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Brill-Noether theory for non-special linear systems

Dedicated to Frans Oort on the occasion of his 60th birthday

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Introduction

Classical Brill-Noether Theory concerns the behaviour of special linear systems on a general smooth curve C of genus g. Because of the Riemann-Roch Theorem this is equivalent to the study of secant space divisors (see the definition below) of the complete canonical linear system $|K_C|$ on C. In this paper we study the secant space divisors for a general non-special (not necessarily complete) linear system on a general smooth curve C.

Let C be a smooth irreducible complete curve defined over an algebraically closed field K of characteristic 0. We use the following notation for linear systems on C. Let $\mathcal{L}$ be an invertible sheaf of degree d on C. An $(n+1)$-dimensional linear subspace V of the space $\Gamma(C;\mathcal{L})$ of global sections of $\mathcal{L}$ defines an n-dimensional linear system on C. We identify this linear system with the projective space $\mathbf{P}(V)$ and we write $g^n_d(V)$. If s is a nonzero element of V, then D_s is the associated divisor on C. If no confusion is possible, we write g^n_d. For an integer $e \geqslant 1$, let $C^{(e)}$ be the eth symmetric product of C. We identify a point on $C^{(e)}$ with the effective divisor E of degree e on C it represents. For $E \in C^{(e)}$ we define

$$V(-E) = \{s \in V: D_s \geqslant E\},$$

$$g^n_d(-E) = \mathbf{P}(V(-E)) = \{D \in g^n_d: D \geqslant E\}.$$

Take a non-negative integer f with $e - f \leqslant n$.

0.1. DEFINITION. $E \in C^{(e)}$ is called an e-secant $(e-f-1)$-space divisor if $\dim(g^n_d(-E)) = n - e + f$. (In case g^n_d defines an embedding $C \subset \mathbf{P}^n$ then the linear span $\langle E \rangle$ has dimension $e - f - 1$. This explains the terminology.)

0.2. NOTATION. $V^{e-f}_e(g^n_d) = \{E \in C^{(e)}: \dim(g^n_d(-E)) \geqslant n - e + f\}$. Those subsets of $C^{(e)}$ have a natural scheme structure (see [1]).

0.3. FACT. (see [1]) If Z is an irreducible component of $V_e^{e-f}(g_d^n)$ then $\dim(Z) \geqslant e - f(n+1-e+f)$. If $e - f(n+1-e+f) \geqslant 0$ and if each irreducible component of $V_e^{e-f}(g_d^n)$ has dimension $e - f(n+1-e+f)$ then $V_e^{e-f}(g_d^n)$ is a Cohen-Macaulay scheme.

0.4. DEFINITION. Assume $e - f(n+1-e+f) \geqslant 0$ and let Z be an irreducible component of $V_e^{e-f}(g_d^n)$. We say that Z has the expected dimension if $\dim(Z) = e - f(n+1-e+f)$. If each irreducible component of $V_e^{e-f}(g_d^n)$ has the expected dimension, then we say that $V_e^{e-f}(g_d^n)$ has the expected dimension.

In this paper, we prove

0.5. THEOREM. *Let C be a general curve of genus g. Let d be an integer with $d \geqslant g+3$ and let n be an integer with $2 \leqslant n \leqslant d-g$. Let g_d^n be a general non-special n-dimensional linear system of degree d on C. Then $V_e^{e-f}(g_d^n)$ is non-empty if and only if $e - f(n+1-e+f) \geqslant 0$. Whenever non-empty, $V_e^{e-f}(g_d^n)$ is a reduced subscheme of $C^{(e)}$ of the expected dimension.*

The reducedness statement is the most interesting part and the deepest statement of the theorem. Let H be the closure in the Hilbert scheme $\mathrm{Hilb}^{d;g}(\mathbf{P}^n)$ of the locus parametrizing smooth irreducible curves of degree d and genus g in $\mathbf{P}^n$ embedded by a non-special linear system (of course $n \leqslant d-g$; also H is an irreducible component of that Hilbert scheme). The reducedness statement of Theorem 0.5 implies that a lot of intersection numbers computed in Chapter VIII of [1] give the number of linear subspaces in $\mathbf{P}^n$ (i.e. each one counted with multiplicity one) satisfying certain conditions with respect to the curve $C \subset \mathbf{P}^n$ corresponding to a general point on H. As an example, the Cayley-number $[(d-2)(d-3)^2 (d-4)/12] - [g(d^2 - 7d + 13 - g)/2]$ is the exact number of 4-secant lines of C if C corresponds to a general element of $H \subset \mathrm{Hilb}^{d;g}(\mathbf{P}^3)$. As far as I know, such kind of result was only known for special types of curves (complete intersection curves; rational curves – see [12]) in $\mathbf{P}^3$. Now we have such results for curves that are general with respect to moduli as abstract curves. In [10], A. Hirschowitz proves – using methods completely different from ours – that C has finitely many 4-secants lines if C corresponds to a general point on $H \subset \mathrm{Hilb}^{d;g}(\mathbf{P}^3)$, but he does not consider the reducedness-problem.

