_{\mathbb{Q}}^{\text{rig}} \rightarrow \mathbb{Q}_{\ell}\text{-Loc}$.

(iii) Both BT^{rig} and $\text{BT}_{\mathbb{Q}}^{\text{rig}}$ are stacks for the étale topology of Rig_K , see [SS, JP].

(iv) Any object $\mathcal{F} \in \mathbb{Z}_{\ell}\text{-Loc}_X$ such that $\mathcal{F} \otimes \mathbb{Q}_p$ is in the essential image of $\text{BT}_{\mathbb{Q},X}^{\text{rig}} \rightarrow \mathbb{Q}_{\ell}\text{-Loc}_X$ is isomorphic to $T_p(G)$ for some $G \in \text{BT}_X^{\text{rig}}$.

Proof. By definition of the procedure $\mathcal{S} \mapsto \mathcal{S}^{\text{rig}}$ it suffices to work in the categories $\text{FS}_{\mathcal{O}}$ and QCRig_K . In the rest of the proof all rigid spaces will be quasi-compact and quasi-separated, all coverings will be finite.

To prove (i) we compare objects. An object of $(\text{BT}^{\text{rig}})_{\mathbb{Q},X}$ is given by a triple $(X = \bigcup U_i, G_i, \phi_{ij})$ where $X = \bigcup U_i$ is a finite covering, $G_i \in \text{BT}_{U_i}^{\text{rig}}$ and ϕ_{ij} is an arrow of $\text{BT}^{\text{rig}} \otimes \mathbb{Q}_p$ over $U_i \cap U_j$ (compare with Section 4). This means that G_i is given by a p -divisible group over a formal model \mathfrak{U}_i of U_i and ϕ_{ij} is given over a formal model \mathfrak{U}_{ij} of $U_i \cap U_j$ dominating both \mathfrak{U}_i and \mathfrak{U}_j . The cocycle condition is fulfilled on a formal model \mathfrak{U}_{ijk} of U_{ijk} . By the lemma below we may assume that the formal schemes $\mathfrak{U}_i, \mathfrak{U}_{ij}$ and \mathfrak{U}_{ijk} are formal open subschemes of the model \mathfrak{X} of X . Thus we get an object of $(\text{BT}_{\mathbb{Q}})^{\text{rig}}_X$. We leave it to the reader to treat morphisms.

Let us prove the second statement. The functor on BT_X^{rig} given by restriction to the reduction of X is faithful, hence it suffices to do the case of reduced X .

Let $X = \text{Sp}(A)$ be a reduced affinoid and let $A^{\circ} \subset A$ denote the subring of power bounded elements. Let G, H be p -divisible groups over the formal model $\mathfrak{X} = \text{Spf}(A^{\circ})$. Thus $A = A^{\circ}[1/p]$. We consider the map

$$\text{Hom}(G, H) \longrightarrow \text{Hom}(G_A, H_A).$$

Working through the definitions we see that in order to prove (ii) we have to show that this map is a bijection. Let A' be the normalization of the ring A ; it is a Tate algebra. We note that $(A')^{\circ}$ is normal; so that by Tate's theorem [T] we have the result for $X' = \text{Sp}(A')$. The result for A follows from the fact that $A^{\circ} = A \cap (A')^{\circ}$.

We come to the third statement. We only indicate how to deal with étale descent for objects. An étale descent datum over X for an object of BT^{rig} is typically given

by a triple (U, G, ϕ) , where $U \rightarrow X$ is a surjective étale morphism of QCRig_K , G is a p -divisible group given over a model \mathfrak{U} of U and ϕ is an isomorphism between the two pullbacks of G over a formal model \mathfrak{U}_2 of $U \times_X U$. The cocycle condition is verified over a formal model \mathfrak{U}_3 of $U \times_X U \times_X U$. By the lemma below we may assume the model \mathfrak{U} to be a flat quasi-finite covering of a model \mathfrak{X} of X . Again by the lemma, we may assume that \mathfrak{U}_2 is actually equal to $\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}$ and similar for \mathfrak{U}_3 . This means that we get descent data on G with respect to the flat quasi-finite covering $\mathfrak{U} \rightarrow \mathfrak{X}$, which is effective.

