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On the sum of consecutive cubes being a perfect square

Dedicated to Frans Oort on the occasion of his 60th birthday

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Abstract. In this paper estimates of linear forms in elliptic logarithms are applied to solve the problem of determining, for given $n \geqslant 2$, all sets of n consecutive cubes adding up to a perfect square. Use is made of a lower bound of linear forms in elliptic logarithms recently obtained by Sinnou David. Complete sets of solutions are provided for all n between 2 and 50, and for $n = 98$.

1. Introduction

Every beginning student of number theory surely must have marveled at the miraculous fact that for each natural number n the sum of the first n positive consecutive cubes is a perfect square, that is

$$\sum_{i=1}^{n} i^3 = \left(\tfrac{1}{2}n(n+1)\right)^2, \quad n \in \mathbb{N}.$$

It seems natural to ask whether this phenomenon keeps occurring when the initial cube is shifted along the number axis over an arbitrary distance. In other words, are there any integral solutions to the Diophantine equation

$$\begin{aligned} y^2 &= x^3 + (x+1)^3 + \cdots + (x+n-1)^3 \\ &= nx^3 + \tfrac{3}{2}n(n-1)x^2 + n(n-\tfrac{1}{2})(n-1)x \\ &\quad + \tfrac{1}{4}n^2(n-1)^2, \quad n \in \mathbb{N} \end{aligned} \tag{1}$$

other than the trivial ones? A simple search reveals that this happens for many values of n. From Table 2 at the end of this paper one may read off that for all odd values of n below 50 such solutions do exist. In the remark immediately following the proof of Lemma 1 an explanation of this observation is given.

Surprisingly for such an obvious question, no more than a handful of relevant references could be found in the literature. Dickson [7, p. 585–588] has a few and in a recent paper of Cassels [4] the case $n = 3$ is solved. On the other hand, no publication was found in which this problem has been addressed in any systematic way.

This paper is about equation (1) and how it may be solved for each integer $n \geqslant 2$. The approach set forth in the following lines provides an algorithmic solution method that – at least in principle – should work for general n.

Clearly, for each $n \geqslant 2$ trivial solutions are given by $x = 0$ and $x = 1$. Also, any solution (x, y) of (1) with $x < 0$ uniquely determines one with $x > 0$ associated with a smaller value of n. So from here on only solutions (x, y) with $x > 1$ will be considered non-trivial.

By means of the transformation

$$(x,y) \longmapsto (X,Y) = \left(nx + \tfrac{1}{2}n(n-1), ny\right), \tag{2}$$

equation (1) is mapped to

$$Y^2 = X^3 + dX, \tag{3}$$

where $d = d_n := \frac{1}{4}n^2(n^2 - 1)$. To solve (1), it clearly suffices to solve (3) for integers X and Y when $d = d_n$ and $n \geqslant 2$ are given.

For general $d \neq 0$ equation (3) is the short Weierstraß representation of an elliptic curve that has been investigated extensively (cf. [1], [2], [3]). In fact, these curves helped to shape this branch of mathematics as it is today by providing numerical evidence for important conjectures. Moreover, the curves (3) with $d = -n^2$ are intimately connected with the congruent number problem (cf. [9], [10]). As a result, quite a bit of information on (3) is available in the literature.

Traditionally, the problem of finding all integral points on an equation like (3) is solved by reducing this equation to a finite number of quartic Thue equations. Subsequently, each individual Thue equation can be solved by an effective procedure based on Diophantine approximation techniques (cf. [20]), or by older methods which depend on explicit knowledge of certain number fields that crop up in the process (cf. [16]). These standard methods are of an ad hoc nature: one never knows beforehand precisely which hurdles have to be taken, it cannot be foreseen which tricks are required in a single case (see also [4]). Explicit examples of the difficulties one may be faced with are given in [18].

