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Purity of the branch locus and Lefschetz theorems

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Purity of local rings has many important applications to commutative algebra and algebraic geometry. These include purity of the branch locus, factoriality, and simple connectedness.

The classical purity theorem is that regular local rings of dimension ≥ 2 are pure. This was proved by Zariski [Z], Auslander and Buchsbaum [AB] and Nagata [N]. Grothendieck showed that complete intersections of dimension ≥ 3 are pure in SGA2. In this paper we show that a much larger class of rings are pure.

A local ring (A, m) is pure if the restriction map of étale covers (finite unramified flat extensions)

$$\text{Et}(\text{spec}(A)) \rightarrow \text{Et}(\text{spec}(A) - m)$$

is an equivalence.

Let R be a regular local ring, $I \subset R$ an ideal, and $A = R/I$. The deviation of A is $\delta(A) = p - (\dim(R) - \dim(A))$, where p is the number of generators of I , R is a regular local ring, and $A = R/I$. In this paper we consider purity for A such that

$$\dim(A) - \delta(A) \geq 3. \tag{*}$$

In the case where I is a complete intersection, this specializes to the purity theorem for complete intersections of Grothendieck. (*) is the best bound that can be hoped for to give purity, since there are complete intersections of dimension two which are not pure.

We prove the following theorem for excellent rings.

THEOREM 19. *Suppose that (A, m) is excellent, a quotient of a regular local ring and equidimensional of dimension ≥ 3 . If A is a complete intersection in codimension $2 + \delta(A)$, then A is pure.*

It follows that purity holds for excellent rings A which are complete intersections away from m and satisfy (*), $\dim(A) - \delta(A) \geq 3$. Some examples (which are not complete intersections) are given by the affine cones over certain Grassmanians and Pfaffians (c.f. Examples 1–3 of Section 3).

As a corollary of Theorem 19, we have an even stronger purity Theorem.

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THEOREM 20. *Suppose that (A, m) is excellent, a quotient of a regular local ring and equidimensional. If A is a complete intersection in codimension $2 + \delta(A)$, then A is pure in codimension ≥ 3 .*

Two applications of Theorem 20 are:

THEOREM 21. *Suppose that (A, m) is excellent, a quotient of a regular local ring, normal and a complete intersection in codimension $2 + \delta(A)$. If B/A is a finite extension with B normal, then $\text{codim}(\text{Ramlocus}(B/A)) \leq 2$.*

THEOREM 22. *Suppose that (A, m) is excellent, equicharacteristic, a quotient of a regular local ring, normal and R_2 . If A is a complete intersection in codimension $2 + \delta(A)$, then the divisor class group of A , $Cl(A)$, has no torsion of order prime to the characteristic of A .*

Our proof of Theorem 19 uses the Lefschetz theory developed by Grothendieck in his proof of purity for complete intersections. Grothendieck developed general techniques to determine purity. However, there is a part of his proof which is difficult to extend beyond complete intersections. In fact, Ogus observes in the introduction to [O] that “this method seems hopeless except for complete intersections, for which the whole problem is trivial”. Our proof is then of interest as it gives a substantial extension of Grothendieck’s method beyond complete intersections.

The problem breaks up into two parts. The first part is to verify the Lefschetz condition Lef when

$$\dim(R) - \text{number of equations defining } I \text{ set theoretically} \geq 2.$$

This was previously shown by Faltings (c.f. Corollary 3 [F]), as the major application of the finiteness theorem he develops in [F]. Our proof (Corollary 16) uses completely different methods.

The second and more difficult part of the proof is to verify the effective Lefschetz condition Leff. This is accomplished in Corollary 18.

The analogue of the problem considered in this paper for étale cohomology was considered by Raynaud in expose XIV of SGA2. Let (A, m) be an excellent local ring of characteristic zero. The étale depth of A is defined to be $\text{etdepth}(A) \geq r$ if $H_m^p(\text{spec}(A), \mathbf{Z}/n\mathbf{Z}) = 0$ for all integers n , and $p < r$. Raynaud proves (Theorem 5.6, expose XIV, SGA2) that

$$\text{etdepth}(A) \geq \dim(A) - \delta(A).$$

By consideration of the exact sequence

$$\begin{aligned} \rightarrow H_m^1(\text{spec}(A), \mathbf{Z}/n\mathbf{Z}) &\rightarrow H^1(\text{spec}(A), \mathbf{Z}/n\mathbf{Z}) \\ \rightarrow H^1(\text{spec}(A) - m, \mathbf{Z}/n\mathbf{Z}) &\rightarrow H_m^2(\text{spec}(A), \mathbf{Z}/n\mathbf{Z}), \end{aligned}$$

and the identity

$$H^1(X, \mathbf{Z}/n\mathbf{Z}) \cong \text{Hom}(\pi_1(X), \mathbf{Z}/n\mathbf{Z}),$$

we see that (for strictly henselian rings of equicharacteristic zero) Raynaud’s theory implies that there are no abelian quotients of $\pi_1(\text{spec}(A) - m)$ if A satisfies $(*)$, $\dim(A) - \delta(A) \geq 3$. In contrast, if A is a complete intersection in codimension $2 + \delta(A)$, Theorem 19 gives the much stronger result that $\pi_1(\text{spec}(A) - m) = 0$. In fact, as a corollary to Theorem 20 we have

COROLLARY. *Suppose that (A, m) is excellent, a quotient of a regular local ring and equidimensional. If A is a complete intersection in codimension $2 + \delta(A)$, then*

$$\pi_1(\text{spec}(A) - Z) = \pi_1(\text{spec}(A))$$

for all closed subsets Z of $\text{spec}(A)$ of codimension ≥ 3 .

