

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 97, n° 1-2 (1995), p. 119-134

http://www.numdam.org/item?id=CM_1995__97_1-2_119_0

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Multicanonical systems of elliptic surfaces in small characteristics*

Dedicated to Frans Oort on the occasion of his 60th birthday

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Received 4 January 1995; accepted in final form 12 April 1995

Abstract. We determine the smallest integer M such that the multicanonical system $|mK_S|$ gives the structure of elliptic surface for any elliptic surface S with Kodaira dimension $\kappa(S) = 1$ and for any integer $m \geq M$. We also investigate the structure of elliptic surfaces of type (p, p, \dots, p) .

0. Introduction

Let k be an algebraically closed field of characteristic p , and $f: S \rightarrow C$ an elliptic surface over k with a nonsingular complete curve C . As is well known, for an elliptic surface S with Kodaira dimension $\kappa(S) = 1$, there exists a positive integer m such that the multicanonical system $|mK_S|$ gives a unique structure of elliptic surface. In this paper, we consider the following problem.

PROBLEM. *Find the smallest integer M such that the multicanonical system $|mK_S|$ gives the structure of elliptic surface for any elliptic surface S with Kodaira dimension $\kappa(S) = 1$ and for any integer $m \geq M$.*

For any *analytic* elliptic surface $f: S \rightarrow C$ with $\kappa(S) = 1$ over the field \mathbb{C} of complex numbers, Iitaka showed that for any $m \geq 86$, the multicanonical system $|mK_S|$ gives a unique structure of elliptic surface $f: S \rightarrow C$. He also showed that the number 86 is best possible, that is, there exists an example of an elliptic surface S with $\kappa(S) = 1$ such that $|85K_S|$ doesn't give the structure of elliptic surface. In the paper [4], we showed that for any *algebraic* elliptic

* Partially supported by Grand-in-Aid for Scientific Research on Priority Areas 231 "Infinite Analysis"

surface $f: S \rightarrow C$ with $\kappa(S) = 1$ over k and for any $m \geq 14$, the multicanonical system $|mK_S|$ gives a unique structure of elliptic surface, and that the number 14 is best possible if $p = 0$ or $p \geq 5$. In this paper, we show, first, that in case $p = 3$, the number 14 is best possible. Secondly, we show that in case $p = 2$, for any $m \geq 12$ $|mK_S|$ gives a unique structure of elliptic surface and that the number 12 is best possible. To prove these results, we use the theory of purely inseparable coverings obtained by p -closed rational vector fields (cf. [2] and [5]), and investigate the structure of elliptic surfaces of type (p, p, \dots, p) (for the definition, see Section 2). In the final section we give some examples of wild fibres of elliptic surfaces constructed by our method.

1. Preliminaries

In this section, we recall some basic facts on rational vector fields and elliptic surfaces. For a non-singular complete algebraic variety X of dimension n defined over an algebraically closed field k of characteristic $p > 0$, we use the following notation:

- \mathcal{O}_X : the structure sheaf of X ,
- K_X : a canonical divisor on X ,
- $\Phi_{|mK_X|}$: the rational mapping associated with the multicanonical system $|mK_X|$,
- $\kappa(X)$: the Kodaira dimension of X ,
- $c_n(X)$: the n th Chern number of X ,
- $\chi(X, \mathcal{O}_X) = \sum_{i=0}^n (-1)^i \dim_k H^i(X, \mathcal{O}_X)$,
- $\text{Pic}^0(X)$: the Picard scheme of X ,

For divisors E_1 and E_2 on X , $E_1 \sim E_2$ (resp. $E_1 \equiv E_2$) means that E_1 is linearly (resp. numerically) equivalent to E_2 . Sometimes, a Cartier divisor and the associated invertible sheaf will be identified. For a group G and an element σ of G , we denote by $\langle \sigma \rangle$ the subgroup generated by σ .

Let D be a non-zero rational vector field on X . D is called a p -closed vector field if there exists a rational function f on X such that $D^p = fD$. In particular, D is called multiplicative if $D^p = D$, and D is called additive if $D^p = 0$. Let $\{\text{Spec} A_i\}_{i \in I}$ be an affine open covering of X . We set $A_i^D = \{\alpha \in A_i \mid D(\alpha) = 0\}$. Then, X^D is defined by $\bigcup_{i \in I} \text{Spec} A_i^D$. It is well known that X^D is normal. If D is p -closed, then the canonical projection $\pi_D: X \rightarrow X^D$ is a purely inseparable morphism of degree p . Conversely, if $\pi: X \rightarrow Y$ is a finite purely inseparable morphism of degree p with a normal variety Y , then there exists a rational p -closed vector field D such that $\pi = \pi_D$ and $Y = X^D$. Moreover, X^D is non-singular if and only if D is divisorial (cf. [5, Section 1]). For a p -closed vector field D , we denote by (D) the divisor associated with D . The

following lemma is a special case in Rudakov and Shafarevich [5, Section 1, Proposition 1].

LEMMA 1.1. *Let X be a non-singular complete algebraic surface over k , and D a divisorial p -closed rational vector field on X . Let $\pi: X \rightarrow X^D$ be the natural morphism. For an irreducible curve C on X , C' denotes the image of C in X^D . If C is an integral curve for D , then $\pi_*C = pC'$ and $\pi^*C' = C$. If C is not an integral curve for D , then $\pi_*C = C'$ and $\pi^*C' = pC$.*

For details of these facts, see [5].