This paper will stay clear of this course of action. Instead, detailed information shall be used on the group structure of the elliptic curve given by (3) with $d = d_n$. In [17] the authors describe how an explicit lower bound of linear forms in elliptic logarithms, recently obtained by S. David in [6], may be employed to solve elliptic equations. This method requires explicit knowledge of the generators for $E(\mathbb{Q})$. Once this information has been obtained, the elliptic logarithm method runs along in a way that is not as curve-dependent as the Thue approach. Clearly, this could be a definite advantage. On the other hand, there are curves for which a set of generators is as good as impossible to compute, because at least one generator is extremely large. For these curves the Thue method remains the more practical one. In [18, Conclusion] the authors elaborate on this point. Here we shall closely follow the method exposed in [17].

In Table 2 all solutions to equation (1) are listed for n in the range $2, \ldots, 50$ and $n = 98$. All related curves are of rank 1, 2 or 3. The extra case is added as it happens to be the smallest value of n for which the corresponding elliptic curve is a rank 4 curve.

Probably the most remarkable of all solutions found is the one which expresses the perfect square 2079^2 as the sum of 33 consecutive cubes, starting with 33^3.

2. Preliminaries on the elliptic curve

Let $E_n/\mathbb{Q}$ be the elliptic curve defined by equation (3) with $d = d_n := \frac{1}{4}n^2(n^2 - 1)$. Further, let $\nu(m)$ denote the number of distinct odd prime divisors of m.

LEMMA 1. *For $n \geqslant 2$, the group $E_n(\mathbb{Q})$ has torsion subgroup $\mathbb{Z}/2\mathbb{Z}$ with non-trivial point $P_0 = (0,0)$ of order 2, and its rank r_n satisfies the inequalities,*

$$1 \leqslant r_n \leqslant 2\{\nu(n-1) + \nu(n) + \nu(n+1)\} + 1.$$

Proof. Most of these assertions are immediate consequences of [11, Chapter X, Proposition 6.1, p. 311]. According to this proposition, $E_n(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z}$ unless $d_n = 4a^4$ for some $a \in \mathbb{Z}$ or $-d_n$ is a perfect square. This forces $E_n(\mathbb{Q})_{\text{tors}} \cong \mathbb{Z}/2\mathbb{Z}$, as neither of these two special cases is possible. Further, Proposition 6.1 also implies that $r_n + 1 \leqslant 2 \cdot \#$ distinct prime divisors of $2d_n$, which gives the required upper bound for r_n. Finally, $E_n(\mathbb{Q})$ contains the point $P_n = (X_n, Y_n) = (\frac{1}{2}n(n+1), \frac{1}{2}n^2(n+1))$, which corresponds to the trivial solution $(1, \frac{1}{2}n(n+1))$ of equation (1). It is not difficult to see that $2P_n = (\frac{1}{4}, \frac{1}{8}(2n^2 - 1))$. Because $X(2P_n) \notin \mathbb{Z}$, the point P_n cannot be of finite order and hence $r_n \geqslant 1$. □

REMARK. It was observed by Henri Cohen that the point $Q_n = -2P_n + (0,0) = (n^2(n^2-1), \frac{1}{2}n^2(n^2-1)(2n^2-1))$ on (3) corresponds to a point on (1) which is integral for odd n. This explains why “non-trivial” integral solutions of (1) exist for all odd values of n.

The upper bound for the rank r_n in Lemma 1 is too large to be of much practical use. However, in each individual case, by checking the homogeneous spaces for solvability, the exact value of the rank may be computed using descent by 2-isogeny as E_n has a rational point of order 2. Unfortunately, this procedure, given by J. Tate in his Haverford lectures in 1961 and reproduced by various authors (see for instance [5, p. 63–68], [14, p. 89–98], and [11, p. 301–304]) is not always effective: if there are points of order 2 in the Tate–Shafarevich group