The case of BT_0^{rig} is a little more tricky. Again the descent datum is given by a triple (U, G, ϕ) as above. In this case G is an object of $\text{BT}_{0,U}^{\text{rig}}$, hence itself given by a triple $(U = \bigcup U_i, G_i, \psi_{ij})$. The morphism ϕ is given over a covering $U \times_X U = \bigcup U_{ij\alpha}$, where $U_i \times_X U_j = \bigcup U_{ij\alpha}$ is a finite covering. Over $U_{ij\alpha}$ we have $\phi_{ij\alpha} = p^{-n} \chi_{ij\alpha}$ where $\chi_{ij\alpha}$ is an isogeny of $pr_1^*(G_i)$ into $pr_2^*(G_j)$. The integer n may be chosen large enough so as to work independently of i, j and α . Thus it is clear that $\chi_{ij\alpha} = \chi_{ij\beta}$ on $U_{ij\alpha} \cap U_{ij\beta}$. This means we get in fact a quasi-isogeny ϕ_{ij} of $pr_1^*(G_i)$ into $pr_2^*(G_j)$ over $U_i \times_X U_j$. Redefining $U = \coprod U_i$, $G = \coprod G_i$ over $\coprod U_i$ and $\phi = \coprod \phi_{ij}$, we see that we get (U, G, ϕ) where $G \in \text{BT}_U^{\text{rig}}$ and ϕ is a quasi-isogeny.

Next, we note that $(U, T_p(G), T_p(\phi))$ gives rise to a local system of \mathbb{Q}_p -vectorspaces \mathcal{V} over X , by definition of $\text{Q}_\ell\text{-Loc}_X$. By Corollary 4.4, there exists a covering $X = \bigcup X_i$ such that \mathcal{V} has a lattice \mathcal{F}_i over each X_i . Over $U_i = U \times_X X_i$ we have a quasi-isogeny $\rho_i : T_p(G)_{U_i} \rightarrow \mathcal{F}_i$, which we may suppose to be an isogeny by replacing \mathcal{F}_i by $p^{-m} \mathcal{F}_i$ for some m sufficiently large.

Suppose G_{U_i} is given over the formal model \mathfrak{U}_i of U_i . The space parametrising finite locally free closed subgroup schemes of $G[p^n]$ is represented by a formal scheme \mathfrak{Z} projective over \mathfrak{U}_i . The subgroup $\text{Ker}(\rho_i) \subset T_p(G)[p^n]_{U_i}$ (here n is taken large enough) determines a morphism $\mathfrak{U}'_i \rightarrow \mathfrak{Z}^{\text{rig}}$. This comes from a morphism $\mathfrak{U}'_i \rightarrow \mathfrak{Z}$ for some admissible blow up $\mathfrak{U}'_i \rightarrow \mathfrak{U}_i$. Hence, the subgroup $\text{Ker}(\rho_i) \subset T_p(G)[p^n]_{U_i}$ extends to a finite flat subgroup scheme $\mathcal{N}_i \subset G[p^n]_{\mathfrak{U}'_i}$. We conclude that there exists a morphism $G_{U_i} \rightarrow G'_i$ in $\text{BT}_{U_i}^{\text{rig}}$ such that $\mathcal{F}_i = T_p(G'_i)$.

The quasi-isogeny ϕ determines a quasi-isogeny ϕ_i of $pr_1^*(G_i)$ into $pr_2^*(G_i)$ over $U_i \times_X U_i$. Since $\mathcal{F}_i = T_p(G'_i)$ we get that $T_p(\phi_i)$ is an isomorphism. We leave it to the reader to show that this implies ϕ_i is actually an isomorphism in BT^{rig} . The cocycle condition is verified on $U_i \times_X U_i \times_X U_i$. The descent datum $(U_i \rightarrow X_i, G'_i, \phi_i)$ so obtained is effective by what has gone before. There results a p -divisible group G_i over X_i .

The quasi-isogeny

$$G_i|_{U_i \cap U_j} \longrightarrow G|_{U_i \cap U_j} \longrightarrow G_j|_{U_i \cap U_j}$$

is compatible with ϕ_i and ϕ_j , hence descends to a quasi-isogeny ϕ_{ij} over $X_i \cap X_j$. This gives the desired object $(X = \bigcup X_i, G_i, \phi_{ij})$ of BT_0^{rig} over X .

The proof of the last statement is the same as the proof above that $\mathcal{F}_i = T_p(G_i)$ for some G_i . □

LEMMA 6.3. [BL] Let $f : U \rightarrow X$ be an étale morphism of QCRig_K . There exists a flat (quasi-finite) formal model $\mathfrak{f} : \mathfrak{U} \rightarrow \mathfrak{X}$. Any other model \mathfrak{U}' of U is dominated by a model of the form $\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{X}'$ for some admissible blowing up $\mathfrak{X}' \rightarrow \mathfrak{X}$.

