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Connectedness results for ℓ-adic representations associated to abelian varieties

Dedicated to Frans Oort on the occasion of his 60th birthday

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1. Introduction

Suppose X is an abelian variety defined over a field F, ℓ is a prime number, and $\ell \neq \text{char}(F)$. Let F^s denote a separable closure of F, let $T_\ell(X) = \varprojlim X_{\ell^r}$ (the Tate module), let $V_\ell(X) = T_\ell(X) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$, and let $\rho_{X,\ell}$ denote the ℓ-adic representation

$$\rho_{X,\ell}\colon \text{Gal}(F^s/F) \to \text{Aut}(T_\ell(X)) \subseteq \text{Aut}(V_\ell(X)) \cong \text{GL}_{2d}(\mathbf{Q}_\ell),$$

where $d = \dim(X)$. If L is an extension of F in F^s, let $G_{L,X}$ denote the image of $\text{Gal}(F^s/L)$ under $\rho_{X,\ell}$. Let $\mathfrak{G}_\ell(F,X)$ denote the algebraic envelope of the image of $\rho_{X,\ell}$, i.e., the Zariski closure in $\text{GL}_{2d}(\mathbf{Q}_\ell)$ of $G_{F,X}$. Let $F_{\Phi,\ell}(X)$ be the smallest extension F' of F such that $\mathfrak{G}_\ell(F',X)$ is connected. If G is an algebraic group, let G^0 denote the identity connected component. Let Φ denote the group of connected components

$$\Phi = \mathfrak{G}_\ell(F,X)/\mathfrak{G}_\ell(F,X)^0.$$

The algebraic group $\mathfrak{G}_\ell(F,X)$, the finite group Φ, and the field $F_{\Phi,\ell}(X)$ were introduced and studied by Serre (see [16] and [17]). Our goal in this paper is to compare the field $F_{\Phi,\ell}(X)$ with other extensions of F (especially those generated by torsion points on X) and to prove sufficient conditions for the connectedness of $\mathfrak{G}_\ell(F,X)$.

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Let $F(\mathrm{End}(X))$ denote the smallest extension of F over which all the endomorphisms of X are defined. We have (see Proposition 2.10)

$$F(\mathrm{End}(X)) \subseteq F_{\Phi,\ell}(X).$$

In Theorem 3.7 (see also Theorem 3.8) we show that if $n \geq 5$, n is not divisible by $\mathrm{char}(F)$, and λ and $\widetilde{X}_n$ are as above, then

$$F(\mathrm{End}(X)) \subseteq F(\widetilde{X}_n, \boldsymbol{\mu}_n, \lambda).$$

Suppose now that F is a finitely generated extension of $\mathbf{Q}$. Serre proved that $F_{\Phi,\ell}(X)$ is independent of ℓ (see Theorem 2.11), so we will denote the field $F_{\Phi,\ell}(X)$ by $F_\Phi(X)$. If n is an integer greater than 2, then (see Remark 3.1)

$$F_\Phi(X) \subseteq F(X_n).$$

A consequence of our main result of Section 3 (see Theorem 3.2) is that if X is an abelian variety defined over a finitely generated extension F of $\mathbf{Q}$, n is an integer greater than 4, λ is a polarization on X, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to the Weil pairing induced by λ, then

$$F_\Phi(X) \subseteq F(\widetilde{X}_n, \boldsymbol{\mu}_n, \lambda).$$

In other words, if F is a field of definition for the polarization λ, the points of $\widetilde{X}_n$, and the nth roots of unity, then $\mathfrak{G}_\ell(F, X)$ is connected. (See also Theorems 3.4 and 3.6 for results for global fields and arbitrary fields, respectively.) This gives a new criterion, in terms of torsion points of X, for the connectedness of $\mathfrak{G}_\ell(F, X)$.

In conversations with Silverberg in 1990, Serre asked whether it is true that $F_\Phi(X) = \bigcap_{p \geq n_0}^{p'} F(X_p)$ for every integer $n_0 \geq 3$. We discuss this question further elsewhere.