Ш $(E_n/\mathbb{Q})$, one may have a problem. In the range $2 \leqslant n \leqslant 50, 98$ only $n = 41$ and $n = 44$ cause trouble in this respect. We used both Ian Connell's Apecs 3.2 (which runs under Maple) and John Cremona's algorithm mwrank.sun to do the rank calculations. In order to determine the rank, one first checks a finite number of quartic equations for rational solutions. Here, these equations are of type

$$z^2 = a_1x^4 + a_2y^4, \quad a_1, a_2 \in \mathbb{Z}, \tag{4}$$

where $a_1a_2 = d_n$ or $-4d_n$ and a_1 is squarefree. If no rational point can be found, maybe one can prove that no such point exists by local solvability techniques. Failing this, one has to do another descent. In our range, this happened for $n = 41$ and $n = 44$, where we found a small number of quartics (4), locally solvable everywhere but without apparent rational point. Our calculations also suggested a rank value of at least 1. We proceeded as follows. If (4) is locally solvable everywhere, then

$$z^2 = a_1X^2 + a_2Y^2$$

must be globally solvable by Hasse–Minkowski. When a rational solution is known – and there is no reason to fear that such a solution might be difficult to find – this conic can be rationally parameterized. Assuming the existence of a rational point on (4), this parameterization yields a system of quadratic equations

$$\begin{aligned} Rx^2 &= Q_1(u, v), \\ Ry^2 &= Q_2(u, v), \end{aligned} \tag{5}$$

with $Rz = Q_3(u, v)$, where Q_1, Q_2, Q_3 are quadratic forms with integer coefficients, and $R \in \mathbb{Z}$ is a squarefree divisor of the resultant of Q_1 and Q_2. For each of the finitely many possible values of R, we found all relevant systems (5) to be locally insoluble at some prime p, confirming our rank 1 suspicion.

A further discussion of the actual computations is postponed to the final section.

The next task is to compute a complete set of generators for $E_n(\mathbb{Q})$ modulo torsion. The first hurdle that has to be taken is the computation of sufficiently many independent points. This may give some problems as is clearly illustrated in [3]. Once r_n independent points have been found, it is easy to obtain 2^{r_n+1} coset representatives of the quotient group $E_n(\mathbb{Q})/2E_n(\mathbb{Q})$; these representatives are chosen in such a way that the maximal value of their canonical heights is minimal. Recall that the canonical height on $E_n(\mathbb{Q})$ is a quadratic form

$$\hat{h}\colon\ E_n(\mathbb{Q}) \to \mathbb{R},$$

which is positive definite on $E_n(\mathbb{Q})/E_n(\mathbb{Q})_{\text{tors}}$, and that the naive logarithmic height (or Weil height) is defined by

$$h(P) = \log\max\{|r|, |s|\},$$

where $P = (X(P), Y(P)) \in E_n(\mathbb{Q})$ and $X(P) = r/s$ with $\gcd(r,s) = 1$. The difference $\hat{h}(P) - \frac{1}{2}h(P)$ is bounded; explicit bounds are given in the following lemma.

LEMMA 2. *Let $P \in E_n(\mathbb{Q})$, then*

$$-\log n - 1.9055 < \hat{h}(P) - \tfrac{1}{2}h(P) < \log n + 1.6915. \tag{6}$$

Proof. From [13, Theorem 1.1] – or rather from [13, Example 2.2] – we have

$$\begin{aligned}\hat{h}(P) - \tfrac{1}{2}h(P) &\leqslant \tfrac{1}{4}h(d_n) + 2.038\\ &= \log n + \tfrac{1}{4}\log\left(1 - \frac{1}{n^2}\right) + 2.038 - \tfrac{1}{2}\log 2\\ &< \log n + 1.6915,\end{aligned}$$

as required. The lower bound may be obtained similarly. □

REMARK. It may be of interest to note that the canonical height of the point $P_n = (\frac{1}{2}n(n+1), \frac{1}{2}n^2(n+1))$ on (3) can be approximated by techniques described in [12] (see also [19]). We found that $\hat{h}(P_n)$ is asymptotic to $\frac{1}{8}\log 2n^2$, which happens to be a good fit, even for small values of n.