Proof. The reference [BL] provides the flat formal model. To prove the last statement, by [BL], we may assume $\mathfrak{U}' \rightarrow \mathfrak{U}$ is an admissible blowing up. By [BL], we may find a admissible blowing up $\mathfrak{X}' \rightarrow \mathfrak{X}$ such that the strict transform \mathfrak{U}'' of \mathfrak{U}' is flat over \mathfrak{X}' . The morphism $\mathfrak{U}'' \rightarrow \mathfrak{U} \times_{\mathfrak{X}} \mathfrak{X}'$ is flat and a blowing up, hence an isomorphism. \square

Next, we relate Isoc^{rig} to flat vectorbundles. The stack FVB of flat vectorbundles over Rig_K has as category of sections FVB_X over X the category of pairs (\mathcal{E}, ∇) where \mathcal{E} is a finite locally free sheaf of \mathcal{O}_X -modules and $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X$ is an integrable connection. This is a K -linear tensor category with tensor product and internal Hom.

Suppose $\mathfrak{X} \in \text{FS}_{\mathcal{O}}$ is affine, say $\mathfrak{X} = \text{Spf}(A)$, with maximal ideal of definition $I \subset A$. The associated affinoid rigid space is $X = \mathfrak{X}^{\text{rig}}$. We can choose a surjection

$$W\{x_1, \dots, x_n\} \longrightarrow A,$$

since $W = W(k) \subset \mathcal{O}$ is finite. Let B be the completion of $W\{x_1, \dots, x_n\}$ in the preimage of the ideal I . (Note: this topological W -algebra is not of topologically finite type over W in general.) The formal scheme $\mathfrak{Y} = \text{Spf}(B)$ is formally smooth over $\text{Spf}(W)$. The rigid space $U = \mathfrak{Y}^{\text{rig}}$ over K_0 (see [B, 2.4; dJ, Section 7]) is a tube for $\mathfrak{X}_{\text{red}} = \text{Spec}(A/I)$. The surjection $B \rightarrow A$ defines a closed immersion

$$j : X = \mathfrak{X}^{\text{rig}} \longrightarrow U = \mathfrak{Y}^{\text{rig}}.$$

Suppose that E is a convergent isocrystal on $\mathfrak{X}_{\text{red}}$. The realization of E on U is a finite locally free \mathcal{O}_U -module \mathcal{E}_U with integrable connection. The pullback $E_X = j^*(\mathcal{E}_U)$ is a flat vectorbundle over X , i.e., an object of FVB_X . The construction $\text{Isoc}(\mathfrak{X}_{\text{red}}) \rightarrow \text{FVB}_X$ just described does not depend on the choice of $\mathfrak{X} \rightarrow \mathfrak{Y}$. Any two of these can be compared to a third, hence we need only compare in the case we have the situation $\mathfrak{X} \rightarrow \mathfrak{Y} \rightarrow \mathfrak{Y}'$. The result follows since $\mathcal{E}_{U'}|_U \cong \mathcal{E}_U$ by definition of convergent isocrystals.

PROPOSITION 6.4. The construction above induces a morphism of stacks

$$\text{Isoc}^{\text{rig}} \longrightarrow \text{FVB}.$$

This functor is denoted $E \mapsto E_X$ on E over X .

Suppose that in the affine situation $\mathfrak{X} = \text{Spf}(A)$ described above, we are given a p -divisible group G over \mathfrak{X} . There are two ways to get a flat vectorbundle over X . The first is to take $\mathbb{B}(G_{\text{red}})_X$, i.e., to take the flat vectorbundle associated to the convergent F -isocrystal of $G_{\text{red}} = G \times \mathfrak{X}_{\text{red}}$. (This is the convergent F -isocrystal associated to the Dieudonné crystal $\mathbb{D}(G_{\text{red}})$ over $\mathfrak{X}_{\text{red}}$, see [B].) The second is to take $(\mathbb{D}(G)_{\mathfrak{X}})^{\text{rig}}$, i.e., to take the flat vectorbundle on X associated to the value over

\mathfrak{X} of the Dieudonné crystal of G , which is a flat vector bundle $\mathbb{D}(G)_{\mathfrak{X}}$ on \mathfrak{X} . The canonical isomorphism

$$\mathbb{B}(G_{\text{red}})_X = (\mathbb{D}(G)_{\mathfrak{X}})^{\text{rig}}$$

is easily constructed, compare [dJ, 5.3].

Note that the value $\mathbb{D}(G)_{\mathfrak{X}}$ over \mathfrak{X} of the Dieudonné crystal of G is endowed with the canonical Hodge-filtration $\omega_G \subset \mathbb{D}(G)_{\mathfrak{X}}$. This means that we may associate to G the pair $(\mathbb{B}(G_{\text{red}}), \omega_G^{\text{rig}})$, consisting of a convergent F -isocrystal on X and a filtration on the associated flat vectorbundle.