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2. Definitions, notation, and lemmas

Let $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote respectively the integers, rational numbers, real numbers, and complex numbers. If F is a field, let $\bar{F}$ denote an algebraic closure and let F^s denote a separable closure. Suppose X is an abelian variety defined over F. Write $\mathrm{End}_F(X)$ for the set of endomorphisms of X which are defined over F, let $\mathrm{End}(X) = \mathrm{End}_{F^s}(X)$, and let $\mathrm{End}^0(X) = \mathrm{End}(X) \otimes_{\mathbf{Z}} \mathbf{Q}$. If λ is a polarization on X, n is a positive integer not divisible by $\mathrm{char}(F)$, and $\boldsymbol{\mu}_n$ is the $\mathrm{Gal}(F^s/F)$-module of nth roots of unity in F^s, then the e_n-pairing induced by the polarization λ

$$e_{\lambda,n} \colon X_n \times X_n \to \boldsymbol{\mu}_n$$

(see Section 75 of [23]), is a skew-symmetric bilinear map which satisfies:

$$\sigma(e_{\lambda,n}(x_1,x_2)) = e_{\sigma(\lambda),n}(\sigma(x_1),\sigma(x_2))$$

for every $\sigma \in \mathrm{Gal}(F^s/F)$ and x_1, $x_2 \in X_n$. If n is relatively prime to the degree of the polarization λ, then the pairing $e_{\lambda,n}$ is nondegenerate. If $\tilde{X}$ is a subset of X_n, then

$$F(\tilde{X}, e_{\lambda,n}(X_n,\tilde{X}),\lambda)$$

denotes the smallest extension of F in F^s which contains the roots of unity in $e_{\lambda,n}(X_n,\tilde{X})$ and which is a field of definition for the polarization λ and the elements of $\tilde{X}$.

We recall some results from [21] and [22], which we extend and apply.

LEMMA 2.1 (Lemma 5.2 of [22]). *Suppose that d and n are positive integers, and for each prime ℓ which divides n we have a matrix $A_\ell \in M_{2d}(\mathbf{Z}_\ell)$ such that the characteristic polynomials of the A_ℓ have integral coefficients independent of ℓ, and such that $(A_\ell - I)^2 \in nM_{2d}(\mathbf{Z}_\ell)$. Then for every eigenvalue α of A_ℓ, $(\alpha-1)/\sqrt{n}$ satisfies a monic polynomial with integer coefficients.*

If k is a positive integer, define a finite set $N(k)$ by

$$N(k) = \{\text{prime powers } \ell^m : 0 \leq m(\ell-1) \leq k\}.$$

If n is a positive integer which is not in $N(k)$, let $R(k,n) = 1$. Let $R(k,1) = 0$. If $1 \neq n = \ell^m \in N(k)$ with ℓ a prime, let

$$R(k,n) = \ell^{r(k,n)} \quad \text{where} \quad r(k,n) = \max\{r \in \mathbf{Z}^+ : m(\ell-1)\ell^{r-1} \leq k\}.$$

THEOREM 2.2 (Corollary 3.3 of [21]). *Suppose n and k are positive integers, $\mathcal{O}$ is an integral domain of characteristic zero such that no rational prime which divides n is a unit in $\mathcal{O}$, $\alpha \in \mathcal{O}$, α has finite multiplicative order, and $(\alpha-1)^k \in n\mathcal{O}$. Then $\alpha^{R(k,n)} = 1$.*

In the case $k = 2$ we have the following corollary.

COROLLARY 2.3. *Suppose n is an integer greater than 4, $\mathcal{O}$ is an integral domain of characteristic zero such that no rational prime divisor of n is a unit in $\mathcal{O}$, $\alpha \in \mathcal{O}$, α has finite multiplicative order, and $(\alpha-1)^2 \in n\mathcal{O}$. Then $\alpha = 1$.*

LEMMA 2.4. *Suppose $\mathcal{O}$ is an integral domain of characteristic zero, n and k are positive integers such that no rational prime which divides n is a unit in $\mathcal{O}$, $A \in \mathrm{GL}_g(\mathcal{O})$ satisfies $(A-I)^k \in nM_g(\mathcal{O})$, and α is a root of unity in the multiplicative group generated by the eigenvalues of A. Then $\alpha^{R(k,n)} = 1$.*

Proof. View the eigenvalues of A as lying in the integral closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ in an algebraically closed field containing $\mathcal{O}$. As shown in Lemma 6.6 of [21], no rational prime divisor of n is a unit in $\bar{\mathcal{O}}$. If μ is an eigenvalue of A, then $\mu \in \bar{\mathcal{O}}$ and $(\mu-1)^k \in n\bar{\mathcal{O}}$. Therefore, the multiplicative group $G = \{\beta \in \bar{\mathcal{O}} : (\beta-1)^k \in n\bar{\mathcal{O}}\}$ contains the multiplicative group generated by the eigenvalues of A. By Theorem 2.2, every root of unity α in G satisfies $\alpha^{R(k,n)} = 1$. □

The following proposition gives a means of verifying the connectedness or disconnectedness of a linear algebraic group. See also [2], especially Section 8 in Chapter III, or [10], especially Chapter VI.