In Section 4, we obtain stronger results for normal rings, essentially of finite type over a field of characteristic zero.

Our proofs in Section 4 use a Theorem from algebraic topology by Hamm [H] and Goresky and MacPherson [GM]. These proofs makes essential use of Morse Theory, and as such make essential use of the assumption of characteristic zero.

Let R be a regular local ring, essentially of finite type over a field k of characteristic zero, $I \subset R$ an ideal, and $A = R/I$. The geometric deviation of A is

$$g\delta(A) = q - (\dim(R) - \dim(A)),$$

where q is the minimum number of generators of an ideal J in R such that $\sqrt{J} = \sqrt{I}$.

We prove the following theorem:

THEOREM 28. *Suppose that A is normal, essentially of finite type over a field of characteristic zero, and that B is a finite extension of A such that B is normal. Then*

$$\text{codim}(\text{Ramlocus}(B/A)) \leq 2 + g\delta(A).$$

This theorem is strictly stronger than the conclusion of Grothendieck’s theorem in equicharacteristic zero, since $g\delta(A) = 0$ if and only if A is set theoretically a complete intersection in R .

Theorem 27 gives a stronger conclusion, by considering the local geometric deviation at localizations of A .

THEOREM 27. *Suppose that A is normal, essentially of finite type over a field of characteristic zero, and that B is a finite extension of A such that B is normal. Suppose that r is a positive integer such that for every prime P of A of height $> r$*

$$\dim(A_P) - g\delta(A_P) \geq 3.$$

Then

$$\text{codim}(\text{Ramlocus}(B/A)) \leq r.$$

As a consequence, we extend the deviation condition in Theorem 21 to geometric deviation in rings essentially of finite type over a field of characteristic zero.

COROLLARY. *Suppose that A is normal, essentially of finite type over a field of characteristic zero, and that B is a finite extension of A such that B is normal. Suppose that A is a complete intersection in codimension $2 + g\delta(A)$. Then*

$$\text{codim}(\text{Ramlocus}(B/A)) \leq 2.$$

In light of the above results and Grothendieck’s Theorem (SGA2 XI 3.14) showing that complete intersections which are factorial in codimension ≤ 3 are factorial, it is natural to consider the condition of factoriality for A such that

$$\dim(A) - \delta(A) \geq 4 \text{ and } A \text{ is factorial in codimension } \leq 3. \tag{**}$$

Hoobler [H] has shown that if A is excellent of equicharacteristic zero normal, S_3 and A is geometrically factorial in codimension $\leq 3 + \delta(A)$ then A is geometrically factorial. Note that Hoobler’s conditions are stronger than those of (**). Hoobler’s proof uses Raynaud’s theorem on etale depth cited above.

In Section 5, as another application of our purity theorem, we obtain some new results on factoriality of excellent rings under conditions much stronger than (**).

In section one we recall some results on purity, and to demonstrate the importance of this concept, prove some simple but very powerful applications of purity which the author has found are not generally known.

In this paper $\pi_1(X)$ will denote the algebraic fundamental group of X and $\pi_1^{\text{top}}(X)$ will denote the topological fundamental group of X . Given a ring extension $A \rightarrow B$, $\text{Ramlocus}(B/A)$ will denote the Zariski closed subset of $\text{spec}(B)$ on which the extension is ramified.

1. Purity

Purity of local rings is defined in the introduction.

THEOREM 1 (Zariski–Auslander–Buchsbaum–Nagata Purity of the Branch Locus). *Suppose A is a regular local ring, B/A is a finite extension with B normal. Then*

$$\text{codim}(\text{Ramlocus}(B/A)) \leq 1.$$

It follows that local rings of dimension ≥ 2 are pure.

THEOREM 2 (Grothendieck)(SGA2 X.3.4). *Suppose A is local and a complete intersection with $\dim(A) \geq 3$. Then A is pure.*

LEMMA 3. *Suppose A is local, normal and excellent and A is pure in codimension $\geq r$ (the completion of A_P at the maximal ideal of A_P is pure if $\text{ht}(P) \geq r$). If B/A is a finite extension with B normal, then*

$$\text{codim}(\text{Ramlocus}(B/A)) \leq r - 1.$$

Proof. Let q be a minimal prime of the branch locus. Suppose that $\text{ht } q \geq r$. Let $p = q \cap A$, $'$ denote p -adic completion. Let $V_q = \text{spec}((B_q)') - q$. $V_q \rightarrow \text{spec}((A_p)') - p$ is étale by SGA1 I9.11. By assumption, V_q extends to an étale cover $\text{spec}(C)$ of $\text{spec}((A_p)')$. C is isomorphic to the normalization of $(A_p)'$ in the quotient field of $(B_q)'$ by SGA1 I10.1. $(B_q)'$ has this same property, so that $(B_q)'$ is étale over $(A_p)'$, a contradiction.