DEFINITION 6.5. A filtered F -isocrystal over $X \in \text{Rig}_K$ is a pair (E, F^\cdot) where $E \in F\text{-isoc}_X^{\text{rig}}$ and $F^\cdot = F^\cdot E_X$ is a decreasing filtration by locally direct summands. The stack on Rig_K this gives rise to is denoted FFC .

The construction above gives rise to a morphism of stacks $\text{BT}_{\mathbb{Q}}^{\text{rig}} \rightarrow \text{FFC}$. Since this is a functor which transforms quasi-isogenies into isomorphisms we get a natural extension

$$(\mathbb{B}, F^\cdot) : \text{BT}_{\mathbb{Q}}^{\text{rig}} \longrightarrow \text{FFC}.$$

PROPOSITION 6.6. The functor (\mathbb{B}, F^\cdot) is fully faithful.

Proof. The question is whether the functor $\text{BT}_{\mathfrak{X}} \rightarrow \text{FFC}_X$ is fully faithful up to isogeny in the affine situation described above. Let us write $S = \mathfrak{X}_{\text{red}}$ and DC_S for the category of Dieudonné crystals over S . We know that the functor $\text{DC}_S \rightarrow F\text{-isoc}_S$ is fully faithful up to isogeny by [B] (see also [dJ, 5.5]). We know that the functor $\text{BT}_S \rightarrow \text{DC}_S$ is fully faithful up to isogeny by [dJ].

Suppose given a homomorphism $\varphi : (\mathbb{B}(G_{2,\text{red}}), \omega_{G_2}^{\text{rig}}) \rightarrow (\mathbb{B}(G_{1,\text{red}}), \omega_{G_1}^{\text{rig}})$, where G_1, G_2 are p -divisible groups over \mathfrak{X} . By the above we get a morphism of p -divisible groups $\psi_{\text{red}} : G_{1,\text{red}} \rightarrow G_{2,\text{red}}$ such that $\mathbb{B}(\psi) = p^n \varphi$ on the isocrystals. Since φ is compatible with the filtrations, we see that some $p^m \psi_{\text{red}}$ will lift (uniquely) to a homomorphism $\psi : G_1 \rightarrow G_2$ over \mathfrak{X} (the argument for this can be found in [dJ, proof of 5.7.8], see also the proof of Lemma 5.4.2 in [dJ]). \square

7. Some examples of étale covering spaces

The motivating example is perhaps the logarithm map

$$\log : \mathcal{D} \longrightarrow \mathbb{A}^1$$

from the open disc D of radius 1 around 1 over \mathbb{Q}_p to the affine-line over \mathbb{Q}_p . This map is of course given by the usual formula $\log(1+z) = z - z^2/2 + z^3/3 \dots$. Notice that it is a homomorphism of analytic groups, if D is seen as a subgroup of \mathbb{G}_m and \mathbb{A}^1 is given the usual additive group structure.

Let E be a sufficiently small closed disc around $0 \in \mathbb{A}^1$. On E the exponential function \exp is an inverse to \log with image a closed disc U around 1. The other

connected components of $\log^{-1}(E)$ can be described as follows: for any $n \in \mathbb{N}$ there is one:

$$U_n = \{ u \cdot \zeta \mid u \in U, \zeta \text{ a } p^n\text{-th primitive root of } 1 \}$$

If we replace \mathbb{Q}_p by \mathbb{C} (the completion of the algebraic closure of \mathbb{Q}_p), then U_n splits into $\varphi(p^n) = (p-1)p^{n-1}$ connected components. Furthermore, each of these components over \mathbb{C} maps isomorphically to E . We have shown that the inverse image $\log^{-1}(E)$ is a disjoint union of affinoids finite étale over E .

Let $E_m = \{x \in \mathbb{A}^1 \mid p^m x \in E\}$. It follows that

$$\log^{-1}(E_m) = \coprod_{n \in \mathbb{N}} \{x \mid x^{p^m} \in U_n\}.$$

It is easily seen that each of the terms on the right maps finite étale onto E_m . Since each point of \mathbb{A}^1 lies in the interior of one of the E_m it follows that \log is an étale covering map.

It is clear from the above that $\log_{\mathbb{C}}$ is a Galois covering with group $\bigcup_n \mu_{p^n}(\mathbb{C})$. We have proved the following result on the fundamental group.