PROPOSITION 2.5. *Suppose φ is an invertible linear operator on a finite-dimensional vector space V over a field of characteristic zero. Then the multiplicative group generated by the eigenvalues of φ contains no non-trivial roots of unity if and only if the smallest algebraic subgroup of* $\mathrm{GL}(V)$ *containing φ is connected.*

Proof. The connectedness or disconnectness of an algebraic group is invariant under extensions of the ground field, so we may assume the ground field k is algebraically closed. The Jordan decomposition (see Section 4 in Chapter I of [2]) gives a unipotent operator u and a semisimple operator s such that $\varphi = su = us$. If $f \in \mathrm{GL}(V)$, let G_f denote the smallest algebraic subgroup of $\mathrm{GL}(V)$ containing f. Let $x = \log(u)$. Then $G_u(k) = \{\exp(tx) : t \in k\}$, a (zero- or one-dimensional) connected algebraic group. Let $\alpha_1, \dots, \alpha_n$ denote the eigenvalues of s, with multiplicity. Then

$$G_s \cong \left\{ \begin{pmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_n \end{pmatrix} : \text{if } \prod \alpha_i^{b_i} = 1 \text{ with } b_i \in \mathbf{Z} \text{ then } \prod \beta_i^{b_i} = 1 \right\}.$$

The multiplication map $G_s \times G_u \to G_\varphi$ is an isomorphism (by the definition of G_φ and the above characterizations of the groups G_s and G_u). Since G_u is connected and the eigenvalues of u are all 1, we can reduce to the case $\varphi = s$. Let $X(G_s) = \mathrm{Hom}(G_s, \mathbf{G}_m)$, the group of characters of G_s. Then $X(G_s) \cong \mathbf{Z}^n/B$, where

$$B = \left\{(b_1, \dots, b_n) \in \mathbf{Z}^n : \prod \alpha_i^{b_i} = 1\right\}.$$

We next show that G_s is connected if and only if $X(G_s)$ has no non-trivial torsion. If G_s is connected then it is a connected commutative algebraic group with no nilpotent radical, so $G_s \cong \mathbf{G}_m^r$ for some r, and so $X(G_s) \cong \mathbf{Z}^r$. Conversely, if G_s is not connected then there is a non-trivial homomorphism $G_s/G_s^0 \to \mathbf{G}_m$, which induces a homomorphism $G_s \to \mathbf{G}_m$ which is a non-trivial torsion element of $X(G_s)$.

Non-trivial torsion elements of $X(G_s)$ correspond to elements $(c_1, \dots, c_n) \in \mathbf{Z}^n$ for which $\prod \alpha_i^{c_i}$ is a non-trivial root of unity in the multiplicative group generated by the eigenvalues of s. We therefore obtain the desired result. □

PROPOSITION 2.6. *Suppose $\mathcal{O}$ is an integral domain of characteristic zero, F is its fraction field, and n and k are positive integers such that no rational prime which divides n is a unit in $\mathcal{O}$. Suppose G is a subgroup of $\mathrm{GL}_g(F)$ generated by elements $A \in \mathrm{GL}_g(\mathcal{O})$ such that $(A - I)^k \in nM_g(\mathcal{O})$. If $n \notin N(k)$, then the Zariski closure of G in $\mathrm{GL}_g(F)$ is connected.*

Proof. By the Corollary on p. 56 of [10], an algebraic group which is generated (as an abstract group) by closed connected subgroups is connected. The Proposition therefore follows from Lemma 2.4 and Proposition 2.5. □

LEMMA 2.7. *If X is an abelian variety over a field F, and L is a finite extension of F in F^s, then $\mathfrak{G}_\ell(L, X) \subseteq \mathfrak{G}_\ell(F, X)$ and $\mathfrak{G}_\ell(L, X)^0 = \mathfrak{G}_\ell(F, X)^0$. In particular, if $\mathfrak{G}_\ell(F, X)$ is connected, then $\mathfrak{G}_\ell(F, X) = \mathfrak{G}_\ell(L, X)$.*

Proof. Since $G_{L,X}$ is a subgroup of finite index in $G_{F,X}$, the group $G_{F,X}$ is a finite disjoint union of cosets of $G_{L,X}$. Therefore $\mathfrak{G}_\ell(F, X)$ is a finite disjoint union of cosets of $\mathfrak{G}_\ell(L, X)$. Thus $\mathfrak{G}_\ell(L, X)$ is a closed subgroup of finite index in $\mathfrak{G}_\ell(F, X)$. By the Proposition on p. 53 of [10], $\mathfrak{G}_\ell(F, X)^0 \subseteq \mathfrak{G}_\ell(L, X)$. Therefore, $\mathfrak{G}_\ell(F, X)^0 = \mathfrak{G}_\ell(L, X)^0$. □