The proof of the theorem is divided in 2 parts. First we prove the theorem in the complete case $(n = d-g)$. In this case, both the existence and the dimension statement follows immediately from classical Brill-Noether Theory. For the reducedness statement we use the Petri-Gieseker Theorem. As E. Ballico pointed out to me, from this case both the existence and dimension statement in the non-complete case follow easily adapting the arguments from [2].

In order to prove the theorem in the non-complete case, we consider the following problem. Let $g_d^n = \mathbf{P}(V)$ be a linear system on C and let $E \in V_e^{e-f}(g_d^n)$

be an e-secant $(e-f-1)$-space divisor for g^n_d. Let W be a codimension 1 linear subspace of V containing $V(-E)$, then $E \in V^{e-f-1}_e(g^{n-1}_d(W))$. Geometrically, if g^n_d embeds C in $\mathbf{P}(V^*)$, then $W \supset V(-E)$ means that W corresponds to a point in the linear span $\langle E \rangle$. The inequality

$$\dim(T_E(V^{e-f}_e(g^n_d))) - \dim(T_E(V^{e-f-1}_e(g^{n-1}_d(W)))) \leqslant n+1-e+f$$

always holds (difference between dimension of Zariski tangent spaces; see Lemma 3.1). We need to find conditions implying equality.

We prove

0.6. PROPOSITION. *Let $E \in V^{e-f}_e(g^n_d)$. Suppose each subdivisor E' of E of degree $n-e+f+1$ imposes independent conditions on $g^n_d(-E)$ (i.e. $g^n_d(-E-E') = \emptyset$) and suppose $E-P \notin V^{e-f-1}_{e-1}(g^n_d)$ for $P \in E$. Then, for a codimension 1 linear subspace W – general under the condition that it contains $V(-E)$ – we have*

$$\dim(T_E(V^{e-f}_e(g^n_d))) - \dim(T_E(V^{e-f-1}_e(g^{n-1}_d(W)))) = n+1-e+f.$$

In case g^n_d defines an embedding $C \subset \mathbf{P}(V^*)$, then the set of points W on $\langle E \rangle$ not satisfying the conclusion of Proposition 0.6, is closely related to the concept of a focal scheme (see e.g. [4]; [5]; [3]). As a matter of fact, the proof of the theorem in the non-complete case is very much influenced by [5]. We prove a stronger statement than the above proposition and as such it becomes a generalisation of Theorem 2.5 in [5].

The organization of the paper is as follows. In Section 1 we recall the description of the Zariski tangent spaces to $V^{e-f}_e(g^n_d)$ obtained in [6]. In Section 2 we prove Theorem 0.5 for the complete linear systems. In Section 3 we prove (the stronger version of) Proposition 0.6. In Section 4 we finish the proof of Theorem 0.5 in the non-complete case.

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1. Secant Space Divisors

In this part, we fix an arbitrary smooth irreducible complete curve C and a linear system $g^n_d = g^n_d(V)$ on C. Here, V is an $(n+1)$-dimensional linear subspace of $\Gamma(C;\mathcal{L})$ with $\mathcal{L}$ an invertible sheaf of degree d on C. We simply write V^{e-f}_e for the scheme $V^{e-f}_e(g^n_d)$ introduced in 0.2. Take $E \in V^{e-f}_e$ and let $T_E(V^{e-f}_e)$ be the Zariski tangent space of V^{e-f}_e at E. In case $E \in V^{e-f-1}_e$ one has $T_E(V^{e-f}_e) = T_E(C^{(e)})$, so now assume $E \notin V^{e-f-1}_e$. We recall the description of $T_E(V^{e-f}_e)$ from [6]. In this description we use the common natural identification between $T_E(C^{(e)})$ and $H^0(E;\mathcal{O}_E(E))$ (see [1]), so we give a description of $T_E(V^{e-f}_e)$ as a subspace of $H^0(E;\mathcal{O}_E(E))$.