PROPOSITION 7.1. The covering $\log_{\mathbb{C}}$ gives rise to a continuous surjection of the fundamental group of $\mathbb{A}_{\mathbb{C}}^1$ onto the discrete group $\bigcup_n \mu_{p^n}(\mathbb{C})$. The fundamental group of \mathbb{A}^1 surjects onto the semidirect product $\bigcup_n \mu_{p^n}(\bar{\mathbb{Q}}_p) \rtimes \text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

The logarithm is an example of a species of rigid analytic period maps for p -divisible groups introduced by Rapoport and Zink [RZ]. In certain cases these give rise to étale covering spaces of projective homogeneous spaces. We only explain this in the case studied by Gross and Hopkin's [HG]. We refer to the articles [RZ, HG] for proofs and references for the following statements. We use freely the terminology from Section 6.

Let X_0 be a divisible group of dimension 1 and height h over $\bar{\mathbb{F}}_p$. Let $W = W(\bar{\mathbb{F}}_p)$ denote the ring of Witt vectors over $\bar{\mathbb{F}}_p$. Consider schemes S over W where p is locally nilpotent. We write $\bar{S} = S \times \bar{\mathbb{F}}_p$. We consider pairs (X, ρ) where X is a p -divisible group over S and $\rho: X_0 \times \bar{S} \rightarrow X \times \bar{S}$ is a quasi-isogeny. If we consider these pairs up to isomorphism we get a functor \mathcal{M} which is representable by a formal scheme

$$\mathcal{M} = \coprod_{n \in \mathbb{Z}} \mathcal{M}^{(n)}$$

over $\text{Spf}(W)$. Thus it carries a universal pair (X, ρ) . The integer n corresponds to $\log_p \deg(\rho)$ and each of the components $\mathcal{M}^{(n)}$ is isomorphic to $\text{Spf}(W[[t_1, \dots, t_{n-1}]])$.

Let K_0 be the quotient field of W . Put $V = \mathbb{D}(X_0)_W \otimes K_0$. This is the rational Dieudonné module of X_0 , it is an F -isocrystal over K_0 . The p -adic period mapping is a rigid analytic étale mapping

$$\pi: M = \mathcal{M}^{\text{rig}} \rightarrow P = \mathbb{P}(V).$$

Let $E \in F\text{-isoc}_P^{\text{rig}}$ denote the constant convergent F -isocrystal over P associated to X_0 , i.e., with value V . Let $F^\cdot \subset E_P = V \otimes \mathcal{O}_P$ be the filtration given by the fact that P is the space of lines in V . Let X^{rig} denote the object of BT_M^{rig} associated to X over \mathcal{M} . The quasi-isogeny ρ determines an isomorphism of F -isocrystals $\mathbb{B}(X^{\text{rig}}) \cong \pi^*(E)$, i.e., with the constant F -isocrystal with fibre V . The morphism π is defined by the property that $\pi^*(E, F^\cdot) = (\mathbb{B}, F^\cdot)(X^{\text{rig}})$. Explicitely, if $m \in M$ is a point then ρ defines an isomorphism $\mathbb{B}(X^{\text{rig}})_m \cong V \otimes \kappa(m)$. Thus the Hodge filtration $\omega_{X,m} \subset \mathbb{B}(X^{\text{rig}})_m$ defines a point $\pi(m)$ of $\mathbb{P}(V)$ over $\kappa(m)$.

PROPOSITION 7.2. There is an object $\mathcal{V} \in \mathcal{Q}_\ell\text{-}\underline{\text{LOC}}_P$ having the following properties.

- (i) It is an object of the full subcategory $\text{BT}_{\mathcal{Q},P}^{\text{rig}} \subset \mathcal{Q}_\ell\text{-}\underline{\text{LOC}}_P$ and $(\mathbb{B}, F^\cdot)(\mathcal{V}) = (E, F^\cdot)$.
- (ii) There is an isomorphism $\pi^*(\mathcal{V}) \cong X^{\text{rig}} \otimes \mathbb{Q}_p$.
- (iii) The space M over P represents the sheaf of lattices in \mathcal{V} over P , see proof of 4.2. In particular M is an étale covering space of P .

Proof. We use the following result of [HG]: there are finitely many affinoids $U_i \subset M$ such that $\bigcup \pi(U_i) = \mathbb{P}$, [HG, Corollary 23.15 & Corollary 23.17]. This means that $M \rightarrow P$ is a covering for the rigid étale topology of P . Since (\mathbb{B}, F^\cdot) is fully faithful by Proposition 6.6 and (E, F^\cdot) lives over P , we get an étale descent datum on $X^{\text{rig}} \otimes \mathbb{Q}_p$. By Proposition 6.2 we get an object \mathcal{V} of $\text{BT}_{\mathcal{Q}}^{\text{rig}}$ over P with $(\mathbb{B}, F^\cdot)(\mathcal{V}) = (E, F^\cdot)$. This proves (i) and (ii).

