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Zagier's Conjecture and Wedge Complexes in Algebraic K-theory

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Abstract. Using a suitable formalism of relative K-theory we construct, for schemes satisfying the Beilinson-Soulé conjecture on weights, wedge complexes whose cohomology maps to the K-theory of those schemes. These complexes contain subcomplexes generated by elements $[x]_n$ for $n \geqslant 2$ with $d[x]_n = x \otimes [x]_{n-1}$. In case the scheme is the spectrum of a number field we can use our construction and results of Suslin, Goncharov and Zagier to prove a version of Zagier's conjecture [31]. In particular, the regulator is given by mapping $[x]_n$ to $P_n(x)$, where P_n is a suitable single valued function obtained from the nth polylogarithm. We also give results about finite generation of the image under the regulator, and give relations satisfied by the elements $[x]_n$.

1. Introduction

Let F be a field. It is a difficult problem to compute the K-theory of F. Of course $K_0(F)$ and $K_1(F)$ are well known, and Matsumoto's theorem (see [23]) tells us that $K_2(F) \cong F^* \otimes F^*/\{x \otimes (1-x) | x \in F \setminus \{0,1\}\}$. The description of $K_n(F)$ for $n \geqslant 3$ and general F – see [25] for finite fields – turns out to be much more difficult. In [6] and [7] Borel computed the rank of $K_n(F)$ for all $n \geqslant 2$ if F is a number field. But in this case an explicit description in terms of generators and relations is unknown. The first result in this direction was published in [4]. For simplicity we tensor all K-groups with $\mathbb{Q}$. Let $F_{\mathbb{Q}}^* = F^* \otimes \mathbb{Q}$ and let $K_j^{(i)}(F)$ be the i-th eigenspace of the Adams operations on $K_j(F) \otimes \mathbb{Q}$ (see, e.g., [27]). Bloch defined a complex $\mathfrak{B}_2(F)$ (in degree one and two), $\mathcal{B}(F) \to F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^*$ together with a map $H^i(\mathfrak{B}_2(F)) \xrightarrow{\psi_i} K_{4-i}^{(2)}(F)$ for $i = 1$ or 2.

$\mathcal{B}(F)$ is an Abelian group generated by elements $(x)_2$ for $x \in F \setminus \{0,1\}$, and the map in the above complex is given by $(x)_2 \mapsto x \otimes (1-x)$. It follows from Matsumoto's theorem that ψ_2 is an isomorphism, and work of Suslin [28] shows that ψ_1 is an isomorphism, at least when F is a number field.

Now let F be a number field, and let $n \geqslant 2$. Inspired by the results of Bloch, Zagier made a conjecture about the K-theory of F in [31]. Here we give a reformulation by Deligne in [10]. Let $\mathcal{L}_1 = F_{\mathbb{Q}}^*$, and let $\{x\}_1 = 1 - x$ for $x \in F \setminus \{0,1\}$. There should exist an Abelian group $\mathcal{L}_n = \mathcal{L}_n(F)$, generated by elements $\{x\}_n$, $x \in F \setminus \{0,1\}$, with a map d_n: $\mathcal{L}_n \to \bigwedge^2(\bigoplus_{l=1}^{n-1} \mathcal{L}_l)$ given by $\{x\}_n \mapsto x \wedge \{x\}_{n-1}$. There should exist a map ϕ_n: $\operatorname{Ker} d_n \to K^{(n)}_{2n-1}(F)$, which should be an isomorphism. Moreover, in [6] and [7] Borel defined regulator maps on the K-groups, and Zagier conjectured a very precise formula for this regulator, given by a suitable single valued function $P_{\mathrm{Zag},n}$ defined by Zagier, based on the n-th polylogarithm.

In [16] Goncharov conjectures an explicit version of complexes that should compute the K-theory of a field F. Namely, for $n \geqslant 2$ let $B_n = B_n(F)$ be the free group on $F \setminus \{0,1\}$ modulo certain relations based on the n-th polylogarithm (see [16]). Define $d_n\{x\}_n = x \otimes \{x\}_{n-1}$ in $F^*_{\mathbb{Q}} \otimes B_n$ for $n \geqslant 3$, and let $d\{x\}_2 = x \wedge (1-x)$ in $\bigwedge^2 F^*_{\mathbb{Q}}$. Then Goncharov conjectures that the i-th cohomology group of the cohomological complex (starting in degree 1 and with the differential determined by its effect on the B_m's)

$$B_n \to F^*_{\mathbb{Q}} \otimes B_{n-1} \to \left(\bigwedge^2 F^*_{\mathbb{Q}}\right) \otimes B_{n-2} \to \cdots \to \left(\bigwedge^{n-2} F^*_{\mathbb{Q}}\right) \otimes B_2 \to \bigwedge^{n} F^*_{\mathbb{Q}}$$

is isomorphic to $K^{(n)}_{2n-i}(F)$.

For an interpretation of this complex involving Tannakian tensor categories and Lie algebra's see [16].

In this paper we use a formalism of multi-relative K-theory to construct complexes of $\mathbb{Q}$-vector spaces for each $n \geqslant 1$,

$$\begin{aligned} L^{\mathrm{alt}}_{(n)} &\to F^*_{\mathbb{Q}} \otimes L^{\mathrm{alt}}_{(n-1)} \to \bigwedge^2 F^*_{\mathbb{Q}} \otimes L^{\mathrm{alt}}_{(n-2)} \to \cdots \\ &\to \bigwedge^{n-2} F^*_{\mathbb{Q}} \otimes L^{\mathrm{alt}}_{(2)} \to \bigwedge^{n-1} F^*_{\mathbb{Q}} \otimes L^{\mathrm{alt}}_{(1)} \end{aligned}$$

for F a field. The p-th cohomology of this complex maps to $K^{(n)}_{2n-p}(F)$ provided $K^{(j)}_m(F) = 0$ if $m - 2j \geqslant 0$ and $m > j$, see Theorem 3.15 below. (Those conditions would follow from a well known conjecture about weights, see, e.g., [27, p. 501].) Each $L^{\mathrm{alt}}_{(p)}$ contains a subspace generated by elements $[x]_p$ (where $x \in F^*$ for $p \geqslant 2$, $[x]_1 = 1-x$ for $x \neq 0, 1$), and under the differential $[x]_p$ gets mapped to $x \otimes [x]_{p-1}$. This gives rise to a subcomplex, based on the elements $[x]_p$, $1 \leqslant p \leqslant n$. It turns out that the subcomplex generated by the $[x]_p + (-1)^p[x]_p$'s is acyclic under suitable assumptions, giving rise to a quotient complex

$$\begin{aligned} \widetilde{M}_{(n)} &\to F^*_{\mathbb{Q}} \otimes \widetilde{M}_{(n-1)} \to \left(\bigwedge^2 F^*_{\mathbb{Q}}\right) \otimes \widetilde{M}_{(n-2)} \to \cdots \\ &\to \left(\bigwedge^{n-2} F^*_{\mathbb{Q}}\right) \otimes \widetilde{M}_{(2)} \to \bigwedge^{n} F^*_{\mathbb{Q}} \end{aligned}$$

that has a shape similar to Goncharov's complex, again with a map from its p-th cohomology to $K^{(n)}_{2n-p}(F)$ under suitable conditions.

The regulator map on H^1 of each of our complexes (which maps to $K^{(n)}_{2n-1}$) is given by sending $[x]_n$ to $P_{\mathrm{Zag},n}(x)$ for $x \in \mathbb{C}$. For number fields all our assumptions are satisfied, so that we actually prove part of Zagier's conjecture for all $n \geqslant 2$. The missing part in general is the surjectivity – injectivity holds by construction in our case. For

$n = 2$ or 3 we can use results by Goncharov and Suslin to prove surjectivity if F is a number field, thereby obtaining a proof of the full conjecture in those cases. In case F is a number field, Zagier's original conjecture contained a clause about the image of the regulator mapping being a finitely generated lattice, see [31]. (Zagier's original conjecture is about Abelian groups, not about $\mathbb{Q}$-vector spaces.) As an approximation we show how to construct complexes of $\mathbb{Z}$-modules rather than $\mathbb{Q}$-vector spaces, with the property that the image under the regulator map of their cohomology groups are finitely generated lattices, see Remark 5.4.

It should be said that similar complexes exist for other schemes, with similar results, but in case the scheme involved is the spectrum of a field, the complexes take on a very nice shape, and the statement of the result is less technical, see the results in section three.

The complexes constructed here can also be used to construct examples in $K_4^{(3)}(E)$ or $K_4^{(3)}(k(E))$ of some elliptic curves E defined over a field k. Together with more computations of the regulator of those elements, and the "tame symbol"

$$K_4^{(3)}(k(E)) \to \coprod_{\substack{x\in E\\ \text{closed}}} K_3^{(2)}(k(x))$$

in case k is a number field, this will be published elsewhere, as the techniques involved are a little different.

Similar results have been found by other people. A result similar to Theorem 5.1 (in case $n = 1$, and with different proofs) was first published in [2], see also [3] for more details. It was shown in [10] and [2] that Theorem 5.1 (in case $n = 1$) can be deduced easily from the existence of an appropriate category of mixed Tate motives. For constructions of candidates for such a category, see [21], [30] and [5].

The organization of the material is as follows. In the second section we construct the necessary formalism in K-theory that is needed for the construction of the complexes in the third section. This is not the first time that statements along those lines were used (see [1]), but lacking proofs in the literature we include them in some detail here. Unfortunately they are somewhat technical. In the third section we construct the complexes, and the computation of the regulator map is carried out in the fourth section. The fifth section uses results from Suslin, Goncharov and Zagier in order to get a proof of a version of Zagier's conjecture in certain cases, and a statement about finite generation of sorts (Remark 5.4).

There is one fundamental result due to Borel ([6] and [7]) to which we will refer as Borel's theorem. Namely, the regulator map on the K-theory of a number field is injective up to torsion. (We will not need the fact that there is a precise relation between the regulator map and certain values of the zeta function of the number field.)

Finally, it should be said that all schemes in this paper are assumed to be noetherian, quasi-projective and separated.

2. Multi-relative K-theory with Weights

In this section we will construct the necessary formalism in K-theory that will be used in the remainder of this paper to construct the complexes. This involves combining relative K-theory with Adams operations and push forwards under suitable assumptions.

We will also define the regulator map on this K-theory to a (relative) analogue in Deligne cohomology, and exhibit an explicit group to which this Deligne cohomology is isomorphic. The construction uses generalized sheaf cohomology as in [13] and [14], and we will use the results and terminology from these two papers. T will be a fixed topos.

2.1. K-COHOMOLOGY

If T is a topos, we will denote by sT the category of pointed simplicial objects in T. $*$ will be the chosen point. In practice T will be either the topos of sheaves on the big Zariski site of all schemes, ZAR, or the topos of sheaves on the Zariski site of schemes over a fixed scheme S, ZAR/S. An object in sT will be called a space. sT is a closed model category in the sense of [24] (see [19]), so that we have the associated homotopy category $\mathrm{Ho}\, sT$. Let $\mathbb{Z}_\infty BGL$ be the sheaf associated to the presheaf $U \mapsto \mathbb{Z}_\infty BGL(U)$, where $\mathbb{Z}_\infty BGL(U)$ is the Bousfield–Kan integral completion of $BGL(U)$. Let $\mathbb{Z} \times \mathbb{Z}_\infty BGL$ be the product of $\mathbb{Z}_\infty BGL$ with the constant sheaf $\mathbb{Z}$, pointed by 0. To simplify notation we will write K for $\mathbb{Z} \times \mathbb{Z}_\infty BGL$ from now on. Then for a space X Gillet and Soulé define its higher K-theory by $H^{-m}(X, K) = [S^m \wedge X, K]$ for $m \geqslant 0$. Here $[\cdot,\cdot]$ is the set of morphisms in $\mathrm{Ho}\, sT$. If $K \to K^\sim$ is a flasque resolution of K in sT (see [14, p. 4]), then this group can be computed as $[S^m \wedge X, K^\sim]$, homotopy classes of actual maps. This group is also isomorphic to $\pi_n \mathrm{Hom}(X, K^\sim)$.

Let X be a regular noetherian finite dimensional scheme. Let $X.$ denote the constant simplicial sheaf $\mathrm{Hom}(\cdot, X)$, with a disjoint basepoint. Then, according to [14],

$$[S^n \wedge X_., K] \cong K_n(X), \tag{1}$$

where the right hand side is the usual Quillen K-theory.

If X and Y are two spaces, $f\colon X \to Y$ a map, let

$$C(X,Y) = Y \coprod X \times I/\sim, \tag{2}$$

where I is the simplicial version of the unit interval, given in degree s by all sequences $\{0,\dots,0,1\dots,1\}$ of length $s+1$, and pointed by $\{1,\dots,1\}$, and $\sim$ are the usual identifications to obtain the reduced mapping cone. Define $H^{-m}(Y,X,K) = H^{-m}(C(X,Y),K)$ for $m \geqslant 0$. We then have long a exact sequence

$$\begin{aligned}\cdots &\to H^{-m}(Y,X,K) \to H^{-m}(Y,K) \to H^{-m}(X,K)\\ &\to H^{-m+1}(Y,X,K) \to \cdots\end{aligned} \tag{3}$$

Maps $X \to Y \to Z$ give rise to a long exact sequence

$$\begin{aligned}\cdots &\to H^{-m}(Z,Y,K) \to H^{-m}(Z,X,K) \to H^{-m}(Y,X,K)\\ &\to H^{-m+1}(Z,Y,K) \to \cdots\end{aligned} \tag{4}$$

For $N \geqslant 1$ let $H^{-m}(X, K^N) = [S^m \wedge X, \mathbb{Z} \times \mathbb{Z}_\infty BGL_N]$.

For a sheaf of groups π (Abelian if $n \geqslant 2$) define the Eilenberg–MacLane space $K(\pi,n)$ by $K(\pi,n) = W^n\pi$, where W is the Moore functor (see [14, p. 4]). If $n = 0$ then $K(\pi,0)$ is set equal to π where π may be either a sheaf of sets, groups, or Abelian groups.

If X is a space, then for each $n \geqslant 0$ and group π in T (Abelian if $n \geqslant 2$) we define $H^n(X,\pi) = [X, K(\pi, n)]$. For a space Y let $\pi_n(Y)$ be the sheaf attached to the presheaf $U \mapsto \pi_n(Y(U))$, for $n \geqslant 0$. A space X is called K-coherent if the natural maps $\varinjlim_N H^{-m}(X, K^N) \to H^{-m}(X,K)$ and $\varinjlim_N H^m(X, \pi_{-n}K^N) \to H^m(X, \pi_{-n}K)$ are isomorphisms for all $m, n \geqslant 0$.

The Loday product $BGL \times BGL \to BGL$ gives rise to a map

$$\mathbb{Z} \times \mathbb{Z}_\infty BGL \wedge \mathbb{Z} \times \mathbb{Z}_\infty BGL \to \mathbb{Z} \times \mathbb{Z}_\infty BGL$$

and we can use this, if U is another space, to get a product

$$H^{-m}(X,K) \times H^{-n}(U,K) \to H^{-m-n}(X \wedge U, K)$$

via

$$S^{m+n} \wedge X \wedge U \cong S^m \wedge X \wedge S^n \wedge U \to K \wedge K \to K. \tag{5}$$

If X and U are K-coherent, this product is associative and graded commutative.

Because mapping cones and smash products commute in our definition this also gives rise to a multiplication

$$H^{-m}(Y,X,K) \times H^{-n}(U,K) \to H^{-m-n}(Y \wedge U, X \wedge U, K) \tag{6}$$

which is compatible with the maps in (3) and (4).

If X is a K-coherent space then Gillet and Soulé in [4] define a special λ-module structure over $H^0(X,K)$ on $H^{-m}(X,K)$. In particular there are Adams operations ψ^k for $k \geqslant 1$ acting on those groups, giving rise to a decomposition

$$H^{-m}(X,K)_{\mathbb{Q}} = \bigoplus_{i \geqslant 0} H^{-m}(X,K)^{(i)}_{\mathbb{Q}} \tag{7}$$

where the superscript (i) denotes the subspace where ψ^k acts as multiplication by k^i for $k \geqslant 1$, and similarly for $H^{-n}(U,K)$. ψ^k is induced from a map ψ^k: $K \to K$, and because the diagram

$$\begin{array}{ccc} K^N \wedge K^N & \longrightarrow & K^M \\ \Big\downarrow{\scriptstyle \psi^k \wedge \psi^k} & & \Big\downarrow{\scriptstyle \psi^k} \\ K \wedge K & \longrightarrow & K \end{array} \tag{8}$$

commutes in Ho sT for M large enough, the multiplication in (6) gives rise to a product $H^{-m}(Y,X,K)^{(i)}_{\mathbb{Q}} \times H^{-n}(U,K)^{(j)}_{\mathbb{Q}} \to H^{-m-n}(Y \wedge U, X \wedge U, K)^{(i+j)}_{\mathbb{Q}}$ for K-coherent spaces X, Y and U. Because the operations in the λ-structure are defined on the sheaf K, the maps in (3) and (4) are compatible with the λ-structure.

2.2. Riemann–Roch

In this subsection we will get the necessary results for the behaviour of weights under push forward under suitable conditions.

Suppose that X and Y are noetherian finite dimensional schemes, and write $X.$ and $Y.$ for the (pointed) spaces they represent. Then the constructions of taking mapping cones and smash products will give rise to spaces all components of which are representable in T, except for one copy of $*$ in each degree. Furthermore those spaces are degenerate above a finite simplicial degree. We will refer to spaces like this loosely as pointed simplicial schemes, and we will call them regular if all scheme components are regular.

LEMMA 2.1. *Let T be either the big Zariski topos of all schemes, ZAR, or the big Zariski topos over a scheme S, or the small Zariski topos over S. Then the element in sT represented by a regular pointed simplicial scheme is K-coherent.*

Proof. Let $X.$ be the regular pointed simplicial scheme. There exists a spectral sequence $H^{-q}(X_p, K) \Rightarrow H^{-p-q}(X., K)$, and similarly for K^N and the other sheaves. Each X_p is K-coherent by [14, Proposition 5, p. 21]. Because $X.$ is degenerate above a certain simplicial degree this implies the K-coherence for $X.$, see [14, p. 8].

There is another way of defining the K-theory for regular pointed simplicial schemes. For any noetherian scheme X let $\Omega BQP(X)$ be the loop space associated to Quillen's Q-construction applied to the category of locally free sheaves on X, and put $\Omega BQP(*) = *$ formally. (We will assume that all categories of sheaves are suitably rigidified in the diagrams we are considering, so that they form real functors.) If $X.$ is a pointed simplicial scheme, put $K_m(X.) = \pi_m(\operatorname{holim} \Omega BQP(X_n))$. If every component of $X.$ is regular and noetherian, then by [14] $H^{-m}(X., K) \cong K_m(X.)$.

Let $i\colon Z. \to X.$ be a map of pointed simplicial schemes which maps $*$ to $*$, and is a closed immersion of schemes on all other components. We want to define a push forward in this context under suitable conditions.

(TC1) Suppose that all maps $Z_k \to Z_l$ and $X_k \to X_l$ are of finite tor-dimension.

(TC2) Suppose that for all k, l

$$\begin{array}{ccc} Z_k & \xrightarrow{\ i_k\ } & X_k \\ \downarrow{\scriptstyle f_{kl}} & & \downarrow{\scriptstyle f_{kl}} \\ Z_l & \xrightarrow{\ i_l\ } & X_l \end{array}$$

is cartesian and that f_{kl} and i_l are tor-independent.

For a pointed simplicial scheme $Z.$ satisfying (TC1) above define

$$M'(Z_l) = \left\{ \begin{array}{l} \text{category of coherent sheaves } N \text{ on } Z_l \text{ such} \\ \text{that } \underline{\operatorname{Tor}_i}^{\mathcal{O}_{Z_l}}(\mathcal{O}_{Z_k}, N) = 0 \text{ for all } k \geqslant 0 \text{ and} \\ i > 0. \end{array} \right\} \tag{9}$$

and similarly for $M'(X_l)$.

By definition the pullbacks $M'(Z_l) \to M'(Z_k)$ and $M'(X_l) \to M'(X_k)$ are exact, so that we can form $\operatorname{holim} \Omega BQM'(Z.)$ and $\operatorname{holim} \Omega BQM'(X.)$. If $Z.$ and $X.$ are degenerate above a certain simplicial dimension there are only finitely many conditions involved in (9). By imposing the conditions one by one and using the resolution theorem [25, Theorem 3, Corollary 3, p. 27] one sees that each $\Omega BQM'(Z_l)$ is weakly equivalent to $\Omega BQM(Z_l)$ where the latter is computed using all coherent sheaves on Z_l, and similarly for X_l. If $Z.$ is regular, then this is again weakly equivalent to $\Omega BQP(Z.)$ by the resolution theorem [25].

Now let be given a closed immersion $Z. \to X.$ of pointed simplicial schemes such that all diagrams

$$\begin{array}{ccc} Z_l & \xrightarrow{i_l} & X_l \\ \downarrow & & \downarrow {\scriptstyle j_{kl}} \\ Z_k & \xrightarrow{i_k} & X_k \end{array}$$

satisfy conditions (TC1) and (TC2). From this we get a push forward $M'(Z_l) \to M'(X_l)$ with M' defined in (9). Therefore we get a map $\operatorname{holim} \Omega BQM'(Z.) \xrightarrow{i_*} \operatorname{holim} \Omega BQM'(X.)$.

Suppose that $Z. \to X.$ satisfies (TC1) and (TC2), and let $U.$ be the localization of $X.$ at $Z.$ at all scheme components. Then by Quillen's localization theorem [25, Theorem 3, Corollary 3, p. 27] and the definition for the $*$ part, we have a homotopy fibration

$$\operatorname{holim} \Omega BQM'(Z.) \to \operatorname{holim} \Omega BQM'(X.) \to \operatorname{holim} \Omega BQM'(U.). \tag{10}$$

Now suppose in addition that $X.$ is regular. Let $K \to K^\sim$ be a fibrant resolution of K in sT. Then for a scheme X there is a natural map $\Omega BQP(X) \to K^\sim(X)$ inducing an isomorphism on homotopy groups. (This is actually more complicated (see [14]), but in order to simplify the diagrams somewhat we pretend it is one map. Because the construction of the weak equivalence is functorial this does not do any harm.) If we let $\operatorname{Hom}_{Z.}(X., K^\sim)$ be the homotopy fibre of $\operatorname{Hom}(X., K^\sim) \to \operatorname{Hom}(U., K^\sim)$ (which is represented by $\operatorname{Hom}(C(X., U.), K^\sim)$) we get a commutative diagram

$$\begin{array}{ccccc} \operatorname{Hom}_Z(X., K^\sim) & \longrightarrow & \operatorname{Hom}(X., K^\sim) & \longrightarrow & \operatorname{Hom}(U., K^\sim) \\ \uparrow & & \uparrow & & \uparrow \\ \text{Fibre} & \longrightarrow & \operatorname{holim} \Omega BQP(X.) & \longrightarrow & \operatorname{holim} \Omega BQP(U.) \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{holim} \Omega BQM'(Z.) & \longrightarrow & \operatorname{holim} \Omega BQM'(X.) & \longrightarrow & \operatorname{holim} \Omega BQM'(U.) \end{array} \tag{11}$$

Because all the vertical maps in the middle and on the right are homotopy equivalences by [25] and [8], the vertical maps on the left are homotopy equivalences too. Introducing the notation

$$K'_n(Z.) = \pi_n(\operatorname{holim} \Omega BQM'(Z.))$$
$$H_{Z.}^{-m}(X., K) = H^{-m}(X., U., K)$$

diagram (11) gives an isomorphism

$$K'_m(Z.) \to H_{Z.}^{-m}(X., K). \tag{12}$$

If $Y. \to X.$ is another regular closed subscheme of $X.$ containing $Z.$, with $Z. \to Y.$ satisfying (TC1) and (TC2), we also have an isomorphism $K'_m(Z.) \to H_{Z.}^{-m}(Y., K)$, and we can combine this with the one for $X.$ to get

$$i_*\colon H_{Z.}^{-m}(Y., K) \to H_{Z.}^{-m}(X., K) \tag{13}$$

to which we shall refer as proper push forward. Those maps satisfy an obvious property for composition $Z_. \to Y' \to Y_. \to X_.$ if all conditions are met.