Let $\beta\colon\ H^0(E;\mathcal{O}_E(E))\otimes V(-E)\to H^0(E;\mathcal{L}\otimes\mathcal{O}_E)$ be the map obtained from the cup-product homomorphism after mapping $V(-E)$ to $\Gamma(C;\mathcal{L}(-E))$ (i.e. locally dividing the elements of $V(-E)$ by the equations of E). Let $\phi_V(E)\colon\ V\to H^0(E;\mathcal{L}\otimes\mathcal{O}_E)$ be the natural restriction map. For $\xi\in V(-E)$, consider

$$\beta_\xi\colon\ H^0(E;\mathcal{O}_E(E))\to H^0(E;\mathcal{L}\otimes\mathcal{O}_E)\colon\ v\to\beta(v\otimes\xi).$$

1.1. PROPOSITION. (see [6]) $T_E(V_e^{e-f})=\cap\{\beta_\xi^{-1}(\operatorname{im}\phi_V(E))\colon\ \xi\in V(-E)\}$.

Now, assume the linear system g_d^n is complete (i.e. $V=\Gamma(C;\mathcal{L})$). From the exact sequence $0\to\mathcal{L}(-E)\to\mathcal{L}\to\mathcal{L}\otimes\mathcal{O}_E\to 0$ we obtain the exact sequence

$$H^0(C;\mathcal{L})\to H^0(E;\mathcal{L}\otimes\mathcal{O}_E)\xrightarrow{\delta}H^1(C;\mathcal{L}(-E)).$$

Because of Proposition 1.1, $v\in H^0(E;\mathcal{O}_E(E))$ belongs to $T_E(V_e^{e-f})$ if and only if $(\delta\circ\beta)(v\otimes V(-E))=0$. From this observation and then repeating the arguments from [1], p. 161, 162, we find the following description for $T_E(V_e^{e-f})$ in the case of complete linear systems.

1.2. PROPOSITION. *Suppose g_d^n is a complete linear system and let $E\in V_e^{e-f}\backslash V_e^{e-f-1}$. Consider the so-called Petri map*

$$\mu\colon\ H^0(C;K_C(E)\otimes\mathcal{L}^{-1})\otimes H^0(C;\mathcal{L}(-E))\to H^0(C;K_C\otimes\mathcal{O}_E)$$

defined as the composition of the ordinary Petri map

$$\mu_0\colon\ H^0(C;K_C(E)\otimes\mathcal{L}^{-1})\otimes H^0(C;\mathcal{L}(-E))\to H^0(C;K_C)$$

(i.e. the cup-product) and the natural restriction to E. Serre duality defines a perfect pairing between $H^0(E;\mathcal{O}_E(E))$ and $H^0(E;K_C\otimes\mathcal{O}_E)$. Let $[\operatorname{im}(\mu)]^*\subset H^0(E;\mathcal{O}_E(E))$ be the orthogonal complement of im(μ) $\subset H^0(E;K_C\otimes\mathcal{O}_E)$ with respect to this pairing. Then

$$T_E(V_e^{e-f})=[\operatorname{im}(\mu)]^*.$$

2. The Complete Case

Let C be a general curve of genus g, let $n\geqslant 3$ be an integer and let $\mathcal{L}$ be a general line bundle of degree $g+n$ on C. Let g_{g+n}^n be the associated complete linear system $\mathbf{P}(\Gamma(C;\mathcal{L}))$ on C. For integers $e;f$ with $e;f\geqslant 1$ and $n-e+f\geqslant 1$ we write V_e^{e-f} instead of $V_e^{e-f}(g_{g+n}^n)$.

2.1. THEOREM. *V_e^{e-f} is not empty if and only if $e\geqslant(n+1-e+f)f$. In case $e\geqslant(n+1-e+f)f$ then V_e^{e-f} is a reduced scheme of the expected dimension.*

2.2. REMARK. Once we know that V_e^{e-f} is of the expected dimension, then the scheme V_e^{e-f} is Cohen–Macaulay. Hence, in order to prove that it is reduced, it is enough to prove reducedness at a general point of each irreducible component.

2.3. NOTATION. Given any smooth irreducible projective curve C, we assume that we fix a base point $P_0 \in C$. Then on $J(C)$ – identified with $\mathrm{Pic}^0(C)$ – we consider the well-known subschemes

$$W_d^r = \{M \in J(C)\colon\ h^0(M(dP_0)) \geqslant r+1\}.$$

If $\mathcal{L}$ is an invertible sheaf on C of degree d, then we write $[\mathcal{L}]$ for the point on $J(C)$ defined by $\mathcal{L}(-dP_0)$. Clearly $\dim(\Gamma(C;\mathcal{L})) \geqslant r+1$ if and only if $[\mathcal{L}] \in W_d^r$. If $E \in C^{(e)}$ then we write $[E]$ instead of $[\mathcal{O}_C(E)]$. If $x \in W_d^r \backslash W_d^{r+1}$ then $g_d^r(x)$ is the complete linear system $\mathbf{P}(\Gamma(C;\mathcal{L}))$ with $\mathcal{L} \in \mathrm{Pic}^d(C)$ and $[\mathcal{L}] = x$.