Now, let $f: S \rightarrow C$ be an elliptic surface defined over k with a non-singular complete curve C . Throughout this paper, we assume that no fibers of f contain exceptional curves of the first kind. Let \mathcal{T} be the torsion part of $R^1f_*\mathcal{O}_S$. Since C is a non-singular curve, we have

$$R^1f_*\mathcal{O}_S \simeq \mathcal{L} \oplus \mathcal{T}, \tag{1.1}$$

where \mathcal{L} is an invertible sheaf. By m_iE_i ($i = 1, 2, \dots, n$), we denote the multiple singular fibers of $f: S \rightarrow C$ with multiplicities m_i . We have the canonical divisor formula,

$$K_S \sim f^*(K_C - \mathcal{L}) + \sum_{i=1}^n a_iE_i, \tag{1.2}$$

where a_i 's are integers such that $0 \leq a_i \leq m_i - 1$, and where

$$-\deg \mathcal{L} = \chi(S, \mathcal{O}_S) + t \quad \text{with } t = \text{length } \hat{\mathcal{T}}. \tag{1.3}$$

We have

$$K_S^2 = 0. \tag{1.4}$$

By Noether's formula and (1.4), we have

$$\chi(S, \mathcal{O}_S) = c_2(X)/12 \geq 0.$$

We set $P_i = f(E_i)$. Then, the following conditions are equivalent.

- (i) $\mathcal{T}_{P_i} = 0$,
- (ii) $\dim_k H^0(m_iE_i, \mathcal{O}_{m_iE_i}) = 1$,
- (iii) $\text{ord } \mathcal{O}_S(E_i)|_{E_i} = m_i$.

In this case, the multiple fiber m_iE_i is called a tame fiber, and we have $a_i = m_i - 1$. If a multiple fiber is not tame, it is called a wild fiber. Namely, a multiple fiber m_iE_i is wild if and only if one of the following equivalent conditions is satisfied.

- (i) $\mathcal{T}_{P_i} \neq 0$,
- (ii) $\dim_k H^0(m_iE_i, \mathcal{O}_{m_iE_i}) \geq 2$,

(iii) $1 \leq \text{ord } \mathcal{O}_S(E_i)|_{E_i} \leq m_i - 1$.

For these facts, see Bombieri and Mumford [1]

Finally, we recall results on the multicanonical systems of elliptic surfaces in [3] and [4]. Using the notation above, we have

$$H^0(S, \mathcal{O}_S(mK_S)) \simeq H^0\left(C, \mathcal{O}_C\left(mK_C - m\mathcal{L} + \sum_{i=1}^n [ma_i/m_i]P_i\right)\right),$$

where $[x]$ is the integral part of a rational number x . We set

$$W_m = mK_C - m\mathcal{L} + \sum_{i=1}^n [ma_i/m_i]P_i.$$

We denote by g the genus of C . Then, $\kappa(S)$ is equal to 1 if and only if

$$2g - 2 + \chi(S, \mathcal{O}_S) + t + \sum_{i=1}^n a_i/m_i > 0. \tag{1.5}$$

If $\text{deg } W_m \geq 2g + 1$, then W_m is very ample, therefore, $\Phi_{|mK_S|}$ gives a unique structure of the elliptic surface. Hence, we must look for the lowest bound M for m such that under the condition (1.5), $\text{deg } W_m \geq 2g + 1$ holds for any $m \geq M$ (cf. [3] and [4]).

2. Elliptic Surfaces of Type (p, p, \dots, p)

In this section, let k be an algebraically closed field of characteristic $p > 0$. Let $f: S \rightarrow \mathbb{P}^1$ be an elliptic surface with $\chi(S, \mathcal{O}_S) = 0$ over the projective line \mathbb{P}^1 . Then, as is easily seen (cf. [4]), $f: S \rightarrow \mathbb{P}^1$ has no singular fibers except multiple fibers $m_i E_i$ ($i = 1, 2, \dots, n$) with elliptic curves E_i . Put $\nu_i = \text{ord } \mathcal{O}_S(E_i)|_{E_i}$.

DEFINITION 2.1 (cf. [4, Definition 3.1]). The elliptic surface $f: S \rightarrow \mathbb{P}^1$ as above is called of type $(m_1, m_2, \dots, m_n \mid \nu_1, \nu_2, \dots, \nu_n)$. The elliptic surface $f: S \rightarrow \mathbb{P}^1$ as above is called of type (m_1, m_2, \dots, m_n) if $\nu_i = m_i$ for all i .

In particular, the multiple fibers of an elliptic surface of type (m_1, m_2, \dots, m_n) are tame. Let \tilde{C} be a non-singular complete algebraic curve over k . The following lemma is an easy exercise of an elementary calculus.

LEMMA 2.2. *Let Δ be a rational vector field on \tilde{C} .*

- (i) *If Δ is multiplicative, then the zero points of Δ are simple.*
- (ii) *If Δ has at least one simple zero point, then Δ is not additive.*

Let E be an ordinary elliptic curve. Then, there exists a non-zero regular vector field δ on E such that $\delta^p = \delta$. Let Δ be a non-zero multiplicative rational vector field on \tilde{C} , and n the number of zero points of Δ . Since $\deg \Delta = 2 - 2g$, the sum of orders of poles of Δ is equal to $n - 2 + 2g$. We set $\tilde{S} = E \times \tilde{C}$. We extend δ and Δ naturally to vector fields on \tilde{S} , which we denote by the same letters. We denote by D the vector field on \tilde{S} defined by $D = \delta + \Delta$. Set

$$S = \tilde{S}^D \quad \text{and} \quad C = \tilde{C}^\Delta.$$

We have an elliptic surface

$$f: S \longrightarrow C, \tag{2.1}$$

where f is the morphism induced from the projection $pr: \tilde{S} = E \times \tilde{C} \rightarrow \tilde{C}$ (cf. [2]).