REMARK 2.8. If X is an abelian variety over a finitely generated extension F of the prime field, and $\ell \neq \mathrm{char}(F)$, then the algebraic group $\mathfrak{G}_\ell(F, X)^0$ is reductive, since the representation $\rho_{X,\ell}$ is semisimple (by Faltings ([7], [8]) in the characteristic zero case, by Zarhin ([25], [26]) in the case of characteristic greater than 2, and by Mori ([11], especially Section 5 of Chapter VI and Section 2 of Chapter XII) in the characteristic 2 case. See also [28].). Note also (see [1]) that if F is a finitely generated extension of $\mathbf{Q}$ then $G_{F,X}$ is an open subgroup of $\mathfrak{G}_\ell(F, X)(\mathbf{Q}_\ell)$.

LEMMA 2.9. *Suppose X is an abelian variety defined over a field F, λ is a polarization of X, n is a positive integer not divisible by the characteristic of F, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to the pairing $e_{\lambda,n}$. Suppose the polarization λ, the points of $\widetilde{X}_n$, and the roots of unity in $e_{\lambda,n}(X_n, \widetilde{X}_n)$ are all defined over F. Then $(\sigma - 1)^2 X_n = 0$ for every $\sigma \in \mathrm{Gal}(F^s/F)$.*

Proof. The pairing $e_{\lambda,n}$ induces a natural homomorphism

$$X_n \to \mathrm{Hom}(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n)),$$

which is $\mathrm{Gal}(F^s/F)$-equivariant since the polarization λ is defined over the field F. Since $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n, $\widetilde{X}_n$ is the kernel of the map, and we can view $X_n/\widetilde{X}_n$ as a $\mathrm{Gal}(F^s/F)$-submodule of $\mathrm{Hom}(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n))$. If $\sigma \in \mathrm{Gal}(F^s/F)$, then $\sigma = 1$ on $\widetilde{X}_n$ and on $e_{\lambda,n}(X_n, \widetilde{X}_n)$. Therefore, $\sigma = 1$

If $\sigma \in \mathrm{Gal}(F^s/F)$, then $\sigma = 1$ on $\widetilde{X}_n$ and on $e_{\lambda,n}(X_n, \widetilde{X}_n)$. Therefore, $\sigma = 1$ on $X_n/\widetilde{X}_n$, i.e., $(\sigma-1)X_n \subseteq \widetilde{X}_n$. Since $(\sigma-1)\widetilde{X}_n = 0$ we have $(\sigma-1)^2X_n = 0$. □

PROPOSITION 2.10. *If X is an abelian variety over a field F, ℓ is a prime, and $\ell \neq \mathrm{char}(F)$, then*

$$F(\mathrm{End}(X)) \subseteq F_{\Phi,\ell}(X).$$

Proof. Without loss of generality we may assume $F = F_{\Phi,\ell}(X)$. It then suffices to show that all the endomorphisms of X are defined over F. Let $V = V_\ell(X)$. If L is a finite extension of F in F^s, we have

$$\mathrm{End}_L(X) \subseteq (\mathrm{End}(V))^{\mathrm{Gal}(F^s/L)} = (\mathrm{End}(V))^{\mathfrak{G}_\ell(L,X)}.$$

Since $\mathfrak{G}_\ell(F, X)$ is connected, by Lemma 2.7 we have $\mathfrak{G}_\ell(F, X) = \mathfrak{G}_\ell(L, X)$. Therefore,

$$\mathrm{End}_L(X) \subseteq (\mathrm{End}(V))^{\mathfrak{G}_\ell(F,X)} = (\mathrm{End}(V))^{\mathrm{Gal}(F^s/F)}.$$

But

$$\mathrm{End}_L(X) \cap (\mathrm{End}(V))^{\mathrm{Gal}(F^s/F)} = \mathrm{End}_F(X).$$

Therefore, $\mathrm{End}_L(X) = \mathrm{End}_F(X)$. Now taking L to be a finite separable extension of F over which all the endomorphisms of X are defined, we have $\mathrm{End}(X) = \mathrm{End}_F(X)$. □

Although we do not make use of the following result in our proofs, we include it because of its importance to the subject of this paper.

THEOREM 2.11 (Serre). *If X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, then the field $F_{\Phi,\ell}(X)$ is independent of the prime ℓ.*

Proof. See [16] (see also Corollary 3.8 of [5], [15], and [18]). □

The following result is an immediate corollary.