Now let $P_0 = (0,0)$ and let $P_1, \ldots, P_{r_n}$ be independent points of $E_n(\mathbb{Q})$ that correspond to a candidate set of generators for $E_n(\mathbb{Q})/E_n(\mathbb{Q})_{\text{tors}}$. Select a complete set S of 2^{r_n+1} representatives for $E_n(\mathbb{Q})$ modulo $2E_n(\mathbb{Q})$ from the set

$$\left\{\sum_{i=0}^{r_n} \varepsilon_i P_i \mid \varepsilon_i \in \{-1,0,1\}\right\},$$

and set $B := \max_{P\in S} \hat{h}(P)$. Then from Lemma 2 and [13, Proposition 7.2] we deduce that

$$S(B) := \{P \in E_n(\mathbb{Q}) \mid h(P) < 2B + 2\log n + 3.811\} \tag{7}$$

generates the free part of $E_n(\mathbb{Q})$. A direct search then should give the required set of generators. Provided, of course, this number B is small. Otherwise some more descent work has to be done (see for instance [15, final section]).

From here on we assume that a complete set of generators of infinite order has been determined. Let us suppose that

$$E_n(\mathbb{Q})/E_n(\mathbb{Q})_{\text{tors}} = \langle P_1, \ldots, P_{r_n}\rangle.$$

For $P \in E_n(\mathbb{Q})$, there exist rational integers $m_1, \ldots, m_{r_n}$ such that

$$P = m_1P_1 + \cdots + m_{r_n}P_{r_n} + \varepsilon P_0, \tag{8}$$

where $P_0 = (0,0)$ and $\varepsilon \in \{0,1\}$.

For integral $P = (X(P), Y(P))$ we intend to estimate the integral vector $\boldsymbol{m} = (m_1, \ldots, m_{r_n})$. If we let $\boldsymbol{m}$ be a column vector, then $\hat{h}(P) = \boldsymbol{m}^T \mathcal{H}_n \boldsymbol{m}$. Here the matrix $\mathcal{H}_n$ is given by $\mathcal{H}_n = \left(\frac{1}{2}\langle P_i, P_j\rangle\right)_{r_n \times r_n}$, where

$$\langle Q, R\rangle := \hat{h}(Q+R) - \hat{h}(Q) - \hat{h}(R)$$

is the Néron–Tate pairing. The matrix $\mathcal{H}_n$ is positive definite and hence

$$\hat{h}(P) \geqslant \lambda_n M_n^2, \tag{9}$$

where λ_n is the smallest eigenvalue of $\mathcal{H}_n$ and $M_n := \max_{1 \leqslant i \leqslant r_n} |m_i|$. It is easy to calculate λ_n once the rank r_n and the set of generators $\langle P_1, \ldots, P_{r_n}\rangle$ are known.

From (9) we see that large coefficients m_i in (8) imply a large canonical height value $\hat{h}(P)$ and by Lemma 2, this forces $h(P)$ to be large, which in turn makes $X(P)$ large, because $h(P) = \log X(P)$ for integral $X(P)$. As the number of integral points on (3) is finite, $X(P)$ is bounded, and this implies that M_n is bounded. In the next sections an upper bound for M_n will be derived.

3. Elliptic logarithms

Again let $P = (X(P), Y(P))$ be an integral point on (3) with $d = d_n$. The boundedness of $X(P)$ can be expressed by saying that P cannot be too close to the identity O of the group $E_n(\mathbb{Q})$. In order to measure the distance between P and O, we use the group isomorphism

$$\phi\colon E_n(\mathbb{R}) \to \mathbb{R}/\mathbb{Z} \quad \text{(circle group)},$$

given by

$$\phi(P) \equiv \begin{cases} 0 \pmod 1 & \text{if } P = O, \\ \dfrac{1}{\omega}\displaystyle\int_{X(P)}^{\infty} \frac{\mathrm{d}t}{\sqrt{t^3 + d_n t}} \pmod 1 & \text{if } Y(P) \geqslant 0, \\ -\phi(-P) \pmod 1 & \text{if } Y(P) \leqslant 0. \end{cases} \tag{10}$$

(see [22, p. 429]). Here

$$\omega = 2\int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{t^3 + d_n t}} = \frac{2}{\sqrt[4]{d_n}} \int_0^{\infty} \frac{\mathrm{d}t}{\sqrt{t^3 + t}}$$

is the fundamental real period of the Weierstraß $\wp$-function associated with (3) for $d = d_n$. There is no loss of generality in assuming that $\phi(P) \in [0, 1)$. Then $\phi(P) \in [0, \frac{1}{2}]$ when $Y(P) \geqslant 0$. We may as well assume this from now on. The quantities $u_i := \omega\phi(P_i)$ for $i = 1, \ldots, r_n$ are known as the elliptic logarithms.