Proof of Theorem 2.1. Clearly $E \in V_e^{e-f}$ if and only if $[\mathcal{L}] - [E] \in W_{g+n-e}^{n-e+f}$, hence $[\mathcal{L}] \in W_{g+n-e}^{n-e+f} + W_e^0$ if V_e^{e-f} is not empty.

Consider the map $\tau\colon\ W_{g+n-e}^{n-e+f} \times W_e^0 \to J(C)\colon\ (x;y) \to x+y$. Then $\mathrm{im}(\tau) = J(C)$ if and only if $\dim(W_{g+n-e}^{n-e+f}) + e \geqslant g$. Since C is general, we know from Brill-Noether Theory that $\dim(W_{g+n-e}^{n-e+f}) = \rho_{g+n-e}^{n-e+f}(g) = g-(n-e+f+1)f$. Since $[\mathcal{L}]$ is a general point on $J(C)$ we obtain both the non-emptiness and the dimension statement of this theorem.

In order to prove the reducedness statement, we use the stronger Gieseker–Petri Theorem (a conjecture of Petri, first proved by Gieseker in [9] for arbitrary fields K, later on there appeared other proofs using however $\mathrm{Char}(K) = 0$: [8]; [11]). This Gieseker–Petri Theorem states that, for a general curve C and for any invertible sheaf $\mathcal{M}$ on C, the cup-producthomomorphism

$$\mu_0\colon H^0(C;\mathcal{M}) \otimes H^0(C;K_C \otimes \mathcal{M}^{-1}) \to H^0(C;K_C)$$

is injective.

Assume $e \geqslant (n+1-e+f)f$ and let Z be an irreducible component of V_e^{e-f}. Let $T \subset V_{e+1}^{e-f} \times C$ be defined by $(E;P) \in T$ if and only if $E - P \geqslant 0$. From the dimension statement, we know that Z is not contained in the image of $\kappa\colon\ T \to C^{(e)}\colon\ (E;P) \to E-P$. Hence, if E is a general point on Z, then $\mathcal{M} = \mathcal{L}(-E)$ defines a complete linear system g_{g+n-e}^{n-e+f} without fixed points. Because of the Gieseker–Petri Theorem, $\mathrm{im}(\mu_0)$ has dimension $(n-e+f+1)f$, hence $\mathcal{M}$ defines a linear subsystem $g_{2g-2}^{(n-e+f+1)f-1}$ of the canonical linear system $|K_C|$ on C. Since E is general with respect to $g_{2g-2}^{(n-e+f+1)f-1}$, it follows that $g_{2g-2}^{(n-e+f+1)f-1}(-E) = \emptyset$ because $e \geqslant (n-e+f+1)f$. Algebraically this means that the natural restriction map $\mathrm{im}(\mu_0) \to H^0(E;K_C \otimes \mathcal{O}_E)$ is injective. Using the notations from Proposition 1.2, it follows that

$$\dim([\mathrm{im}\,\mu]^*) = e - (n-e+f+1)f$$

and then from Proposition 1.2, it follows that V_e^{e-f} is smooth at E.

3. The focal set

As mentioned in the introduction, we make a little disgression now, proving a sharper version of Proposition 0.6. Let $\mathcal{L}$ be a line bundle of degree d on a smooth irreducible complete curve C, let V be an $(n+1)$-dimensional linear subspace of $\Gamma(X;\mathcal{L})$ and write $g_d^n = g_d^n(V)$. Let $E \in C^{(e)}$ be an e-secant $(e-f-1)$-space divisor for g_d^n. Take a codimension 1 linear subspace W of V containing $V(-E)$. Then E is an e-secant $(e-f-2)$-space divisor for the linear system $g_d^{n-1}(W)$. Geometrically, if g_d^n defines an embedding of C in $\mathbf{P}^n = \mathbf{P}(V^*)$ then W is a point on the linear span $\langle E\rangle$; if $W \notin C$ and we take the projection $\mathbf{P}^n - \to \mathbf{P}^{n-1} = \mathbf{P}(W^*)$, then the linear span of the image of E has dimension $\dim(\langle E\rangle) - 1$.