If we have $Z_. \to Z'_. \to X_.$ with both $Z_. \to X_.$ and $Z'_. \to X_.$ cartesian, we have a long exact sequence

$$\cdots \to H_{Z_.}^{-m}(X_., K) \to H_{Z'_.}^{-m}(X_., K) \to H_{Z'_.\setminus Z_.}^{-m}(X_. \setminus Z_., K) \to H_{Z_.}^{-m+1}(X_., K) \to \cdots.$$

If we have $Z_. \to Z'_. \to Y_. \to X_.$ these maps fit into a commutative diagram

$$\begin{array}{ccccccc} \longrightarrow H_{Z'_.}^{-m}(Y_., K) & \longrightarrow & H_{Z'_.\setminus Z_.}^{-m}(Y_. \setminus Z_., K) & \longrightarrow & H_{Z_.}^{-m+1}(Y_., K) \longrightarrow \\ \downarrow i_* & & \downarrow i_* & & \downarrow i_* \\ \longrightarrow H_{Z'_.}^{-m}(X_., K) & \longrightarrow & H_{Z'_.\setminus Z_.}^{-m}(X_. \setminus Z_., K) & \longrightarrow & H_{Z_.}^{-m+1}(X_., K) \longrightarrow \end{array} \tag{14}$$

Assume we have $Z_. \to Y_. \to X_.$ closed immersions satisfying (TC1) and (TC2), with $Y_.$ and $X_.$ regular. We have an isomorphism

$$H_{Z_.}^{-m}(Y_., K) \xrightarrow{i_*} H_{Z_.}^{-m}(X_., K).$$

After tensoring with $\mathbb{Q}$ both groups can be decomposed by (7), and we want to compare the different decompositions under this push forward. We want to apply this only under simplifying conditions, so that we can prove a special case of a Riemann-Roch theorem for the K-theory of (pointed) simplicial schemes. The reader should bear in mind that the Riemann-Roch theorem for closed immersions $Z \to Y \to X$ is largely a statement about the action of $K_0(Y)$ on $K_n^Z(Y)$, using the deformation to the normal cone, cf. [29]. In the simplicial context, this gets replaced – under suitable assumptions – with the action of $K_0(Y_0)$ on $K_n^{Z_.}(Y_.)$, if the deformation to the normal cone can still be carried out.

DEFINITION 2.2. We will say that $Y_. \to X_.$ is defined by an effective cartier divisor if $Y_l \to X_l$ is defined by an effective cartier divisor meeting all $X_k \to X_l$ transversally. We will say that $Y_. \to X_.$ is defined by a system of effective cartier divisors if this can be written as $Y_. \to Y_{1.} \to \cdots \to Y_{n.} \to X_.$ with each of those inclusions defined by an effective cartier divisor.

We recall some facts about chern characters and todd classes, for which we refer to [29]. Let X_0 be regular and quasi projective over a field. Let $Y_0 \to X_0$ be a closed immersion of regular schemes, of codimension d. Let $F^{\cdot}K_0(X_0)$ be the usual Grothendieck filtration on $K_0(X_0)$. Let $Gr^{\cdot}K_0(X_0) = \oplus_n F^n K_0(X_0)/F^{n+1}K_0(X_0)$. Let N be the normal bundle of Y_0 in X_0, and let $td(N^\vee) \in Gr^{\cdot}K_0(X_0)_{\mathbb{Q}}$ be the usual todd class. Let ch: $K_0(X_0)_{\mathbb{Q}} \to Gr^{\cdot}K_0(X_0)_{\mathbb{Q}}$ be the usual chern character. It is well known that ch is an isomorphism.

PROPOSITION 2.3. *Assume $Z_.$, $Y_.$ and $X_.$ are pointed simplicial schemes over a field k. Suppose $Z_. \to Y_.$ and $Y_. \to X_.$ are two closed immersions over k satisfying (TC1) and (TC2). Suppose that $Y_.$ and $X_.$ are regular, and that $Y_.$ is defined in $X_.$ by a system* *of effective cartier divisors. Moreover, suppose that every simplical map $X_{l+1} \to X_l$ on irreducible components is either the identity, or the inclusion of the zero locus of an effective cartier divisor. Let d be the codimension of Y_0 in X_0, and hence the codimension of Y_l in X_l for all l. Then the map*

$$\begin{aligned} i_\#\colon\ H_{Z.}^{-n}(Y., K)_{\mathbb{Q}} &\to H_{Z.}^{-n}(X., K)_{\mathbb{Q}} \\ y &\mapsto i_*(ch^{-1}(td(N^\vee))y) \end{aligned}$$

maps $H_{Z.}^{-n}(Y., K)_{\mathbb{Q}}^{(j)}$ to $H_{Z.}^{-n}(X., K)_{\mathbb{Q}}^{(j+d)}$.

Remark 2.4. We will usually apply this proposition under the following conditions. Suppose that $Z. \to Y. \to X.$ are two cartesian closed immersions of pointed simplicial schemes, with $Y.$ and $X.$ regular, and $Y. \to X.$ defined by a system of effective cartier divisors. Assume that for all $k \geqslant 0$ $X_{k+1} \to X_k$ on the irreducible components is either the identity, or is defined by an effective cartier divisor x. If x restricted to Y_k remains a cartier divisor, and also when restricted to Z_k, defining Y_{k+1} resp. Z_{k+1}, then (TC1) and (TC2) are satisfied.

Proof of Proposition 2.3. We follow [29] closely. In order to stress the analogy we will write $K_n^{Z.}(Y.)$ for $H_{Z.}^{-n}(Y., K)$ and $K_n^{Z.}(X.)$ for $H_{Z.}^{-n}(X., K)$ and similarly with weights, so that we want to prove that $y \mapsto i_*(ch^{-1}(td(N^\vee))y)$ maps $K_n^{Z.}(Y.)_{\mathbb{Q}}^{(j)}$ to $K_n^{Z.}(X.)_{\mathbb{Q}}^{(j+d)}$.

Let μ be a natural transformation of λ-rings such that $\mu(0) = 0$. Let N be the normal bundle of Y_0 in X_0, and write N for the class $[N] \in K_0(Y_0)$. As in [29 Lemma 1.1, p. 124] for every $y \in K_n^{Z.}(Y.)$ there exists an element $\mu(N, y) \in K_n^{Z.}(Y.)$ which is a universal polynomial in the $\lambda(N)$ and the $\lambda(y)$ with integer coefficients, depending only on μ, such that $\mu(\lambda_{-1}(N)y) = \lambda_{-1}(N)\mu(N, y)$. We need

PROPOSITION 2.5. *The diagram*

$$\begin{array}{ccc} K_n^{Z.}(Y.) & \xrightarrow{\mu(N,\cdot)} & K_n^{Z.}(Y.) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ K_n^{Z.}(X.) & \xrightarrow{\mu} & K_n^{Z.}(X.) \end{array}$$

commutes, i.e.

$$\mu(i_*(y)) = i_*(\mu(N, y)) \tag{15}$$

for every $y \in K_n^{Z.}(Y.)$.

Remark 2.6. As in [29], this implies that with $\theta^k(N) \overset{def}{=} \psi^k(N, 1)$ in $K_0(Y_0)$, $\psi^k(N, y) = \theta^k(N)\psi^k(y)$, and hence

$$\psi^k(i_*(y)) = i_*(\theta(N)\psi^k(y)). \tag{16}$$

Proof of Proposition 2.5. We need a lemma, cf. [29, Lemma 2.2, p. 127].

LEMMA 2.7. *Suppose*

$$\begin{array}{ccccc} Z_\cdot & \longrightarrow & Y_\cdot & \xrightarrow{i} & X_\cdot \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Z'_\cdot & \longrightarrow & Y'_\cdot & \xrightarrow{i'} & X'_\cdot \end{array}$$

is a cartesian diagram with closed immersions of pointed simplicial schemes, with $Y_\cdot$, $X_\cdot$, $Y'_\cdot$ *and* $X'_\cdot$ *all regular, and suppose all horizontal maps satisfy* (TC1) *and* (TC2). *Suppose moreover that for every* k *the maps* $Y'_k \to X'_k$ *and* $X_k \to X'_k$ *are tor independent, and similarly for the maps* $Z'_k \to Y'_k$ *and* $Y_k \to Y'_k$. *Finally, suppose that* $Z_\cdot \to Z'_\cdot$ *is defined by an effective cartier divisor. Then the diagram*

$$\begin{array}{ccc} K_n^{Z'_\cdot}(Y'_\cdot) & \xrightarrow{f^*} & K_n^{Z_\cdot}(Y_\cdot) \\ {\scriptstyle i'_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ K_n^{Z'_\cdot}(X'_\cdot) & \xrightarrow{g^*} & K_n^{Z_\cdot}(X_\cdot) \end{array}$$

commutes.

Proof. Let $F_{Z_\cdot}(Y_\cdot)$ be the fibre of $\operatorname{holim}\Omega BQP(Y_\cdot) \to \operatorname{holim}\Omega BQP(Y_\cdot \setminus Z_\cdot)$ and similarly for $Z'_\cdot$ and $Y'_\cdot$. We have a diagram

$$\begin{array}{ccc} \mathrm{Hom}_{Z'_\cdot}(Y'_\cdot, K^\sim) & \longrightarrow & \mathrm{Hom}_{Z_\cdot}(Y_\cdot, K^\sim) \\ \uparrow & & \uparrow \\ F_{Z'_\cdot}(Y'_\cdot) & \longrightarrow & F_{Z_\cdot}(Y_\cdot) \\ \downarrow & & \downarrow \\ \operatorname{holim}\Omega BQM'(Z'_\cdot) & \longrightarrow & \operatorname{holim}\Omega BQM'(Z_\cdot) \\ \uparrow & & \uparrow \\ F_{Z'_\cdot}(X'_\cdot) & \longrightarrow & F_{Z_\cdot}(X_\cdot) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{Z'_\cdot}(X'_\cdot, K^\sim) & \longrightarrow & \mathrm{Hom}_{Z_\cdot}(X_\cdot, K^\sim) \end{array}$$

The upper and lower squares commute by functoriality, and that the middle squares commute can be seen as in [29, Lemma 2.2, p. 127]. Because the arrow in the top row defines f^*, the bottom row defines g^*, and the two columns define i_* and i'_* this proves the lemma.

For a pointed simplicial scheme $S_\cdot$, let $\mathbb{A}^1_{S_\cdot}$ be the pointed simplicial scheme obtained by taking the affine line over all scheme components. We will use scheme terminology from now on when speaking about pointed simplicial schemes, where the constructions are supposed to apply only to the scheme components of the pointed simplicial schemes involved, leaving $*$ untouched.

Consider $i\colon \mathbb{A}^1_{Y_\cdot} \to \mathbb{A}^1_{X_\cdot}$. Let $0, 1\colon Y_\cdot \to \mathbb{A}^1_{Y_\cdot}$ be the sections at zero and one. Let $W_\cdot \to \mathbb{A}^1_{X_\cdot}$ be the blow up of $\mathbb{A}^1_{X_\cdot}$ along the closed immersion $Y_\cdot \xrightarrow{0} \mathbb{A}^1_{Y_\cdot} \xrightarrow{i} \mathbb{A}^1_{X_\cdot}$. Because every irreducible component of X_l meets Y_0 transversally in X_0, this blow up is obtained from the blow up of $Y_0 \xrightarrow{0} \mathbb{A}^1_{Y_0} \xrightarrow{i} \mathbb{A}^1_{X_0}$ by pulling back to every component of $\mathbb{A}^1_{X_\cdot}$ via the unique map $X_l \to X_0$. Let N be the normal bundle of the embedding $Y_0 \to X_0$. Let $P_\cdot = \mathbb{P}(N_\cdot \oplus \mathcal{O}_{Y_\cdot})$ be the exceptional divisor. ($N_\cdot$ is the pullback of N to the components of $Y_\cdot$ via the unique map $Y_\cdot \to Y_0$.) $P_\cdot$ is a pointed simplicial scheme over $Y_\cdot$.

Because $Y_\cdot$ is defined in $\mathbb{A}^1_{Y_\cdot}$ by an effective cartier divisor, we get a closed immersion $\mathbb{A}^1_{Y_\cdot} \to W_\cdot$, and a cartesian diagram of closed immersions

$$\begin{array}{ccc} Y_\cdot & \xrightarrow{i'} & P_\cdot \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle j'} \\ \mathbb{A}^1_{Y_\cdot} & \longrightarrow & W_\cdot \end{array}$$

For the section $1\colon X_\cdot \to \mathbb{A}^1_{X_\cdot}$ we get, as the blow up $W_\cdot \to \mathbb{A}^1_{X_\cdot}$ is an isomorphism away from its center, a cartesian diagram

$$\begin{array}{ccc} Y_\cdot & \xrightarrow{i} & X_\cdot \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle j} \\ \mathbb{A}^1_{Y_\cdot} & \longrightarrow & W_\cdot \end{array}$$

We have to check (TC1) and (TC2) for the embeddings $\mathbb{A}^1_{Y_\cdot} \to W_\cdot$ and $Y_\cdot \to P_\cdot$. On every irreducible component the map $X_{k+1} \to X_k$ is either the identity or the inclusion of the zero locus of an effective cartier divisor, and Y_k meets X_{k+1} transversally in X_k. Hence the effective cartier divisor defines an effective cartier divisor on W_k defining W_{k+1}, which restricts to an effective cartier divisor on $\mathbb{A}^1_{Y_k}$, defining $\mathbb{A}^1_{Y_{k+1}}$. This implies that the maps $\mathbb{A}^1_{Y_k} \to W_k$ and $W_{k+1} \to W_k$ are tor independent. The inclusion $Y_\cdot \to P_\cdot$ is a section of a vector bundle, so this will certainly satisfy the conditions.

Hence we can apply Lemma 2.7 to the above diagrams (completed with the inclusions of $Z_\cdot$ and $\mathbb{A}^1_{Z_\cdot}$), and find the following. (For the necessary tor independence in each degree see [29, p. 130].) For the section $0\colon X_\cdot \to \mathbb{A}^1_{X_\cdot}$ we get a commutative diagram

$$\begin{array}{ccc} K_n^{Z_\cdot}(Y_\cdot) & \xrightarrow{i'_*} & K_n^{Z_\cdot}(P_\cdot) \\ {\scriptstyle 0^*}\uparrow & & \uparrow{\scriptstyle j'^*} \\ K_n^{\mathbb{A}^1_{Z_\cdot}}(\mathbb{A}^1_{Y_\cdot}) & \longrightarrow & K_n^{\mathbb{A}^1_{Z_\cdot}}(W_\cdot) \end{array}$$

and similarly for the section at 1,

$$\begin{array}{ccc} K_n^{Z_\cdot}(Y_\cdot) & \xrightarrow{i_*} & K_n^{Z_\cdot}(X_\cdot) \\ {\scriptstyle 1^*}\uparrow & & \uparrow{\scriptstyle j^*} \\ K_n^{\mathbb{A}^1_{Z_\cdot}}(\mathbb{A}^1_{Y_\cdot}) & \longrightarrow & K_n^{\mathbb{A}^1_{Z_\cdot}}(W_\cdot) \end{array}$$

Because the pointed simplicial schemes $Y_\cdot$ and $Y_\cdot \setminus Z_\cdot$ are regular and noetherian, the K-theory satisfies the homotopy property, and hence

$$s^*\colon\ K^{\mathbb{A}^1_{Z_\cdot}}(\mathbb{A}^1_{Y_\cdot}) \xrightarrow{\sim} K^{Z_\cdot}(Y_\cdot)$$

for any section $s\colon\ Y_0 \to \mathbb{A}^1_{Y_0}$. Because the maps j^* and j'^* are λ-morphisms, it follows that it suffices to prove the formula (15) for $K^{Z_\cdot}(Y_\cdot) \xrightarrow{i'_*} K^{Z_\cdot}(P_\cdot)$, cf. [29, p. 128].

Write $X'_\cdot$ for $P_\cdot$, and let $p\colon\ X'_\cdot \to Y_\cdot$ be the projection. Let $Z'_\cdot = p^{-1}(Z_\cdot)$.

LEMMA 2.8. *Let $s\colon\ Y_\cdot \to X'_\cdot$ be a section. Then the composed map $K_n^{Z_\cdot}(Y_\cdot) \xrightarrow{s_*} K_n^{Z_\cdot}(X'_\cdot) \to K_N^{Z'_\cdot}(X'_\cdot)$ is injective.*

Proof. In fact, we will show the following. Let $z \in K_0(X'_0)$ be the class of the canocical line bundle $\mathcal{O}_P(-1)$. Because the pullback of z to X'_l represents the canonical class of $\mathcal{O}_{X'_l}(-1)$, z acts on $K_n^{Z'_\cdot}(X'_\cdot)$. We then claim that we have an isomorphism

$$\begin{aligned} (K_n^{Z_\cdot}(Y_\cdot))^{d+1} &\xrightarrow{\sim} K_n^{Z'_\cdot}(X'_\cdot) \\ \{a_i\}_{0\leqslant i\leqslant d} &\mapsto \sum_{i=0}^{d} z^i \cdot p^*(a_i). \end{aligned} \tag{17}$$

Because of the long exact sequence

$$\cdots \to K_n(X'_\cdot) \to K_n(X'_\cdot \setminus Z'_\cdot) \to K_{n-1}^{Z'_\cdot}(X'_\cdot) \to K_{n-1}(X'_\cdot) \to \cdots$$

and similarly for $K_n^{Z_\cdot}(Y_\cdot)$, it suffices to prove (17) for $K_n(X'_\cdot)$ and $K_n(X'_\cdot \setminus Z'_\cdot)$. There exists a spectral sequence [14, 1.2.3]

$$K_{-q}(X'_p) \Rightarrow K_{-p-q}(X'_\cdot)$$

and similarly for $Y_\cdot$. The construction of this spectral sequence is compatible with the action of $K_0(X'_0)$. It suffices therefore to check that the map $K_n(Y_k)^{d+1} \xrightarrow{\sim} K_n(X'_k)$ given by $\{a_i\} \mapsto \sum_{i=0}^d z^i \cdot p^*(a_i)$ induces an isomorphism on each component, which is the statement of [25, p. 58, Theorem 2.1]. In the same way one proves (17) for $X'_\cdot \setminus Z'_\cdot$. So the statement of the lemma comes down to checking that $s_*\colon\ K_0(Z_0) \to K_0(X'_0)$ is injective, which is well known.

Because the map $K_n^{Z_\cdot}(X'_\cdot) \to K_n^{Z'_\cdot}(X'_\cdot)$ is a λ-morphism, this lemma reduces us to proving the assertion in Proposition 2.5 for the composition $K_n^{Z_\cdot}(Y_\cdot) \to K_n^{Z_\cdot}(X'_\cdot) \to K_n^{Z'_\cdot}(X'_\cdot)$. Consider the cartesian diagrams

$$\begin{array}{ccc} Z_\cdot & \xrightarrow{i'} & Z'_\cdot \\ \downarrow & & \downarrow \\ Y_\cdot & \xrightarrow{i'} & X'_\cdot \end{array} \quad \text{and} \quad \begin{array}{ccc} Z'_\cdot & \xrightarrow{p} & Z_\cdot \\ \downarrow & & \downarrow \\ X'_\cdot & \xrightarrow{p} & Y_\cdot \end{array}$$

From this we get pullbacks

$$i'^*\colon K^{Z'_\cdot}(X'_\cdot) \to K^{Z_\cdot}(Y_\cdot)$$
$$p^*\colon K^{Z_\cdot}(Y_\cdot) \to K^{Z'_\cdot}(X'_\cdot)$$

satisfying $i'^* \circ p^* = id$, so $i'^*\colon K^{Z'_\cdot}(X'_\cdot) \to K^{Z_\cdot}(Y_\cdot)$ is surjective.

We need one more lemma.

LEMMA 2.9 (Projection formula). *The diagram*

$$\begin{array}{ccc} K_0(Y_0)\times K_n^{Z_\cdot}(Y_\cdot) & \longrightarrow & K_n^{Z_\cdot}(Y_\cdot) \\ {\scriptstyle i'_*}\downarrow \qquad \uparrow{\scriptstyle i'^*} & & \downarrow{\scriptstyle i'_*} \\ K_0(X'_0)\times K_n^{Z'_\cdot}(X'_\cdot) & \longrightarrow & K_n^{Z'_\cdot}(X'_\cdot) \end{array}$$

commutes.

Proof. Let $C_\cdot = C(Y_\cdot \setminus Z_\cdot \to Y_\cdot)$ and $C' = C(X'_\cdot \setminus Z'_\cdot \to X'_\cdot)$. Then with our isomorphisms in (12) this comes down to checking the commutativity of the diagram

$$\begin{array}{ccc} \operatorname{holim} \Omega BQP(Y_0)\times \operatorname{holim} \Omega BQP(C) & \longrightarrow & \operatorname{holim} \Omega BQP(C_\cdot) \\ \downarrow \qquad\qquad\qquad \uparrow & & \downarrow \\ \operatorname{holim} \Omega BQPM'(X'_0)\times \operatorname{holim} \Omega BQP(C_\cdot) & \longrightarrow & \operatorname{holim} \Omega BQM'(C'_\cdot) \end{array}$$

This is immediate from the fact that for the closed immersion i' in every simplicial degree we have $i'_*(M \otimes i'^*(N)) \cong i'_*(M) \otimes N$ canonically, for $M \in P(Z_l)$, and $N \in M'(X_l)$, where M' is defined in (9).

According to [29, p. 132], we have $i'_*(1) = \lambda_{-1}(H)$ in $K_0(X'_0)$, where $i'^*(H) = N$. So if we let, for $y \in K^{Z_\cdot}(Y_\cdot)$, $x \in K^{Z'_\cdot}(X'_\cdot)$ be such that $i'^*(x) = y$, we find as in [29, p. 133]

$$\begin{aligned} \mu(i'_*(y)) &= \mu(i'_*(i'^*(x))) \\ &= \mu(i'_*(1)x) \\ &= \mu(\lambda_{-1}(H)x) \\ &= \lambda_{-1}(H)\mu(H,x) \\ &= i'_*(1)\mu(H,x) \\ &= i'_*(i'^*(\mu(H,x))) \\ &= i'_*(\mu(i'^*(H),i'^*(x))) \\ &= i'_*(\mu(N,y)) \end{aligned}$$

because i'^* is a λ-morphism. This proves Proposition 2.5.

We are now in a position to prove Proposition 2.3. According to [29, p. 137, Lemma 2.1] we have an identity in $K_0(Y_0)_{\mathbb{Q}}$ for a locally free module of rank d,

$$\theta^k(N)\psi^k(ch^{-1}(td(N^\vee))) = k^d ch^{-1}(td(N^\vee)).$$

From this we deduce as in [29, p. 139, Lemma 2.2] the statement of Proposition 2.3. For an element y of $K^{Z\cdot}(Y.)^{(j)}_{\mathbb{Q}}$ we have that, using (16)

$$\begin{aligned}\psi^k(i_*(ch^{-1}(td(N^\vee))y)) &= i_*(\theta^k(N)\psi^k(ch^{-1}(td(N^\vee)))\psi^j(y)) \\ &= i_*(k^d ch^{-1}(td(N^\vee))k^j y) \\ &= k^{j+d} i_*(ch^{-1}(td(N^\vee))y),\end{aligned}$$

so that $i_\#(y) \in K^{Z\cdot}(X.)^{(d+j)}_{\mathbb{Q}}$. This finishes the proof of Proposition 2.3.

Remark 2.10. This is the only place where one uses the fact that we tensored with $\mathbb{Q}$. If we lift a multiple of $ch^{-1}(td(N^\vee))$ to $\alpha \in K_0(Y_0)$ such that $\theta^k(N)\psi^k(\alpha) = k^d\alpha$ in $K_0(Y_0)$, then we get a map from $K^{Z\cdot}(Y.)^{(d+j)}$ to $K^{Z\cdot}(X.)^{(d+j)}$ that induces a multiple of the old map after tensoring with $\mathbb{Q}$.

2.3. MULTI-RELATIVE K-THEORY

We want to apply the material of Sections 2.1 and 2.2 in the following situation. Let $Y_1, \ldots, Y_s$ be closed subschemes of a finite dimensional noetherian scheme X, and assume that X and all finite intersections of the Y_i's are regular. We will write $X.$ for the constant simplicial sheaf represented by X together with a disjoint basepoint, and similarly for the other schemes. Let, inductively, $C(X, \{Y_1, \ldots, Y_s\})$ be defined by (see (2))

$$\begin{aligned}C(X,\{Y_1\}) &= C(Y_1., X.) \\ C(X,\{Y_1,\ldots,Y_{n+1}\}) &= C(C(Y_{n+1},\{Y_{1,n+1},\ldots,Y_{n,n+1}\}), \\ &\qquad C(X,\{Y_1,\ldots,Y_n\}))\end{aligned} \tag{18}$$

where $Y_{i,j} = Y_i \cap Y_j$. Explicitly, the space $C.$ one finds for $X, Y_1, \ldots, Y_s$ is as follows.

$$C_n = * \coprod \coprod_{\alpha_1,\ldots,\alpha_s} Y_{\alpha_1,\ldots,\alpha_s}$$

with $\alpha_i \in \{(0,\ldots,0),(0,\ldots,0,1),\ldots,\overbrace{(0,1,\ldots,1)}^{n+1}\}$, $Y_{\alpha_1\ldots,\alpha_s} = \bigcap_{\alpha_i \neq (0,\ldots,0)} Y_i$ and $\bigcap_\emptyset Y_i = X$. The boundary and degeneracy maps are the natural maps coming from the inclusions and the identity, which we get by deleting or doubling the i-th place in the zeroes and ones, with the convention that we identify $Y_{\alpha_1\ldots\alpha_s}$ with $*$ if at least one of the α's consists of only 1's.