Applying ϕ to (8) yields

$$\phi(P) \equiv m_1\phi(P_1) + \cdots + m_{r_n}\phi(P_{r_n}) + \tfrac{1}{2}\varepsilon \pmod 1, \quad \varepsilon \in \{0,1\}$$

and hence a rational integer m_0 exists such that

$$\phi(P) = m_0 + \tfrac{1}{2}\varepsilon + m_1\phi(P_1) + \cdots + m_{r_n}\phi(P_{r_n}). \tag{11}$$

Clearly, $|m_0| \leqslant 1 + |m_1| + \cdots |m_{r_n}| \leqslant 1 + r_n M_n$.

Multiplying (11) by ω and setting $L(P) := \omega\phi(P)$ yields

$$L(P) = (m_0 + \tfrac{1}{2}\varepsilon)\omega + m_1 u_1 + \cdots + m_{r_n} u_{r_n}. \tag{12}$$

An upper bound for $|L(P)|$ in terms of n and M_n is quickly obtained.

LEMMA 3. *Let* $P = (X(P), Y(P)) \in E_n(\mathbb{Q})$ *be an integral point on* (3) *with* $d = d_n$ *and* $X(P) > 0$. *Assume further that* P *satisfies* (8) *and let* $L(P)$ *be as in* (12). *Then*

$$|L(P)| \leqslant 2n \exp\left(1.9055 - \lambda_n M_n^2\right). \tag{13}$$

Proof. From Lemma 2 and (9) it follows that

$$\begin{aligned} |L(P)| &= |\omega\phi(P)| = \int_{X(P)}^{\infty} \frac{\mathrm{d}t}{\sqrt{t^3 + d_n t}} \leqslant \int_{X(P)}^{\infty} t^{-3/2}\mathrm{d}t \leqslant 2X(P)^{-1/2} \\ &< 2\exp\left(\log n + 1.9055 - \hat{h}(P)\right) \leqslant 2n\exp(1.9055 - \lambda_n M_n^2), \end{aligned}$$

as required. □

This upper bound for $|L(P)|$, combined with Sinnou David's lower bound produces an upper bound for M_n. We shall reproduce this lower bound for $|L(P)|$ as it can be found in [17], omitting the details.

LEMMA 4. (S. David) *Let* $P \in E_n(\mathbb{Q})$ *be given as in Lemma* 3. *Further, let* $h_n = \max\{\log 1728, 4\log n\}$, *and for* $i = 0, \ldots, r_n$, *let* A_i *be a positive number, satisfying* $A_i \geqslant \max\{\hat{h}(P_i), h_n, 6\pi(u_i/\omega)^2\}$ *($u_0 = \omega$ by definition). If*

$$B_n \geqslant \max\{\exp(A_0), \ldots, \exp(A_{r_n}), 2|m_0| + 1, |m_1|, \ldots, |m_{r_n}|, 16\},$$

then a lower bound for $|L(P)|$ *is given by*

$$|L(P)| \geqslant \exp\left(-c_n(\log B_n + 1)(\log\log B_n + 1 + h_n)^{r_n+2}\right), \tag{14}$$

where

$$c_n = 2 \cdot 10^{7r_n+15}\left(\frac{2}{e}\right)^{2(r_n+1)^2}(r_n+2)^{4r_n^2+18r_n+14}\prod_{i=0}^{r_n} A_i.$$

This is a special case of [6, Theorem 2.1]). From [17, Appendix] it can be seen that we may take $\tau = i$ and $\omega_1 = \frac{1}{2}\omega(1+i)$.

4. The computation of solutions

If we take $B_n = 2r_nM_n + 3$ in Lemma 4, then combining (13) and (14), leads to the following inequality

$$\lambda_n M_n^2 < c_n(\log B_n + 1)(\log\log B_n + 1 + h_n)^{r_n+2} + \log 2n + 1.9055, \tag{15}$$

which provides an upper bound for M_n. This upper bound generally is very large – up to 10^{60} in the range of n-values we consider. This is way out of reach of any practical search method. So we apply the LLL-reduction process described in detail in [17, Section 5]; see also [21, Chapter 3]. A brief breakdown of this procedure should suffice at this point.