3.1. LEMMA. $\dim(T_E(V_e^{e-f}(g_d^n)))$-$\dim(T_E(V_e^{e-f-1}(g_d^{n-1}(W)))) \leqslant n+1-e+f$.

Proof. (We use the notations introduced in Section 1.) Since $\ker(\phi_V(E)) = \ker(\phi_W(E))$, $\mathrm{im}(\phi_W(E))$ is a hyperplane in $\mathrm{im}(\phi_V(E))$. For $v \in T_E(V_e^{e-f}(g_d^n))$ and $\xi \in V(-E)$ we have $\beta_\xi(v) \in \mathrm{im}(\phi_V(E))$. Then $v \in T_E(V_e^{e-f-1}(g_d^{n-1}(W)))$ if and only if $\beta_\xi(v) \in \mathrm{im}(\phi_W(E))$. So, $V(-E)$ defines an $(n-e+f+1)$-dimensional linear family of linear functions

$$p_\xi\colon\ T_E(V_e^{e-f}(g_d^n)) \to K = \mathrm{im}(\phi_V(E))/\mathrm{im}(\phi_W(E))$$

($p_\xi(v)$ is the class of $\beta_\xi(v)$). The intersection of the kernels of those linear functions is exactly $T_E(V_e^{e-f-1}(g_d^{n-1}(W)))$. This proves the lemma.

3.2. NOTATION. There is a bijection between codimension 1 subspaces W of V containing $V(-E)$ and the projective space defined by the dual vectorspace $(V/V(-E))^*$. Each $W \in (V/V(-E))^*$ defines a linear map

$$p_E(W)\colon\ T_E(V_e^{e-f}(g_d^n)) \to (V(-E))^*\colon\ v \to (\xi \to p_\xi(v))$$

(the notation p_ξ comes from the proof of Lemma 3.1). We obtain a linear family of linear maps

$$p_E\colon\ (V/V(-E))^* \to \mathrm{Hom}_K(T_E(V_e^{e-f}(g_d^n)); (V(-E))^*).$$

Let $W \in (V/V(-E))^*$. From the proof of Lemma 3.1 we obtain

$$\dim(T_E(V_e^{e-f}(g_d^n))) - \dim(T_E(V_e^{e-f-1}(g_d^{n-1}(W)))) = n+1-e+f$$

if and only if $p_E(W)$ is surjective.

3.3. DEFINITION. The focal set F_E of E with respect to g^n_d is defined by

$$F_E = \{W \in (V/V(-E))^* : p_E(W) \text{ is not surjective}\}.$$

This terminology comes from a comparable definition of focal sets in e.g. [4]; [5]; [3].

The family p_E can be considered as being defined by a linear map

$$\lambda\colon T_E(V^{e-f}_e(g^n_d)) \otimes V(-E) \to V/V(-E).$$

The following terminology comes from [7].

3.4. DEFINITION. The family p_E is 1-generic if for each $v \in T_E(V^{e-f}_e(g^n_d))$ and for each $s \in V(-E)$ – both nonzero – we have

$$\lambda(v \otimes s) \neq 0.$$

(One also says: there are no non-trivial pure tensors in ker(λ).)

REMARK. For $v \in T_E(V^{e-f}_e(g^n_d))$ and $s \in V(-E)$ both nonzero, the equation $\lambda(v\otimes s) = 0$ is equivalent to the following statement. For each $W \in (V/V(-E))^*$ one has $[(p_E(W))(v)](s) = p_s(v) = 0$ and this is equivalent to $\beta_s(v) = 0$, hence s is a nonzero element of the kernel of the map $V(-E) \to \operatorname{im}(\phi_V(E)) : \xi \to \beta_\xi(v)$.

From Theorem 2.1 in [7], we obtain

3.5. PROPOSITION. If the family p_E is 1-generic then F_E has codimension $\dim(T_E(V^{e-f}_e(g^n_d))) - (n - e + f)$.

Now Proposition 0.6 is implied by the following stronger statement. It is a generalization of Theorem 2.5 in [5].

3.6. THEOREM. *Let E be an e-secant $(e - f - 1)$-space divisor for some linear system g^n_d on C. Suppose each subdivisor E' of degree $n - e + f + 1$ of E imposes independent conditions on $g^n_d(-E)$ (i.e. $g^n_d(-E - E')$ is empty).*

(a) *If $n - e + f + 1 = 1$, then p_E is 1-generic.*

(b) *If $n-e+f+1 \geqslant 2$, then p_E is 1-generic if and only if $E - P \notin V^{e-(f+1)}_{e-1}(g^n_d)$ for each $P \in E$.*