For $n \geqslant 0$ we define $K_n(X, \{Y_1, \ldots, Y_s\}) = H^{-n}(C(X, \{Y_1, \ldots, Y_s\}), K)$ and $K_n^{(i)}(X, \{Y_1, \ldots, Y_s\}) = H^{-n}(C(X, \{Y_1, \ldots, Y_s\}), K)_{\mathbb{Q}}^{(i)}$. So we get a long exact sequence for $n \geqslant 1$:

$$\begin{aligned} \cdots &\to K_n(X, \{Y_1, \ldots, Y_s\}) \to K_n(X, \{Y_1, \ldots, Y_{s-1}\}) \\ &\to K_n(Y_s, \{Y_{1,s}, \ldots, Y_{s-1,s}\}) \to K_{n-1}(X, \{Y_1, \ldots, Y_s\}) \to \cdots \end{aligned} \tag{19}$$

and similar with weights.

We define maps

$$C(X, \{Y_1, \ldots, Y_{r+s}\}) \to C(X, \{Y_1, \ldots, Y_s\}) \wedge C(X, \{Y_{s+1}, \ldots, Y_{s+r}\})$$

by the diagonal embedding $Y_{\alpha_1, \ldots, \alpha_{r+s}} \to Y_{\alpha_1, \ldots, \alpha_s} \times Y_{\alpha_{s+1}, \ldots, \alpha_{r+s}}$, and identifying $.. \times *$ and $* \times ..$ with $*$ as element of the right hand side. Those maps satisfy an obvious associativity property, and composing this map with the map (5) we get, considering (8), a multiplication

$$\begin{aligned} &K_m^{(i)}(X, \{Y_1, \ldots, Y_s\}) \times K_n^{(j)}(X, \{Y_{s+1}, \ldots, Y_{s+r}\}) \\ &\quad \to K_{m+n}^{(i+j)}(X, \{Y_1, \ldots, Y_{s+r}\}) \end{aligned}$$

Suppose $X = W_0 \supset W_1 \supset \ldots \supset W_{n-1} \supset W_n \supset W_{n+1} = \emptyset$, is a stratification of X with $W_i \subset X$ closed of codimension i. Let $U_i = X \setminus W_i$, $C = C(X, \{Y_1, \ldots, Y_s\})$, and write $C \cap U_i$ for the pointed simplicial scheme obtained by intersecting all scheme components of C with U_i. Then we have long exact sequences for $m \geqslant 0$:

$$\begin{aligned} \cdots &\to H_{W_{i-1}}^{-m-1}(C, K) \to H_{W_{i-1} \setminus W_i}^{-m-1}(C \cap U_i, K) \\ &\to H_{W_i}^{-m}(C, K) \to H_{W_{i-1}}^{-m}(C, K) \to \cdots \end{aligned}$$

and similarly for the weight j-part, after tensoring with $\mathbb{Q}$. Define $H_{W_i}^1(C, K)$ as the cokernel of the map $H_{W_{i-1}}^0(C, K) \to H_{W_{i-1} \setminus W_i}^0(C \cap U_i, K)$, and let all other $H^n = 0$ for $n \geqslant 0$. Letting

$$\begin{aligned} D^{p,q} &= H_{W_p}^{p+q}(C, K) \\ E^{p,q} &= H_{W_p \setminus W_{p+1}}^{p+q}(C \setminus W_{p+1}, K) \end{aligned}$$

we get an exact couple

$$\begin{array}{ccc} \bigoplus_{p,q} D_{p,q} & \longrightarrow & \bigoplus_{p,q} D_{p,q} \\ & \nwarrow \quad \swarrow & \\ & \bigoplus_{p,q} E_{p,q} & \end{array}$$

and hence a spectral sequence

$$E_1^{p,q} = H_{W_p \setminus W_{p+1}}^{p+q}(C \cap U_{p+1}, K)_{\mathbb{Q}}^{(j)} \Rightarrow H^{p+q}(C, K)_{\mathbb{Q}}^{(j)}.$$

Suppose that all schemes are defined over a field. If all scheme components of $C \cap (W_p \setminus W_{p+1})$ are regular and satisfy the transversality condition in Proposition 2.3, we can apply Proposition 2.3 to get an isomorphism $H^{-m}_{W_p\setminus W_{p+1}}(C\cap U_{p+1},K)^{(j)}_{\mathbb{Q}} \cong H^{-m}(C\cap(W_p\setminus W_{p+1}),K)^{(j-p)}_{\mathbb{Q}}$. Writing

$$K_m^{(j-p)}(W_p\setminus W_{p+1},\{Y_1,\dots,Y_s\}) = H^{-m}(C\cap(W_p\setminus W_{p+1}),K)^{(j-p)}_{\mathbb{Q}},$$

this becomes

$$E_1^{p,q} = K^{(j-p)}_{-p-q}(W_p\setminus W_{p+1},\{Y_1,\dots,Y_s\}) \Rightarrow K^{(j)}_{-p-q}(X,\{Y_1,\dots,Y_s\}). \tag{20}$$

Remark 2.11. By the compatibility of the push forward (14), we have maps of spectral sequences under suitable conditions.

For $\sigma\in S_n$, we have a canonical isomorphism

$$C(X,\{Y_1,\dots,Y_n\}) \xrightarrow{\sim} C(X,\{Y_{\sigma(1)},\dots,Y_{\sigma(n)}\}).$$

Hence, if ϕ is an automorphism of X permuting the Y_i's, then we get an action of ϕ on $C(X,\{Y_1,\dots,Y_n\})$, and hence on $K_n^{(j)}(X,\{Y_1,\dots,Y_n\})$. If all the W_j are invariant under the action of ϕ we get an action of ϕ on the spectral sequence (20) too.

We need one more lemma for the construction of the complexes in the next section.

LEMMA 2.12. *If Y is a regular noetherian finite dimensional scheme, $X = \mathbb{P}^1\setminus\{t=1\}$, and $X^n_Y = X\times_Y\times\cdots\times_Y X$, then the action of S_n on $K_p(X^n_Y;\{\{t_1=0,\infty\},\dots,\{t_n=0,\infty\}\})$ induced from permuting the t-coordinates, is alternating.*

Proof. Let $C = C(X^n_Y;\{\{t_1=0,\infty\},\dots,\{t_n=0,\infty\}\})$. There is a spectral sequence ([14, 1.2.3])

$$K_{-q}(C_p) \Rightarrow K_{-p-q}(C)$$

where $K_{-q}(*)=0$. Because all scheme components of C are products of the regular noetherian scheme Y with affine spaces, $K_{-q}(C_p)$ is isomorphic to a direct sum of copies of $K_{-q}(Y)$. Considering the configuration of the components of C, one sees that the associated chain complex of the E^2-term of this spectral sequence is therefore the same as the chain complex that computes the reduced cohomology of an n-dimensional sphere, tensored with $K_{-q}(Y)$. (It is reduced because the contribution of one of the simplices has been replaced with zero.) For this chain complex it is well known that the alternating part is the only contributing to the cohomology, so the same must hold for the spectral sequence and hence for the limit $K_p(X^n_Y;\{\{t_1=0,\infty\},\dots,\{t_n=0,\infty\}\})$.

2.4. REGULATOR MAPS

Gillet and Soulé also define regulators for suitable cohomology theories. Let V be a category of noetherian finite dimensional schemes over a base S, and let T be the topos of sheaves on the big Zariski site of V. Let $\Gamma = \{\Gamma^*(i), i\in\mathbb{Z}\}$ be a graded complex of Abelian sheaves in T satisfying the axioms of [13, 1.1]. If A is a sheaf of homological chain complexes, let $\Gamma(A)$ be the sheaf obtained by applying the Dold-Puppe construction to the homological chain complex $A(U)$ for every U (see [11]).

Gillet and Soulé define chern classes which give rise to regulator maps

$$H^{-m}(X,K)^{(i)} \to H^{-m}(X,K(\Gamma(i),\delta i)) \tag{21}$$

for $m \geqslant 0$. Here $\delta = 1$ or 2, depending on the cohomology theory, and $K(\Gamma(i),\delta i)$ is the simplicial Abelian sheaf obtained by applying the Dold–Puppe construction to the homological complex $\cdots \to \Gamma(i)_{\delta i-2} \to \Gamma(i)_{\delta i-1} \to \mathrm{Ker}(d_{\delta i})$ where the last group has degree zero.

There is a total chern class ch_Γ defined at the level of sheaves, such that the diagram

$$\begin{array}{ccc} K^N \wedge K^N & \longrightarrow & K^M \\ \downarrow{\scriptstyle ch_\Gamma} & & \downarrow{\scriptstyle ch_\Gamma} \\ \left(\prod_{i\geqslant 0} K(\Gamma(i),\delta i)\otimes\mathbb{Q}\right) \wedge \left(\prod_{i\geqslant 0} K(\Gamma(i),\delta i)\otimes\mathbb{Q}\right) & \xrightarrow{\phi} & \left(\prod_{i\geqslant 0} K(\Gamma(i),\delta i)\otimes\mathbb{Q}\right) \end{array} \tag{22}$$

where ϕ is the map that induces multiplication on the Γ-cohomology, commutes in Ho sT for M large enough. Hence, for K-coherent spaces, the regulator map transforms the product in K-theory into the product in the cohomology theory.

For explicit computations we want to identify $[S^p \wedge X, \Gamma(A)]$ as follows, where A is a sheaf of homological chain complexes. For a space X, let $C_*(X)$ be the reduced chain complex of X, and let $N_*(X)$ be the normalized reduced chain complex of X. We have to introduce the Alexander–Whitney map

$$\begin{aligned} C_*(X\wedge Y) &\to C_*(X)\otimes C_*(Y) \\ X_n \wedge Y_n &\mapsto \sum_{i=0}^{n} d_{i+1}\cdots d_{n-1}d_n X_n \otimes d_0^i Y_n. \end{aligned}$$

The Alexander–Whitney map gives a quasi isomorphism between $C_*(X\wedge Y)$ and $C_*(X)\otimes C_*(Y)$. This is proved for the map of non-reduced chain complexes

$$C(X\times Y) \to C(X)\otimes C(Y),$$

in [22], but this statement follows easily. Furthermore, the Alexander–Whitney map satisfies an obvious associativity property for $C_*(X\wedge Y\wedge Z) \to C_*(X)\otimes C_*(Y)\otimes C_*(Z)$ ([22, p. 242, Proposition 8.7]). The Alexander–Whitney map factors through $C_*(X) \to N_*(X)$ to give

$$N_*(X\wedge Y) \to N_*(X)\otimes N_*(Y). \tag{23}$$

For two sheaves of Abelian chain complexes A and B in T, let $D_T[A,B]$ denote the maps in the derived category of Abelian chain complexes in T. Applying the Alexander–Whitney map repeatedly, and using the projection $C_*(S)$ onto $\{0,1\}$ in degree 1, we get isomorphisms

$$\begin{aligned} [S^p\wedge X,\Gamma(A)] &= D_T[N_*(S^p\wedge X), N(\Gamma(A))] = D_T[C_*(S^p\wedge X), A] \\ &= D_T[C_*(S^p)\otimes C_*(X), A] \xleftarrow{\sim} \cdots \\ &\xleftarrow{\sim} D_T[C_*S\otimes\ldots\otimes C_*(S)\otimes C_*(X), A] \\ &\xleftarrow{\sim} D_T[C_*(X)[p], A]. \end{aligned} \tag{24}$$

If we assume that all A_i are injective and the complex A is bounded below, these classes are represented by actual maps up to chain homotopy ([17, p. 67]). Using that $C_*(X)_p$ is the free Abelian group generated by X_p, one sees that the groups in (24) are computed by the p-th homology group of the complex $\mathfrak{C}.(X)$, with

$$\begin{aligned} \mathfrak{C}_q(X,A) &= \bigoplus_{t-s=q} \mathrm{Hom}(X_s, A_t) \\ d^{(s,t)} &= (-1)^{t-s}(d_s^{C_*(X)})^* + d_t^A, \end{aligned} \tag{25}$$

cf. [17, p. 64]. Here we identify $\{\phi_{st}\}$: $C_*(X) \to A$ with $\{\phi_{st}\} \in \mathrm{Hom}(X_s A_t)$ for $t-s$ odd, and with $\{(-1)^s \phi_{st}\} \in \mathrm{Hom}(X_s, A_t)$ for $t-s$ even. This yields an identification

$$[S^p \wedge X, \Gamma(A)] \cong H_p(\mathfrak{C}.(X,A)). \tag{26}$$

If we have $X \to Y$ a map of spaces with cone $C = C(X \to Y)$, we want to identify the maps in the long exact sequence

$$\begin{aligned} \cdots \to\ & H_{p+1}(\mathfrak{C}.(Y,A)) \to H_{p+1}(\mathfrak{C}.(X,A)) \xrightarrow{\phi} H_p(\mathfrak{C}.(C,A)) \\ \to\ & H_p(\mathfrak{C}.(Y,A)) \to \cdots \end{aligned} \tag{27}$$

corresponding to

$$\begin{aligned} \to [S^{p+1} \wedge Y, \Gamma(A)] \ \to\ & [S^{p+1} \wedge X, \Gamma(A)] \to [S^p \wedge C, \Gamma(A)] \\ \to\ & [S^p \wedge Y, \Gamma(A)] \to \cdots. \end{aligned}$$

All maps are induced from the maps in $X \to Y \to C$, except for ϕ. In order to identify this, we have to apply the Alexander-Whitney map repeatedly to get

$$C_*(S^{p+1} \wedge X) \to \cdots \to C_*(S) \otimes \cdots \otimes C_*(S) \otimes C_*(X)$$

and then project onto the $\{0,1\}$-component in each $C_*(S_1)$. But the Alexander–Whitney map is associative, from which is follows that this map is completely determined by the natural map $C \to S \wedge X$ followed by the Alexander–Whitney map and the projection $C_*(S) \to \mathbb{Z}[1]$. Looking at the explicit shape of the Alexander–Whitney map this means that the only non-zero component of this map in simplicial degree $s+1$ is

$$\{0, \underbrace{1, \ldots, 1}_{s+1}\} \times X_{s+1} \tag{28}$$

which is mapped to $d_0 X_{s+1}$ in chain complex degree s.

We want to say something about products. Suppose that A, B and C are homological chain complexes with a map $A \otimes B \to C$. This gives rise to a map $\Gamma(A) \wedge \Gamma(B) \xrightarrow{\phi} \Gamma(A \otimes B) \to \Gamma(C)$ where ϕ is such that $N_*(\phi)$: $(N_*(\Gamma(A) \wedge \Gamma(B)) \to N_*(A) \otimes N_*(B)$ is the Alexander–Whitney map. This gives rise to a multiplication

$$[S^p \wedge X, \Gamma(A)] \times [S^q \wedge Y, \Gamma(B)] \to [S^{p+q} \wedge X \wedge Y, \Gamma(C)]. \tag{29}$$

We want to identify this map in terms of our identification (24). For this, let $\tilde{\alpha}$ in $[S^p \wedge X, \Gamma(A)]$ be such that it corresponds to $\alpha \in [N_*(S^p \wedge X), A]$, and similarly for $\tilde{\beta}$ and $\beta \in [N_*(S^q \wedge Y), B]$. Then consider the following commutative diagram.

$$\begin{array}{ccccc}
S^p \wedge X \wedge S^q \wedge Y & \xrightarrow{\tilde{\alpha}\wedge\tilde{\beta}} & \Gamma(A) \wedge \Gamma(B) & \xrightarrow{\phi} & \Gamma(A \otimes B) \\
\downarrow & & \downarrow & & \downarrow \\
N_*(S^p \wedge X \wedge S^q \wedge Y) & \xrightarrow{N(\tilde{\alpha}\wedge\tilde{\beta})} & N(\Gamma(A) \wedge \Gamma(B)) & \xrightarrow{AW} & N(\Gamma(A \otimes B)) \\
{\scriptstyle AW}\downarrow & & {\scriptstyle AW}\downarrow & & \| \\
N_*(S^p \wedge X) \otimes N_*(S^q \wedge Y) & \xrightarrow{N(\tilde{\alpha})\otimes N(\tilde{\beta})} & A \otimes B & & A \otimes B
\end{array}$$

But $N(\tilde{\alpha}) = \alpha$ and similarly for β, so that the result in the first step of (24) is nothing but the composition of the Alexander–Whitney map and the tensor product of α and β.

To go down the rest of the identifications, we first have to use the projection $C_*(X) \to N_*(X)$, then apply the Alexander–Whitney map repeatedly and project down onto suitable subcomplexes. Because the Alexander–Whitney map factors through this projection (23), we can replace N_* with C_*. Because the Alexander–Whitney map is associative we have a commutative diagram

$$\begin{array}{ccccc}
C_*(S^p\wedge X)\otimes C_*(S^q\wedge Y) & \longrightarrow & C_*(S)^{\otimes p+q}\otimes C_*(X)\otimes C_*(Y) & \xrightarrow{\text{proj}\otimes\text{id}} & \\
{\scriptstyle AW}\uparrow & & {\scriptstyle \text{id}^{\otimes p+q}\otimes AW}\uparrow & & \\
C_*(S^p\wedge X\wedge S^q\wedge Y) & \xrightarrow{AW} & C_*(S)^{\otimes p+q}\otimes C_*(X\wedge Y) & \xrightarrow{\text{pro}\otimes\text{id}} &
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\text{proj}\otimes\text{id}} & \mathbb{Z}[1]^{p+q}\otimes C_*(X)\otimes C_*(Y) & \longrightarrow & (C_*(X)\otimes C_*(Y))[p+q] \\
& {\scriptstyle \text{id}^{\otimes p+q}\otimes AW}\uparrow & & {\scriptstyle AW}\uparrow \\
\xrightarrow{\text{proj}\otimes\text{id}} & \mathbb{Z}[1]^{p+q}\otimes C_*(X\wedge Y) & \longrightarrow & C_*(X\wedge Y)[p+q]
\end{array}$$

This shows that the product

$$[C_*(X)[p], A] \times [C_*(Y)[q], B] \to [C_*(X \wedge Y)[p+q], (A \otimes B)]$$

that corresponds to (29) under the identification (6) is given by the composition of

$$C_*(X \wedge Y)[p+q] \xrightarrow{AW} C_*(X)[p] \otimes C_*(Y)[q] \to A \otimes B \tag{30}$$

with the map $A \otimes B \to C$. In particular, on the degree zero component of $X \wedge Y$ this is given by the normal tensor product

$$\text{Hom}(X_0, A_p) \times \text{Hom}(Y_0, B_q) \to \text{Hom}(X_0 \wedge Y_0, A_p \otimes B_q) \tag{31}$$

because at this level the Alexander–Whitney map is given by

$$X_0 \wedge Y_0 \to X_0 \otimes Y_0.$$

Note that if X_s is represented by a scheme also denoted X_s, then by the Yoneda lemma we have $\mathrm{Hom}(X_s, A_t) \cong \Gamma(A_t)$. In particular, if $I_0 \to I_1 \to \cdots$ is an injective resolution of $\Gamma(i)$, then we can rewrite (26) as

$$[S^p \wedge X, K(\Gamma(i), \delta i)] \cong H^{\delta i - p}(C^{\cdot}(X, \Gamma(i))) \tag{32}$$

where $C^q(X, \Gamma(i)) = \bigoplus_{s+t=q} \Gamma(X_s, I_t)$ and the differential is given by $d = (-1)^q d_X^* + d^A$. (The shift in indices comes from (21), together with the fact that I is now a cohomological complex as opposed to homological.)

We have to define a push forward for the complexes in (32). To avoid sign problems in the identifications involved, we shall assume from now on that the δ for our cohomology theory equals 2.

Let i: $Z \to X$ be a closed immersion of smooth schemes where Z is of codimension d. There exists a quasi isomorphism

$$i_!\colon\ R\Gamma(Z, \Gamma(i)) \to R\Gamma_Z(X, \Gamma(i+d)[2d]). \tag{33}$$

Furthermore, if Y is another smooth closed subscheme of X that intersects Z transversally, we have a commutative diagram in the derived category (see [1, 2.3.2, p. 2050])

$$\begin{array}{ccc} R\Gamma(Z, \Gamma(i)) & \xrightarrow{i_!} & R\Gamma_Z(X, \Gamma(i+d))[2d] \\ \downarrow & & \downarrow \\ R\Gamma(Y \cap Z, \Gamma(i)) & \xrightarrow{i_!} & R\Gamma_{Y\cap Z}(Y, \Gamma(i+d))[2d] \end{array}$$

There exists a commutative diagram in the derived category,

$$\begin{array}{ccc} R\Gamma(Z, \Gamma(i)) \overset{L}{\bigotimes} R\Gamma(X, \Gamma(j)) & \xrightarrow{i_!\otimes \mathrm{id}} & R\Gamma_Z(X, \Gamma(i+d))[2d] \overset{L}{\bigotimes} R\Gamma(X, \Gamma(j)) \\ \downarrow & & \downarrow \\ R\Gamma(Z, \Gamma(i+j)) & \xrightarrow{i_!} & R\Gamma_Z(X, \Gamma(i+j+d))[2d] \end{array} \tag{34}$$

Now let $Z. \xrightarrow{i} X.$ be a closed cartesian immersion of smooth pointed simplicial schemes. Let $U. \to X.$ be its complement on each scheme component. Assume that all maps in $X.$ are closed immersions. Let $I(i)$ be an injective resolution of $\Gamma(i)$ in T. If all Z_l intersect X_k transversally in X_l, we can apply (33) for every scheme component $Z_l \to X_l$ and put zeroes at the components coming from $U.$, to get a map

$$C^q(Z., I(i)) \xrightarrow{i_!} C^q(C(U. \to X.), I(i+d)[2d]) = C^{q+2d}(C(U. \to X.), I(i+d)).$$

This gives rise to a map

$$\begin{aligned} H^q(C^{\cdot}(Z., \Gamma(i))) &\to H^{q+2d}(C^{\cdot}(C(U. \to X.), \Gamma(i+d))) \\ &\overset{\mathrm{def}}{=} H^{q+2d}_{Z.}(C^{\cdot}(X., \Gamma(i+d))) \end{aligned}$$

which we will also denote by $i_!$. As a consequence of (34) we get the projection formula $i_!(\alpha) \cup \beta = i_!(\alpha \cup i^!\beta)$ for $\alpha \in H^q(C^{\cdot}(Z., \Gamma(i)))$ and $\beta \in H^r(C^{\cdot}(C(X.)), \Gamma(j))$.

(Note that the cup product on $H^*(\mathcal{C}^\cdot)$ is determined by the map (30) together with our identifications.)

As for the regulator map, we now have that the exact sequence corresponding to two cartesian closed immersions $Z. \to Z'_. \to X.$,

$$\begin{aligned}\cdots &\to H_{Z.}^{-n}(X., K)_{\mathbb{Q}}^{(j)} \to H_{Z'_.}^{-n}(X., K)_{\mathbb{Q}}^{(j)} \\ &\to H_{Z'_.\setminus Z.}^{-n}(X. \setminus Z., K)_{\mathbb{Q}}^{(j)} \to H_{Z.}^{-n+1}(X., K)_{\mathbb{Q}}^{(j)} \to \cdots\end{aligned}$$

under the regulator map ch_Γ is mapped to (see (21) and (32))

$$\begin{aligned}\cdots &\to H_{Z.}^{2j-n}(\mathcal{C}^\cdot(X., \Gamma(j))) \to H_{Z'_.}^{2j-n}(\mathcal{C}^\cdot(X., \Gamma(j))) \\ &\to H_{Z'_.\setminus Z.}^{2j-n}(\mathcal{C}^\cdot(X. \setminus Z., \Gamma(j))) \to H_{Z.}^{2j-n+1}(\mathcal{C}^\cdot(X., \Gamma(j))) \to \cdots\end{aligned}$$

LEMMA 2.13. *Let $Z. \to X.$ be a closed immersion of regular simplicial schemes over a field satisfying* (TC1) *and* (TC2). *Suppose $Z.$ is defined in $X.$ by a system of cartier divisors (see Definition* 2.2). *Suppose that every simplical map $X_{l+1} \to X_l$ on irreducible components is either the identity, or the inclusion of the zero locus of an effective cartier divisor. Let d be the codimension of Z_0 in X_0. Then we have a commutative diagram*

$$\begin{array}{ccc} K_n^Z(Z)_{\mathbb{Q}}^{(j)} & \xrightarrow{i_\#} & K_n^Z(X)_{\mathbb{Q}}^{(j+d)} \\ \downarrow{\scriptstyle ch_\Gamma} & & \downarrow{\scriptstyle ch_\Gamma} \\ H_{Z.}^{2j-n}(\mathcal{C}^\cdot(Z., \Gamma(j))) & \xrightarrow{i_!} & H_{Z.}^{2j+2d-n}(\mathcal{C}^\cdot(X., \Gamma(j+d))) \end{array}$$

i.e. $ch_\Gamma(i_*(ch^{-1}(td(N^\vee))y)) = i_!(ch_\Gamma(y))$. *(See Proposition* 2.3 *for the definition of the map $i_\#$.)*

Proof. As in the proof of Proposition 2.3, because the Γ-cohomology satisfies the homotopy property, our conditions allow us to deform the problem to the immersion $Z. \to X'_.$, the zero section of a a projective bundle over $Z.$ Let p: $X'_. \to Z.$ denote the projection. Let $ch_\Gamma^{X'}$ be the chern character on $H_{Z.}^*(X'_., K)$, and ch_Γ^Z be the chern character on $H_{Z.}^*(Z., K) = H^*(Z., K)$. Then using the Riemann–Roch theorem for $Z_0 \to X'_0$ we find that

$$\begin{aligned} ch_\Gamma^{X'}(i_*(ch^{-1}(td(N^\vee))y)) &= ch_\Gamma^{X'}(p^*(ch^{-1}(td(N^\vee))y) \cup i_*(\mathcal{O}_{Z_0})) \\ &= ch_\Gamma^{X'}(p^*(ch^{-1}(td(N^\vee)y))) \cup ch^{X'_0}(i_*(\mathcal{O}_{Z_0})) \\ &= p^*(td_\Gamma(N^\vee) \cup ch_\Gamma^{Z.}(y)) \cup ch^{X'_0}(i_*(\mathcal{O}_{Z_0})) \\ &= p^*(td_\Gamma(N^\vee) \cup ch_\Gamma^{Z.}(y)) \cup i_!(td_\Gamma(N^\vee)^{-1}) \\ &= i_!(td_\Gamma(N^\vee) \cup ch_\Gamma^{Z.}(y) \cup td_\Gamma(N^\vee)^{-1}) \\ &= i_!(ch_\Gamma^{Z.}(y)) \end{aligned}$$

as desired.