THEOREM 2.3. *Under the notation as above, the singular fibers of the elliptic surface in (2.1) consist of n tame multiple fibers pE_i ($i = 1, 2, \dots, n$) with multiplicity p , where E_i 's are elliptic curves. The canonical divisor of S is given by*

$$K_S \sim f^*(K_C - \mathcal{L}) + \sum_{i=1}^n (p - 1)E_i, \tag{2.2}$$

where the divisor \mathcal{L} is given as in (1.2) with $\deg \mathcal{L} = 0$.

Proof. Let \tilde{P}_i ($i = 1, 2, \dots, n$) be zero points of Δ . We have a diagram

$$\begin{array}{ccc} S & \xleftarrow{\pi} & E \times \tilde{C} = \tilde{S} \\ f \downarrow & & \downarrow pr \\ C & \xleftarrow{F} & \tilde{C} \end{array}$$

where F (resp. π) is the natural morphism given by Δ (resp. D). F is nothing but the Frobenius morphism. By Lemma 1.1, multiple fibers of f exist only over $P_i = F(\tilde{P}_i)$ ($i = 1, 2, \dots, n$). We set $pr^*(\tilde{P}_i) = \tilde{E}_i (\simeq E)$ and $f^*(P_i) = m_i E_i$ ($i = 1, 2, \dots, n$). Since $\pi^* \circ f^*(P_i) = pr^* \circ F^*(P_i)$ and $\pi^*(E_i) = \tilde{E}_i$ by Lemma 1.1, we have

$$m_i \tilde{E}_i = p \tilde{E}_i.$$

Therefore, we have $m_i = p$. The canonical divisor formula of S is given by

$$K_S \sim f^*(K_C - \mathcal{L}) + \sum_{i=1}^n a_i E_i$$

with integers a_i ($0 \leq a_i \leq p - 1$). Since π is radicial, we have $c_2(S) = c_2(\tilde{S}) = 0$. Since $\chi(S, \mathcal{O}_S) = c_2(S)/12 = 0$, we have

$$-\deg \mathcal{L} = \chi(S, \mathcal{O}_S) + t = t,$$

where t is the length of the torsion part of $R^1 f_* \mathcal{O}_S$. Since $K_{\tilde{S}} \sim \pi^* K_S + (p-1)(D)$, we have

$$K_{\tilde{S}} \equiv pr^*(p\{(2g-2)+t\}P) + \sum_{i=1}^n a_i \tilde{E}_i - (p-1)(n-2+2g)pr^*(P)$$

with a point P of \tilde{C} . Taking the degrees of both sides, we have

$$2g-2 = p(2g-2+t) + \sum_{i=1}^n a_i - (p-1)(n-2+2g).$$

Therefore, we have

$$pt + \sum_{i=1}^n a_i = n(p-1). \quad (2.3)$$

Suppose that there exist some wild fibers of f . We may assume that pE_i ($i = 1, 2, \dots, \ell$) are wild with $\ell \geq 1$, and pE_i ($i = \ell + 1, \dots, n$) are tame. Then, we have $t \geq \ell$, and $a_i = p-1$ ($i = \ell + 1, \dots, n$). Since $a_i \geq 0$ for all i , the left-hand side of (2.3) is greater than the right-hand side of (2.3). A contradiction. Therefore, all multiple fibers pE_i ($i = 1, \dots, n$) are tame, that is, we have $t = 0$ and $a_i = p-1$ ($i = 1, \dots, n$). Hence, the canonical divisor formula of S is given by

$$K_S \sim f^*(K_C - \mathcal{L}) + \sum_{i=1}^n (p-1)E_i \quad \text{with } \deg \mathcal{L} = 0.$$

□

As a special case of Theorem 2.3, we have the following result.

COROLLARY 2.4. *Under the same notation as in Theorem 2.3 except for $\tilde{C} = \mathbb{P}^1$, the elliptic surface $f: S \rightarrow \mathbb{P}^1$ constructed as in (2.1) is of type (p, p, \dots, p) with n multiple fibers. The canonical divisor K_S of S is given by*

$$K_S \sim f^*(-2P) + \sum_{i=1}^n (p-1)E_i.$$

The following lemma is well known. We give here an easy direct proof for the reader's convenience.

LEMMA 2.5. *Let $f: S \rightarrow \mathbb{P}^1$ be an elliptic surface without degenerate fibers. Then, S is isomorphic to $E \times \mathbb{P}^1$ with an elliptic curve E , and f is given by the projection $pr: E \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$.*

Proof. Since $K_S \sim f^*(-2P)$ for a point P of \mathbb{P}^1 , we have the geometric genus $p_g(S) = 0$ and $\kappa(S) = -\infty$. Therefore, the Picard scheme of S is reduced and we have $\dim_k H^1(S, \mathcal{O}_S) = 1$ by $\chi(S, \mathcal{O}_S) = 0$. Therefore, the dimension of the Albanese variety $\text{Alb}(S)$ of S is equal to one. Since $\kappa(S) = -\infty$, S is a ruled surface. Using the argument by the Stein factorization, we see that $g: S \rightarrow \text{Alb}(S)$ gives the structure of ruled surface. We set $E = \text{Alb}(S)$. Let G be a general fiber of g . Then, $G^2 = 0$. Since G is isomorphic to \mathbb{P}^1 , the virtual genus of G is equal to 0. Therefore, we have $(G \cdot K_S) = -2$. Therefore we have