Proof. Suppose $n-e+f+1 \geqslant 2$ and suppose for some $P \in E$ one has $E-P \in V^{e-(f+1)}_{e-1}(g^n_d)$. We are going to prove that p_E is not 1-generic. Let $E_1 = E - P$. We can find $s \in V$ with $E_1 \subset D_s$ but $E \not\subset D_s$. Take a base $\xi_1, \ldots, \xi_{n-e+f+1}$ for $V(-E)$ with $E+P \not\subset D_{\xi_1}$; $E+P \subset D_{\xi_i}$ for $2 \leqslant i \leqslant n-e+f+1$. Take a nonzero element v of $H^0(E;\ O_E(E))$ with $E_1 \subset Z(v)$. (Here $Z(v)$ is the zero-scheme of v; it is a closed subscheme of E and we consider it as an effective divisor on C.) Then $\beta_{\xi_1}(v) \in \langle \phi_V(E)(s) \rangle$ and $\beta_{\xi_i}(v) = 0$ for $i \geqslant 2$, hence $v \in T_E(V_e^{e-f}(g_d^n))$. Since $n - e + f + 1 \geqslant 2$, we find $\beta_{\xi_2}(v) = 0$, hence $\lambda(v \otimes \xi_2) = 0$ and so the family p_E is clearly not 1-generic.

Next, assume $n-e+f+1 \geqslant 1$ and in case $n-e+f+1 \geqslant 2$ assume $E-P \notin V_e^{e-(f+1)}$ for each $P \in E$. We are going to prove that p_E is 1-generic. Suppose $v \in T_E(V_e^{e-f}(g_d^n))$, nonzero, does not induce an injection $V(-E) \to \operatorname{im}(\phi_V(E))$. Take a nonzero element $\xi \in V(-E)$ with $\beta_\xi(v) = 0$. Write $E_1 = Z(v) \subset E$ and $E_2 = E - E_1$. Then $E_2 + E \subset D_\xi$, hence $g_d^n(-E - E_2) \neq \emptyset$. Because of our assumptions $\deg(E_2) \leqslant n - e + f$. In case $n - e + f + 1 = 1$, we obtain a contradiction. Now, assume $n - e + f + 1 \geqslant 2$. Take $Q \in E_2$. Because of our assumptions, we can find $\xi \in V(-E)$ with $(D_\xi - E) \cap E_2 = E_2 - Q$. Because $v \in T_E(V_e^{e-f}(g_d^n))$, we have $\beta_\xi(v) \in \operatorname{im}(\phi_V(E))$. But $Z(\beta_\xi(v)) = E - Q$. Since $E - Q \notin V_{e-1}^{e-(f+1)}(g_d^n)$ we obtain a contradiction.

Further on, we also need the following lemma.

3.7. LEMMA. *Let $g_d^n = g_d^n(V)$ be a linear system on a smooth curve C and take $E \in C^{(e)} (e \leqslant n)$ with* $\dim(V(-E)) = n + 1 - e$. *Let* $f = \min(\{e; n + 1 - e\})$ *and assume for some $F = P_1 + \cdots + P_f \leqslant E$ we have* $\dim(V(-E - F)) = n + 1 - e - f$. *Take a codimension 1 linear subspace W of V with $W \supset V(-E)$ and $W(-(E - P_i)) = W(-E) = V(-E)$ for $1 \leqslant i \leqslant f$. Then*

$$\dim(T_E(V_e^{e-1}(g_d^{n-1}(W)))) = e - f.$$

(Geometrically, if g_d^n defines an embedding of C in $\mathbf{P}^n = \mathbf{P}(V^*)$ then W is a point on the linear span $\langle E \rangle$ but for $1 \leqslant i \leqslant f$, W is not a point of the linear span $\langle E - P_i \rangle$, a hyperplane in $\langle E \rangle$.)

Proof. Because $T_E(V_e^e(g_d^e)) = T_E(C^{(e)}) = H^0(E; O_E(E))$, the inequality

$$\dim(T_E(V_e^{e-1}(g_d^{n-1}(W)))) \geqslant e - f$$

is proven as in Lemma 3.1.

Write $E_0 = 0 \leqslant E_1 = P_1 \leqslant E_2 = P_1 + P_2 \leqslant \cdots \leqslant E_f = P_1 + P_2 + \cdots + P_f = F$. Because of the assumptions, we can find $\xi_0, \ldots, \xi_{f-1}$ in $V(-E)$ with $D_{\xi_i} \cap (E + F) = E + E_i$. Choose $v_1, \ldots, v_f$ in $H^0(E; O_E(E))$ with $Z(v_i) = E - E_i$. Take $v = \Sigma_{i=1}^k c_i v_i$ with $c_k \neq 0$ and $k \leqslant f$. Then $Z(v) \supset E - E_k$ but $E - E_{k-1} \not\subset Z(v)$. It follows that $E - P_k \subset Z(\beta_{\xi_{k-1}}(v))$ but $E \not\subset Z(\beta_{\xi_{k-1}}(v))$. The assumption $W(-(E - P_k)) = W(-E)$ implies $\beta_{\xi_{k-1}}(v) \notin \operatorname{im}(\phi_W(E))$. It follows that $v \notin T_E(V_e^{e-1}(g_d^{n-1}(W)))$, hence $T_E(V_e^{e-1}(g_d^{n-1}(W)))$ has nonzero intersection with $\langle v_1, \ldots, v_f \rangle$. This implies $\dim(T_E(V_e^{e-1}(g_d^{n-1}(W)))) = e - f$, hence the lemma is proved.