2.5. An explicit version of the relative Deligne cohomology groups

We need a more computational version for the cohomology groups in (32) in case the cohomology theory is Deligne cohomology. For an algebraic variety X over $\mathbb{C}$ let $\mathbb{R}(n)_{\mathcal{D}} = \mathrm{Cone}(F_D^n \xrightarrow{-\pi_{n-1}} j_* S^{\cdot}_X(n-1))[-1]$. Here $j\colon X \to \overline{X}$ is a compactification with complement D, a divisor with normal crossings; $S^{\cdot}_X(n-1)$ is the complex of sheaves of $\mathbb{R}(n-1)$-valued C^∞-forms on X; F_D^n is the complex of holomorphic p-forms on $\overline{X}$ with logarithmic poles along D with $p \geqslant n$, tensored with the C^∞-forms on $\overline{X}$; and π_p is the projection onto $\mathbb{R}(p) \subset \mathbb{C}$. It is well known that the $\mathbb{R}(n)$-valued Deligne cohomology of X can be computed using the cohomology of the complex of global sections of $\mathbb{R}(n)_{\mathcal{D}}$. This means that the natural map $\Gamma(X, \mathbb{R}(i)_{\mathcal{D}}) \to \Gamma(X, I(i))$ when applied to all components in (32) will induce an isomorphism on cohomology

$$H^p(C^{\cdot}(X_{\cdot}, \Gamma(i))) \cong H^p(C^{\cdot}(X_{\cdot}, \mathbb{R}(i)_{\mathcal{D}})). \tag{35}$$

Together with the isomorphism (1) this is the usual regulator map. We note here that the elements in $\mathrm{Cone}(F_D^n \xrightarrow{-\pi_{n-1}} j_* S^{\cdot}_X(n-1))[-1]$ can be described by pairs (ω, s) with $\omega \in F_D^n$ and $s \in j_* S^{\cdot}_X(n-1)$. In this description the boundary operator is given by $d(\omega, s) = (d\omega, \pi_{n-1}(\omega) - ds)$.

There are products

$$\begin{gathered}\mathbb{R}(i)_{\mathcal{D}} \otimes \mathbb{R}(j)_{\mathcal{D}} \to \mathbb{R}(i+j)_{\mathcal{D}} \\ (\omega_n, s_n) \cup (\omega_m, s_m) = (\omega_n \wedge \omega_m, s_n \wedge \pi_m \omega_m + (-1)^{\deg \omega_n} (\pi_n \omega_n) \wedge s_m).\end{gathered}$$

Because this product coincides up to chain homotopy with the product $I(i) \otimes I(j) \to I(i+j)$ ([12, Lemma 3.11, p. 68]), this gives the same product on cohomology groups (using (30)).

We want to turn to the regulator maps on the K-cohomology of iterated cones, (18). For a completely explicit version of $H^p(C(C(X, \{Y_1, \ldots, Y_s\}), \mathbb{R}(n)_{\mathcal{D}}))$, let $\mathbb{R}(n)_{\mathcal{D},0} = \mathbb{R}(n)_{\mathcal{D}}$ on X and define $\mathbb{R}(n)_{\mathcal{D},s} = \ker(\mathbb{R}(n)_{\mathcal{D},s-1} \to i_s^*(\mathbb{R}(n)_{\mathcal{D},s-1}))$ inductively, with i_s the inclusion of Y_s into X.

From now on assume that $Y_1, \ldots, Y_s$ are smooth divisors in X. Assume that we have a smooth compactification $\overline{X}$ of X, with complement a divisor with normal crossings D, such that the union $D \cup \{Y_1, \ldots, Y_s\}$ is still a divisor with normal crossings in $\overline{X}$. Under these conditions one easily checks that the restriction map to Y_s is surjective, and remains surjective after taking global sections. (This is essentially a local problem because we can use partitions of unity.) Hence we have a long exact sequence

$$\begin{aligned}\cdots &\to H^p(\Gamma(X, \mathbb{R}(n)_{\mathcal{D},s}) \to H^p(\Gamma(X, \mathbb{R}(n)_{\mathcal{D},s-1}) \\ &\to H^p(\Gamma(Y_s, \mathbb{R}(n)_{\mathcal{D},s-1})) \to H^{p+1}(\Gamma(X, \mathbb{R}(n)_{\mathcal{D},s}) \to \cdots.\end{aligned} \tag{36}$$

Mapping the complex $\Gamma(X, \mathbb{R}(n)_{\mathcal{D},s})$ to the complex $C^{\cdot}(C(X, \{Y_1, \ldots, Y_s\}), \mathbb{R}(n)_{\mathcal{D}})$ by placing its components on X in degree 0, and zero elsewhere, we get a diagram

$$\begin{array}{ccccc} \longrightarrow H^p(\Gamma(X, \mathbb{R}(n)_{\mathcal{D},s-1})) & \longrightarrow & H^p(\Gamma(Y_s, \mathbb{R}(n)_{\mathcal{D},s-1})) & \longrightarrow \\ \downarrow & & \downarrow & \\ \longrightarrow H^p(C^{\cdot}(C(X, \{Y_1, \ldots, Y_{s-1}\}), \Gamma(n))) & \longrightarrow & H^p(C^{\cdot}(C(Y_s, \{Y_1, \ldots, Y_{s-1}\}), \Gamma(n))) & \longrightarrow \end{array}$$

$$\begin{array}{ccc}
\longrightarrow & H^{p+1}(\Gamma(X,\mathbb{R}(n)_{\mathcal{D},s})) & \longrightarrow \cdots \\
 & \downarrow & \\
\longrightarrow & H^{p+1}(C^{\cdot}(C(X,\{Y_1,\ldots,Y_s\}),\Gamma(n))) & \longrightarrow \cdots
\end{array}$$

If (ω, s) is a section on Y_s, then the boundary map in the top row is given by first lifting to $(\widetilde{\omega}, \tilde{s})$ on X, and then considering $d(\widetilde{\omega}, \tilde{s})$ on X. The boundary map in the bottom row is given by putting (ω, s) on $Y_s \times \{0,1\}$ in $C(X, \{Y_1, \ldots, Y_s\})$ (see (28)), possibly with a minus sign due to our identifications. The two classes (up to signs) differ by $d(\widetilde{\omega}, \tilde{s})$ with d as in (25), so that the diagram commutes up to sign.

Because of (27) the bottom row is exact, so by the five lemma we get an isomorphism

$$H^p(C^{\cdot}(C(X, \{Y_1, \ldots, Y_s\}), \Gamma(n))) \cong H^p(\Gamma(X, \mathbb{R}(n)_{\mathcal{D},s})). \tag{37}$$

We introduce the more familiar notation

$$H^p_{\mathcal{D}}(X; \{Y_1, \ldots, Y_s\}; \mathbb{R}(n)) = H^p(\Gamma(X, \mathbb{R}(n)_{\mathcal{D},s})) \tag{38}$$

so that we have a regulator

$$K_n^{(j)}(X, \{Y_1, \ldots, Y_s\}) \to H^{2j-n}_{\mathcal{D}}(X; \{Y_1, \ldots, Y_s\}; \mathbb{R}(j)). \tag{39}$$

We will use this representation for the regulator map from now on. Note that an element in the right hand side is given by the class of a pair (ω, s) with s a $\mathbb{R}(j-1)$-valued $(2j-n-1)$-form, ω a global section of the degree $2j-n$ part of $F^n_{\mathcal{D}}$, and $ds = \pi_{n-1}\omega$. Under all our identifications the product in Γ-cohomology is now given by the formula in (36) together with the result of the computations of the Alexander–Whitney map. Because the elements involved live only in the degree zero component, we get the formula (see (31)) for a $\mathbb{R}(n-1)$-valued form s_n with $ds_n = \pi_{n-1}\omega_n$ and a s_m-valued form s_m with $ds_m = \pi_{m-1}\omega_m$:

$$(\omega_n, s_n) \cup (\omega_m, s_m) = (\omega_n \wedge \omega_m, s_n \wedge \pi_m\omega_m + (-1)^{\deg \omega_n}(\pi_n\omega_n) \wedge s_m). \tag{40}$$

If $Z. \to X.$ satisfies the conditions in Lemma 2.13 with codimension d, we have a commutative diagram

$$\begin{array}{ccccc}
H^{-n-1}(X.\backslash Z.,K)^{(j)} & \longrightarrow & H^{-n}_{Z.}(X.,K)^{(j)} & \xleftarrow{\ i_\# \ } & H^{-n}(Z.,K)^{(j-d)} \\
\downarrow ch & & \downarrow ch & & \downarrow ch \\
H^{2j-n-1}(C^{\cdot}(X.\backslash Z;\mathbb{R}(j))) & \xrightarrow{\ \phi\ } & H^{2j-n}_{Z.}(C^{\cdot}(X.;\mathbb{R}(j))) & \xleftarrow{\ i_!\ } & H^{2j-2d-n}_{Z.}(C^{\cdot}(Z.;\mathbb{R}(j-d)))
\end{array}$$

For explicit computations we make some remarks about the complexes involved. We use notation as in [18]. Let X be a smooth proper analytic space over $\mathbb{C}$ of dimension d, and let $\Omega^p = \Omega^p_{X^\infty}$ be the sheaf of C^∞ p-forms on X. Let $'\Omega^p = {'\Omega}^p_{X^\infty}$ be the sheaf of distributions over Ω^{-p}. There is a map $\Omega^p \to {'\Omega}^{p-2d}$, given by $\omega \mapsto D(\omega)$, with

$$D(\omega)(\phi) = (-1)^{\deg \omega} \frac{1}{2\pi\sqrt{-1}^{\,d}} \int_X \phi \wedge \omega.$$

If the boundary map δ on $'\Omega^{\cdot}$ is defined by $(\delta D)(\phi) = (-1)^{\deg D+1} D(\mathrm{d}\phi)$ the map from forms to distribution above is a morphism of complexes. We can get a map $'\Omega^{\cdot} \otimes \Omega^{\cdot} \to {'\Omega^{\cdot}}$ by $(D \otimes \omega)(\phi) = D(\omega \wedge \phi)$. This is a map of complexes, and it is compatible with the map from forms to distributions.

The above normalizations can of course also be applied to non proper analytic spaces. Using those normalizations, and applying them in simplicial context, it follows from an explicit computation for $X_{\cdot} = C(X, \{Y_1, \dots, Y_s\})$ and $Z_{\cdot}$ of codimension 1 that under the isomorphism (37) the map

$$i_!^{-1} \circ \phi\colon\ H_{\mathcal{D}}^{2j-n-1}(C(X, \{Y_1, \dots, Y_s\}) \setminus Z_{\cdot}, \Gamma(j)) \to H_{\mathcal{D}}^{2j-n-2}(Z_{\cdot}, \Gamma(j-d))$$

is given by $(\omega, s) \mapsto (-\mathrm{res}_{Z_0}(\omega), \tilde{s})$ for some $\tilde{s}$ with $\mathrm{d}\tilde{s} = -\mathrm{res}_{Z_0}(\omega)$ and

$$(0, s) \mapsto (0, \mathrm{res}_{Z_0}(s)). \tag{41}$$

Here, if Z_0 is defined by $z = 0$ locally, and the forms involved have only logarithmic poles,

$$\mathrm{res}_{z=0} \frac{\mathrm{d}z}{z} \wedge \alpha = \alpha_{|z=0}$$

if α is a form without poles along $z = 0$.

3. Construction of the Complexes

NOTATION 3.1. Throughout this section, Y will be a regular noetherian scheme of pure dimension d, defined over a field of characteristic zero. We let $X_S = \mathbb{P}^1_S \setminus \{1\}$ for a scheme S, and $X_S^n = X_S \times_S \dots \times_S X_S$. The standard affine coordinate on X will be called t. We will abbreviate $\{t_1 = 0, \infty\}, \dots, \{t_n = 0, \infty\}$ by $\square^n$. We will suppress intersections in the relative part of the notations for K-theory.

3.1. Some preliminary complexes

DEFINITION 3.2. A scheme Y has no low weight K-theory if $K_m^{(j)}(Y) = 0$ for $m - 2j \geqslant 0$ and $m \geqslant 1$.

Remark 3.3. This is in fact a well-known conjecture by Beilinson and Soulé, see [1] and [27, 2.9].

Note that from the long exact sequence (which is (19) in this situation)

$$\cdots \to K_{m+1}^{(j)}(\{t_n = 0, \infty\}; \square^{n-1}) \to K_m^{(j)}(X_Y^n; \square^n) \to K_m^{(j)}(X_Y^n; \square^{n-1}) \to \cdots$$

and the homotopy property for K-theory for regular noetherian schemes, it follows that $K_m^{(j)}(X_Y^n; \square^n) \cong K_{m+1}^{(j)}(X_Y^{n-1}; \square^{n-1})$ for $m \geqslant 0$. So in general $K_m^{(j)}(X_Y^n; \square^n) \cong K_{m+n}^{(j)}(Y)$.

LEMMA 3.4. *Let $u_{11}, \dots, u_{1i_1}, \dots, u_{n1}, \dots, u_{ni_n}$ be elements in $\Gamma(Y, \mathcal{O}_Y^*) \setminus \{1\}$, such that $u_{jl} - u_{jk} \in \Gamma(Y, \mathcal{O}_Y^*)$ for all l and k, $l \neq k$. Write $X_{Y,\mathrm{loc}}^n$ for $X_Y^n \setminus \bigcup_{l,k} \{t_l = u_{lk}\}$. If Y has no low weight K-theory, then $K_m^{(j)}(X_{Y,\mathrm{loc}}^n; \square^n) = 0$ for $m + n - 2j \geqslant 0$ and $m > n$.*

Proof. Induction on n, the case $n = 0$ being true from the definitions. Consider the stratification of $X^n_Y = W_0 \supset W_1 \supset \ldots \supset W_{n-1} \supset W_n \supset \emptyset$, where $W_i = \bigcup i$-fold intersections of hyperplanes $t_j = u_{jk}$. By our assumptions, $W_p \setminus W_{p+1}$ is a disjoint union of $X^{n-p}_{Y,\mathrm{loc}}$'s, where the localization takes place at the $n-p$ coordinates on X^{n-p}_Y.

By (20) we have a spectral sequence

$$\coprod K^{(r-p)}_{-p-q}(X^{n-p}_{Y,\mathrm{loc}}; \square^{n-p}) \Rightarrow K^{(r)}_{-p-q}(X^n_Y; \square^n).$$

which more visually is

$$\begin{array}{llll} K^{(r)}_{-q-1}(X^n_{Y,\mathrm{loc}}; \square^n) & \coprod K^{(r-1)}_{-q-2}(X^{n-1}_{Y,\mathrm{loc}}; \square^{n-1}) & \coprod K^{(r-2)}_{-q-3}(X^{n-2}_{Y,\mathrm{loc}}; \square^{n-2}) & \\ K^{(r)}_{-q}(X^n_{Y,\mathrm{loc}}; \square^n) & \coprod K^{(r-1)}_{-q-1}(X^{n-1}_{Y,\mathrm{loc}}; \square^{n-1}) & \coprod K^{(r-2)}_{-q-2}(X^{n-2}_{Y,\mathrm{loc}}; \square^{n-2}) & (42) \\ K^{(r)}_{-q+1}(X^n_{Y,\mathrm{loc}}; \square^n) & \coprod K^{(r-1)}_{-q}(X^{n-1}_{Y,\mathrm{loc}}; \square^{n-1}) & \coprod K^{(r-2)}_{-q-1}(X^{n-2}_{Y,\mathrm{loc}}; \square^{n-2}) & \end{array}$$

The terms contributing to $K^{(r)}_m(X^n_Y; \square^n)$ are $K^{(r-p)}_m(X^{n-p}_{Y,\mathrm{loc}}; \square^{n-p})$'s, which are zero for $p \geqslant 1$ by induction. The boundary maps in the spectral sequence leaving $K^{(r)}_m(X^n_{Y,\mathrm{loc}}; \square^n)$ land in $K^{(r-p)}_{m-1}(X^{n-p}_{Y,\mathrm{loc}}; \square^{n-p})$'s for $p \geqslant 1$, which are zero as well. So $K^{(r)}_m(X^n_{Y,\mathrm{loc}}; \square^n) \cong K^{(r)}_m(X^n_Y; \square^n) \cong K^{(r)}_{m+n}(Y) = 0$.

From now on, assume given a collection $U = \{u_1, \ldots, u_s\} \subset \Gamma(Y, \mathcal{O}^*_Y)$, such that $u_i - u_j \in \Gamma(Y, \mathcal{O}^*_Y)$ for all i and j, $i \neq j$, and $1 - u_i \in \Gamma(Y, \mathcal{O}^*_Y)$ for all $u_i \neq 1$. Put $U' = U \setminus \{1\}$. Assume Y has no low weight K-theory. Then Lemma 3.4 applies to $X^n_{Y,\mathrm{loc}} \overset{\mathrm{def}}{=} X^n_Y \setminus \bigcup_{i,j}\{t_i = u_j\}$, so in the spectral sequence (see (42))

$$\Bigg(\coprod_{U'^p} K^{(n-p)}_{-p-q}(X^{n-p-1}_{Y,\mathrm{loc}}; \square^{n-p-1})\Bigg)^{\oplus\binom{n-1}{p}} \Rightarrow K^{(n)}_{-p-q}(X^{n-1}_Y; \square^{n-1})$$

(where $\binom{n-1}{p}$ corresponds to the different directions for p intersections of $t_i = u_j$ in X^{n-1}_Y) the terms vanish if

$$-p - q + n - p - 1 - 2n + 2p = -q - n - 1 \geqslant 0$$

and

$$-p - q > n - p - 1.$$

That means that the lowest non vanishing row is the row where $q = -n$. Introducing the notation $K_{(p)} = K^{(p)}_p(X^{p-1}_{Y,\mathrm{loc}}; \square^{p-1})$ for $1 \leqslant p \leqslant n$, this means that this row is the cohomological complex (starting in degree 1)

$$K_{(n)} \to \Bigg(\coprod_{U'} K_{(n-1)}\Bigg)^{\oplus\binom{n-1}{1}} \to \Bigg(\coprod_{U'^2} K_{(n-2)}\Bigg)^{\oplus\binom{n-1}{2}} \to \ldots$$

$$\to \Bigg(\coprod_{U'^{n-1}} K_{(1)}\Bigg)^{\oplus\binom{n-1}{n-1}}$$

We will denote this complex by $C^{\cdot}_{(n)} = C^1_{(n)} \to \ldots \to C^n_{(n)}$. All differentials d_r in the spectral sequence leaving $C^i_{(n)}$ are zero for $r \geqslant 2$, so we have the following

PROPOSITION 3.5. *If Y has no low weight K-theory, there is a map*

$$\phi_p\colon H^p(C^{\cdot}_{(n)}) \to K^{(n)}_{n+1-p}(X^{n-1}_Y; \square^{n-1}) \cong K^{(n)}_{2n-p}(Y).$$

This map is an isomorphism for $i = 1$ and an injection for $i = 2$.

Remark 3.6. Because of Lemma 2.12 the only contributions from the spectral sequence are coming from the alternating part under the action of S_{n-1}, so that we have a map on the alternating part

$$\phi^{\text{alt}}_i\colon H^i(C^{\cdot\,\text{alt}}_{(n)}) \to K^{(n)}_{n+1-i}(X^{n-1}_Y; \square^{n-1}),$$

which is an isomorphism for $i = 1$ and an injection for $i = 2$.

We want to change the complex $C^{\cdot}_{(n)}$ into a complex where the disjoint union over U' is replaced with a factor $\otimes U$. Following [4] we do this by taking the quotient of the complex by a suitable subcomplex.

Consider the following commutative diagram with exact column and row.

$$\begin{array}{ccccc}
 & & K^{(1)}_2(\{t=0,\infty\}) \cong K^{(1)}_2(Y)^{\oplus 2} = 0 & & \\
 & & \downarrow & & \\
K^{(1)}_1(X^1_Y; \square^1) & \longrightarrow & K^{(1)}_1(X^1_{Y,\text{loc}}; \square^1) & \longrightarrow & \coprod_{U'} K^{(0)}_0(Y) \\
\| & & \downarrow & & \| \\
K^{(1)}_2(Y) = 0 & & K^{(1)}_1(X^1_{Y,\text{loc}}) & \longrightarrow & \coprod_{U'} K^{(0)}_0(Y) \\
 & & \downarrow & & \\
 & & K^{(1)}_1(\{t=0,\infty\}) \cong K^{(1)}_1(Y)^{\oplus 2} \cong \Gamma(Y, \mathcal{O}^*_Y)^{\oplus 2}_{\mathbb{Q}} & &
\end{array}$$

It follows that if $\prod u_j^{n_j} = 1$, $f(t) = \prod(\frac{t-u_j}{t-1})^{n_j} \in K^{(1)}_1(X^1_{Y,\text{loc}}; \square^1)$. Following the notation in [4] we will denote the subspace generated by these elements f by $(1+I)^*$. We have the products $(1+I)^* \times K_{(p-1)} \to K_{(p)}$ given by $(f, \alpha) \mapsto f \cdot \alpha$, for $p \geqslant 2$, where we pull $(1+I)^*$ back along one of the projections $X^{p-1} \to X$, $K_{(p-1)}$ along the complementary $p-2$ coordinates, to get elements in $K^{(1)}_1(X^{p-1}_{Y,\text{loc}}; \square^1)$ respectively $K^{(p-1)}_{p-1}(X^{p-1}_{Y,\text{loc}}; \square^{p-2})$ and then take the product there. Doing this for all coordinates, we denote by $I_p \subset K_{(p)}$ the subspace generated by image of all those products. Note that this product is compatible with d: $d(f \cdot \alpha) = (df) \cdot \alpha - f \cdot (d\alpha)$.

Now define a complex $J^{\cdot}_{(n)} \subset C^{\cdot}_{(n)}$ by putting $J^1_{(n)} = I_n$, $J^{p+1}_{(n)} = dJ^p_{(n)} + \coprod_{U'^p} I_{n-p}$ for $1 \leqslant p \leqslant n-2$, and $J^n_{(n)} = dJ^{n-1}_{(n)}$, that is

$$I_n \to dI_n + \left(\coprod_{U'} I_{n-1}\right)^{\oplus\binom{n-1}{1}} \to d(\cdots) + \left(\coprod_{U'^2} I_{n-2}\right)^{\oplus\binom{n-1}{2}} \to \cdots$$

$$\ldots \to d(\ldots) + \left(\coprod_{U'^{n-2}} I_2\right)^{\oplus\binom{n-1}{n-2}} \to d(\ldots).$$

LEMMA 3.7. *$J^{\cdot}_{(n)}$ is acyclic.*

We will prove this in two steps.