The last assertion follows from (iv) of Proposition 6.2 and the definition of \mathcal{M} . □

According to Theorem 4.2 the local system \mathcal{V} corresponds to a continuous homomorphism

$$\pi_1(P, \bar{x}) \longrightarrow \text{GL}_h(\mathbb{Q}_p).$$

Let us base change to the algebraically closed field \mathbb{C} . Note that $M_{\mathbb{C}}$ is still the space of lattices in $\mathcal{V}_{\mathbb{C}}$. Of course $M_{\mathbb{C}} = \coprod M(n)_{\mathbb{C}}$ is a countable union of open polydiscs. The action of p on lattices by multiplying $(\mathcal{F} \mapsto p\mathcal{F})$ shifts the integer n by h . This already implies that the fundamental group of $P_{\mathbb{C}}$ is nontrivial.

LEMMA 7.3. The fundamental group of $P_{\mathbb{C}}$ does not have any abelian quotients.

Proof. We have to show that $H_{\text{ét}}^1(P_{\mathbb{C}}, M)$ is zero for any abelian group M . By the usual arguments it suffices to do this when either M is a finite group or $M \cong \mathbb{Z}$. The case of a finite group is OK since there are no nontrivial finite étale covering spaces of $P_{\mathbb{C}}$. Consider the spectral sequence associated to the morphism of sites $\pi : (P_{\mathbb{C}})_{\text{ét}} \rightarrow |P_{\mathbb{C}}|$. By [B2, Proposition 4.2.4], one has $R^1\pi_*(\mathbb{Z}_{P_{\mathbb{C}}}) = 0$, and therefore $H_{\text{ét}}^1(P_{\mathbb{C}}, \mathbb{Z}) = H^1(|P_{\mathbb{C}}|, \mathbb{Z})$. The latter group is zero by [B1, Theorem 6.1.5]. □

The lemma implies that the homomorphism $\pi_1(P_C, \bar{x}) \rightarrow \mathrm{GL}_h(\mathbb{Q}_p)$ maps into $\mathrm{SL}_h(\mathbb{Q}_p)$. This could have been seen also by the following argument: the diagram

$$\begin{array}{ccc} \pi_1(P, \bar{x}) & \longrightarrow & \mathrm{GL}_h(\mathbb{Q}_p) \\ \downarrow & & \downarrow \text{det} \\ \mathrm{Gal}(\mathbb{C}/K_0) & \xrightarrow{\chi} & \mathbb{Q}_p^* \end{array}$$

is commutative. This is true since, by Raynaud, the Galois group acts on the determinant of the Tate module of a p -divisible group G by the character $\chi^{\dim G}$. (Here χ is the Teichmüller character.) We see from this that $\pi_1(P, \bar{x})$ maps into the subgroup $\mathrm{GL}'_h(\mathbb{Q}_p) \subset \mathrm{GL}_h(\mathbb{Q}_p)$ consisting of elements g having $\mathrm{ord}_p \det g = 0$.

PROPOSITION 7.4. The continuous map $\pi_1(P_C, \bar{x}) \rightarrow \mathrm{SL}_h(\mathbb{Q}_p)$ constructed above is a surjection. It identifies $\mathrm{SL}_h(\mathbb{Q}_p)$ with a topological quotient of the fundamental group of $P_C \cong \mathbb{P}^{h-1}$.

Proof. We first prove an analogous statement over the field K_0 . Let us take a point $x \in M$ and let us choose our geometric point \bar{x} lying over x . We may also see \bar{x} as a geometric point of P . We define Γ as the image of the homomorphism $\pi_1(P, \bar{x}) \rightarrow \mathrm{GL}_h(\mathbb{Q}_p)$. We endow it with the quotient topology coming from the surjection $\pi_1(P, \bar{x}) \rightarrow \Gamma$. Hence we have a continuous injection $i : \Gamma \rightarrow \mathrm{GL}_h(\mathbb{Q}_p)$. We claim the set

$$i(\Gamma) \backslash \mathrm{GL}_h(\mathbb{Q}_p) / p^{\mathbb{Z}} \cdot \mathrm{GL}_h(\mathbb{Z}_p) \tag{1}$$

has h elements. This follows from the remark above that the space M has h connected components modulo the action of $p^{\mathbb{Z}}$, and the identification of M with the space of lattices in \mathcal{V} .