From Lemma 3 we deduce

$$|\phi(P)| < K_1 \exp(-K_2 M_n^2) \quad \text{and} \quad M_n < K_3, \tag{16}$$

where $K_1 = 2n\omega^{-1}\exp(1.9055)$, $K_2 = \lambda_n$, and K_3 is a (usually large) upper bound for M_n. Let $\mathcal{L}$ be the (r_n+1)-dimensional lattice, generated by the columns of the matrix

$$\mathcal{A}_{\mathcal{L}} := \begin{pmatrix} 1 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ [K_0\phi(P_1)] & \dots & [K_0\phi(P_r)] & K_0 \end{pmatrix},$$

where K_0 will be a large integer that will be conveniently chosen later.

If the vector $(m_1, \dots, m_r, m_0) \in \mathbb{Z}^{r_n+1}$ satisfies $|m_i| < K_3$ for $i = 0, 1, \dots, r_n$, and

$$\ell := \mathcal{A}_{\mathcal{L}} \begin{pmatrix} 2m_1 \\ \vdots \\ 2m_{r_n} \\ 2m_0 + \varepsilon \end{pmatrix} = \begin{pmatrix} 2m_1 \\ \vdots \\ 2m_{r_n} \\ t \end{pmatrix},$$

with

$$t := \sum_{i=1}^{r_n} 2m_i[K_0\phi(P_i)] + (2m_0 + \varepsilon)K_0,$$

then, because of (11),

$$||\ell||^2 = 4\sum_{i=1}^{r_n} m_i^2 + t^2 \leqslant 4r_nK_3^2 + 4\left\{K_0|\phi(P)| + r_nK_3\right\}^2. \tag{17}$$

On the other hand, if $\{b_1, \ldots, b_{r_n+1}\}$ is an LLL-reduced basis of $\mathcal{L}$, then

$$||b_1||^2 \leqslant 2^{r_n}||\ell||^2.$$

Combining this with (17) yields

$$K_0|\phi(P)| \geqslant \sqrt{2^{-r_n-2}||b_1||^2 - r_n K_3^2} - r_n K_3. \tag{18}$$

From (16) and (18) we now obtain a new upper bound for M_n, implicitly given by the inequality

$$M_n^2 \leqslant K_2^{-1}\left(\log(K_0K_1) - \log(\sqrt{2^{-r_n-2}||b_1||^2 - r_n K_3^2} - r_n K_3)\right), \tag{19}$$

provided the right-hand-side of (18) is positive, i.e.

$$||b_1|| > 2^{1+r_n/2}K_3\sqrt{r_n^2 + r_n}. \tag{20}$$

It is reasonable to expect that $||b_1|| \approx (\det \mathcal{A}_{\mathcal{L}})^{1/(r_n+1)} = K_0^{1/(r_n+1)}$. Therefore, if we choose K_0 such that $K_0^{1/(r_n+1)}$ is slightly larger than the right-hand side of (20), then most likely this inequality is satisfied. In that case equation (19) produces a new upper bound which is of the size of $\sqrt{\log M_n}$, a considerable improvement. In most cases, the reduction process can be applied a few times before no further improvement is obtained. In Table 1, the successive values of M_n are listed. The size of the original M_n is determined almost exclusively by the rank of the curve. Two reduction steps were sufficient to bring the ultimate M_n-value down to 6 or below; only for $n = 98$ an extra reduction step was required. This means that a final direct search for points that may have been missed is always feasible. In order to execute the reduction process we need to know the $\phi(P_i)$ values correctly to a great number of decimal places. In [22] Don Zagier describes a very efficient algorithm to compute these values to the precision needed. We programmed his algorithm in the very fast UBasic 8.30 language. The LLL-reduction step was carried out by using Pari GP 1.38's integral LLL algorithm.