$$(G \cdot f^{-1}(P)) = 1. \tag{2.4}$$

By the universality of fiber product, we have the morphism

$$(f, g): S \longrightarrow \mathbb{P}^1 \times E.$$

By (2.4), we see that (f, g) is an isomorphism. The rest is clear. □

THEOREM 2.6. *Let $f: S \rightarrow \mathbb{P}^1$ be an elliptic surface of type (p, p, \dots, p) . Then, there exist an ordinary elliptic curve E and a rational multiplicative vector field $D = \delta + \Delta$ on $E \times \mathbb{P}^1$ such that*

$$S \simeq (E \times \mathbb{P}^1)^D, \quad \mathbb{P}^1 \simeq (\mathbb{P}^1)^\Delta$$

and such that f is the natural morphism from $(E \times \mathbb{P}^1)^D$ to $(\mathbb{P}^1)^\Delta$. Here, δ (resp. Δ) is a non-zero regular (resp. rational) multiplicative vector field on E (resp. \mathbb{P}^1).

Proof. Let pE_i ($i = 1, 2, \dots, n$) be the multiple fibers of $f: S \rightarrow \mathbb{P}^1$. Since the multiple fiber pE_i is tame by assumption, $\mathcal{O}_S(E_i)|_{E_i}$ is of order p . Since $\mathcal{O}_S(E_i)|_{E_i}$ is an element of $\text{Pic}^0(E_i)$, we see that E_i is an ordinary elliptic curve.

First, we show that, taking the base change by the Frobenius morphism $F: \mathbb{P}^1 \rightarrow \mathbb{P}^1$, and taking the normalization of the fiber product, we have the following diagram:

$$\begin{array}{ccc} S & \xleftarrow{\pi} & \tilde{S} \\ f \downarrow & & \downarrow \tilde{f} \\ \mathbb{P}^1 & \xleftarrow{F} & \mathbb{P}^1 \end{array}, \tag{2.5}$$

where π is a purely inseparable finite morphism of degree p , and where $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ is an elliptic surface without multiple fibers. Take a wild fiber pE_i , and set $P_i = f(pE_i)$. Let x be a local parameter at P_i , and let V be a small affine open neighborhood of P_i such that $x - x(P)$ is a local parameter at any point P of V . We consider the invertible sheaf $\mathcal{O}_S(E_i)|_{f^{-1}(V)}$ on $f^{-1}(V)$. Since pE_i is

linearly equivalent to zero on $f^{-1}(V)$, we see that $\mathcal{O}_S(E_i)|_{f^{-1}(V)}$ is of order p . Take an affine open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of $f^{-1}(V)$. Let x_λ be a local equation of E_i on U_λ ($\lambda \in \Lambda$). Then, there exists a unit u_λ on U_λ such that

$$f^*(x) = u_\lambda x_\lambda^p \quad \text{on each } U_\lambda \tag{2.6}$$

The transition functions $f_{\lambda\mu} = x_\lambda/x_\mu$ on $U_\lambda \cap U_\mu$ ($\lambda, \mu \in \Lambda$) give the invertible sheaf $\mathcal{O}_S(E_i)|_{f^{-1}(V)}$. Using

$$f_{\lambda\mu}^p = x_\lambda^p/x_\mu^p = (x/u_\lambda)/(x/u_\mu) = u_\mu/u_\lambda,$$

we can construct the non-trivial purely inseparable covering of $f^{-1}(V)$ of degree p defined by

$$u_\lambda = z_\lambda^p \quad \text{on } U_\lambda, \quad z_\mu = f_{\lambda\mu} z_\lambda \quad \text{on } U_\lambda \cap U_\mu. \tag{2.7}$$

We denote this covering by $\rho: \widetilde{f^{-1}(V)} \rightarrow f^{-1}(V)$. Now, we consider a regular 1-form ω on $f^{-1}(V)$ defined by

$$\omega = du_\lambda/u_\lambda \quad \text{on } U_\lambda \ (\lambda \in \Lambda).$$

Since the cocycle $\{f_{\lambda\mu}\}$ is non-trivial, we see that ω is not zero. By (2.7), we have $\rho^*\omega = 0$. By (2.6), we have

$$\rho^* f^*(dx/x) = \rho^*(du_\lambda/u_\lambda) = \rho^*\omega = 0.$$

Therefore, there exists an element y of the function field $k(\widetilde{f^{-1}(V)})$ such that $\rho^* f^*(x) = y^p$. This means that $k(\mathbb{P}^1) = k(x)$ is not algebraically closed in $k(\widetilde{f^{-1}(V)})$, and by the base change by the Frobenius morphism, the fiber product W is birationally equivalent to $\widetilde{f^{-1}(V)}$:

$$\begin{array}{ccc} f^{-1}(V) & \xleftarrow{\theta} & W \xleftarrow{\eta} \widetilde{f^{-1}(V)} \\ f \downarrow & & \downarrow g \\ V & \xleftarrow{F} & V \end{array}, \tag{2.8}$$

where $\rho = \theta \circ \eta$. Set $\tilde{f} = g \circ \eta$ and take the point \tilde{Q} such that $F(\tilde{Q}) = Q$. Since $\mathcal{O}_S(E_i)|_{E_i}$ is of order p , the restriction of this covering $\rho: \widetilde{f^{-1}(V)} \rightarrow f^{-1}(V)$ on E_i is non-trivial. Therefore, $\widetilde{f^{-1}(V)}$ is non-singular on $\tilde{f}^{-1}(Q)$ (cf. [4, p. 312]), and $\tilde{f}^{-1}(Q)$ is a regular fiber. Since the fiber of f is an elliptic curve or a multiple elliptic curve, we get the diagram (2.5) by (2.8).