Furthermore, let \mathcal{F} be the lattice of $\mathcal{V}|_M$ over M given by Proposition 7.2. Let us take a basis of $\mathcal{F}_{\bar{x}}$. If we compute the monodromy of \mathcal{V} using this basis, then we see that $i^{-1}(\mathrm{GL}_h(\mathbb{Z}_p))$ equals the image of the composition $\pi_1(M, \bar{x}) \rightarrow \pi_1(P, \bar{x}) \rightarrow \mathrm{GL}_h(\mathbb{Q}_p)$. In particular, we can look at the image Γ_x of the composition

$$\pi_1(x, \bar{x}) \longrightarrow \pi_1(M, \bar{x}) \longrightarrow \mathrm{GL}_h(\mathbb{Z}_p).$$

Note that since the first group is profinite, the group Γ_x is a closed subgroup of $\mathrm{GL}_h(\mathbb{Z}_p)$. We claim that we can choose $x \in M(K_0)$ such that Γ_x has finite index in $\mathrm{GL}_h(\mathbb{Z}_p)$. For this it suffices that Γ_x is Zariski dense in $\mathrm{GL}_{h, \mathbb{Q}_p}$.

This just means that there exists a p -divisible group G over $W(\bar{\mathbb{F}}_p)$, whose special fibre is isogeneous to X_0 and such that the Galois representation on the Tate module has the corresponding property. Such may be constructed, for example by using Fontaine’s theory: we give a filtered Dieudonné module (V, Φ, F^1) over K_0 such that the \mathbb{Q}_p -linear tensor category generated by it is isomorphic to $\mathrm{Rep}(\mathrm{GL}_h)$.

We give a sufficient criterium for (V, Φ, F^1) to have this property. The σ -linear map Φ has slope $1/h$ hence we can find a basis v_1, \dots, v_h of V consisting of elements such that $\Phi^h(v_i) = pv_i$. (This basis is unique up to a base change give by a matrix with coefficients in \mathbb{Q}_p .) It suffices to take $F^1 = K_0(v_1 + a_2v_2 + \dots + a_nv_n)$, where $a_i \in K_0$ are algebraically independent over \mathbb{Q}_p .

From all of this we conclude that $i^{-1}(\mathrm{GL}_h(\mathbb{Z}_p)) \subset \Gamma$ contains a profinite subgroup Γ_x which maps to a subgroup of finite index in $\mathrm{GL}_h(\mathbb{Z}_p)$. It follows that the map i is an open and closed immersion and that the image is not discrete. We conclude that i induces an isomorphism of topological groups $\Gamma \rightarrow \mathrm{GL}'_h(\mathbb{Q}_p)$: use the remarks preceding the proposition, the number of elements of the set (1) and a theorem on non-discrete closed subgroups of finite covolume [Ma, page 95].

At this point we would be done if we knew that the sequence $\pi_1(P_c, \bar{x}) \rightarrow \pi_1(P, \bar{x}) \rightarrow \mathrm{Gal}(\mathbb{C}/K_0)$ were exact, but we don't know this (see Remark 2.15). We do know the corresponding statement for the algebraic fundamental group of M , and this is what we are going to use.

Let us define Γ_c as the image of the map $\pi_1(P_c, \bar{x}) \rightarrow \mathrm{SL}_h(\mathbb{Q}_p)$, endowed with the quotient topology. There is a continuous injection i_c of Γ_c into $\mathrm{SL}_h(\mathbb{Q}_p)$.

First, remark that the preceding result implies that the monodromy over M gives a surjection $\pi_1(M, \bar{x}) \rightarrow \mathrm{GL}_h(\mathbb{Z}_p)$. This factors through $\pi_1^{\mathrm{alg}}(M, \bar{x})$, see Theorem 2.10. Let $N \subset \mathrm{SL}_h(\mathbb{Z}_p)$ be the image of $\pi_1^{\mathrm{alg}}(M_c, \bar{x})$ in the monodromy representation given by the local system of \mathbb{Z}_p -lattices \mathcal{F}_c over M_c . By Proposition 2.13 we get that N is a (closed) normal subgroup of $\mathrm{SL}_h(\mathbb{Z}_p)$. Hence, it is either a finite subgroup acting diagonally, or it has finite index. Let $M(n) \subset M$ be the connected component \bar{x} maps into. The first possibility would imply that all Galois representations associated to the p -divisible groups corresponding to $(K_0$ -valued) points of $M(n)$ are isomorphic as representations into $\mathrm{PGL}_h(\mathbb{Z}_p)$. Clearly, this is absurd. Hence, we get that N has finite index. We want to show that $\pi_1(M_c, \bar{x})$ maps onto N .