The results of our calculations are gathered in two tables, from which the computational process can be directly read off. Only for $n = 29, 41, 44$, and 47 in the range of chosen n-values, Apecs could not determine the rank unconditionally. Fortunately, John Cremona's algorithms – especially mwrank.sun – turned out to be very useful in these exceptional cases. We remind the reader of our discussion of section 2, just before Lemma 2. Also for $n = 29, 38, 46$, and 47, the bound (7) is larger than 20, which is rather big for a direct search. Here an extra computational effort was needed. The fact that we encountered few hard cases is very encouraging and may count as a modest success for the method of elliptic logarithms in general.

The final table contains all solutions to the original problem for $n = 2, \ldots, 50, 98$.

TABLE I. Rank and generators of $E_n(\mathbb{Q})$ ($n = 2, \ldots, 50, 98$)

$P = m_1P_1 + \cdots + m_{r_n}P_{r_n} + \varepsilon P_0$ with $\varepsilon \in \{0,1\}$, $P_0 = (0,0)$ and $\lvert m_i\rvert \leqslant M_n$						
n	rank r_n	generators $P_1, \ldots, P_{r_n}$ on $Y^2 = X^3 + \frac{1}{4}n^2(n^2-1)X$	λ_n	M_n before and after LLL–reduction steps		
2	1	$(1,2)$	0.25059	1.236×10^{23}	15	5
3	1	$(3,9)$	0.32737	1.033×10^{23}	12	5
4	1	$(6,24)$	0.43104	9.394×10^{22}	11	4
5	2	$(10,50), (24,132)$	0.44721	1.609×10^{38}	20	6
6	1	$(15,90)$	0.53363	8.433×10^{22}	10	4
7	1	$(21,147)$	0.57242	8.654×10^{22}	10	4
8	2	$(4,64), (28,224)$	0.55957	1.961×10^{38}	18	5
9	1	$(36,324)$	0.63552	1.002×10^{23}	9	3
10	1	$(45,450)$	0.66194	1.029×10^{23}	9	3
11	1	$(55,605)$	0.68583	1.074×10^{23}	9	3
12	2	$(66,792), (22,352)$	0.57673	2.692×10^{38}	18	5
13	2	$(78,1014), (168,2436)$	0.68710	2.653×10^{38}	16	5
14	2	$(91,1274), (2625/4, 135975/8)$	0.73786	2.734×10^{38}	16	4
15	2	$(105,1575), (480,10800)$	0.68660	3.011×10^{38}	16	5
16	1	$(120,1920)$	0.77965	1.256×10^{23}	8	3
17	2	$(136,2312), (72,1368)$	0.74054	3.224×10^{38}	16	4
18	3	$(153,2754), (9,486), (441/4, 16443/8)$	0.72966	6.955×10^{58}	24	6
19	2	$(171,3249), (10051/25, 1104299/125)$	0.82216	3.351×10^{38}	15	4
20	1	$(190,3800)$	0.83549	1.369×10^{23}	8	3
21	2	$(210,4410), (504,12348)$	0.72983	3.853×10^{38}	16	4
22	1	$(231,5082)$	0.85933	1.418×10^{23}	8	3
23	1	$(253,5819)$	0.87045	1.441×10^{23}	8	3
24	1	$(276,6624)$	0.88109	1.463×10^{23}	8	3
25	1	$(300,7500)$	0.89130	1.485×10^{23}	8	3
26	2	$(325,8450), (101439/49, 32734962/343)$	0.85791	4.182×10^{38}	14	4
27	1	$(351,9477)$	0.91055	1.526×10^{23}	8	3
28	2	$(378,10584), (58,3016)$	0.72904	4.794×10^{38}	16	5
29	2	$(406,11774), (5793649/67600, 69787713143/17576000)$	0.90273	4.416×10^{38}	14	4
30	2	$(435,13050), (625,19250)$	0.90509	4.519×10^{38}	14	4
31	2	$(496,15376), (20956/81, 6396416/729)$	0.94471	4.528×10^{38}	14	4
32	3	$(496,15872), (16,2048), (1984,91264)$	0.83995	1.163×10^{59}	22	5
33	2	$(528,17424), (1617,68607)$	0.90380	4.839×10^{38}	14	4
34	3	$(561,19074), (289,10982), (14875/9, 1921850/27)$	0.81568	1.249×10^{59}	22	5
35	2	$(595,20825), (8470,781550)$	0.93915	4.945×10^{38}	14	4
36	1	$(630,22680)$	0.98249	1.681×10^{23}	7	3