By Lemma 2.5, there exists an elliptic curve E such that $\tilde{S} \simeq E \times \mathbb{P}^1$ and such that \tilde{f} is the second projection. Since $\pi^{-1}(E_i) \simeq E$, E must be an ordinary elliptic curve. We fix a non-zero regular multiplicative vector field δ on E . Since

π is a purely inseparable finite morphism of degree p , there exists a p -closed divisorial vector field D on $E \times \mathbb{P}^1$ such that $S = (E \times \mathbb{P}^1)^D$ (cf. Rudakov and Shafarevich [5, Section 1]). Since D is divisorial, by the argument in Ganong and Russell [2, 3.7.1], we may assume that D is a non-zero p -closed vector field of the form

$$D = \delta + h(x) \frac{\partial}{\partial x},$$

where x is a local coordinate of an affine line in \mathbb{P}^1 . We set $\Delta = h(x) \frac{\partial}{\partial x}$. Since D is p -closed and δ is multiplicative, we conclude that D is multiplicative. Therefore, Δ is multiplicative. Hence, by Lemma 2.2, the zero points of Δ are simple. The rest is clear by Lemma 1.1. \square

COROLLARY 2.7. *Let k be an algebraically closed field of characteristic 2, and let $f: S \rightarrow \mathbb{P}^1$ be an elliptic surface of type $(2, 2, \dots, 2)$ over k . Then, the number of multiple fibers is even.*

Proof. Under the notation similar to that in Theorem 2.6, the elliptic surface $f: S \rightarrow \mathbb{P}^1$ is expressed as

$$f: S \simeq (E \times \mathbb{P}^1)^D \longrightarrow (\mathbb{P}^1)^\Delta \simeq \mathbb{P}^1,$$

where Δ is a rational vector field on \mathbb{P}^1 whose zero points are all simple. The number of zero points of Δ is equal to the number of multiple fibers of $f: S \rightarrow \mathbb{P}^1$ by Theorem 2.3. Using a local coordinate x of an affine line in \mathbb{P}^1 , we write Δ as

$$\Delta = h(x) \frac{\partial}{\partial x},$$

where $h(x)$ is a non-zero rational function on \mathbb{P}^1 . Since $\Delta^2 = \Delta$, we have a differential equation in characteristic 2:

$$\frac{\partial}{\partial x} h(x) = 1.$$

Solving the differential equation in characteristic 2 on the affine line in \mathbb{P}^1 , we have $h(x) = x + g(x)^2$ with an arbitrary rational function $g(x)$. Hence, the order of each pole of $h(x)$ is even. Let $y = 1/x$ be a local coordinate of the point at infinity. Then, we have

$$\frac{\partial}{\partial x} = -y^2 \frac{\partial}{\partial y} = y^2 \frac{\partial}{\partial y}.$$

Therefore the order of pole of Δ at the point at infinity is also even. Hence, the sum of the orders of poles of Δ is even. Since the degree of the divisor given by Δ is equal to 2, we conclude that the number of zero points of Δ is even. \square

3. Characteristic 3

In this section, let k be an algebraically closed field of characteristic 3. Let E be an ordinary elliptic curve, and δ a non-zero regular multiplicative vector field on E . We take two different points a, b of order two of E , and consider two translations T_a and T_b defined by a and b , respectively. Let x be a local coordinate of an affine line in \mathbb{P}^1 . We set

$$\Delta = (x^3 - 1/x) \frac{\partial}{\partial x}.$$

Then, we have $\Delta^3 = \Delta$ and Δ has four simple zero points on \mathbb{P}^1 :

$$\Omega = \{1, -1, \sqrt{-1}, -\sqrt{-1}\}.$$

We consider two automorphisms of \mathbb{P}^1 defined by

$$\sigma': x \mapsto -x, \quad \tau': x \mapsto 1/x.$$

These automorphisms are of order 2 and induce permutations of the elements of Ω . The vector field Δ is invariant under σ' and τ' . We set

$$D = \delta + \Delta$$

and

$$\sigma = T_a \times \sigma', \quad \tau = T_b \times \tau'.$$

Then, D is a multiplicative vector field on $E \times \mathbb{P}^1$, and σ, τ are automorphisms of order 2 of $E \times \mathbb{P}^1$ which have no fixed points on $E \times \mathbb{P}^1$. σ and τ commute with each other, and D is invariant under σ, τ . We set

$$S_1 = (E \times \mathbb{P}^1)^D, \quad C_1 = (\mathbb{P}^1)^\Delta \quad (\simeq \mathbb{P}^1).$$

Then we have a diagram

$$\begin{array}{ccc} S_1 & \xleftarrow{\pi_1} & E \times \mathbb{P}^1 \\ f_1 \downarrow & & \downarrow pr \\ C_1 & \xleftarrow{F_1} & \mathbb{P}^1 \end{array}, \tag{3.1}$$

where π_1 and F_1 are natural morphisms, and where f_1 is the morphism induced from the projection pr . We set

$$\begin{aligned} P_1 &= F_1(1), & P_2 &= F_1(-1), & P_3 &= F_1(\sqrt{-1}), \\ P_4 &= F_1(-\sqrt{-1}), & P_5 &= F_1(0), & P_6 &= F_1(\infty). \end{aligned}$$