LEMMA 3.8. *d: $I_n \to \coprod_{U'^{n-1}} K_{(n-1)}$ is injective for $n \geqslant 2$.*

Proof. Order U'. For $f, g \in (1+I)^*$, say that $(f) > (g)$ if every index j occurring in $(g) = \sum n_j(u_j)$ is smaller than the biggest occurring in (f). Suppose $\alpha \in I_n$ with $d\alpha = 0$. Write $\alpha = \alpha_1 + \ldots + \alpha_{n-r}$ where $\alpha_i \in ((1+I)^*)^r \cdot K_{(n-r)}$ for some fixed $r \geqslant 1$, and each $(1+I)^*$ occurring in α_i has coordinate t_j, $j \geqslant i$, and $j = i$ for at least one factor. (For $r = 1$ this can be done by definition.) We will show that α can be written in similar shape with r replaced by $r+1$. Write $\alpha_1 = \sum_j f_j(t_1) \cdot \beta_j$ with $(f_1) > (f_2) > \cdots$. Note that this can always be done: if the largest element in (f_1) and (f_2) occurs in (f_1), then $f_1 \cdot \beta_1 + f_2 \cdot \beta_2 = f_1 \cdot (\beta_1 + a\beta_2) + f_2/f_1^a \cdot \beta_2$ for $a \in \mathbb{Q}$, and we can choose a such that (f_2/f_1^a) does not contain the largest element of (f_1) and (f_2). Because $d\alpha = 0$, it follows that $\beta_1 \in ((1+I)^*)^r$ because the factors $(1+I)^*$ in $\alpha_2, \ldots, \alpha_{n-r}$ wil survive to the biggest $u_i \in (f_1)$, and $f_2 \cdot \beta_2$ etc. do not contribute. It now follows that $\beta_2 \in ((1+I)^*)^r$, etc. So in fact $\alpha_1 \in ((1+I)^*)^{r+1} \cdot K_{(n-r-1)}$. It then follows by writing α_2 in a similar way that $\alpha_2 \in ((1+I)^*)^{r+1} \cdot K_{(n-r-1)}$, etc. Repeating this, it follows that $\alpha \in ((1+I)^*)^{n-1} \cdot K_{(1)}$, where the same procedure now shows that $\alpha = 0$.

LEMMA 3.9. *If*

$$\alpha \in \left(\coprod_{U'^p} I_{n-p}\right)^{\oplus\binom{n-1}{p}}$$

for $1 \leqslant p \leqslant n-2$, *and* $d\alpha = 0$, *then* $\alpha \in dJ^p_{(n)}$.

Proof. We have a filtration

$$0 = F_0^{n-p} \subset F_1^{n-p} \subset \ldots \subset F_{n-1}^{n-p} = \left(\coprod_{U'^p} I_{n-p}\right)^{\oplus\binom{n-1}{p}}$$

where

$$F_r^{n-p} = \left\{ \begin{array}{l} \text{subspace generated by expressions } f(t_j) \cdot \beta \text{ on those} \\ \text{components where } t_j \text{ is a coordinate (i.e. non constant),} \\ j \leqslant r, \text{ and } \beta \in K_{(n-p-1)},\ f \in (1+I)^*. \end{array} \right\}$$

We will show that if $\alpha \in F_s^{n-p}$, then α is homologous to $\alpha' \in F_{s-1}^{n-p}$. For $f, g \in (1+I)^*$, $(f) > (g)$ will have the same meaning as in the proof of Lemma 3.8.

Note that if t_s is constant on a component, then F_s^{n-p} and F_{s-1}^{n-p} coincide on that component, so we only have to worry about components on which t_s is non constant. Fix a direction for components where t_s is non constant, i.e. a subset Q of p elements of $\{1, \ldots, n\}$ not containing s. The components in this direction correspond to the equations $\{t_q = u_q\}$ for $q \in Q$ and $u_q \in U'$. The direction of the divisor where t_s is constant corresponds to the $p+1$ indices $Q \cup \{s\}$.

Now fix a component D in the Q-direction. If C is any divisor on D in the $Q \cup \{s\}$ direction, then

$$d_C F^{n-p}_{s-1|D} \subset F^{n-p-1}_{s-1} \tag{43}$$

because the factors $(1+I)^*$ must survive in this case as C is defined in D by setting t_s constant. But modulo F^{n-p}_{s-1} we can write

$$\alpha_{|D} = \sum_{i=1}^{m} f_i(t_s) \cdot \beta_i$$

with $(f_1) > (f_2) > \cdots$. If $\beta_1 \in F^{n-p-1}_{s-1}$ then $f_1 \cdot \beta_1 \in F^{n-p}_{s-1}$. If C is the largest element in (f_1), which is a divisor in the $Q \cup \{s\}$-direction, then $d_C\alpha_{|D}$ modulo F^{n-p-1}_{s-1} is a multiple of β_1 by (43). So $d_C\alpha_{|D}$ modulo F^{n-p-1}_{s-1} is the obstruction for $f_1 \cdot \beta_1$ to be in F^{n-p}_{s-1}. Hence it suffices to change α in such a way that $d_C\alpha_{|D}$ is in F^{n-p}_{s-1} for all Q-components D, and all $Q \cup \{s\}$-divisors C, where Q runs through all subsets of p indices not containing s. Unfortunately, in order to get there we have to do slightly more. Namely, we will have to change α in such a way that $d_C\alpha_{|E} \in F^{n-p-1}_{s-1}$ for all components E having a divisor C in the $Q \cup \{s\}$ directions. This forces us to do some calculations for components in non-Q-directions.

Fix a direction Q as above, i.e. not containing s. Let C be a divisor in the $Q \cup \{s\}$-direction. We identify the possible C's with $(U')^{p+1}$, and we order them by ordering this last set lexicographically. Let E be any component passing through C. The possible directions of such E correspond to the subsets of $Q \cup \{s\}$ obtained by deleting one of the elements. Then C is defined on E by an equation $t_q = u$ where q is the deleted index, and $u \in U'$. We want to distinguish three cases.

1. If $q > s$, $F^{n-p}_s = F^{n-p}_{s-1}$ on E because t_s is then necessarily constant. Now $d_C\alpha_{|E} \in F^{n-p-1}_{s-1}$ because the factors $(1+I)^*$ must survive to C.
2. If $q = s$ we are in the case were we have been doing our computations, and we saw that if C is the largest divisor in E in the $Q \cup \{s\}$-direction then $d_C\alpha_{|E}$ modulo F^{n-p-1}_{s-1} is the obstruction for one term to be in F^{n-p}_{s-1}.
3. $q < s$. In this case F^{n-p}_s and F^{n-p}_{s-1} coincide because t_s is constant. In this case $d_C\alpha_{|E}$ is not necessarily in F^{n-p-1}_{s-1}. However, because $q < s$ we can write $\alpha_{|E} = \sum_{j=1}^{m} f_j(t_q) \cdot \beta_j$ modulo terms whose contribution to $d_C\alpha|E$ lies in F^{n-p-1}_{s-1}, with $(f_1) > (f_2) > \ldots$. (C is defined by putting t_q equal to a constant, so all factors $(1+I)^*$ corresponding to other t_q's will survive to C.)

Now let C be the largest divisor in the $Q \cup \{s\}$-direction for which $d_C\alpha_{|E} \notin F^{n-p-1}_{s-1}$ for some component E passing through C. Note that this places us in the last two of the above possibilities for E. Because $d_C\alpha = 0$, there are at least two such E's. Call one of them E, and let $E_1, \ldots, E_r$ be the others. Write

$$\alpha_{|E} = \sum_{j=1}^{m_E} f_j(t_{q_E}) \cdot \beta^E_j$$

modulo terms whose contribution to $d_C\alpha_{|E}$ lies in F^{s-1}_{n-p-1}, and with $(f_1) > (f_2) > \cdots$. Proceed similarly for the E_j's. (This can be done as we are in case 2 or 3 above.) Note that because C is maximal we must have that C corresponds to the largest element in (f_1) on E and all E_k's.

Let

$$\tilde{\alpha} = \alpha - d\sum_{k=1}^{r} \frac{1}{\mathrm{ord}_C(f_1^E)} f_1^E(t_{q_E}) \cdot f_1^{E_k}(t_{q_{E_k}}) \cdot \beta_1^{E_k}{}_{|\langle E, E_k\rangle},$$

where the indices indicate from which component we took the respective elements, and $\langle E, E_k\rangle$ stands for the linear span of E and E_k, that is the codimension $(p-1)$ plane containing both. (This corresponds to something in the $Q \cup \{s\} \setminus \{q_E, q_{E_k}\}$-direction. The expression is an element in $K_{(n-p+1)}$ because E and E_k are different components passing through C, so q_E and q_{E_k} are different too.) Because both q_E and q_{E_k} are less than or equal to s, (44) below shows that this is in F^{n-p}_s. For each of the terms in the sum, d gives a contribution

$$\begin{aligned}
&-\frac{1}{\mathrm{ord}_C(f_1^E)}(df_1^E(t_{q_E})) \cdot f_1^{E_k}(t_{q_{E_k}}) \cdot \beta_1^{E_k} \\
&\quad \text{on components in the } Q \cup \{s\} \setminus \{q_{E_k}\}\text{-direction} \\
&+\frac{1}{\mathrm{ord}_C(f_1^E)} f_1^E(t_{q_E}) \cdot \left(df_1^{E_k}(t_{q_{E_k}})\right) \cdot \beta_1^{E_k} \\
&\quad \text{on components in the } Q \cup \{s\} \setminus \{q_E\}\text{-direction} \\
&-\frac{1}{\mathrm{ord}_C(f_1^E)} f_1^E(t_{q_E}) \cdot f_1^{E_k}(t_{q_{E_k}}) \cdot (d\beta_1^{E_k}) \\
&\quad \text{on components in the } Q \cup \{s, q\} \setminus \{q_E, q_{E_k}\}\text{-direction} \\
&\quad \text{for some } q \notin Q \cup \{s, q_E, q_{E_k}\}.
\end{aligned} \tag{44}$$

(The coordinates on a component corresponding to Q are t_q where $q \notin Q$.) We now examine the contributions. The last term will never contribute to d_C for any component, because on C both t_{q_E} and $t_{q_{E_k}}$ are constant and in this case at least one of them will survive. The first one lives on components parallel to E_k. It kills $d_C\alpha_{|E_k}$ (modulo F^{n-p-1}_{s-1}). Moreover, it contributes nothing to $d_{C'}\tilde{\alpha}_{|E_k}$ for C' parallel to C but $C' > C$ because C is defined by the maximal component of $(f_1^{E_k})$. Now $d_C\tilde{\alpha}_{|E_k} = 0$ modulo F^{n-p-1}_{s-1} for $k = 1, \ldots, r$, and because $d_C\tilde{\alpha} = 0$ this holds for $d_C\tilde{\alpha}_{|E}$ also. The second contribution never contributes anything to $d_{C'}\tilde{\alpha}$ for some C' parallel to C with $C' > C$, again by the maximality of C in (f_1). So $d_C\tilde{\alpha}_{|E} \in F^{n-p-1}_{s-1}$ for all components E passing through C, and $d_{C'}\tilde{\alpha} = 0$ modulo F^{n-p-1}_{s-1} for C' parallel to C but bigger than C. By applying this procedure repeatedly we can change α in such a way that $d_C\alpha_{|E} = 0$ modulo F^{n-p-1}_{s-1} for all components E having a $Q \cup \{s\}$-divisor C. Then our previous computations show that this new α must be in F^{n-p}_{s-1} for all Q-components D.

We now have to see what happens if we switch to a different direction for D, say Q'. But inspection of the indices involved in (44) show that the only components where the expression in (44) is not in F^{n-p}_{s-1} is in the Q-direction. This means that we can move α into F^{n-p}_{s-1} direction by direction.

Remark 3.10. Lemma 3.7 in fact holds for any row in the spectral sequence, and any subcomplex closed under multiplication by $(1+I)^*$. This is clear from the elements used in the proof of Lemma 3.9, see (44). So the subcomplex generated by elements with at least one factor in $(1+I)^*$ is acyclic.

Remark 3.11. We shall need the following for later reference. Let $Y = \mathrm{Spec}\,(F)$, F a field containing only a finite number of roots of unity. Let V be the subgroup of F^* generated by U, and assume V is generated by $s_1, \ldots, s_k \in U$ with s_k torsion if V contains torsion, and the remaining s's independent. Let $(1+I)^* = \{f \in K_1(X_{loc})$ such that $f(0) = f(\infty) = 1\}$, so that the old $(1+I)^*$ is this one tensored with $\mathbb{Q}$. Suppose that, in the definition of $J^{\cdot}_{(n)}$ preceding Lemma 3.7, we replace $C^{\cdot}_{(n)}$ with any complex of $\mathbb{Z}$-modules closed under multiplication by the new $(1+I)^*$ as in Remark 3.10. We want to show that the cohomology of the resulting complex is torsion of bounded exponent. Order U' such that $s_1, \ldots, s_k$ are the smallest elements, and are in decreasing order. If s_k is a root of unity, let d be its order in F^*. If $x \in U'$, write $x = s_1^{m_1} \cdots s_k^{m_k}$ with $0 \leqslant m_k < d$ if s_k is a root of unity, and let $f_x(t) = (\frac{t-x}{t-1}) \prod_i (\frac{t-1}{t-s_i})^{m_i}$, so that $f_x \in (1+I)^*$. If $s_k^d = 1$ let $f_{s_k}(t) = (\frac{t-s_k}{t-1})^d \in (1+I)^*$. Then $(1+I)^*$ is generated by the f_x, and all the functions f_j occurring in the proofs of Lemmas 3.8 and 3.9 can be replaced by f_x's, $x \in U' \setminus \{s_1, \ldots, s_{k-1}\}$, because $\prod_i \frac{t-x_i}{t-1} = (\prod_i f_{x_i}) f_{s_k}^m$ if $\prod_i x_i = 1$, for some $m \in \mathbb{Z}$. (Simply write all x_i in terms of the s_j.) Using the f_x's, one sees that actual division (by the order of a zero or pole) in the proofs occurs only if f_{s_k} is involved. This happens if we are looking at the smallest element in U' only, namely s_k, if at all. In the proofs of Lemmas 3.8 and 3.9 this happens only a finite number of times, depending on n and p only, i.e., depending on the number of directions involved and the length of the filtration used in the proof of Lemma 3.9. This shows that in this case $H^*(J^{\cdot}_{(n)})$ is torsion of bounded exponent, namely some power of the number of roots of unity in the subgroup V of F^* generated by U.

Lemma 3.7 and Proposition 3.5 imply that the cohomology of the complex $C^{\cdot}_{(n)}/J^{\cdot}_{(n)}$ maps to the K-theory of Y. Because $J^{\cdot}_{(n)}$ is stable under the action of S_{n-1} a similar result holds for the alternating part. With the notations

$$L_{(p)} = K_{(p)}/I_p \quad \text{and} \quad U_{\mathbb{Q}} = \text{subspace of } \Gamma(Y, \mathcal{O}_Y^*)_{\mathbb{Q}} \text{ generated by } U$$

it is easy to see that

$$\mathcal{L}^{\cdot}_{(n)} \stackrel{\text{def}}{=} C^{\cdot}_{(n)}/J^{\cdot}_{(n)} \cong L_{(n)} \to (U_{\mathbb{Q}} \otimes L_{(n-1)})^{\oplus \binom{n-1}{1}} \to (U_{\mathbb{Q}}^{\otimes 2} \otimes L_{(n-2)})^{\oplus \binom{n-1}{2}} \to \cdots \to (U_{\mathbb{Q}}^{\otimes (n-2)} \otimes L_{(2)})^{\oplus \binom{n-1}{n-2}} \to U_{\mathbb{Q}}^{\otimes (n-1)} \otimes L_{(1)}$$

and

$$\mathcal{L}^{\cdot\,\text{alt}}_{(n)} \stackrel{\text{def}}{=} (C^{\cdot}_{(n)}/J^{\cdot}_{(n)})^{\text{alt}} \cong L^{\text{alt}}_{(n)} \to U_{\mathbb{Q}} \otimes L^{\text{alt}}_{(n-1)} \to \bigwedge^2 U_{\mathbb{Q}} \otimes L^{\text{alt}}_{(n-2)} \to \cdots \to \bigwedge^{n-2} U_{\mathbb{Q}} \otimes L^{\text{alt}}_{(2)} \to \bigwedge^{n-1} U_{\mathbb{Q}} \otimes L^{\text{alt}}_{(1)}.$$

Here all the tensor and exterior products are taken over $\mathbb{Q}$, and $L^{\mathrm{alt}}_{(p)} = (K_{(p)}/I_p)^{\mathrm{alt}}$ for the action of S_{p-1}. So we have the following

THEOREM 3.12. *Suppose Y is a regular, noetherian scheme, and suppose Y has no low weight K-theory. Let $U = \{u_1, \dots, u_s\} \subset \Gamma(Y, \mathcal{O}^*_Y)$ be such that if $u, v \in U$, then $u - v$ and $1 - u$ are in $\Gamma(Y, \mathcal{O}^*_Y)$ if $u \neq v$, $u \neq 1$. Then we have maps*

$$\phi_p\colon\ H^p(\mathcal{L}^{\cdot}_{(n)}) \to K^{(n)}_{n-p+1}(X^{n-1}_Y; \square^{n-1}) \cong K^{(n)}_{2n-p}(Y)$$

and

$$\phi^{\mathrm{alt}}_p\colon\ H^p(\mathcal{L}^{\cdot\,\mathrm{alt}}_{(n)}) \to K^{(n)}_{n-p+1}(X^{n-1}_Y; \square^{n-1})^{\mathrm{alt}} \cong K^{(n)}_{2n-p}(Y).$$

These maps are isomorphisms for $p = 1$, and are injective for $p = 2$.

3.2. CONSTRUCTION OF THE ELEMENTS $[S]_n$

We will now, for any $u \in \Gamma(Y, \mathcal{O}^*_Y)$, construct an element $[u]_p \in L^{\mathrm{alt}}_{(p)}$ for $p \geqslant 2$. We do this by constructing an element $[S]_p \in K_{(p)}(\mathrm{Spec}(\mathbb{Q}[S, S^{-1}]))$, and then pulling it back to Y. For $p = 1$, we put $[u]_1 = 1 - u \in \Gamma(Y, \mathcal{O}^*_Y)_{\mathbb{Q}} = K^{(1)}_1(Y)$ if $1 - u$ is a unit.

Let S be the coordinate on $\mathbb{G}_m$, and let

$$\begin{aligned} T^p &= X^{p-1}_{\mathbb{G}_m} & T^p_{\mathrm{loc}} &= X^{p-1}_{\mathbb{G}_m} \setminus \textstyle\bigcup_{i=1}^{p-1}\{t_i = S\} \\ R^p &= X^{p-1}_{\mathbb{G}_m\setminus\{1\}} & R^p_{\mathrm{loc}} &= X^{p-1}_{\mathbb{G}_m\setminus\{1\}} \setminus \textstyle\bigcup_{i=1}^{p-1}\{t_i = S\} \end{aligned}$$

Using the stratification $T^p = W_0 \supset W_1 \supset \dots \supset W_p = \emptyset$, where $W_j = \cup j$-fold intersections of the $\{t_i = S\}$'s, we get a spectral sequence

$$K^{(j-l)}_{-l-q}(W_l \setminus W_{l+1}; \square^{p-1}) \Rightarrow K^{(j)}_{-l-q}(T^p; \square^{p-1})$$

Note that $W_0 \setminus W_1 = T^p_{\mathrm{loc}}$, $W_l \setminus W_{l+1} = \coprod_{\binom{p-1}{l}} R^{p-l}_{\mathrm{loc}}$ for $l \geqslant 1$, and that the schemes $\mathbb{G}_m$ and $\mathbb{G}_m \setminus \{1\}$ have no low weight K-theory by Borel and Quillen. So Lemma 3.4 applies, and we get that the i-th cohomology of

$$K^{(p)}_p(T^p_{\mathrm{loc}}; \square^{p-1}) \to \coprod K^{(p-1)}_{p-1}(R^{p-1}_{\mathrm{loc}}; \square^{p-2}) \to \dots$$

maps to $K^{(p)}_{2p-i}(\mathbb{G}_m)$, injectively for $i = 2$, isomorphically for $i = 1$. For $p = 1$, put $[S]_1 = 1 - S \in K^{(1)}_1(\mathbb{G}_m \setminus \{1\})$. Now let $p \geqslant 2$. Suppose that by induction we have an element $[S]_{p-1}$ such that $\alpha = \sum_{i=1}^{p-1}(-1)^i[S]_{p-1|t_i=S}$ satisfies $d\alpha = 0$ and $\alpha = 0$ in $K^{(p)}_{2p-2}(\mathbb{G}_m) \cong K^{(p-1)}_{2p-3}(\mathbb{G}_m) \cdot S$. To check this for $p = 2$, note that we prove something stronger if we leave out the weights by (7). In that case we can take $\mathrm{Spec}(\mathbb{Z})$ as our base scheme rather than $\mathrm{Spec}(\mathbb{Q})$. Then $[S]_{1|t=S}$ lands in $K_1(X_{\mathbb{G}_m}; \square) \cong K_2(\mathbb{G}_m) \cong K_2(\mathbb{Z}) \oplus K_1(\mathbb{Z})$, which is torsion.

Let $[S]^*_p \in K^{(p)}_p(T^p_{\mathrm{loc}}; \square^{p-1})$ be such that $d[S]^*_p = \alpha$. Then $[S]^*_p$ is determined up to an element in $K^{(p)}_p(T^p; \square^{p-1}) \cong K^{(p)}_{2p-1}(\mathbb{G}_m) \cong K^{(p)}_{2p-1}(\mathbb{Q})$. $\sum_{i=1}^{p}(-1)^i[S]^*_{p|t_i=S}$ has boundary zero and hence will map to an element $\beta \in K^{(p+1)}_p(T^{p+1}; \square^p) \cong K^{(p+1)}_{2p}(\mathbb{G}_m)$. It follows from the lemma below that there is a unique $[S]^*_p$ such that its image $\beta = 0$. This unique element will be $[S]_p$.

LEMMA 3.13. *Let $i_{\# j}$ be the proper pushforward associated to the inclusion of T^p into T^{p+1} as the hyperplane $t_j = S$; see* (13). *Then*

$$K_p^{(p)}(T^p; \square^{p-1}) \xrightarrow{\sum_{j=1}^p (-1)^j i_{\# j}} K_p^{(p+1)}(T^{p+1}; \square^p)$$

is an isomorphism.

Proof. We start with the push forward $i_{\# p}$. It is not hard to check that the push forward (11) i_*: $K_p(T^p; \square^{p-1}) \to K_p(T^{p+1}; \square^p)$ is compatible with the isomorphisms $K_p(T^p; \square^{p-1}) \cong K_{2p-1}(\mathbb{G}_m)$ and $K_p(T^{p+1}; \square^p) \cong K_{2p-1}(X_{\mathbb{G}_m}; \square^1)$ obtained by using the relativity with respect to the first $p-1$ coordinates. And because the normal bundle is the pullback of the normal bundle of $\{t = S\} \to X_{\mathbb{G}_m}$ via projection onto the last coordinate, the same holds for the maps $i_\#$: $K_p^{(p)}(T^p; \square^{p-1}) \to K_p^{(p+1)}(T^{p+1}; \square^p)$. So we want to look at $i_\#$: $K_{2p-1}^{(p)}(\mathbb{G}_m) \to K_{2p-1}^{(p+1)}(X_{\mathbb{G}_m}; \square^1)$. Note that those groups are isomorphic to $K_{2p-1}^{(p)}(\mathbb{Q})$ and $K_{2p-1}^{(p)}(\mathbb{Q}) \cdot S$ respectively. Because those are finite dimensional spaces, it suffices to prove the map is surjective. We now have a commutative diagram

$$\begin{array}{ccccc}
K_{2p-1}^{(p)}(\{t=S\}) & \xrightarrow{i_\#} & K_{2p-1}^{(p+1)}(X_{\mathbb{G}_m};\square) & \xrightarrow{\sim} & K_{2p}^{(p+1)}(\mathbb{G}_m) \\
\uparrow l & & \uparrow & & \uparrow \\
K_{2p-1}^{(p)}(\mathbb{Q})\otimes K_0^{(0)}(\{t=S\}) & \xrightarrow{i_\#} & K_{2p-1}^{(p)}(\mathbb{Q})\otimes K_0^{(1)}(X_{\mathbb{G}_m};\square) & \xrightarrow{\sim} & K_{2p-1}^{(p)}(\mathbb{Q})\otimes K_1^{(1)}(\mathbb{G}_m)
\end{array}$$

So if we prove that $K_0^{(0)}(\{t = S\}) \xrightarrow{i_\#} K_0^{(1)}(X_{\mathbb{G}_m}; \square) \xrightarrow{\sim} K_1^{(1)}(\mathbb{G}_m)$ maps 1 to $S^{\pm 1}$, then $i_{\# p}$ is an isomorphism. Postponing this to the following lemma for the moment, for the other push forwards we then have the commutative diagram

$$\begin{array}{ccc}
K_p^{(p)}(T^p; \square^{p-1}) & \xrightarrow[\sim]{i_{\# l+1}} & K_p^{(p+1)}(T^{p+1}; \square^p) \\
\uparrow & & \uparrow \sigma^* \\
K_p^{(p)}(T^p; \square^{p-1}) & \xrightarrow[\sim]{i_{\# l}} & K_p^{(p+1)}(T^{p+1}; \square^p)
\end{array}$$

where σ is the transposition $(l+1\ l)$ interchanging the $l+1$-st and l-th coordinate. Because the action of S_p is alternating on the K-theory by Lemma 2.12, σ^* acts as multiplication by -1, so $(-1)^j i_{\# j} = (-1)^{j+1} i_{\# j+1} = \ldots = (-1)^p i_{\# p}$, and the map in the statement of the lemma is just a non zero multiple of the isomorphism $i_{\# p}$.