Let us make a general remark about the algebraic fundamental group of the open polydisc $D^{h-1} \cong M(n)_c$ over \mathbb{C} . For any closed polydisc $E^{h-1} \subset D^{h-1}$ we get a continuous homomorphism $\pi_1^{\mathrm{alg}}(E^{h-1}, \bar{x}) \rightarrow \pi_1^{\mathrm{alg}}(D^{h-1}, \bar{x})$. Furthermore, for any surjection $\pi_1^{\mathrm{alg}}(D^{h-1}, \bar{x}) \rightarrow G$ onto a finite group G , we can choose our closed polydisc such that the composition $\pi_1^{\mathrm{alg}}(E^{h-1}, \bar{x}) \rightarrow \pi_1^{\mathrm{alg}}(D^{h-1}, \bar{x}) \rightarrow G$ is surjective still. This follows as the open polydisc is an increasing union of closed polydiscs.

Now note that there exists a finite quotient $N \rightarrow G$ having the following property: any closed subgroup $N' \subset N$ surjecting onto G is equal to N . The proof of this statement is left to the reader. Combining the above, we see that there exists a closed polydisc $E \subset M(n)_c$ whose algebraic fundamental group maps onto N . By Proposition 7.5 there exists a profinite subgroup $K \subset \pi_1(E, \bar{x})$ mapping onto $\pi_1^{\mathrm{alg}}(E, \bar{x})$. Mapping K into $\pi_1(M_c, \bar{x})$ we get a profinite subgroup $K' \subset \pi_1(M_c, \bar{x})$ which surjects onto N . Conclusion: the topological group Γ_c

contains a profinite subgroup K'' mapping isomorphically onto a subgroup of finite index in $\mathrm{SL}_h(\mathbb{Z}_p)$.

The rest of the argument is similar to the arguments we gave to deal with the group Γ . For example the analogue of the set (1) in this case is also finite since the connected components of M_c agree with the connected components of M . \square

In the above proof we used some results on the fundamental groups of the space $X = E(0, 1)^n$, the closed n -dimensional unit disc. We work over the complete nontrivially valued field k . The maximal point x_{\max} of X is the point $x_{\max} \in X = \mathcal{M}(k\langle T_1, \dots, T_n \rangle)$ corresponding to the supremum norm $|\cdot|$ on the algebra $k\langle T_1, \dots, T_n \rangle$. We consider the homomorphism of profinite groups

$$\alpha : \pi_1(x_{\max}, \bar{x}) \longrightarrow \pi_1^{\mathrm{alg}}(X, \bar{x}),$$

where \bar{x} is a geometric point lying over x_{\max} . We remark that composing α with the natural map to $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ gives a surjection; this follows as the field extension $k \subset \mathcal{H}(x)$ is separable and k is algebraically closed in $\mathcal{H}(x)$. The following proposition holds more generally for any strictly affinoid X over k with nonsingular reduction over $k^\circ/k^{\circ\circ}$.

PROPOSITION 7.5. The map α is surjective. It follows that the canonical map $\pi_1(X, \bar{x}) \rightarrow \pi_1^{\mathrm{alg}}(X, \bar{x})$ is surjective.

Proof. We have to show that any finite étale covering $f : Y \rightarrow X$ of degree $d \geq 2$ such that $Y_{x_{\max}} = \{p_1, \dots, p_d\}$ splits. This implies that f is a local isomorphism at each point p_i . Consider first the case $n = 1$. We can find an affinoid domain $U \subset X$, which is a neighbourhood of x_{\max} such that $f^{-1}(U) = \coprod_{i=1}^d V_i$ each V_i mapping isomorphically to U . Put $\mathcal{V} = X \setminus U$. We note that $\mathcal{V} \subset \mathbb{P}^1$ is open and that $\mathbb{P}^1 \setminus \mathcal{V}$ is a closed analytic domain of \mathbb{P}^1 . The reason for this is that U contains x_{\max} and $\partial(X/\mathbb{P}^1) = \{x_{\max}\}$. We glue Y with d copies of the space $\mathbb{P}^1 \setminus \mathcal{V}$ via the isomorphisms $V_i \rightarrow U$, see [B2, Proposition 1.3.3 b)]. The space Y' so obtained maps to \mathbb{P}^1 . The morphism $f' : Y' \rightarrow \mathbb{P}^1$ is finite étale, since it is true over the members of the admissible G-covering $\mathbb{P}^1 = X \cup (\mathbb{P}^1 \setminus \mathcal{V})$. Hence, the finite étale covering $Y' \rightarrow \mathbb{P}^1$ comes from a finite étale covering $T \rightarrow \mathrm{Spec}(k)$. We get the same for Y and by the remark before the proposition we see that Y is split.

In the case $n \geq 2$, consider the projection $pr : X \rightarrow X_1$ to the first $n - 1$ coordinates. By the first case, f is split over the fibre X_{x_1} of pr at the maximal point x_1 of X_1 . It follows, by induction, that f is split over the fibres of the projection to the n th coordinate, and we are done. \square

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