COROLLARY 2.12 (Serre). *If X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, then*

(i) *if the algebraic group $\mathfrak{G}_\ell(F, X)$ is connected for one prime ℓ then it is connected for every prime ℓ,*
(ii) *the group Φ of connected components is independent of the prime ℓ.*

3. Field inclusions

REMARK 3.1. If X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, and n is an integer greater than 2, then

$$F_\Phi(X) \subseteq F(X_n)$$

(see [4], [3], and Proposition 3.6 of [5]).

In the result below we replace the n-torsion by a maximal isotropic subgroup.

THEOREM 3.2. *Suppose X is an abelian variety defined over a finitely generated extension F of $\mathbf{Q}$, λ is a polarization on X, n is an integer, $n \geq 5$, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$. Then*

$$F_{\Phi}(X) \subseteq F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda).$$

Proof. Suppose ℓ is a prime number. Without loss of generality, we may assume

$$F = F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda).$$

It then suffices to show that $\mathfrak{G}_\ell(F, X)$ is connected. Let R be a finitely generated smooth sub-$\mathbf{Z}$-algebra of F whose fraction field is F, and such that X is the generic fiber of an abelian scheme over $\mathrm{Spec}(R)$. Let $S = \mathrm{Spec}(R[\frac{1}{n\ell}])$, and let $\pi_1(S)$ denote the étale fundamental group of S with respect to the geometric point $\mathrm{Spec}(\bar{F})$. Then $\pi_1(S)$ is a quotient of $\mathrm{Gal}(\bar{F}/F)$, and the action of $\mathrm{Gal}(\bar{F}/F)$ on $V_\ell(X)$ factors through $\pi_1(S)$. To each closed point $y \in S$ we can associate a conjugacy class Fr_y of a Frobenius element in $\pi_1(S)$ (see p. 206 of [8]). By the Chebotarev density theorem (see Theorem 12 on p. 289 of [24] in the number field case, and see the Theorem on p. 206 of [8] for the Chebotarev density theorem in the generality of finitely generated extensions of $\mathbf{Q}$), the Fr_y are dense in $\pi_1(S)$. Let $\sigma \in \mathrm{Gal}(\bar{F}/F)$ be an element which maps to an element of a Frobenius conjugacy class associated to a closed point $y \in S$. By Lemma 2.9, we have $(\sigma - 1)X_n = 0$, and therefore for all prime numbers q we have

$$(\rho_{X,q}(\sigma) - I)^2 \in n\,\mathrm{End}(T_q(X)) \cong nM_{2d}(\mathbf{Z}_q),$$

where d is the dimension of X. If q is a prime not equal to the residue characteristic of y, then the characteristic polynomial of $\rho_{X,q}(\sigma)$ has integer coefficients which are independent of q. Note that the residue characteristic p of y does not divide ℓn. Let $\bar{\mathbf{Z}}$ denote the ring of algebraic integers. The eigenvalues of $\rho_{X,\ell}(\sigma)$ are in $1 + \sqrt{n}\bar{\mathbf{Z}}$ by Lemma 2.1, and are in $(\bar{\mathbf{Z}}[\frac{1}{p}])^\times$ by Weil's theorem. The multiplicative group generated by the eigenvalues of $\rho_{X,\ell}(\sigma)$ is a subset of the multiplicative semi-group $1 + \sqrt{n}\bar{\mathbf{Z}}[\frac{1}{p}]$, and therefore by Corollary 2.3 contains no non-trivial root of unity. By Proposition 2.5 and the Chebotarev density theorem, $\mathfrak{G}_\ell(F, X)$ is connected. (We again use that an algebraic group which is generated by closed connected subgroups is connected.) □

The following result is an immediate corollary.

COROLLARY 3.3. *Suppose X is an abelian variety defined over a finitely generated extension F of $\mathbf{Q}$, λ is a polarization on X, n is an integer, $n \geq 5$, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$. Then*

$$F_{\Phi}(X) \subseteq F(\widetilde{X}_n, \mu_n, \lambda).$$

THEOREM 3.4. *Suppose X is an abelian variety defined over a global field F of positive characteristic p, ℓ is a prime number different from p, λ is a polarization on X, n is an integer not divisible by p, $n \geq 5$, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$. Then*

$$F_{\Phi,\ell}(X) \subseteq F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda) \subseteq F(\widetilde{X}_n, \mu_n, \lambda).$$

Proof. The proof is the same as the proof of Theorem 3.2. For the Chebotarev density theorem for global fields, see Theorem 12 on p. 289 of [24]. □

REMARK 3.5. Theorem 3.2 and the result stated in Remark 3.1 should also hold for F a finitely generated extension of a finite field, using Theorem 3.4 and Mori's technique (see [12]) for inducting on the transcendence degree of F.