LEMMA 3.14. *The composed map $K_0^{(0)}(\{t = S\}) \xrightarrow{i_\#} K_0^{(1)}(X_{\mathbb{G}_m}; \square) \cong K_1^{(1)}(\mathbb{G}_m)$ maps 1 to $S^{\pm 1}$.*

Proof. If we forget about weights, the above maps are localizations to $\mathrm{Spec}(\mathbb{Q})$ of maps defined with $\mathrm{Spec}(\mathbb{Z})$ as our base scheme. So we will prove something stronger if we prove it in this context. We then have a commutative diagram

$$\begin{array}{ccccccc}
K_1(X_{\mathbb{G}_m}; \square) & \longrightarrow & K_1(X_{\mathbb{G}_{m,\mathrm{loc}}}; \square) & \xrightarrow{d} & K_0(\mathbb{G}_m \setminus \{1\}) & \longrightarrow & K_0(X_{\mathbb{G}_m}; \square) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_1(X_{\mathbb{G}_m}) & \longrightarrow & K_1(X_{\mathbb{G}_{m,\mathrm{loc}}}) & \longrightarrow & K_0(\mathbb{G}_m \setminus \{1\}) & \longrightarrow & K_0(X_{\mathbb{G}_m})
\end{array} \qquad (45)$$

Now $K_1(X_{\mathbb{G}_m}) \cong K_1(\mathbb{G}_m) \cong \pm S^{\mathbb{Z}}$ and $K_0(\mathbb{G}_m \setminus \{1\}) \cong \mathbb{Z}$ hence $K_1(X_{\mathbb{G}_m,\mathrm{loc}})$ is generated by ± 1, S, and $\frac{t-S}{t-1}$. The image of $K_1(X_{\mathbb{G}_m,\mathrm{loc}};\square)$ must restrict to 1 where $t = 0, \infty$, and hence must be 1. It follows that the map d in the above diagram must be zero. Hence the composition ϕ: $\mathbb{Z} = K_0(\mathbb{G}_m \setminus \{1\}) \to K_0(X_{\mathbb{G}_m};\square) \cong \pm S^{\mathbb{Z}}$ is an injection, so must map 1 to $\pm S^n$ for some $n \neq 0$. Considering $K_0(\mathbb{G}_m \setminus \{1\}) \to K_0(X_{\mathbb{G}_m};\square) \to K_0(X_{\mathbb{G}_m,\mathrm{loc}};\square)$ and combining this with the restriction to $S = 1$, we see that ± 1 must survive into the last group (because it comes from the base $\mathbb{Z}$ via pull back). Hence ϕ maps 1 to S^n for some $n \neq 0$. Now suppose that $|n| \neq 1$. Let ζ be a primitive $|n|$-th root of unity. We can then specialize S to ζ, and then ϕ is the trivial map $1 \to \zeta^n$. On the other hand, one checks as before using the bottom row of the above diagram that there is no $f \in K_1(X_{\mathbb{Z}[\zeta],\mathrm{loc}};\square)$ with $df = 1$. Hence $|n| = 1$.

Now let $C^{\cdot}_{(n),\log}$ be the subcomplex of $C^{\cdot}_{(n)}$ consisting of elements of the form $f_1 \cdots f_l \cdot [u]_{p-l}$, $0 \leqslant l \leqslant p-1$, on every component X^{p-1}, with $u \in U$ if $p - l \geqslant 2$, $u \in U'$ if $p - l = 1$. Let $J^{\cdot}_{(n),\log}$ be the subcomplex generated by those elements for which $l \geqslant 1$. (Here $f_1, \ldots, f_l \in (1+I)^*$ are pulled back along different coordinates, and $[u]_{p-l}$ along the remaining $p-l-1$.) By Remark 3.10 $J^{\cdot}_{(n),\log}$ is acyclic, and it is clearly stable under the action of S_{n-1}. Therefore the cohomology of the quotient complex will map to the appropriate K-theory. More precisely, let $M_{(1)} = \{[u]_1, u \in U'\}$, and for $p \geqslant 1$ $M_{(p+1)} = \{[u]_{p+1}$ for $u \in U$, $(1+I)^* \cdot M_{(p)}\}/(1+I)^* \cdot M_{(p)}$. ($\{\ldots\}$ means the $\mathbb{Q}$-vector space generated by those elements.) Then with

$$\mathcal{M}^{\cdot}_{(n)} = C^{\cdot}_{(n),\log}/J^{\cdot}_{(n),\log}$$

we have the following result.

THEOREM 3.15. *Let Y be a regular, irreducible noetherian scheme, with no low weight K-theory. Let $U \subset \Gamma(Y, \mathcal{O}_Y^*)$ be such that for $u, v \in U$, $u \neq v$, $u - v \in \Gamma(Y, \mathcal{O}_Y^*)$ and $1 - u \in \Gamma(Y, \mathcal{O}_Y^*)$ if $u \neq 1$. Let $\mathcal{M}^{\cdot}_{(n)}$ be the complex*

$$M_{(n)} \to U_{\mathbb{Q}} \otimes M_{(n-1)} \to \bigwedge^2 U_{\mathbb{Q}} \otimes M_{(n-2)} \to \ldots \to \bigwedge^{n-1} U_{\mathbb{Q}} \otimes M_{(1)}$$

where $d(u_1 \wedge \ldots \wedge u_p \otimes [u]_{n-p}) = u_1 \wedge \ldots \wedge u_p \wedge u \otimes [u]_{n-p-1}$ for $n - p \geqslant 2$. Then there is map

$$H^p(\mathcal{M}^{\cdot}_{(n)}) \to K^{(n)}_{2n-p}(Y)$$

for $p = 1, \ldots, n$, which is an injection for $p = 1$.

Remark 3.16. $\mathcal{M}^{\cdot}_{(n)}$ is actually the subcomplex of $\mathcal{L}^{\cdot\mathrm{alt}}_{(n)}$ generated by the symbols $[u]_p$. To see this, we prove by induction on n that $M_{(n)}$ is a subgroup of $L_{(n)}$. For $n = 1$ this is clear. For higher n, consider the commutative diagram

$$\begin{array}{ccc} L_{(n)} & \longrightarrow & U_{\mathbb{Q}} \otimes L_{(n-1)} \\ \uparrow & & \uparrow \\ M_{(n)} & \longrightarrow & U_{\mathbb{Q}} \otimes M_{(n-1)} \end{array}$$

If $\alpha \in M_{(n)}$ goes to zero in $L_{(n)}$, then $d\alpha = 0$ by induction, and α defines a class in $H^1(\mathcal{M}^{\cdot}_n)$. But we have

$$\begin{array}{ccc} H^1(\mathcal{L}_{(n)}) & = & H^1(C_{(n)}) \\ \uparrow & & \uparrow \\ H^1(\mathcal{M}_{(n)}) & = & H^1(C_{(n),\log}) \end{array}$$

and the right vertical arrow is an inclusion. So $\alpha = 0$ already in $H^1(\mathcal{M}^{\cdot}_{(n)}) \subset M_{(n)}$. Similarly, adjoining more elements to U will give an inclusion of subcomplexes, provided we do not change Y.

Remark 3.17. If we know that for a map of schemes $f\colon Y \to Y'$ with no low weights the induced map $K^{(p)}_{2p-1}(Y') \to K^{(p)}_{2p-1}(Y)$ is injective, for all $p \leqslant n$, the complex $\mathcal{M}^{\cdot}_{(n)}$ for Y injects into that for Y'. This follows as above, because now by assumption the map $H^1(\mathcal{M}^{\cdot}_{(n)}(Y')) \to H^1(\mathcal{M}^{\cdot}_{(n)}(Y))$ is an injection as both groups are subgroups of $K^{(n)}_{2n-1}(Y')$ resp. $K^{(n)}_{2n-1}(Y)$.

EXAMPLE 3.18. If $Y = \mathrm{Spec}(F)$ for a field F, any finite subset U of F^* will satisfy the assumptions of Proposition 3.15. Because the map to the K-theory of Y is compatible with adding more elements to U, by taking direct limits over U we get a map of the p-th cohomology of the complex

$$M_{(n)} \to F^*_{\mathbb{Q}} \otimes M_{(n-1)} \to \bigwedge^2 F^*_{\mathbb{Q}} \otimes M_{(n-2)} \to \dots \to \bigwedge^{n-1} F^*_{\mathbb{Q}} \otimes M_{(1)}$$

to $K^{(n)}_{n-p}(F)$ which is injective for $p = 1$. (Here $M_{(n)}$ now stands for the direct limit over U of our previous $M_{(n)}$, which depended on U.)

We will need the following explicit relation between elements for the computation of the regulator map for $[1]_n$.

LEMMA 3.19. *We have* $[1]_n = 2^{n-1}[1]_n + 2^{n-1}[-1]_n$ *in* $M_{(n)}(\mathbb{Q}) \subset L^{\mathrm{alt}}_{(n)}(\mathbb{Q})$ *for all* $n \geqslant 2$.

Proof. We do this by showing that we have a universal relation

$$[S^2]_n - 2^{n-1}[S]_n - 2^{n-1}[-S]_n = 0$$

in $L_{(n)}(\mathbb{G}_m \setminus \{\pm 1\})$, and then extend this to $S = 1$. Inductively, we can assume that $d([S^2]_n - 2^{n-1}[S]_n - 2^{n-1}[-S]_n) = 0$ in $U_{\mathbb{Q}} \otimes L_{(n-1)}(\mathbb{G}_m \setminus \{\pm 1\})$. (For $n = 2$ this is clear.) We can lift this element to an element α in $H^1(C^1_{(n)}(\mathbb{G}_m \setminus \{\pm 1\}))$ which injects into $K^{(n)}_{2n-1}(\mathbb{G}_{m,\mathbb{Q}} \setminus \{\pm 1\}) \cong K^{(n)}_{2n-1}(\mathbb{Q})$ by Borel and Quillen. So we want to show that its image, say β, is zero. The computations in Lemma 3.13 show that $S \otimes \alpha \in H^2(C^{\cdot}_{(n+1)})$ will be mapped to a non zero multiple of $\beta \cdot S$ under

$$H^2(C^{\cdot}_{(n+1)}) \to K^{(n+1)}_n(X^n_{\mathbb{G}_m \setminus \{\pm 1\}}; \square^n) \cong K^{(n+1)}_{2n}(\mathbb{G}_m \setminus \{\pm 1\})$$

But obviously

$$\begin{aligned}&[S^2]_{n+1} - 2^n[S]_{n+1} - 2^n[-S]_{n+1} \mapsto 2S \otimes \alpha \\ &\quad = 2S \otimes ([S^2]_n - 2^{n-1}[S]_n - 2^{n-1}[-S]_n)\end{aligned}$$

in $H^2(\mathcal{L}^{\cdot\text{alt}}_{(n+1)}) \cong H^2(C^{\cdot}_{(n+1)})$ which shows that α must equal zero.

By Remarks 3.16 and 3.10 α will involve only elements in $(1+I)^*$ and symbols $[S^2]_p$, $[S]_p$, $[-S]_p$, $p \leqslant n$. The only type of elements in α that cannot be specialized to 1 directly are elements involving $[S]_1$, i.e. $((1+I)^*)^{n-1} \cdot [S]_1$. Write

$$0 = \alpha = g \cdot [S]_1 + \tilde{\alpha} \tag{46}$$

with $g \in ((1+I)^*)^{n-1}$, and $\tilde{\alpha}$ not containing $[S]_1$. $\tilde{\alpha}$ has no contribution to the boundary map at the divisor $S = 1$, so $g \cdot [S]_1$ cannot have one either.

Now $(1+I)^*$ is generated by the two functions

$$f_1 = \frac{(t-S^2)(t-1)}{(t-S)^2} \quad \text{and} \quad f_2 = \frac{(t-S^2)(t-1)}{(t+S)^2}.$$

f_1 specializes to 1 as S specializes to 1, so under the boundary map $d_{S=1}$ any term involving f_1 goes to zero. So the only term in $((1+I)^*)^{n-1} \cdot [S]_1$ that can give a contribution to $d_{S=1}$ is $f_2 \cdot \ldots \cdot f_2 \cdot [S]_1$. But $(f_2 \cdot \ldots \cdot f_2)_{|S=1}$ is not trivial as one sees by take repeated boundaries at $t_i = -1$. So this term cannot occur. Therefore it suffices to show that $f_1 \cdot [S]_1$ specializes to zero at $S = 1$.

This element is already defined in $K_2^{(2)}(X_{\mathbb{G}_m \setminus \{1\}} \setminus \{t = S, S^2\})$, and the boundary map $d_{S=1}$ lands in $K_1^{(1)}(X_{\mathbb{Q}}; \square^1) \cong K_2^{(1)}(\mathbb{Q}) = 0$, so $f_1 \cdot [S]_1$ is actually the restriction of an element in $K_2^{(2)}(X_{\mathbb{G}_m} \setminus \{t = S, S^2\}; \square^1)$ and can be specialized at $S = 1$ to an element in $K_2^{(2)}(X; \square^1) \cong K_3^{(2)}(\mathbb{Q}) = 0$ by Borel. This shows that if we want to specialize (46) to $S = 1$ we can ignore $g \cdot [S]_1$ and only specialize S to 1 in $\tilde{\alpha}$. Up to terms involving $(1+I)^*$ this will give us $[1]_n - 2^{n-1}[1]_n - 2^{n-1}[-1]_n = 0$, which shows that this relation holds in $M_{(n)}(\mathbb{Q}) \subset L^{\text{alt}}_{(n)}(\mathbb{Q})$, which is what we wanted to show.

The complex in Example 3.18 has $\bigwedge^{n-1} F^*_{\mathbb{Q}} \otimes M_{(1)}$ as its n-th term, with $M_{(1)} \cong F^*_{\mathbb{Q}}$, whereas Goncharov's complex has $\bigwedge^n F^*_{\mathbb{Q}}$ in the corresponding place. We therefore construct a subcomplex of $\mathcal{M}_{(n)}$ that will turn out to be acyclic if $Y = \operatorname{Spec}(F)$, where F is a field, such that the quotient complex has the required term on the right.

PROPOSITION 3.20. *Suppose that Y is a regular noetherian scheme with no low weight K-theory. Assume that there are no low weights in the K-theory of finite dimensional regular noetherian schemes defined over cyclotomic fields. Assume U is as in Theorem* 3.15, *and is closed under inversion. Write $\alpha_p(u)$ for $[u]_p + (-1)^p[u^{-1}]_p$ for $p \geqslant 2$, and write N_p for the subspace of $M_{(p)}$ generated by those elements. Then the subcomplex*

$$N_n \to U_{\mathbb{Q}} \otimes N_{n-1} \to \cdots \to \bigwedge^{n-2} U_{\mathbb{Q}} \otimes N_2 \to d\Big(\bigwedge^{n-2} U_{\mathbb{Q}} \otimes N_2\Big)$$

of $\mathcal{M}^{\cdot}_{(n)}$ *is isomorphic to the complex*

$$S^n \to U_{\mathbb{Q}} \otimes S^{n-1} \to \cdots \to \bigwedge^{n-2} U_{\mathbb{Q}} \otimes S^2 \to d\Big(\bigwedge^{n-2} U_{\mathbb{Q}} \otimes S^2\Big)$$

where S^p *is the subspace of* $\mathrm{Sym}^p(U_{\mathbb{Q}})$ *generated by the elements* $u \otimes \ldots \otimes u$, $u \in U$.

Proof. To prove that $d\colon N_p \to U_{\mathbb{Q}} \otimes N_{p-1}$ is injective, we can assume by induction that $N_{p-1} \to \mathrm{Sym}^{p-1}(U_{\mathbb{Q}})$ is injective and check that $\alpha_p(u) \mapsto u \otimes \ldots \otimes u \in \mathrm{Sym}^p(U_{\mathbb{Q}})$ is an injection. (For $p = 2$ this gets the induction started.) But choosing a basis of the subgroup generated by U in $\Gamma(Y, \mathcal{O}_Y^*)$, we see that any element α with $d\alpha = 0$ must be a pullback of an element $\beta = \sum_j \alpha_p(\zeta_j T_1^{n_{1,j}} \ldots T_m^{n_{m,j}})$ on an open part Z_{loc} of $Z \stackrel{\mathrm{def}}{=} \mathrm{Spec}(F[T_1, \ldots, T_m, (T_1 \cdots T_m)^{-1}])$, and $d\beta = 0$ in the corresponding $U_{\mathbb{Q}} \otimes \mathrm{Sym}$ there, so β defines an element in $H^1(\mathcal{M}_{(n)}(Z_{\mathrm{loc}})) \subset K_{2n-1}^{(n)}(Z_{\mathrm{loc}})$. (Here the ζ_j's are roots of unity, Z_{loc} is such that the differences of all the elements that appear in β, and their differences with 1 are invertible, and F is a cyclotomic extension of $\mathbb{Q}$ containing all the ζ_j's.) Consider the localizaton sequence [27, Théorème 4, p. 521]

$$\cdots \to K_{2p-1}^{(p)}(Z) \to K_{2p-1}^{(p)}(Z_{\mathrm{loc}}) \to K_{2p-2}^{(m-p)'}(Z \setminus Z_{\mathrm{loc}}) \to \cdots.$$

LEMMA 3.21. $K_{2p-2}^{(m-p)'}(Z \setminus Z_{\mathrm{loc}}) = 0$.

Proof. Write $Z \setminus Z_{\mathrm{loc}} = W_{m-1} \supset \ldots \supset W_0 \supset \emptyset$ with $W_{i+1} \setminus W_i$ regular of dimension $i+1$, then in the spectral sequence

$$K_{-l-q}^{(p-l-1)}(W_{m-l-1} \setminus W_{m-l-2}) \Rightarrow K_{-l-q}^{(m-p)'}(Z \setminus Z_{\mathrm{loc}})$$

all terms contributing to $K_{2p-2}^{(m-p)'}(Z \setminus Z_{\mathrm{loc}})$ vanish.

Because of the lemma, β is actually in the image of $K_{2p-1}^{(p)}(Z) \cong K_{2p-1}^{(p)}(F)$. Therefore, to determine that it vanishes, we can adjoin more roots of unity so that we can specialize all T_i's to roots of unity at a point in Z_{loc}. We get an element that is a sum of $\alpha_p(\zeta)$'s (where ζ is a root of unity). It now follows from the computation of the regulator map Proposition 4.1 below that the regulator vanishes on this element for every embedding of the cyclotomic field into $\mathbb{C}$. By Borel's theorem the element must then be zero.

COROLLARY 3.22. *Write* $\widetilde{M}_{(p)}$ *for* $M_{(p)}/N_p$ *if* $p \geqslant 2$. *Assume that* U *is as in Theorem* 3.20, *and such that* $\mathrm{Sym}^p(U_{\mathbb{Q}})$ *is generated by* $u \otimes \ldots \otimes u$ *for* $p = 2, \ldots, n$. *Let* $\widetilde{\mathcal{M}}^{\cdot}_{(n)}$ *be the complex*

$$\widetilde{M}_{(n)} \to U_{\mathbb{Q}} \otimes \widetilde{M}_{(n-1)} \to \bigwedge^{2} U_{\mathbb{Q}} \otimes \widetilde{M}_{(n-2)} \to \ldots \to \bigwedge^{n-2} U_{\mathbb{Q}} \otimes \widetilde{M}_{(2)} \to \bigwedge^{n} U_{\mathbb{Q}}.$$

where the last differential maps $[u]_2$ *to* $u \wedge (1-u)$ *if* $u \neq 1$. ($[1]_2 = 0$ *anyway.*) *Then under the assumptions of Theorem* 3.15, *and the additional assumptions in Proposition* 3.20, *we have a map*

$$H^p(\widetilde{\mathcal{M}}^{\cdot}_{(n)}) \to K_{2n-p}^{(n)}(Y),$$

which is an injection for $p = 1$.

Proof. The complex

$$\mathrm{Sym}^n(U_{\mathbb{Q}}) \to U_{\mathbb{Q}} \otimes \mathrm{Sym}^{n-1}(U_{\mathbb{Q}}) \to \cdots$$
$$\cdots \to \bigwedge^{n-2} U_{\mathbb{Q}} \otimes \mathrm{Sym}^2(U_{\mathbb{Q}}) \to d\Big(\bigwedge^{n-2} U_{\mathbb{Q}} \otimes \mathrm{Sym}^2(U_{\mathbb{Q}})\Big)$$

is well known to be acyclic, a chain homotopy being

$$K_n\colon x_1 \wedge \ldots \wedge x_{n-1} \otimes y_1 \otimes \ldots \otimes y_{m+1}$$
$$\mapsto \sum_{i=1}^{n} (-1)^i x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_n \otimes \mathrm{sym}(x_i, y_1, \ldots, y_m)$$

where $\mathrm{sym}(z_1, \ldots, z_{m+1}) = \sum_{\sigma \in S_{m+1}} z_{\sigma(1)} \otimes \ldots \otimes z_{\sigma(m+1)}$. K_n satisfies

$$mK_{n+1}d_n - d_{n-1}K_n = (-1)^{n+1}(m+1)!$$

Remark 3.23. The known facts about the weights for the K-theory of regular noetherian finite dimensional schemes ($K_n^{(1)} = 0$ for $n \geqslant 2$, see [27, Théorème 4, p. 521]) show that in fact $H^{n-1}(N_n) = H^n(N_n) = 0$, so that without any additional assumptions $H^p(\widetilde{\mathcal{M}}_{(n)}) \cong H^p(\mathcal{M}_{(n)})$ for $p = n-1$ or n, and the map in Corollary 3.22 always exists for those values of p.

EXAMPLE 3.24. The prime example for applying Corollary 3.22 is of course again the case when $Y = \mathrm{Spec}(F)$ for a field F having no low weight K-theory. If we take the direct limit over finite subsets of F^* as in Example 3.18, we get a complex

$$\widetilde{M}_{(n)} \to F^*_{\mathbb{Q}} \otimes \widetilde{M}_{(n-1)} \to \bigwedge^2 F^*_{\mathbb{Q}} \otimes \widetilde{M}_{(n-2)} \to \cdots \to \bigwedge^{n-2} F^*_{\mathbb{Q}} \otimes \widetilde{M}_{(2)} \to \bigwedge^n F^*_{\mathbb{Q}}$$

Under the additional assumptions about weights in Proposition 3.20 we have a map from the p-th cohomology of this complex to $K^{(n)}_{2n-p}(F)$. This map is an injection for $p = 1$.

4. Computation of the Regulator Map

In this section we want to compute the regulator map on $H^1(\mathcal{M}_{(n)})$ and $H^1(\widetilde{\mathcal{M}}_{(n)})$ for number fields.

Recall notation from the beginning of Section 3.2. Let $T^n_{\mathrm{loc}} = X^{n-1}_{\mathbb{C}} \times \mathbb{G}^1_{m,\mathbb{C}} \setminus \bigcup\{t_j = S\}$. Write T for short for T^n_{loc} and $\overline{T}$ for a compactification with complement a divisor with normal crossings D, with j the inclusion $T \to \overline{T}$. By (38) the regulator of $[S]_n$ lands in the group

$$H^n_{\mathcal{D}}(T^n_{\mathrm{loc}}; \square^{n-1}; \mathbb{R}(n)) \cong$$
$$\left\{ \begin{array}{l} \phi \in \Gamma(j_* S^{n-1}_T(n-1)),\ d\phi = \pi_{n-1}\omega \\ \text{for some } \omega \in \Omega^n_{\overline{T}}(\log D) \text{ such that} \\ \phi_{|\square^{n-1}} \equiv 0 \end{array} \right\} \Big/ \left\{ \begin{array}{l} d\psi,\ \psi \in \Gamma(j_* S^{n-2}_T(n-1)) \\ \text{such that } \psi_{|\square^{n-1}} \equiv 0 \end{array} \right\}.$$

Denote the image of $[S]_n$ for $n \geqslant 2$ in the right hand side by $\varepsilon_n(t_1, \ldots, t_{n-1}, S)$. As it stands those forms are not suitable for computations which involve integration because the integral might not even converge. But if we specialize S to a value in $\mathbb{C}^*$ we get the restriction of ε_n as an element of $H_{\mathrm{dR}}^{n-1}(X_{\mathrm{loc}}^{n-1}; \square^{n-1}; \mathbb{R}(n-1))$ because there are no holomorphic n-forms on X_{loc}^{n-1}.