Then, f_1 has four tame multiple fibers

$$f_1^{-1}(P_i) = 3E_i \quad (i = 1, 2, 3, 4)$$

with elliptic curves E_i . Automorphisms σ and τ induce automorphisms σ_1 and τ_1 , respectively, of order 2 of S_1 which have no fixed points on S_1 . Automorphisms σ' and τ' induce automorphisms σ'_1 and τ'_1 , respectively, of order 2 of C_1 . We have

$$\begin{aligned} \tau_1(E_1) &= E_1, & \tau_1(E_2) &= E_2, & \tau_1(E_3) &= E_4, & \tau_1(E_4) &= E_3, \\ \tau'_1(P_1) &= P_1, & \tau'_1(P_2) &= P_2, & \tau'_1(P_3) &= P_4, & \tau'_1(P_4) &= P_3, \end{aligned}$$

and

$$\begin{aligned} \sigma_1(E_1) &= E_2, & \sigma_1(E_2) &= E_1, & \sigma_1(E_3) &= E_4, & \sigma_1(E_4) &= E_3, \\ \sigma'_1(P_1) &= P_2, & \sigma'_1(P_2) &= P_1, & \sigma'_1(P_3) &= P_4, & \sigma'_1(P_4) &= P_3. \end{aligned}$$

The automorphisms σ'_1 has two fixed points P_5 and P_6 on C_1 . Automorphisms σ_1 and τ_1 commute with each other. We set

$$S_2 = S_1 / \langle \sigma_1 \rangle, \quad C_2 = C_1 / \langle \sigma'_1 \rangle \quad (\simeq \mathbb{P}^1).$$

Then, we have a diagram

$$\begin{array}{ccc} S_2 & \xleftarrow{\pi_2} & S_1 \\ f_2 \downarrow & & \downarrow f_1 \\ C_2 & \xleftarrow{F_2} & C_1 \end{array}, \tag{3.2}$$

where morphisms π_2 , F_2 and f_2 are similar to those in (3.1). We have

$$F_2(P_1) = F_2(P_2) \quad \text{and} \quad F_2(P_3) = F_2(P_4).$$

We set

$$Q_1 = F_2(P_1), \quad Q_3 = F_2(P_3), \quad Q_5 = F_2(P_5), \quad Q_6 = F_2(P_6).$$

Then, f_2 has four tame multiple fibers

$$\begin{aligned} f_2^{-1}(Q_1) &= 3F_1, & f_2^{-1}(Q_3) &= 3F_3, \\ f_2^{-1}(Q_5) &= 2F_5, & f_2^{-1}(Q_6) &= 2F_6. \end{aligned}$$

τ_1 (resp. τ'_1) induces an automorphism τ_2 (resp. τ'_2) of order 2 of S_2 (resp. C_2). τ_2 has no fixed points on S_2 . We have

$$\tau_2(F_1) = F_1, \quad \tau_2(F_3) = F_3, \quad \tau_2(F_5) = F_6, \quad \tau_2(F_6) = F_5,$$

$$\tau'_2(Q_1) = Q_1, \quad \tau'_2(Q_3) = Q_3, \quad \tau'_2(Q_5) = Q_6, \quad \tau'_2(Q_6) = Q_5.$$

We set

$$S = S_2 / \langle \tau_2 \rangle, \quad C = C_2 / \langle \tau'_2 \rangle \quad (\simeq \mathbb{P}^1).$$

Then, we have a diagram

$$\begin{array}{ccc}
 S & \xleftarrow{\pi_3} & S_2 \\
 f \downarrow & & \downarrow f_2 \\
 C & \xleftarrow{F_3} & C_2
 \end{array} , \tag{3.3}$$

where morphisms π_3, F_3 and f are similar to those in (3.1). We have

$$F_3(Q_5) = F_3(Q_6).$$

We set

$$R_1 = F_3(Q_1), \quad R_3 = F_3(Q_3), \quad R_5 = F_3(Q_5).$$

Then, f has three tame multiple fibers:

$$f^{-1}(R_1) = 6G_1, \quad f^{-1}(R_3) = 6G_3, \quad f^{-1}(R_5) = 2G_5,$$

where G_1, G_3 and G_5 are ordinary elliptic curves. Hence, we have in characteristic 3 an elliptic surface of type $(2, 6, 6)$. For this elliptic surface, $|mK_S|$ gives a unique structure of the elliptic surface for any $m \geq 14$, and $|13K_S|$ does not give the structure of elliptic surface. Hence, using results in [4, Section 5], we have the following theorem.

THEOREM 3.1. *Let k be an algebraically closed field of characteristic 3. If S is an elliptic surface defined over k with $\kappa(S) = 1$, then $\Phi_{|mK_S|}$ gives a unique structure of elliptic surface for any $m \geq 14$. Moreover, the number 14 is best possible.*

4. Characteristic 2

In this section, let k be an algebraically closed field of characteristic 2. First, we prove that there doesn't exist an elliptic surface of type $(2, 6, 6)$ in characteristic 2.

Suppose there exists an elliptic surface $f: S \rightarrow \mathbb{P}^1$ of type $(2, 6, 6)$. We denote by $2E_1, 6E_2, 6E_6$ the multiple fibers, and we set $P_i = f(E_i)$ ($i = 1, 2, 3$). We consider the invertible sheaf $L = \mathcal{O}_S(2E_2 - 2E_3)$. Since $L^{\otimes 3} \simeq \mathcal{O}_S$, we have an étale covering \tilde{S} of S of degree 3 by a standard method. We denote by E the fiber on a general point P of \mathbb{P}^1 . Since the restriction of L on E (resp. E_1) is

trivial, we see that the étale covering restricted on E (resp. E_1) splits into three pieces of E (resp. E_1). Hence, taking the normalization of the base curve \mathbb{P}^1 in the function field of \tilde{S} , we have a diagram