TABLE I. – *contd.*

$P = m_1P_1 + \cdots + m_{r_n}P_{r_n} + \varepsilon P_0$ with $\varepsilon \in \{0,1\}$, $P_0 = (0,0)$ and $\lvert m_i \rvert \leqslant M_n$						
n	rank r_n	generators $P_1, \ldots, P_{r_n}$ on $Y^2 = X^3 + \frac{1}{4}n^2(n^2-1)X$	λ_n	M_n before and after LLL–reduction steps		
37	1	$(666, 24642)$	0.98934	1.696×10^{23}	7	3
38	3	$(703, 26714), (841/4, 87203/8),$ $(12103/9, 1511146/27)$	0.91224	1.307×10^{59}	21	5
39	2	$(741, 28899), (234, 12168)$	0.77297	5.872×10^{38}	15	5
40	2	$(780, 31200), (625, 25375)$	0.98120	5.297×10^{38}	14	4
41	1	$(820, 33620)$	1.0150	1.753×10^{23}	7	3
42	2	$(861, 36162), (3549, 217854)$	0.87058	5.812×10^{38}	14	4
43	1	$(903, 38829)$	1.0269	1.779×10^{23}	7	3
44	1	$(946, 41624)$	1.0326	1.792×10^{23}	7	3
45	2	$(990, 44550), (8910, 846450)$	1.0357	5.573×10^{38}	13	4
46	3	$(1035, 47610), (441, 24066),$ $(500940, 354551670)$	1.0219	1.462×10^{59}	20	5
47	2	$(1081, 50807), (395641/5776,$ $4019566003/438976)$	0.96654	5.935×10^{38}	14	4
48	3	$(504, 28224), (1128, 54144),$ $(564, 30456)$	0.62355	1.944×10^{59}	26	6
49	1	$(1176, 57624)$	1.0595	1.853×10^{23}	7	3
50	2	$(1225, 61250), (25, 6250)$	1.0612	5.892×10^{38}	13	4
98	4	$(1617, 203742), (4753, 465794),$ $(2352, 259308)\ (19404, 2784474)$	0.62186	1.872×10^{85}	36	8|6

TABLE II. Consecutive cubes adding up to a square ($n = 2, \ldots, 50, 98$)

Solutions $(x, y) \in \mathbb{N}^2$ of $y^2 = x^3 + \cdots + (x + n - 1)^3$ with $x > 1$. No entry indicates that no such solution exists	
n	(x, y)
3	(23,204)
5	(25,315), (96,2170), (118,2940)
7	(333, 16296)
8	(28,504)
9	(716, 57960)
11	(1315, 159060)
12	(14,312)
13	(144, 6630), (2178, 368004)
15	(25,720), (3353, 754320), (57960, 54052635)
17	(9,323), (120, 5984), (4888, 1412496)
18	(153, 8721), (680, 76653)
19	(6831, 2465820)
21	(14,588), (144, 8778), (9230, 4070220)
23	(12133, 6418104)
25	(15588, 9742200)
27	(19643, 14319396)
28	(81,4914)
29	(24346, 20474580)
31	(29745, 28584480)
32	(69,4472), (133, 10296), (496, 65472)
33	(33,2079), (35888, 39081504)
35	(225, 22330), (42823, 52457580)
37	(50598, 69267996)
39	(111, 9360), (59261, 90135240)
40	(3276, 1196520)
41	(68860, 115752840)
42	(64,5187)
43	(79443, 146889204)
45	(176, 18810), (91058, 184391460)
47	(103753, 229189296)
48	(64,5880), (410, 62628), (19881, 19455744), (60040, 101985072)
49	(117576, 282298800)
50	(1225, 312375)
98	(25,7497), (97, 18333), (216, 43309), (745, 221697), (760, 227997) (3961, 2513511), (4753, 3293829)

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