We use the embedding $H_{\mathrm{dR}}^{n-1}(X_{\mathrm{loc}}^{n-1}; \mathbb{R}(n-1))$ into $H_{\mathrm{dR}}^{n-1}(X_{\mathrm{loc}}^{n-1}; \square^{n-1}; \mathbb{C})$. This last group can be computed using a complex of forms with more modest behaviour at infinity: let $\overline{X}^{n-1}$ be a compactification of ${X_{\mathrm{loc}}}^{n-1}$ with complement a divisor with normal crossings. For a variety Y let $A(Y)$ be the complex of complex valued C^∞-forms on Y. Let $A^{\cdot}_{\overline{X}^{n-1}}(\log D)$ be the complex on $\overline{X}^{n-1}$ generated by $A^{\cdot}_{\overline{X}^{n-1}}$ and the complex of sheaves of holomorphic forms on $\overline{X}^{n-1}$ with logarithmic poles along D. Then $A^{\cdot}_{\overline{X}^{n-1}}(\log D) \to A^{\cdot}_{X^{n-1}}$ is a quasi isomorphism ([9, 3.2.3 b)]), and this still holds true if we impose the vanishing conditions on $\square^{n-1}$. So we can represent ε_n by an element in the complex $A^{\cdot}_{\overline{X}^{n-1}}(\log D)$, and this ε_n differs from the restriction of the original ε_n by the boundary of a (possibly not so well behaved) $(n-2)$-form.

In order to get a number out of ε_n for a specific value of $z \in \mathbb{C}^*$, we compute

$$\frac{1}{(2\pi i)^{2n-2}} \int_{X^{n-1}} \varepsilon_n \wedge \mathrm{d}\, i \arg t_1 \wedge \ldots \wedge \mathrm{d}\, i \arg t_{n-1}. \tag{48}$$

which converges due to the behaviour at infinity and the vanishing of ε_n on $\square^{n-1}$. Using Stokes' theorem one checks that the value of (48) is independent of the representative of the class of ε_n.

We have to justify why we compute (48). Consider the long exact sequence

$$\begin{aligned} \cdots &\to H_{\mathrm{dR}}^p(X^n; \square^{n-1}; \mathbb{R}(q)) \to H_{\mathrm{dR}}^p(\{t_n = 0, \infty\}; \square^{n-1}; \mathbb{R}(q)) \\ &\to H_{\mathrm{dR}}^{p+1}(X^n; \square^n; \mathbb{R}(q)) \to H_{\mathrm{dR}}^{p+1}(X^n; \square^{n-1}; \mathbb{R}(q)) \to \cdots \end{aligned}$$

(which is (36) together with (37)). Because of the homotopy property of the Deligne cohomology we get an isomorphism

$$H_{\mathrm{dR}}^p(X^{n-1}; \square^{n-1}; \mathbb{R}(q)) \to H_{\mathrm{dR}}^{p+1}(X^n; \square^n; \mathbb{R}(q)) \tag{49}$$

and hence $H_{\mathrm{dR}}^n(X^n; \square^n; \mathbb{R}(q)) \cong H_{\mathrm{dR}}^0(\mathrm{Spec}(\mathbb{C}); \mathbb{R}(q)) \cong \mathbb{R}(q)$. This isomorphism can be made explicit as follows. Let f be an $\mathbb{R}(1)$-valued function on X such that $f(\infty) - f(0) = 2\pi i$. Then integrating the class of $\mathrm{d}f$ along a path from 0 to ∞ shows that $\mathrm{d}f$ generates $H_{\mathrm{dR}}^1(X; \square; \mathbb{R}(1))$, and it shows that in fact $\mathrm{d}f$ is in the image of $H^1(X; \square; \mathbb{Z}(1))$. The isomorphism in (49) is given by $\psi \mapsto \psi \cup \mathrm{d}f$, where ψ is pulled back along the projection onto the first $n-1$ coordinates, and $\mathrm{d}f$ along the projection onto the last coordinate. If we let $\mathrm{d}f_i$ denote the pullback of $\mathrm{d}f$ along the projection of X^{n-1} onto the i-th coordinate, then we see that $\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_{n-1}$ is a generator of $H_{\mathrm{dR}}^{n-1}(X^{n-1}; \square^{n-1}; \mathbb{R}(n-1))$, and lies in fact in the image of $H^{n-1}(X^{n-1}; \square^{n-1}; \mathbb{Z}(n-1))$. Note that $\int_X \mathrm{d}f \wedge \mathrm{d}i \arg t = 2\pi i(f(\infty) - f(0)) = (2\pi i)^2$. Therefore, the integral in (48) will map the generators of $H^{n-1}(X^{n-1}; \square^{n-1}; \mathbb{Z}(n-1)) \subset H_{\mathrm{dR}}^{n-1}(X^{n-1}; \square^{n-1}; \mathbb{R}(n-1))$ to ± 1.

Consider the commutative diagram

$$\begin{array}{ccc} K_n^{(n)}(X^{n-1};\square^{n-1}) & \longrightarrow & K_n^{(n)}(X_{\mathrm{loc}}^{n-1};\square^{n-1}) \\ \downarrow{\scriptstyle \mathrm{reg}} & & \downarrow{\scriptstyle \mathrm{reg}} \\ H_{\mathrm{dR}}^{n-1}(X^{n-1};\square^{n-1};\mathbb{R}(n-1)) & \longrightarrow & H_{\mathrm{dR}}^{n-1}(X_{\mathrm{loc}}^{n-1};\square^{n-1};\mathbb{R}(n-1)). \end{array} \tag{50}$$

Let $\alpha \in K_n^{(n)}(X^{n-1};\square^{n-1})$ be represented by $\beta \in H^1(C^{\cdot}_{(n),\log})$ under the map in Proposition 3.5, which is nothing but the inverse of the localization map in (50) in this case. The localization of the regulator of α is the same class as the regulator of β, and because of the independence of the integral of the representative of the class in cohomology we can compute the regulator of α by computing the integral in (48) for the regulator of β. Note that from the definition of $C^{\cdot}_{(n),\log}$ (see just before Theorem 3.15) β is a sum of elements of type $[x]_n$ for $x \in \mathbb{C}^*$ or $f \cdot \gamma$ where $f \in (1+I)^*$ and $\gamma \in C^{\cdot}_{(n-1),\log}$. We therefore want to compute two types of integrals. For the description of the result we need to introduce some well known functions.

For $k \in \mathbb{N}$ and $|z| < 1$, $z \in \mathbb{C}$, let $Li_k(z) = \sum_{n=1}^{\infty} z^n/n^k$. It is well known that those functions can be analytically continued to multi-valued functions on $\mathbb{C} \setminus \{0,1\}$. It turns out that the result of the integration can be expressed in the function

$$P_n(z) = \mathfrak{R}_{n-1} \sum_{k=0}^{n-1} \frac{1}{k!}(-\log|z|)^k Li_{n-k}(z).$$

Here $\mathfrak{R}_m$ means that we take the real part for m even, the imaginary part for m odd. It is well known that this is single valued function on $\mathbb{C}^*$.

PROPOSITION 4.1. *1. Let $z \in \mathbb{C}^*$. Let $\varepsilon_n(t_1,\dots,t_{n-1},z) \in H^n_{\mathcal{D}}(X^{n-1}_{\mathbb{C},\mathrm{loc}};\square^{n-1};\mathbb{R}(n))$ be the image of $[z]_n$, $n \geqslant 2$. Then*

$$\frac{1}{(2\pi i)^{2n-2}} \int_{(\mathbb{P}^1)^{n-1}} \varepsilon_n \wedge d\, i \arg t_1 \wedge \ldots \wedge d\, i \arg t_{n-1} = \pm \frac{1}{(2\pi)^{n-1}} (n-1) P_n(z).$$

2. If ε_n is the class of an element in $(1+I)^* \cdot K_{(n-1)}$, the integral vanishes.

Remark 4.2. Because the function $P_m(z)$ satisfies the functional equation

$$\begin{aligned} &P_m(z) + P_m(z^{-1}) = 0 && \text{for } m \text{ even,} \\ &P_m(z) - P_m(z^{-1}) = \tfrac{\log^m |z|}{m!} && \text{for } m \text{ odd,} \end{aligned}$$

in order to get a well defined regulator map on $\widetilde{\mathcal{M}^{\cdot}_{(n)}}$ for n odd, one has to take a suitable linear combination of $\log^d |z| \cdot P_{n-d}(z)$'s where $0 \leqslant d \leqslant n$, see Remark 5.2.

Proof. 1. We can restrict the row in the spectral sequence in which we constructed the elements $[S]_n$ to a fixed value for S in $\mathbb{C}^*$, z. Then the boundary map in the spectral sequence corresponds to the residue in the de Rham cohomology groups in which the ε_n's live, with the signs as in (41). By construction ε_n satisfies (see (41)):

1. $\varepsilon_1(S) = \log|1-S|$,

2. $\text{res}_{t_j=S}(\varepsilon_n) = (-1)^j \varepsilon_{n-1}$,
3. $\varepsilon_n \equiv 0$ for $t_i = 0, \infty$.

To start with pick z not in $(-\infty, 0]$ and $z \neq 1$. Let I be a path from $-\infty$ to 0 in $[-\infty, 0]$. Let $k, l-1 \geqslant 0$, and let C_r be $\mathbb{P}^1$ minus discs of radius r around $t = 0, 1, z, \infty$ and slit open from $-\infty$ to 0. Write $t_j = x_j + iy_j$. We want to compute

$$\int_{X^l} \mathrm{d}\, i \arg t_l \wedge \ldots \wedge \mathrm{d}\, i \arg t_1 \wedge \varepsilon_{l+1}(z)$$

for $l = n-1$ but it turns out that it is convenient to start with a slightly more general integral. Using the orientation given by $\mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_l \wedge \mathrm{d}y_l \wedge \mathrm{d}x_{l+1} \wedge \ldots \wedge \mathrm{d}x_{k+l}$ on $X^l \times I^{k-l-1}$ one finds by applying Stokes' theorem that

$$\begin{aligned}
&\int_{X^l \times I^k} \mathrm{d}i \arg t_l \wedge \ldots \wedge \mathrm{d}i \arg t_1 \wedge \varepsilon_{k+l+1}(z) \\
&\quad = \lim_{r \to 0} \int_{X^{l-1} \times C_r \times I^k} \mathrm{d}\big(i \arg t_l \, \mathrm{d}i \arg t_{l-1} \wedge \ldots \wedge \mathrm{d}i \arg t_1 \wedge \varepsilon_{k+l+1}(z)\big) \\
&\quad = 2\pi i \int_{X^{l-1} \times I^{k+1}} \mathrm{d}i \arg t_{l-1} \wedge \ldots \wedge \mathrm{d}i \arg t_1 \wedge \varepsilon_{k+l+1}(z) + \\
&\qquad + 2\pi i (i \operatorname{Arg} z) \int_{X^{l-1} \times I^k} \mathrm{d}i \arg t_{l-1} \wedge \ldots \wedge \mathrm{d}i \arg t_1 \wedge \varepsilon_{k+l}(z)
\end{aligned}$$

because the only residue that contributes is at $t_l = z$. Hence one finds by induction on l that

$$\begin{aligned}
&\int_{X^l \times I^k} \mathrm{d}i \arg t_l \wedge \ldots \wedge \mathrm{d}i \arg t_1 \wedge \varepsilon_{k+l+1}(z) \\
&\quad = (2\pi i)^l \sum_{p=0}^{l} \binom{l}{p} (i \operatorname{Arg} z)^{l-p} \int_{I^{k+p}} \varepsilon_{k+p+1}(z).
\end{aligned}$$

In particular we find

$$\begin{aligned}
&\int_{X^l} \mathrm{d}i \arg t_l \wedge \ldots \wedge \mathrm{d}i \arg t_1 \wedge \varepsilon_{l+1}(z) \\
&\quad = (2\pi i)^l \sum_{p=0}^{l} \binom{l}{p} (i \operatorname{Arg} z)^{l-p} \int_{I^p} \varepsilon_{p+1}(z). \qquad (51)
\end{aligned}$$

In order to evaluate this pick two points $z_1, z_2 \notin (-\infty, 0]$ and $z_1, z_2 \neq 1$. Pick a path γ from z_1 to z_2 in $\mathbb{C} \setminus (-\infty, 0)$ avoiding 1. Note that the integrals $\int_{I^p} \varepsilon_{p+1}(z)$ do not depend on the explicit representative of ε_{p+1} in the relative de Rham cohomology. In fact they will have the same value if we use the restriction of the original $\varepsilon_{p+1}(S)$ to $S = z$, because those two forms differ by a boundary vanishing on the components of $\square^p$, say $\mathrm{d}\alpha$, and then $\int_{I^p} \mathrm{d}\alpha = \int_{\partial I^p} \alpha = 0$. Hence we can use a form $\varepsilon_{p+1}(S)$ with $\mathrm{d}\varepsilon_{p+1}(S) = \pi_p \omega_{p+1}$ for some holomorphic $p+1$-form ω_{p+1} with logarithmic poles on X^p_{loc}. In order to determine ω_{p+1} observe that the boundary maps in the row of the spectral sequence where we construct $[S]_{p+1}$ correspond to the residue map on holomorphic forms. We see that ω_{p+1} is determined entirely by its residue at the $t_j = S$,

$\mathrm{res}_{t_j=S}\omega_{p+1} = (-1)^{j+1}\omega_p$. (For the extra minus sign, see (41).) By induction one finds, because $\omega_1 = \mathrm{d}\log(1-S)$, that

$$\omega_{p+1} = \mathrm{d}\log\frac{t_1 - S}{t_1 - 1} \wedge \ldots \wedge \mathrm{d}\log\frac{t_p - S}{t_p - 1} \wedge \mathrm{d}\log(1-S).$$

Using this and once again applying Stokes' theorem, we get

$$\int_{\partial(I^p\times\gamma)} \varepsilon_{p+1} = \int_{I^p\times\gamma} \pi_p\omega_{p+1} = \pi_p \int_\gamma (\mathrm{Log}\, S)^p\, \mathrm{d}\log(1-S).$$

Because of the orientations involved the left hand side of this equation equals

$$(-1)^p\left(\int_{I^p}\varepsilon_{p+1}(z_2) - \int_{I^p}\varepsilon_{p+1}(z_1)\right)$$

with the orientation on I^p as before.

We can assume by induction that $\int_{I^p}\varepsilon_{p+1}(z_1)$ approaches zero as z_1 approaches zero for $p \leqslant l-1$. (For $p=0$ this is certainly true because then this integral is simply $\log|1-z|$.) Write C_l for this limit in the case $p = l$. Then combining this result with (51) above we find that

$$\begin{aligned}&\int_{X^l} \mathrm{d}\, i\arg t_l \wedge \ldots \wedge \mathrm{d}\, i\arg t_1 \wedge \varepsilon_{l+1}(z)\\ &= (2\pi i)^l \sum_{p=0}^{l}\binom{l}{p}(i\mathrm{Arg}\, z)^{l-p}(-1)^p\pi_p\int_\gamma(\mathrm{Log}\, S)^p\,\mathrm{d}\log(1-S) + (2\pi i)^l C_l.\end{aligned}$$

Ignoring the term involving C_l for the moment, we can rewrite this as follows. Because

$$(\mathrm{Log}\, S)^p\,\mathrm{d}\log(1-S) = \mathrm{d}\sum_{k=1}^{p+1}(-1)^k\frac{p!}{(p-k+1)!}(\mathrm{Log}\, S)^{p+1-k}Li_k(S)$$

(where we continue the Li_k's analytically along γ), the integral becomes

$$\begin{aligned}&(2\pi i)^l\sum_{p=0}^{l}\sum_{k=1}^{p+1}\binom{l}{p}(i\mathrm{Arg}\, z)^{l-p}(-1)^{p+k}\frac{p!}{(p-k+1)!}\pi_p\big((\mathrm{Log}\, z)^{p-k+1}Li_k(z)\big)\\ &= -(2\pi i)^l\pi_l\sum_{k=1}^{l+1}\sum_{p=k-1}^{l}\frac{l!}{(l-k+1)!}\binom{l-k+1}{l-p}(i\mathrm{Arg}\, z)^{l-p}(-\mathrm{Log}\, z)^{p+1-k}Li_k(z)\\ &= -(2\pi i)^l l!\pi_l\sum_{k=1}^{l+1}\frac{1}{(l-k+1)!}(-\log|z|)^{l-k+1}Li_k(z)\\ &= \pm(2\pi)^l l! P_{l+1}(z)\end{aligned}$$

For $z \in (-\infty, 0)$ one can change the previous situation by replacing I with a path from $-\infty$ to 0 avoiding z and 1, and γ a path from z_1 to z_2 not intersecting I, so that the terms $t_i - z$ and $t_i - 1$ involved in the integrals are everywhere defined. One gets the same result. Keeping track of what the value of arg and log is one concludes that the formulas hold for those values of z also. But this means in particular that C_l must be zero. Namely, otherwise the formula in (51) for $l+1$ instead of l would be changed with $(2\pi i)^{l+1}((-1)^{l+1}C_{l+1} + (-1)^l(i\mathrm{Arg}\,z)C_l)$ which would depend on the path chosen to compute the integral. This shows that the integral in (48) is $P_n(z)$ for $z \in \mathbb{C}$, $z \neq 1$.

Finally, for $z = 1$, one uses the identity $[S^2]_n = 2^{n-1}[S]_n + 2^{n-1}[-S]_n$ which also holds for $S = 1$ as shown in Lemma 3.19, and the fact that the integral vanishes on terms involving factors $(1+I)^*$ by 2 below.

2. By the formula for products in Deligne cohomology, if the regulator of an element $\alpha \in K_{(n-1)}$ is represented by $\psi_{n-1} \in H^{n-2}(X^{n-2}_{\mathrm{loc}}; \square^{n-2}; \mathbb{R}(n-2))$ and $f \in (1+I)^*$, then $f \cdot \alpha$ is represented by $\pm \mathrm{d}i \arg f \wedge \psi_{n-1}$. We can assume that ψ is already represented by an element of $S^{\cdot}_{\overline{X}^{n-2}}(\log D)$, so replacing $\mathrm{d}i \arg f \wedge \psi_{n-1}$ by $\mathrm{d} \log f \wedge \psi_{n-1}$ gives us a representative of the required shape. As for the integral, using Fubini, it suffices to show that $\int_{\mathbb{P}^1} \mathrm{d} \log f \wedge \mathrm{d}\, i \arg t = 0$. This follows easily from $f(\infty) = f(0) = 1$.

5. The Number Field Case

If F is a number field, it is known that the K-theory of F satisfies the conditions about weights in Definition 3.2. So if we let $\mathcal{M}_{(n)}$ denote the complex defined in Example 3.18, then we have a map $H^p(\mathcal{M}_{(n)}) \to K^{(n)}_{2n-p}(F)$, which is an injection for $p = 1$. We begin by showing that this also holds for the complex $\widetilde{\mathcal{M}}^{\cdot}_{(n)}$ defined in Example 3.24, without any further assumptions about weights.

PROPOSITION 5.1. *If F is a number field there are maps*

$$H^p(\widetilde{\mathcal{M}}^{\cdot}_{(n)}) \to K^{(n)}_{2n-p}(F)$$

for $1 \leqslant p \leqslant n$, and for $p = 1$ this map is an injection.

Proof. As this statement is a corollary to the statement of Proposition 3.20, we change the proof of this proposition using Borel's theorem instead of an assumption about no low weights. The only part that has to be modified is the statement that if $\alpha = \sum_j \alpha_p(x_j)$ satisfies $d\alpha = 0$ in $F^*_{\mathbb{Q}} \otimes \mathrm{Sym}^{p-1}(F^*_{\mathbb{Q}})$, then $\alpha = 0$ in $L^{\mathrm{alt}}_{(p)}$. But this follows easily, because the functions P_m satisfy the functional equation (which follows from [31, §1, Proposition 1, p. 411]

$$\begin{aligned} P_m(z) + P_m(z^{-1}) &= 0 && \text{for } m \text{ even,} \\ P_m(z) - P_m(z^{-1}) &= \tfrac{\log^m |z|}{m!} && \text{for } m \text{ odd.} \end{aligned}$$

So the regulator map vanishes for each embedding of F into $\mathbb{C}$: for p even, it vanishes identically on the symbols α_p, for p odd the condition $d\alpha = 0$ ensures that the terms involving logarithms cancel. Therefore by Borel's theorem α must be zero.

Remark 5.2. According to Proposition 4.1 the regulator map on $H^1(\mathcal{M}^{\cdot}_{(n)})$ is given by $[x]_n \mapsto \{P_n(\sigma(x))\}_\sigma$ where σ runs through the embeddings of F into $\mathbb{C}$ up to complex conjugation. As the functions P_m satisfy the functional equation

$$\begin{aligned} P_m(x) + P_m(x^{-1}) &= 0 && \text{for } m \text{ even,} \\ P_m(x) - P_m(x^{-1}) &= \tfrac{\log^m |x|}{m!} && \text{for } m \text{ odd,} \end{aligned}$$

this has to be modified in order to get a regulator that factors through $H^1(\widetilde{\mathcal{M}}^{\cdot}_{(n)})$. Consider the composition of maps

$$\begin{aligned} M_{(n)} &\xrightarrow{d} F_{\mathbb{Q}}^* \otimes M_{(n-1)} \xrightarrow{\mathrm{id}\otimes d} F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^* \otimes M_{(n-2)} \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes d} \cdots \\ &\longrightarrow (F_{\mathbb{Q}}^*)^{\otimes p} \otimes M_{(n-p)}. \end{aligned}$$

We can compose this with the map

$$\begin{aligned} (F_{\mathbb{Q}}^*)^{\otimes p} \otimes M_{(n-p)} &\to \mathbb{R} \\ x_1 \otimes \ldots \otimes x_p \otimes [y]_{n-p} &\mapsto \log|x_1| \cdots \log|x_p| P_{n-p}(y) \end{aligned}$$

to get a map $M_{(n)} \to \mathbb{R}$ given by $[x]_n \mapsto \log^p|x|P_{n-p}(x)$. By construction this map vanishes on H^1, so that we can change the function $P_n(x)$ with linear combinations of $\log^p|x|P_{n-p}(x)$ for $p \geqslant 1$ without changing its value on H^1. We will therefore look at the function considered by Zagier (and others):

$$P_{\mathrm{Zag},n}(x) = \mathfrak{R}_{n-1} \sum_{j=0}^{n-1} \frac{2^j B_j}{j!} \log^j|x| Li_{n-j}(x),$$

which satisfies $P_{\mathrm{Zag},n}(x) + (-1)^n P_{\mathrm{Zag},n}(x^{-1}) = 0$. (Here B_j is the j-th Bernoulli number.) We will consider this function as the regulator map on $H^1(\mathcal{M}^{\cdot}_{(n)})$ and $H^1(\widetilde{\mathcal{M}}^{\cdot}_{(n)})$.

We now want to show, using work of Suslin, Goncharov and Zagier, that the maps from H^1 to $K_{2n-1}^{(n)}(F)$ are isomorphisms in certain cases.

We begin with the case $n = 2$. Note that the complex $\mathcal{M}^{\cdot}_{(2)}$ is already essentially in [4]. We refer to [15] for the proofs (and some details) of the statements here, which were proved by Suslin but not published.

Put

$$B_2(F) = \left\{ \begin{array}{l} \text{free } \mathbb{Q}\text{-vector space on } F^* \setminus \{1\} \\ \text{modulo certain relations} \end{array} \right\}$$

(see [15], we tensor the construction there with $\mathbb{Q}$). We will denote the generators of this vector space with $\{x\}_2$ for $x \in F^* \setminus \{1\}$. There is a map

$$\begin{aligned} d\colon B_2(F) &\to F_{\mathbb{Q}}^* \wedge F_{\mathbb{Q}}^* \\ \{x\}_2 &\mapsto x \wedge (1-x) \end{aligned}$$

and the kernel of d is isomorphic to $K_3^{(2)}(F)$. The map $\{x\}_2 \mapsto P_2(x) = P_{\mathrm{Zag},2}(x)$ factors through the relations in $B_2(F)$, and after embedding F into $\mathbb{C}$, the regulator of $\alpha = \sum_i x_i \in \operatorname{Ker} d$ for this embedding is given by $\sum_i P_2(x_i) = \sum_i P_{\mathrm{Zag},2}(x_i)$.