$$\begin{array}{ccc} S & \xleftarrow{\pi} & \tilde{S} \\ f \downarrow & & \downarrow \tilde{f} \\ \mathbb{P}^1 & \xleftarrow{\phi} & \mathbb{P}^1 \end{array},$$

where π is the étale covering of degree 3 which we constructed above, and where ϕ is the morphism of degree 3 given by the normalization which ramifies only at P_2 and P_3 . Since $\phi^{-1}(P_1)$ consists of three points, and $\phi^{-1}(P_2)$ (resp. $\phi^{-1}(P_3)$) consists of one point, the elliptic surface $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ has five multiple fibers with multiplicity 2. Since $2E_1$ (resp. $6E_2$, resp. $6E_3$) is tame, the order of the normal bundle $\mathcal{O}_S(E_1)|_{E_1}$ (resp. $\mathcal{O}_S(E_2)|_{E_2}$, resp. $\mathcal{O}_S(E_3)|_{E_3}$) is two (resp. six, resp. six). Using these facts, we see that all multiple fibers of \tilde{f} are tame. Hence, the elliptic surface $\tilde{f}: \tilde{S} \rightarrow \mathbb{P}^1$ is of type $(2, 2, 2, 2, 2)$. This contradicts Corollary 2.7. Hence, there doesn't exist an elliptic surface of type $(2, 6, 6)$.

Now, we construct an elliptic surface of type $(2,5,10)$ in characteristic 2. Let E be an ordinary elliptic curve, and δ a non-zero regular multiplicative vector field on E . We take a point a of order 5 of E , and consider the translation T_a given by a . Let x be a local coordinate of an affine line in \mathbb{P}^1 . We consider a rational vector field given by

$$\Delta = (x - 1/x^4) \frac{\partial}{\partial x}$$

on \mathbb{P}^1 . We have $\Delta^2 = \Delta$. Let ζ be a primitive fifth root of unity. We take the automorphism σ' of order 5 defined by

$$\sigma': x \mapsto \zeta x.$$

Then, Δ is invariant under σ' . We set

$$D = \delta + \Delta, \quad \sigma = T_a \times \sigma'.$$

Then, D is a rational multiplicative vector field on $E \times \mathbb{P}^1$, and σ is an automorphism of order 5 of $E \times \mathbb{P}^1$ which has no fixed points on $E \times \mathbb{P}^1$. D is invariant under σ . As in Section 3, we have an elliptic surface $\tilde{f}: (E \times \mathbb{P}^1)^D \rightarrow (\mathbb{P}^1)^\Delta \simeq \mathbb{P}^1$ of type $(2, 2, 2, 2, 2, 2)$. On the elliptic surface, we have the automorphism $\tilde{\sigma}$ of order 5 which is induced from σ . Then, taking the quotient by $\langle \tilde{\sigma} \rangle$, we have an elliptic surface

$$f: S \longrightarrow \mathbb{P}^1$$

of type $(2, 5, 10)$. Since $|mK_S|$ gives a unique structure of the elliptic surface for any $m \geq 12$, and $|11K_S|$ doesn't give the structure of elliptic surface. Hence, by careful calculations as in [4, Section 5] in characteristic 2, we have the following theorem.

THEOREM 4.1. *Let k be an algebraically closed field of characteristic 2. If S is an elliptic surface defined over k with $\kappa(S) = 1$, then $\Phi_{|mK_S|}$ gives a unique structure of elliptic surface for any $m \geq 12$. Moreover, the number 12 is best possible.*

5. Examples of Multiple Fibers

In this section, let k be an algebraically closed field of characteristic $p > 0$. We give some examples of elliptic surfaces which have multiple fibers obtained by vector fields. We denote by x a local coordinate of an affine line in \mathbb{P}^1 .

EXAMPLE 5.1. We fix a supersingular elliptic curve E , and a non-zero regular additive vector field δ on E . Let Δ be a rational vector field on \mathbb{P}^1 given by

$$\Delta = x^{pn} \frac{\partial}{\partial x}$$

with a positive integer n . We set $D = \delta + \Delta$. This gives an additive vector field on $E \times \mathbb{P}^1$. We set $S = (E \times \mathbb{P}^1)^D$ and $C = (\mathbb{P}^1)^\Delta$. C is isomorphic to \mathbb{P}^1 , and we have a diagram

$$\begin{array}{ccc} S & \xleftarrow{\pi} & E \times \mathbb{P}^1 \\ f \downarrow & & \downarrow pr \\ \mathbb{P}^1 & \xleftarrow{F} & \mathbb{P}^1 \end{array} ,$$

where F, π are natural morphisms, and where f is the morphism induced from the projection pr . The morphism F is nothing but the Frobenius morphism. We set $P_0 = F(0)$, $f^{-1}(P_0) = pG_0$ and $pr^{-1}(\infty) = E_\infty$. Then we have

$$(D) = -(pn - 2)E_\infty.$$

The elliptic surface $f: S \rightarrow \mathbb{P}^1$ has only one multiple fiber $f^{-1}(P_0) = pG_0$. Since G_0 is a supersingular elliptic curve, the Picard variety of G_0 is also supersingular, therefore, it has no points of order p . Therefore, the normal bundle $\mathcal{O}_S(G_0)|_{G_0}$ is trivial. Hence, $f^{-1}(P_0)$ is a wild fiber. The canonical divisor formula of S is given by a form

$$K_S \sim f^*((t - 2)P) + aE_0$$

with non-negative integers t, a ($0 \leq a \leq p - 1$) and a point P of \mathbb{P}^1 . Let \tilde{P} be the point of \mathbb{P}^1 such that $F(\tilde{P}) = P$. Since we have