THEOREM 3.6. *Suppose X is an abelian variety defined over an arbitrary field F, λ is a polarization on X, n is a positive integer relatively prime to* char(F), *and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$. Suppose ℓ is a prime divisor of n, and either*

(i) *$\ell \geq 5$, or*
(ii) *$\ell = 3$ and n is divisible by 9, or*
(iii) *$\ell = 2$ and n is divisible by 8.*

Then

$$F_{\Phi,\ell}(X) \subseteq F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda) \subseteq F(\widetilde{X}_n, \mu_n, \lambda).$$

Proof. Without loss of generality, we may assume

$$F = F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda).$$

It then suffices to show that $\mathfrak{G}_\ell(F, X)$ is connected. Let ℓ^w be the highest power of ℓ which divides n. By Lemma 2.9, if $\sigma \in \mathrm{Gal}(F^s/F)$ then

$$(\rho_{X,\ell}(\sigma) - I)^2 \in nM_{2d}(\mathbf{Z}_\ell) = \ell^w M_{2d}(\mathbf{Z}_\ell).$$

By Proposition 2.6, $\mathfrak{G}_\ell(F, X)$ is connected. □

We now give a direct proof, valid over an arbitrary field F, that $F(\mathrm{End}(X)) \subseteq F(\widetilde{X}_n, e_{\lambda,n}(X_n, \widetilde{X}_n), \lambda)$. Theorems 3.7 and 3.8 extend earlier results in [19]; see also [20].

THEOREM 3.7. *Suppose (X, λ) is a polarized abelian variety defined over a field F, n is a positive integer which is greater than 4 and is not divisible by the characteristic of F, $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to the pairing $e_{\lambda,n}$, and the points of $\widetilde{X}_n$ and the roots of unity in $e_{\lambda,n}(X_n, \widetilde{X}_n)$ are all defined over F. Then every endomorphism of X is defined over F.*

Proof. The action of $\mathrm{Gal}(F^s/F)$ on X induces a representation

$$\rho\colon \mathrm{Gal}(F^s/F) \to \mathrm{Aut}(\mathrm{End}(X)).$$

Suppose $\sigma \in \mathrm{Gal}(F^s/F)$ and α is an eigenvalue of $\rho(\sigma)$. Then α is an algebraic integer. Since the endomorphisms of X are defined over a finite separable extension of F, $\rho(\sigma)$ has finite order and α is a root of unity. Let $p = \mathrm{char}(F)$ and let ℓ be a prime number different from p. Using the injections

$$\mathrm{End}(X) \hookrightarrow \mathrm{End}(X) \otimes \mathbf{Z}_\ell \hookrightarrow \mathrm{End}(T_\ell(X)),$$

we can view ρ as a map from $\mathrm{Gal}(F^s/F)$ to $\mathrm{Aut}(\mathrm{End}(T_\ell(X)))$. Then ρ is the adjoint representation of $\rho_{X,\ell}$. Let $\bar{\mathbf{Z}}$ and $\bar{\mathbf{Z}}_\ell$ denote integral closures of $\mathbf{Z}$ and $\mathbf{Z}_\ell$, respectively. For every embedding of $\bar{\mathbf{Z}}$ into $\bar{\mathbf{Z}}_\ell$, we can write $\alpha = a/b$ with a and b eigenvalues of $\rho_{X,\ell}(\sigma)$. By Lemma 2.9, we have $(\rho_{X,\ell}(\sigma) - I)^2 \in nM_{2d}(\mathbf{Z}_\ell)$. Therefore, $(a-1)/\sqrt{n}$ and $(b-1)/\sqrt{n}$ satisfy monic polynomials over $\mathbf{Z}_\ell$, i.e., $a, b \in 1 + \sqrt{n}\bar{\mathbf{Z}}_\ell$. Thus, $\alpha \in 1 + \sqrt{n}\bar{\mathbf{Z}}_\ell$, i.e., every embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_\ell$ sends $(\alpha - 1)/\sqrt{n}$ into $\bar{\mathbf{Z}}_\ell$, for every prime $\ell \neq p$. Therefore $(\alpha - 1)/\sqrt{n} \in \bar{\mathbf{Z}}[\frac{1}{p}]$, so $(\alpha - 1)^2 \in n\bar{\mathbf{Z}}[\frac{1}{p}]$. By Corollary 2.3, if $n \geq 5$ then $\alpha = 1$. Therefore $\rho(\sigma) = 1$ and all the endomorphisms of X are defined over F. □