4. The non-complete case

4.1. DEFINITION. Let $g^n_d = g^n_d(V)$ be a linear system on a smooth curve C. Let $e; f$ be positive integers with $n - e + f \geqslant 1$. Suppose E is an e-secant $(e - f - 1)$-space divisor.

(i) We say that E is a good secant divisor if the following two conditions are satisfied. For each subdivisor E' of E of degree $n - e + f + 1$ the linear system $g^n_d(-E - E')$ is empty; for each $P \in E$ one has $E - P \notin V^{e-(f+1)}_{e-1}(g^n_d)$.

(ii) Assume $e - (n + 1 - e + f)f \geqslant 0$. We say that g^n_d is of general secant type for $V^{e-f}_e(g^n_d)$ if the following conditions hold: $V^{e-f}_e(g^n_d)$ is not empty; it is of the expected dimension; it is reduced as a scheme and for each irreducible component Z of $V^{e-f}_e(g^n_d)$, a general point E of Z is a good secant divisor.

(iii) We say that g^n_d is of general secant type if for all integers $e; f \leqslant 1$ satisfying $n+1-e+f \geqslant 2$ one of the following two possibilities occur. If $e < (n+1-e+f)f$ then $V^{e-f}_e(g^n_d)$ is empty; if $e \geqslant (n + 1 - e + f)f$ then g^n_d is of general secant type for $V^{e-f}_e(g^n_d)$.

4.2. PROPOSITION. *Let $g^n_d = g^n_d(V)$ be a linear system on a smooth curve C. Suppose $n \geqslant 3$ and assume g^n_d is of general secant type. Let $W \in \mathbf{P}(V^*)$ be a general codimension 1 linear subspace of V. Then the linear system $g^{n-1}_d(W)$ is of general secant type.*

Proof. Let $W \in \mathbf{P}(V^*)$. Elements $E \in V^{e-f}_e(g^{n-1}_d(W))(n-e+f \geqslant 2;\ f \geqslant 1)$ can be obtained in two ways:

(i) $E \in V^{e-f}_e(g^n_d)$;
(ii) $E \in V^{e-(f-1)}_e(g^n_d)$ and $W \supset V(-E)$.

The irreducible components of $V^{e-f}_e(g^{n-1}_d(W))$ have dimension at least $e - (n - e + f)f$. But $V^{e-f}_e(g^n_d)$ is empty if $e < (n + 1 - e + f)f$ and $V^{e-f}_e(g^n_d)$ has the expected dimension if $e \geqslant (n + 1 - e + f)f$. It follows that for a general point E of some irreducible component of $V^{e-f}_e(g^{n-1}_d(W))$ possibility (ii) occurs.

Let $U^{e-(f-1)}_e = V^{e-(f-1)}_e(g^n_d) \backslash V^{e-f}_e(g^n_d)$ and consider

$$T \subset U^{e-(f-1)}_e \times \mathbf{P}(V^*)$$

defined by: $(E; W) \in T$ if and only if $W \supset V(-E)$. Consider the projection morphism $p' : T \to U^{e-(f-1)}_e$. The fibres of this morphism have dimension $n - (n - e + f)$, hence $\dim(T) = e - f(n - e + f) + n$. So, if $e < f(n - e + f)$, then T does not dominate $\mathbf{P}(V^*)$. In that case we conclude $V^{e-f}_e(g^{n-1}_d(W)) = \emptyset$ for a general $W \in \mathbf{P}(V^*)$. We also conclude that, in case $e \geqslant f(n - e + f)$ and if $V^{e-f}_e(g^{n-1}_d(W))$ is not empty for a general $W \in \mathbf{P}(V^*)$, then each irreducible component of it has dimension $e - f(n - e + f)$.