$$K_{E \times \mathbb{P}^1} \sim \pi^*K_S + (p - 1)(D) \tag{5.1}$$

and

$$\pi^*K_S \sim pr^{-1}((t - 2)p\tilde{P}) + a(pr^{-1}(0)), \tag{5.2}$$

taking the degrees of both sides in (5.1) and (5.2), we have

$$-2 = p(t - 2) + a - (p - 1)(pn - 2).$$

Therefore, we have

$$a = p(-t + (p - 1)n). \tag{5.3}$$

Since $0 \leq a \leq p - 1$, we conclude

$$a = 0 \quad \text{and} \quad t = (p - 1)n.$$

Hence, the canonical divisor of S is given by

$$K_S \sim f^{-1}(((p - 1)n - 2)P).$$

It is easy to see that if $p = 2$ and $n = 1$, then S is a ruled surface over an elliptic curve ($\kappa(S) = -\infty$), if either $p = 2, n = 2$, or $p = 3, n = 1$, then S is a hyperelliptic surface ($\kappa(S) = 0$), and otherwise, S is an elliptic surface with $\kappa(S) = 1$ (see also [2]).

EXAMPLE 5.2. We give here an example of an elliptic surface with one tame fiber and one wild fiber. We fix an ordinary elliptic curve E , and a non-zero regular multiplicative vector field δ on E . Let Δ be a rational vector field on \mathbb{P}^1 given by

$$\Delta = (x^p - x) \frac{\partial}{\partial x}.$$

It is easy to see that Δ is multiplicative. We take a point a of order p of E , and consider the translation T_a given by a . We consider the automorphism of order p of \mathbb{P}^1 defined by

$$\sigma': x \mapsto x + 1.$$

We set $D = \delta + \Delta$ and $\sigma = T_a \times \sigma'$. Then D is a p -closed rational vector field of $\mathbb{P}^1 \times E$, and σ is an automorphism of order p of $\mathbb{P}^1 \times E$ which has no fixed points on $\mathbb{P}^1 \times E$. D is invariant under σ . We set $\tilde{S} = (E \times \mathbb{P}^1)^D$ and $\tilde{C} = (\mathbb{P}^1)^\Delta (\simeq \mathbb{P}^1)$. Then, the elliptic surface $\tilde{f}: \tilde{S} \rightarrow \tilde{C}$ has p tame multiple fibers with multiplicity p , where \tilde{f} is the natural morphism. We denote the multiple fibers by pE_i ($i = 1, 2, \dots, p$). The automorphism σ (resp. $\tilde{\sigma}'$) induces an automorphism $\tilde{\sigma}$ (resp. $\tilde{\sigma}'$) of order p of \tilde{S} ; $\tilde{\sigma}$ is fixed-point-free. The group $\langle \tilde{\sigma} \rangle$ acts transitively on the set of multiple fibers. We set $S = \tilde{S} / \langle \tilde{\sigma} \rangle$ and $C = \tilde{C} / \langle \tilde{\sigma}' \rangle$. Then, we have an elliptic surface $f: S \rightarrow C \simeq \mathbb{P}^1$ and a diagram

$$\begin{array}{ccc} S & \xleftarrow{\pi} & \tilde{S} \\ f \downarrow & & \downarrow \tilde{f} \\ C & \xleftarrow{\phi} & \tilde{C} \end{array} ,$$

where ϕ and π are natural morphisms, and where π is an étale morphism of degree p . The elliptic surface $f: S \rightarrow C \simeq \mathbb{P}^1$ has two multiple fibers of multiplicity p : one comes from multiple fibers of \tilde{f} which is tame, and the other comes from the fixed point of $\tilde{\sigma}'$. We denote by pE_0 the former tame multiple fiber, and by pE_∞ the later multiple fiber. We take a point P of C , and a point \tilde{P} of \tilde{C} such that $\phi(\tilde{P}) = P$. Then, the canonical divisor formula of S is given by

$$K_S \sim f^{-1}((t-2)P) + (p-1)E_0 + aE_\infty$$

with non-negative integers t, a ($0 \leq a \leq p-1$). We set $P_\infty = \tilde{f}(E_\infty)$, and denote by \tilde{P}_∞ the unique point of \tilde{C} such that $\phi(\tilde{P}_\infty) = P_\infty$. We set $\tilde{E}_\infty = \tilde{f}^{-1}(\tilde{P}_\infty)$. This fiber is a regular fiber. Since π is étale, we have $K_{\tilde{S}} \sim \pi^*K_S$. Therefore, by the canonical divisor formula, we have

$$\begin{aligned} \tilde{f}^{-1}(-2\tilde{P}) + (p-1)\tilde{E}_1 + \dots + (p-1)\tilde{E}_p \\ \sim \tilde{f}^{-1}((t-2)pP) + (p-1)\tilde{E}_1 + \dots + (p-1)\tilde{E}_p + a\tilde{E}_\infty. \end{aligned}$$

Therefore, taking the degrees of both sides, we have

$$-2 = (t-2)p + a.$$

Since $0 \leq a \leq p-1$, we have

$$a = p-2 \quad \text{and} \quad t = 1.$$

Hence, we have the canonical divisor formula

$$K_S \sim f^{-1}(-P) + (p-1)E_0 + (p-2)E_\infty.$$

This gives an example of an elliptic surface which has one tame fiber and one wild fiber. Incidentally, if $p = 2$, then S is a ruled surface over an elliptic curve ($\kappa(S) = -\infty$), if $p = 3$, then S is a hyperelliptic surface ($\kappa(S) = 0$, $3K_S \sim 0$), and otherwise, S is an elliptic surface with $\kappa(S) = 1$.

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