THEOREM 3.8. *Suppose (X, λ) and (Y, μ) are polarized abelian varieties defined over a field F, and n is a positive integer which is greater than 4 and is not divisible by the characteristic of F. Suppose $\widetilde{X}_n$, respectively $\widetilde{Y}_n$, is a maximal isotropic subgroup of X_n, respectively Y_n, with respect to the pairing $e_{\lambda,n}$, respectively $e_{\mu,n}$. Suppose the points of $\widetilde{X}_n$ and $\widetilde{Y}_n$ and the roots of unity in $e_{\lambda,n}(X_n, \widetilde{X}_n)$ and $e_{\mu,n}(Y_n, \widetilde{Y}_n)$ are all defined over F. Then every homomorphism between X and Y is defined over F.*

Proof. Apply Theorem 3.7 to the polarized abelian variety $(X \times Y, \lambda \times \mu)$ with maximal isotropic subgroup $\widetilde{X}_n \times \widetilde{Y}_n \subseteq (X \times Y)_n$. □

4. Mumford–Tate groups

Next we define the Mumford–Tate group of a complex abelian variety X (see Section 2 of [14] or Section 6 of [27]). If X is a complex abelian variety, let $V = H_1(X(\mathbf{C}), \mathbf{Q})$ and consider the Hodge decomposition $V \otimes \mathbf{C} = H_1(X(\mathbf{C}), \mathbf{C}) = H^{-1,0} \oplus H^{0,-1}$. Define a homomorphism $\mu\colon \mathbf{G}_m \to \mathrm{GL}(V)$ as follows. For $z \in \mathbf{C}$, let $\mu(z)$ be the automorphism of $V \otimes \mathbf{C}$ which is multiplication by z on $H^{-1,0}$ and is the identity on $H^{0,-1}$.

DEFINITION 4.1. The *Mumford–Tate group* MT_X of X is the smallest algebraic subgroup of $\mathrm{GL}(V)$, defined over $\mathbf{Q}$, which after extension of scalars to $\mathbf{C}$ contains the image of μ.

It follows from the definition that MT_X is connected.

REMARK 4.2. Define a homomorphism $\varphi\colon \mathbf{G}_m \times \mathbf{G}_m \to \mathrm{GL}(V)$ as follows. For $z, w \in \mathbf{C}$, let $\varphi(z,w)$ be the automorphism of $V \otimes \mathbf{C}$ which is multiplication by z on $H^{-1,0}$ and is multiplication by w on $H^{0,-1}$. Then MT_X can also be defined as the smallest algebraic subgroup of $\mathrm{GL}(V)$, defined over $\mathbf{Q}$, which after extension of scalars to $\mathbf{C}$ contains the image of φ. The equivalence of the definitions follows easily from the fact that $H^{-1,0}$ is the complex conjugate of $H^{0,-1}$. (See Section 3 of [15], where MT_X is called the Hodge group. See also Section 6 of [27].)

If X is an abelian variety over a subfield F of $\mathbf{C}$, we fix an embedding of $\bar{F}$ in $\mathbf{C}$. This gives an identification of $V_\ell(X)$ with $H_1(X, \mathbf{Q}) \otimes \mathbf{Q}_\ell$, and allows us to view $MT_X \times \mathbf{Q}_\ell$ as a linear $\mathbf{Q}_\ell$-algebraic subgroup of $\mathrm{GL}(V_\ell(X))$. Let $MT_{X,\ell} = MT_X \times_{\mathbf{Q}} \mathbf{Q}_\ell$. Then $MT_X(\mathbf{Q}_\ell) = MT_{X,\ell}(\mathbf{Q}_\ell)$.

REMARK 4.3. The Mumford–Tate conjecture for abelian varieties (see [15]) may be reformulated as the equality of $\mathbf{Q}_\ell$-algebraic groups, $\mathfrak{G}_\ell(F,X)^0 = MT_{X,\ell}$.