Now, first, assume $f \geqslant 2$, hence $f - 1 \geqslant 1$. Let $T' \subset T$ be defined by $(E;\ W) \in T'$ if and only if $W \in F_E$. For a general element of some irreducible component of $V_e^{e-(f-1)}(g_d^n)$ we can apply Theorem 3.6, hence $\dim(T') < n + e - f(n-e+f)$. It follows from Fact 0.3 that for $W \in \mathbf{P}(V^*)$ general no irreducible component of $V_e^{e-f}(g_d^{n-1}(W))$ is contained in the fibre of T' over W. Hence, if Z is an irreducible component of $V_e^{e-f}(g_d^{n-1}(W))$ and if E is a general element of Z then $W \notin F_E$. Since E is a general element of some irreducible component of $V_e^{e-(f-1)}(g_d^n)$ and moreover g_d^n is of good secant type, it follows that $\dim(T_E(V_e^{e-f}(g_d^{n-1}(W)))) = e-(n-e+f)f$. This proves that, for $W \in \mathbf{P}(V^*)$ general, each irreducible component is reduced and of the expected dimension (remember Remark 2.2). Because $T \neq T'$, it also proves that a general non-empty fibre of the projection $T \to \mathbf{P}(V^*)$ has dimension $e - f(n-e+f)$, hence T dominates $\mathbf{P}(V^*)$. This proves the non-emptiness of $V_e^{e-f}(g_d^{n-1}(W))$ for $W \in \mathbf{P}(V^*)$ general. For $W \in \mathbf{P}(V^*)$ general, we still have to prove: if E is a general point of some irreducible component of $V_e^{e-f}(g_d^{n-1}(W))$, then E is a good secant divisor. Remember that, as an element of $V_e^{e-(f-1)}(g_d^n)$, E is a good secant divisor. In particular, each subdivisor E' of E of degree $n-e+f$ imposes independent conditions on $V(-E)$. But $W(-E) = V(-E)$, so this condition still holds. On the other hand, for each $P \in E$, $E - P \notin V_e^{e-(f-1)}(g_d^n)$. This means $V(-(E-P)) = V(-E)$ for $P \in E$. Since $W \supset V(-E)$, it is clear that $W(-(E-P)) = W(-E)$ too.

Next, we consider the case $f = 1$. We already proved that $V_e^{e-1}(g_d^{n-1}(W)) = \emptyset$ if $2e < n+1$ for a general $W \in \mathbf{P}(V^*)$, so we assume $2e \geqslant n+1$. For $W \in \mathbf{P}(V^*)$ general, a general element E of some irreducible component of $V_e^{e-1}(g_d^{n-1}(W))$ is a general element on $C^{(e)}$. So, we can consider E as a general element of $C^{(e)}$ and W as a general element of $\mathbf{P}((V/V(-E))^*)$. A general $E \in C^{(e)}$ satisfies the assumption of Lemma 3.7 for the linear system g_d^n (here we use $\mathrm{char}(K) = 0$). From Lemma 3.7 we conclude $\dim[T_E(V_e^{e-1}(g_d^{n-1}(W)))] = e - (n-e+1)$. This proves the reducedness statement. We still need to prove that E is a good secant divisor for $g_d^{n-1}(W)$. Since $E \in C^{(e)}$ is general, each subdivisor E' of degree $n+1-e$ of E imposes independent conditions on $V(-E)$ ($\mathrm{char}(K) = 0$). Since $V(-E) = W(-E)$, this claim also holds for $W(-E)$. Moreover if $P \in E$ then $\dim(V(-(E-P))) = \dim(V(-E)) + 1$. But W is a general element of $\mathbf{P}((V/V(-E))^*)$, hence $V(-(E-P)) \not\subset W$. This means $E - P \notin V_{e-1}^{e-2}(g_d^{n-1}(W))$ and we proved that E is a good secant divisor for $g_d^{n-1}(W)$.

4.3. PROOF OF THEOREM 0.5. Because of Proposition 4.2 and the fact that we proved Theorem 0.5 already in the complete case (Theorem 2.1), we only need to prove the following statement. Let C be a general curve of genus g and let g_{g+n}^n ($n \geqslant 3$) be a general complete linear system on C. Let E be a general point on some irreducible component of $V_e^{e-f}(g_{g+n}^n)$. Then E is a good secant divisor.

From the proof of Theorem 2.1, we know that $g^n_{g+n} - E$ corresponds to a general element of W^{n-e+f}_{g+n-e}. So, take a general g^{n-e+f}_{g+n-e} on C and E a general element of $C^{(e)}$. Then any subdivisor E' of E of degree $n-e+f+1$ imposes independent conditions on g^{n-e+f}_{g+n-e}. Moreover, a point $P \in E$ is a general point on C. Since g^{n-e+f}_{g+n-e} is a special linear system on C, it follows that $\dim|g^{n-e+f}_{g+n-e}+P| = n-e+f$. This proves $E-P \notin V^{e-(f+1)}_{e-1}(g^n_{g+n})$. So we proved that E is a good secant divisor for g^n_{g+n}.

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