Define a map ψ_2 fitting into the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & K_3^{(2)}(F) & \longrightarrow & L_{(2)}(F)/N_2 & \longrightarrow & F_{\mathbb{Q}}^*\wedge F_{\mathbb{Q}}^* \\
\uparrow & & \uparrow & & \uparrow & & \| \\
0 & \longrightarrow & K_3^{(2)}(F)\cap\widetilde{M}_{(2)} & \longrightarrow & \widetilde{M}_{(2)} & \longrightarrow & F_{\mathbb{Q}}^*\wedge F_{\mathbb{Q}}^* \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \psi_2} & & \| \\
0 & \longrightarrow & K_3^{(2)}(F) & \longrightarrow & B_2(F) & \xrightarrow{d} & F_{\mathbb{Q}}^*\wedge F_{\mathbb{Q}}^*
\end{array}$$

by $\psi_2(\{x\}_2) = [x]_2$. This is well defined: on an element in the relations d vanishes, so it lands in $K_3^{(2)}(F) \cap \widetilde{M}_{(2)}$. But after embedding F into $\mathbb{C}$, also the regulator, given by $\sum_k \{x_k\}_2 \mapsto \sum P_{\mathrm{Zag},2}(x_k)$ vanishes because it is an element in the relations. Hence the image of this relation defines the trivial element in $K_3^{(2)}(F) \cap \widetilde{M}_{(2)}$. Because the regulator factors (up to a non zero factor) as $K_3^{(2)}(F) = \mathrm{Ker}\,(d) \to \widetilde{M}_{(2)} \xrightarrow{P_{\mathrm{Zag},2}} \mathbb{R}$ for each embedding of F into $\mathbb{C}$, the induced map in the diagram above must be injective. Because $K_3^{(2)}(F)$ is a finite dimensional $\mathbb{Q}$-vector space, it follows that $K_3^{(2)}(F)$ must be isomorphic to $K_3^{(2)}(F) \cap \widetilde{M}_{(2)}$. Because we can extend all rows in the diagram with the cokernel, $K_2^{(2)}(F)$, it follows from the five lemma that $B_2(F) \cong \widetilde{M}_{(2)}(F) \cong L_{(2)}(F)/N_2$. Because $\widetilde{M}_{(2)} = M_{(2)}/N_2$, it follows that $M_{(2)}(F) = L_{(2)}(F)$.

For $n = 3$, Goncharov in [15] proves that, with

$$B_3(F) = \left\{ \begin{array}{l} \text{free } \mathbb{Q}\text{-vector space on } F^* \text{ modulo} \\ \text{certain relations} \end{array} \right\}$$

there is a map

$$\begin{aligned} d\colon\ B_3(F) &\to F_{\mathbb{Q}}^* \otimes B_2(F) \\ \{x\}_3 &\mapsto x \otimes \{x\}_2 \\ \{1\}_3 &\mapsto 0. \end{aligned}$$

There is a map from $H_5(GL_5(F))$ to $\mathrm{Ker}\, d$, and the regulator on $K_5^{(3)}(F)$ can be factored

$$K_5^{(3)}(F) \subset H_5(GL_5(F)) \otimes \mathbb{Q} \to \mathrm{Ker}\, d \xrightarrow{P_{\mathrm{Zag},3}} \mathbb{R} \tag{52}$$

where the last map is given by a non zero multiple of $\{x\}_3 \mapsto P_{\mathrm{Zag},3}(x)$. This last map vanishes on the relations involved in the definition of $B_3(F)$.

We can define a similar diagram as above with ψ_3:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_5^{(3)}(F) & \longrightarrow & L_{(3)}(F)/N_3 & \longrightarrow & F_{\mathbb{Q}}^* \otimes L_{(2)}(F)/N_2 & \longrightarrow & \bigwedge^3 F_{\mathbb{Q}}^* \\
 & & \uparrow & & \uparrow & & \| & & \| \\
0 & \longrightarrow & K_5^{(3)}(F) \cap \widetilde{M}_{(3)}(F) & \longrightarrow & \widetilde{M}_{(3)}(F) & \longrightarrow & F_{\mathbb{Q}}^* \otimes \widetilde{M}_{(2)}(F) & \longrightarrow & \bigwedge^3 F_{\mathbb{Q}}^* \\
 & & \uparrow & & \uparrow \psi_3 & & l \uparrow \mathrm{id} \otimes \psi_2 & & \| \\
0 & \longrightarrow & \operatorname{Ker} d & \longrightarrow & B_3(F) & \xrightarrow{d} & F_{\mathbb{Q}}^* \otimes B_2(F) & \longrightarrow & \bigwedge^3 F_{\mathbb{Q}}^* \\
 & & \uparrow & & & & & & \\
 & & K_5^{(3)}(F) & & & & & &
\end{array}$$

where $\psi_3(\{x\}_3) = [x]_3$. (The map $K_5^{(3)}(F) \to \operatorname{Ker} d$ is the composite map occurring in (52).) One checks as before that this is well defined. And as we factored the regulator map through $K_5^{(3)}(F) \cap \widetilde{M}_{(3)}(F)$ (i.e. both are given by $P_{\mathrm{Zag},3}$), the map induced on $K_5^{(3)}(F) \to K_5^{(3)}(F) \cap \widetilde{M}_{(3)}(F)$ must be an injection. Hence it must be an isomorphism as both spaces are finite dimensional.

Finally, let $F = \mathbb{Q}(\zeta)$ where ζ is a primitive N-th root of unity, $(N > 2)$, and let $n \geqslant 2$. Zagier's computations in [31] show that the elements $[\zeta^j]_n$ where $0 < j < \frac{N}{2}$ and $(j, N) = 1$ (which obviously satisfy $d[\zeta^j]_n = 0$) must be linearly independent in $\widetilde{M}_{(n)}$. Because the rank of $K_{2n-1}^{(n)}(F)$ is $\frac{1}{2}\,\phi(N)$, it follows that

$$H^1(\mathcal{M}^{\cdot}_{(n)}) \to K_{2n-1}^{(n)}(F)$$

is an isomorphism for all $n \geqslant 2$, and similarly for $H^1(\widetilde{\mathcal{M}}^{\cdot}_{(n)}) \to K_{2n-1}^{(n)}(F)$.

For $F = \mathbb{Q}$ it is easy to check that those maps are isomorphisms too due to the presence of $[1]_n$, and we summarize all this as follows:

THEOREM 5.3. *Let F be a number field. Then the maps*

$$H^1(\mathcal{M}^{\cdot}_{(n)}) \to K_{2n-1}^{(n)}(F)$$

and hence

$$H^1(\widetilde{\mathcal{M}}^{\cdot}_{(n)}) \to K_{2n-1}^{(n)}(F)$$

are isomorphisms for $n = 2, 3$. Moreover, those maps are isomorphisms for all $n \geqslant 2$ if F is a cyclotomic field.

Remark 5.4. For the map $H^1(\mathcal{M}^{\cdot}_{(n)}) \to K_{2n-1}^{(n)}(F)$ (respectively $H^1(\widetilde{\mathcal{M}}^{\cdot}_{(n)}) \to K_{2n-1}^{(n)}(F)$) to exist, there are several steps in its construction at which place we have to tensor with $\mathbb{Q}$. However, in case F is a number field Zagier conjectured (see [31, Section 8]) that the image of what Zagier calls the higher Bloch groups under the conjectured regulator map is a finitely generated Abelian group. It is possible to construct everything we need without tensoring with $\mathbb{Q}$ if we are willing to work up to torsion of bounded exponent at every step. For this, let in the remainder of this section $K_n^{(j)}$ denote the subset of K_n on which all Adams operations ψ^k act as multiplication by k^j, i.e., without tensoring with $\mathbb{Q}$. Let $(1+I)^*$ be the subgroup of $K_1(X_{\mathrm{loc}})$ consisting of al functions $f(t)$ such that

$$f(t) = \prod_i \frac{t - x_i}{t - 1}$$

for which $\prod x_i = 1$. (So our old $(1+I)^*$ is the new one tensored with $\mathbb{Q}$.) We shall also indicate how we can construct the necessary complexes as $\mathbb{Z}$-modules rather than as $\mathbb{Q}$-vector spaces, which allows us to obtain results about finite generation of the image under the regulator map, which are closer to Zagier's original conjecture.

The point is that if we have an exact sequence of Abelian groups

$$0 \to A \to B \xrightarrow{\xi} C \to D \to 0$$

with $M \cdot A = 0$, $N \cdot D = 0$ for some $M, N \in \mathbb{N}$, then there is a morphism from C to B defined as follows. If $\alpha \in C$, $\xi(\beta) = N \cdot \alpha$, then $M \cdot \beta \in B$ is well defined. Using this, one can "invert" the maps pointing from left to right in

$$\begin{aligned} K_{2n-1}^{(n)}(F) &\xrightarrow{\xi_1} H^1(C^{\cdot}_{(n)}(F)) \leftarrow H^1(C_{(n),\log}) \\ &\xrightarrow{\xi_2} H^1(\mathcal{M}^{\cdot}_{(n)}(F)) \xrightarrow{\xi_3} H^1(\widetilde{\mathcal{M}}^{\cdot}_{(n)}(F)) \end{aligned}$$

because each of these maps will be shown to be of the above type (without tensoring with $\mathbb{Q}$). This yields the following statements: for a number field F and a given integer $n \geqslant 2$ there exist a natural number N and a map ξ such that the following diagram commutes

$$\begin{array}{ccc} H^1(\mathcal{M}^{\cdot}_{(n)}(F)) & \xrightarrow{\quad\quad\xi\quad\quad} & K_{2n-1}^{(n)}(F) \\ & \searrow^{\varphi_1} \qquad \swarrow_{m_N} & \\ & K_{2n-1}^{(n)}(F)_{\mathbb{Q}} & \end{array}$$

where m_N is multiplication by N and ϕ_1 is the map of Theorem 3.15. Similarly for $\widetilde{\mathcal{M}}^{\cdot}_{(n)}$. Because $K_{2n-1}^{(n)}(F)$ is a finitely generated group (see 2 below), it follows that the image of the regulator map of the integral version of $\mathcal{M}^{\cdot}_{(n)}$ and $\widetilde{\mathcal{M}}^{\cdot}_{(n)}$ is a finitely generated lattice.

We shall now go through the different steps of the construction of the complexes and the map to see that indeed for a number field F the complexes can be defined integrally, and give rise to a finitely generated group under the regulator map as computed in 4.1. Following [27], we will say "mod $\mathcal{S}$" for "up to torsion of bounded exponent."

The different steps in the construction of the complexes are:

1. the construction of spectral sequence (42);
2. the vanishing of this spectral sequence below a certain row, Lemma 3.4;
3. the acyclicity lemmas 3.7 and 3.9;
4. the construction of the element $[S]_n$ with $d[S]_n = \sum_{i=1}^{n-1}(-1)^i [S]_{n-1|t_i=S}$;

5. for the construction of the complex $\widetilde{\mathcal{M}}_{(n)}(F)$, the acyclicity of the complex generated by the elements $[x]_n + (-1)^n[x^{-1}]_n$ (see Proposition 3.20, Proposition 5.1 and Corollary 3.22).

We now turn to the proofs of the mod $\mathcal{S}$ versions of 1 through 5.

1. The spectral sequence $\coprod K^{(r-p)}_{-p-q}(X^{n-p}_{Y,\mathrm{loc}};\square^{n-p}) \Rightarrow K^{(r)}_{-p-q}(X^n_Y;\square^n)$. (see (42)) exists, and converges, mod $\mathcal{S}$. We use the notation at the beginning of Section 2.3. So $C.$ is the space associated to X and $Y_1,\dots,Y_s$, see (2). This spectral sequence is constructed in two steps, both of which will be shown to be true up to torsion of bounded exponent. If $X \supset W_0 \supset \cdots \supset W_n \supset W_{n+1} = \emptyset$ is a stratification with each W_i closed of codimension i, we get a long exact sequence

$$\begin{aligned}\cdots &\to H^{-m-1}_{W_{i-1}}(C,K) \to H^{-m-1}_{W_{i-1}\setminus W_i}(C\cap U_i,K)\\ &\to H^{-m}_{W_i}(C,K) \to H^{-m}_{W_{i-1}}(C,K) \to \cdots.\end{aligned} \tag{53}$$

Then we want to show that this is exact mod $\mathcal{S}$ if we look at the weight j part. Moreover, we need that $H^{-m}_{W_p\setminus W_{p+1}}(C\cap U_{p+1},K)^{(j)} \cong H^{-m}(C\cap(W_p\setminus W_{p+1}),K)^{(j-p)}$ mod $\mathcal{S}$ under the assumptions of Proposition 2.3. (U_p is the complement of W_p in $X.$, see Section 2.3.)

For the first we shall show that for a regular pointed simplicial scheme $X.$ with all scheme components noetherian of finite Krull dimension

$$H^{-m}(X.,K) = \bigoplus_{\substack{i\geqslant 1\\ \text{finite}}} H^{-m}(X.,K)^{(i)} \qquad (\mathrm{mod}\,\mathcal{S})$$

if $m \geqslant 1$. From this one sees that the long exact sequence (53) is exact (mod $\mathcal{S}$) on the weight j part.

According to [27, page 510] for a regular noetherian scheme X of finite Krull dimension, for every $m \leqslant -1$, there exists an $N \in \mathbb{N}$ such that $H^m(X,\mathbb{Z}\times\mathbb{Z}_\infty BGL_N) = H^m(X,\mathbb{Z}\times\mathbb{Z}_\infty BGL)$, i.e., $H^m(X,K^N) = H^m(X,K)$. Moreover, for this N $H^0(X,K^N) \to H^0(X,K)$ is an injection. From the spectral sequence

$$H^p(X_q,K^N) \Rightarrow H^{p+q}(X.,K^N) \qquad (p+q\leqslant 0),$$

which is compatible with the map $K^N \to K$, it follows that there exists M such that $H^{-m}_{Z.}(X.,K^M) = H^{-m}_{Z.}(X.,K)$ if $m \geqslant -1$. From the proof on pages 35 and 36 of [14] if now follows that $H^{-m}_{Z.}(X.,K) = \oplus_{\substack{j\geqslant 1\\ \text{finite}}} H^{-m}_{Z.}(X.,K_M)^{(j)}$ mod $\mathcal{S}$ if $m \geqslant 1$.

For the second part we need a version of the Riemann–Roch statement (Proposition 2.3), without tensoring with $\mathbb{Q}$. For the isomorphism $H^{-m}_{Z.}(X.,K)^{(j)} \cong H^{-m}_{Z.}(Z.,K)^{(j-d)}$ (mod $\mathcal{S}$) we proceed as follows. As pointed out in Remark 2.10 we can find a multiple α of $ch^{-1}(td(N^\vee))$ in $K_0(Y_0)$, such that $\theta^k(N)\psi^k(\alpha) = k^d\alpha$ in $K_0(Y_0)$. This yields a map $i_\#$: $H^{-n}_{Z.}(Y.,K) \to H^{-n}_{Z.}(X.,K)$ given by $y \mapsto i_*(\alpha y)$, which maps $H^{-m}_{Z.}(Y.,K)^{(j)}$ to $H^{-m}_{Z.}(X.,K)^{(j+d)}$ by the proof of Proposition 2.3, see (2.2). Because we also have that

$$\begin{aligned}\oplus_j H^{-m}_{Z.}(Y.,K)^{(j)} &\cong H^{-m}_{Z.}(Y.,K)\\ \oplus_j H^{-m}_{Z.}(X.,K)^{(j)} &\cong H^{-m}_{Z.}(X.,K)\end{aligned}$$

(mod $\mathcal{S}$), if follows that we have

$$i_{\#}\colon H_{Z.}^{-m}(Y.,K)^{(j)} \xrightarrow{\sim} H_{Z.}^{-m}(X.,K)^{(j+d)}$$

mod $\mathcal{S}$.

2. This depends only on $K_n^{(j)}(F)_{\mathbb{Q}} = 0$ for $n-2j \geqslant 0$, $n \geqslant 1$, which in this case is to be replaced with the fact that $K_n^{(j)}(F)$ is of bounded exponent for those indices. Let $\mathcal{O}_F \subset F$ be the ring of algebraic integers. For n even, $n = 2m$, $K_n(\mathcal{O}_F)$ is a finite group. We have an exact sequence (up to two torsion, see [26])

$$0 \to K_n^{(j)}(\mathcal{O}_F) \to K_n^{(j)}(F) \to \coprod_{\text{finite fields } \mathbb{F}} K_{n-1}^{(j-1)}(\mathbb{F}) \to 0.$$

By [20] $K_{2m-1}(\mathbb{F})$ is pure of weight m, so that $K_{2m-1}^{(j-1)}(F)$ is a group of $(k^m - k^{j-1})$-torsion for all $k \geqslant 1$. (Note: $n-2j = 2(m-j) \geqslant 0$, so $m \neq j-1$.) Therefore $K_n^{(j)}(F)$ has bounded exponent in this case.

As $K_1(F) = K_1^{(1)}(F)$ it follows that $K_1^{(0)}(F) = 1$. For n odd bigger than 1, say $n = 2m-1$, $m > 0$, because $K_{2m}(\mathbb{F}) = 0$ for a finite field $\mathbb{F}$ ($m > 0$), we get a surjection $K_n(\mathcal{O}_F) \to K_n(F)$. $K_{2m-1}(\mathcal{O}_F)$ is a finitely generated group, and $K_{2m-1}(\mathcal{O}_F)_{\mathbb{Q}}$ is pure of weight m by [1]. Therefore $K_n^{(j)}(F)$ is a finite group in this case.

3. This is Remark 3.11.

4. $[S]_n$ was constructed in $K_n^{(n)}(T_{\text{loc}}^n; \square^{n-1})$ (see the beginning of Section 3.2 for the notation, etc.). Hence a suitable multiple of it will exist integrally, and have boundary a multiple of $\sum_{i=1}^{n-1} [S]_{n-1|t_i=S}$, which exists integrally by induction.

5. We recall the proof consists of two parts:

a) Let N_p be the subgroup of $M_{(p)}$ generated by the elements $[u]_p + (-1)^p[u^{-1}]_p$. Then the map
$$[u]_p + (-1)^p[u^{-1}]_p \mapsto u \otimes \cdots \otimes u \otimes (-u)^2 \in F^* \otimes \cdots \otimes F^*$$
gives an isomorphism (mod $\mathcal{S}$) with the subgroup S^p of $F^* \otimes \cdots \otimes F^*$ generated by $u \otimes \cdots \otimes u \otimes u^2$, $u \in U$. (This is actually twice the map used in Proposition 3.20.)

b) The complex
$$\begin{aligned}\mathrm{Sym}^n(F^*) &\to F^* \otimes \mathrm{Sym}^{n-1}(F^*) \to \cdots \\ &\to \bigwedge^{n-2} F^* \otimes \mathrm{Sym}^2(F^*) \to d\Big(\bigwedge^{n-2} F^* \otimes \mathrm{Sym}^2(F^*)\Big)\end{aligned}$$
has cohomology of bounded exponent.

(Here $\wedge^p F^*$ is the part of $F^{*\otimes p}$ on which S_p acts alternatingly.) The latter is immediate from the chain homotopy given in the proof of Corollary 3.22. For the first statement, note that by 1 through 4 above and induction on n, we have a map

$$\mathrm{Kernel}(N_n \to \mathrm{Sym}^n(F)) \to K_{2n-1}^{(n)}(F)$$

which is injective up to torsion of bounded exponent. Moreover, $K_{2n-1}^{(n)}(F)$ is finitely generated, and the regulator vanishes on this kernel, see the proof of Proposition 5.1. Because the regulator is injective modulo torsion, this kernel must have finite image and hence must be of bounded exponent.

6. Some Explicit Relations Between Elements

In this section we give an example, of a relation that exists between elements $[x]_n$, and consider how to use specialization of elements in $H^1(\widetilde{M}_{(n)}(F(C))$ for a curve C defined over a number field F, to get other relations.

PROPOSITION 6.1. *If F contains the field $\mathbb{Q}(\zeta)$ for a p-th root of unity ζ, then*

$$[x^p]_n = \sum_{\zeta^p=1} p^{n-1}[\zeta x]_n$$

for all $n \geqslant 1$.

Proof. We argue universally for $X \in \Gamma(\mathbb{G}_m, \mathcal{O}^*)$, and then pull back. For $n = 1$ this is trivial. Assume by induction that it holds for $n-1$. Then we have that

$$\begin{aligned} d\Big([X^p]_n - \sum_{\zeta^p=1} p^{n-1}[\zeta X]_n\Big) &= X^p \otimes [X^p]_{n-1} - \sum_{\zeta^p=1} p^{n-1}\zeta X \otimes [\zeta X]_{n-1} \\ &= p \cdot X \otimes \Big([X^p]_{n-1} - \sum_{\zeta^p=1} p^{n-2}[\zeta X]_{n-1}\Big) \\ &= 0 \end{aligned}$$

by induction. So $[X^p]_n - \sum_{\zeta^p=1} p^{n-1}[\zeta X]_n \in K_{2n-1}^{(n)}(\mathbb{Q}(\zeta)(X)) \cong K_{2n-1}^{(n)}(\mathbb{Q}(\zeta))$. So by Borel, it suffices to show that the regulator map vanishes for $X \in \mathbb{Q}(\zeta) \subset \mathbb{C}$. But letting $X \to 0$ in $\mathbb{C}$ it is clear that that is the case. So our relation holds over $\mathbb{G}_m/\mathbb{Q}(\zeta)$, and by pull back will hold for fields containing $\mathbb{Q}(\zeta)$.

In applications where one has elements in $H^1(\mathcal{M}^{\cdot}_{(n)}(K))$ where K is the function field of a smooth curve defined over a number field one would like to be able to specialize those elements to points in the curve. There is no problem if all the elements involved specialize nicely, but one has to be more careful if any of them has a pole or assumes the value 0 or 1 at the given point. We will use Borel's theorem and continuity to get specialization of an element for $n \geqslant 2$, at points where no poles or zeroes occur.

Let F be a number field, C a regular connected curve defined over F. Note that by Remarks 3.16 and 3.17, and the localization sequence, for $V \subset C$ open, we have $M_{(n)}(V) \subset L_{(n)}(V) \subset L_{(n)}(F(C))$. We will always regard elements in this biggest group $L_{(n)}(F(C))$. Suppose we have $\alpha \in M_{(n)}(F(C))$ with $d\alpha = 0$, so that α defines an element in $K_{2n-1}^{(n)}(F(C)) \cong K_{2n-1}^{(n)}(C) \subset L_{(n)}(F(C))$. In order to specialize, there is no problem to just restrict α to a point Q unless $f(Q) = 0, 1, \infty$ for some $[f]_n$ occurring in α. We assume $f(Q) \neq 0, \infty$ for all $[f]_n$ occurring in α. Because $[S]_n$ also makes sense for $S = 1$ if $n \geqslant 2$, the only problem that occurs in this case is when we try to specialize $[S]_1$ at $S = 1$.

Lift α to an element β in $C^1_{(n),\log}(V) \subset C^1_{(n)}$, with $d\beta = 0$. By Remarks 3.10 and 3.16, if $\alpha = \sum_j [x_j]_n$, $\beta \in \sum_{0\leqslant l\leqslant n-1}((1+I)^*)^l \cdot \{[x_j]_{n-l}\}$. Fix some $x \in F(C)^*$ with a pole at Q. We can collect all elements $[x_j]_1$ with poles at Q in one term $\gamma \in ((1+I)^*)^{n-1} \cdot x$. Write $\delta = \beta - \gamma$, so that δ consists of terms that can be specialized at Q. β itself, defining an element in $K_{2n-1}^{(n)}(C)$, can be specialized. It follows that $\gamma = \beta - \delta$ can also be specialized at Q. To see that $\gamma_{|Q}$ equals zero, we look at all the embeddings of F into $\mathbb{C}$. Considering $Q \in C(\mathbb{C})$, we can argue by continuity. The elements in δ extend across Q, and so does the regulator, in fact in a continuous way. β is a global element and the regulator of this element will certainly extend in a continuous way across Q. So the regulator of their difference γ extends in a continuous way across Q. Because $n \geqslant 2$, each term in γ contains a factor in $(1+I)^*$, so that by Proposition 4.1 the regulator vanishes around Q, so it must vanish at Q as well. This shows that we have a specialization map for $n \geqslant 2$,

$$\begin{aligned} \mathrm{spec}_Q\colon \operatorname{Ker} d \subset M_{(n)}(F(C)) &\to K^{(n)}_{2n-1}(F) \subset M_{(n)}(F) \\ [x]_n &\mapsto [x(Q)]_n \end{aligned}$$

provided all x's are defined at Q. The same result holds for $\widetilde{M}_{(n)}$ instead of $M_{(n)}$, because the lift of an element α in $\widetilde{M}_{(n)}$ with $d\alpha = 0$ to $M_{(n)}$ will not have zeroes or poles at points where the symbols in α didn't have them. (See the explicit form of the chain homotopy in the proof of Corollary 3.22.)

It is also clear that if $Q_1, Q_2 \in C(F)$ are two points of C where we can specialize a given element, then $\mathrm{spec}_{Q_1} = \mathrm{spec}_{Q_2}$: by Borel it suffices to check that the regulators coincide for all embeddings of F into $\mathbb{C}$, which follows easily from ([31, § 1, Proposition 1, p. 411]). (Zagier's proof never uses that the curve is a $\mathbb{P}^1$: he uses $\left\{\frac{1}{x-\alpha},\ \alpha \in \mathbb{C}\right\}$, but any multiplicative basis of the group in $\mathbb{C}(C)^*$ generated by the x_j's will play the same role.)

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