THEOREM 4.4 (Piatetski-Shapiro [13], Deligne [6], Borovoi [3]). *If X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, then $MT_{X,\ell}(\mathbf{Q}_\ell)$ contains an open subgroup of finite index in $G_{F,X}$.*

COROLLARY 4.5. *If X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, then $\mathfrak{G}_\ell(F,X)^0 \subseteq MT_{X,\ell}$.*

Proof. By Theorem 4.4, we can find a finite algebraic extension L of F such that $G_{L,X} \subseteq MT_{X,\ell}(\mathbf{Q}_\ell)$. Then $\mathfrak{G}_\ell(L,X) \subseteq MT_{X,\ell}$. By Lemma 2.7, $\mathfrak{G}_\ell(F,X)^0 = \mathfrak{G}_\ell(L,X)^0 \subseteq \mathfrak{G}_\ell(L,X)$. □

In [4] (see also [3]) Borovoi showed that if X is an abelian variety over a finitely generated extension F of $\mathbf{Q}$, n is an integer greater than 2, and $F = F(X_n)$, then $G_{F,X}$ is contained in $MT_{X,\ell}(\mathbf{Q}_\ell)$, i.e., $\mathfrak{G}_\ell(F,X) \subseteq MT_{X,\ell}$. We have the following strengthening of Borovoi's result.

THEOREM 4.6. *Suppose (X, λ) is a polarized abelian variety over a finitely generated extension F of $\mathbf{Q}$, n is an integer greater than 4, and $\widetilde{X}_n$ is a maximal isotropic subgroup of X_n with respect to $e_{\lambda,n}$. Suppose the points of $\widetilde{X}_n$ and the roots of unity in $e_{\lambda,n}(X_n, \widetilde{X}_n)$ are all defined over F. Then $\mathfrak{G}_\ell(F,X) \subseteq MT_{X,\ell}$.*

Proof. By Theorem 3.2, we have $\mathfrak{G}_\ell(F,X) = \mathfrak{G}_\ell(F,X)^0$. The result now follows from Corollary 4.5. □

5. Semistable reduction and connectedness

Suppose X is an abelian variety over a field F and v is a discrete valuation on F. Let $\bar{v}$ be an extension of v to F^s, and let I_v denote the corresponding inertia subgroup of $\mathrm{Gal}(F^s/F)$. For a definition of semistable reduction, see p. 349 of [9] or Section 3 of [22] (or define it from the following theorem).

THEOREM 5.1 (Grothendieck, Proposition 3.5 and Corollaire 3.8 of [9]). *Suppose X is an abelian variety over a field F, v is a discrete valuation on F, and ℓ is a prime number different from the residue characteristic of v. Let $V = V_\ell(X)$. Then the following statements are equivalent:*

(i) *X has semistable reduction at v,*
(ii) *there is a subspace W of V such that I_v acts as the identity on W and on V/W,*
(iii) *I_v acts by unipotent operators on V.*

The definition of *motif semi-stable* on p. 396 of [18] suggests that the following result is already known. Since it follows easily from the techniques used in this paper, we have included it here.

THEOREM 5.2. *Suppose X is an abelian variety over a field F, v is a discrete valuation on F, and ℓ is a prime number different from the residue characteristic of v. Then X has semistable reduction at v if and only if the Zariski closure of $\rho_{X,\ell}(I_v)$ is connected.*

Proof. Let $\mathfrak{G}$ denote the Zariski closure of $\rho_{X,\ell}(I_v)$ in $\mathrm{GL}(V_\ell(X))$. If X has semistable reduction at v, then I_v acts on V by unipotent operators by Theorem 5.1, so 1 is the only eigenvalue of elements of $\rho_{X,\ell}(I_v)$. By Proposition 2.5, $\mathfrak{G}$ is connected.

Conversely, suppose $\mathfrak{G}$ is connected. Let L be a finite Galois extension of F over which X has semistable reduction above v, let w denote the restriction of $\bar{v}$ to L, and let I_w be the inertia subgroup for $\bar{v}$ over w. Let $W = V^{I_w}$, the subspace of V on which I_w acts as the identity. Then I_w is the identity on V/W, by Theorem 5.1. Let $\mathfrak{G}_w$ denote the Zariski closure of $\rho_{X,\ell}(I_w)$. Then $\mathfrak{G}_w$ acts as the identity on W and on V/W. Since I_w is an open subgroup of finite index in I_v, $\rho_{X,\ell}(I_w)$ is an open subgroup of finite index in $\rho_{X,\ell}(I_v)$. Therefore $\mathfrak{G}_w \subseteq \mathfrak{G}$, and $\mathfrak{G}$ is a finite disjoint union of cosets of $\mathfrak{G}_w$. Since $\mathfrak{G}$ is connected, $\mathfrak{G} = \mathfrak{G}_w$. Therefore, the subgroup $\rho_{X,\ell}(I_v)$ of $\mathfrak{G}(\mathbf{Q}_\ell)$ acts as the identity on W and on V/W. By Theorem 5.1, X has semistable reduction at v. □

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