LEMMA 4. *Suppose A is excellent normal local equicharacteristic and pure in codimension ≥ 2 (the completion of A_P at the maximal ideal of A_P is pure if $\text{ht}(P) \geq 2$). Then the divisor class group $Cl(A)$ has no torsion of order prime to the characteristic of A .*

Proof. Let $M \in Cl(A)$ such that M has order s prime to p . There exists an A module isomorphism $\sigma : (M^{\otimes s})^{**} \rightarrow A$. Consider the normal A algebra $B = \bigoplus_{n=0}^{s-1} (M^{\otimes n})^{**}$ obtained by identifying $(M^{\otimes s})^{**}$ with A by σ . Suppose $P \in \text{spec}(A)$ has height one. M is locally free in codimension one, hence $B \otimes_{A_P} \cong A_P[z]/z^s - u$ where u is a unit in A_P . It follows that $\text{spec}(B) \rightarrow \text{spec}(A)$ is unramified in codimension one, hence is flat by Lemma 3, so that $M \cong A$ and $s = 1$.

The following Theorem is immediate from Theorem 2 and Lemma 3.

THEOREM 5. *Suppose that A is normal excellent local and a complete intersection. If B/A is a finite extension with B normal then $\text{codim}(\text{Ramlocus}(B/A)) \leq 2$.*

Another result follows from a purity theorem of Faltings.

THEOREM 6. *Suppose that (R, m, k) is an excellent normal equicharacteristic local domain, such that k is perfect. Suppose that S is a normal, finite separable extension of R . Then*

$$\text{codim}(\text{Ramlocus}(S/R)) \leq 1 + \text{embcodim}(R).$$

Proof. Let $f : \text{spec}(S) \rightarrow \text{spec}(R)$ be the natural map. Let b be an ideal defining the branch locus, and let $a = b \cap R$. Then

$$\text{ht}(a) = \text{codim}(\text{Ramlocus}(S/R)).$$

Let $U = \text{spec}(R) - \text{spec}(R/a)$. Suppose that

$$\text{ht}(a) > 1 + \text{embcodim}(R).$$

Then

$$\dim_k(m/m^2) < \dim(R) + \text{ht}(a) - 1,$$

and

$$\dim_k(m/m^2) \leq 2 \dim(R) - \dim(R/a) - 2.$$

By a Theorem of Faltings', Corollary 3 to Theorem 2 of [F], $\text{spec}(S)|f^{-1}(U)$ extends to an étale cover $\text{spec}(C)$ of $\text{spec}(R)$. $C = S$ since both C and S are isomorphic to the integral closure of R in the quotient field of S . Hence S is étale over R , a contradiction.

The following two Lemmas show that local purity theorems give global theorems over projective varieties.

LEMMA 7. *Suppose that k is a field, $X \subset \mathbf{P}_k^n$ is geometrically connected and the local ring of the vertex of the coordinate ring of X is pure. Then $\pi_1(X) \cong \text{Gal}(\bar{k}/k)$ where \bar{k} is an algebraic closure of k .*

Proof. First suppose that k is algebraically closed. Let $A = k[x_0, \dots, x_n]$, $m = (x_0, \dots, x_n)$, I be the ideal of X . By assumption, $(A/I)_m$ is pure.

Give A/I the natural grading. Let $C_X = \text{spec}(A/I)$, $E_X = C_X - m$. Let $\alpha : A/I \rightarrow \bigoplus_{-\infty < n < \infty} \mathcal{O}_X(n)$ be the natural graded map. By EGA II 8.4, 8.6 $E_X \cong \text{spec}(\bigoplus_{-\infty < n < \infty} \mathcal{O}_X(n))$ and there is a commutative diagram

$$\begin{array}{ccc} E_X & \xrightarrow{\text{spec}(\alpha)} & C_X \\ \downarrow \pi & & \downarrow \\ X & \longrightarrow & \text{spec}(k) \end{array}$$

$\mathcal{O}_{C_X, m} = (A/I)_m$ is pure, so that (C_X, m) is pure by SGA2 X 3.3, and $Et(C_X) \cong Et(E_X)$. Let $G = k^*$. It follows from this equivalence that G equivariant étale covers of E_X extend to G equivariant étale covers of C_X .

Let $Y = \text{spec}(R)$ be a G -equivariant étale cover of C_X . Let $R = \bigoplus_{n \in \mathbf{Z}} R_n$ with graded map $\phi : A/I \rightarrow R$. ϕ finite implies there exists a constant m_0 such that $R_n = 0$ for $n < m_0$. ϕ unramified implies that R/mR is reduced. Since there is an inclusion $R_{m_0} \rightarrow R/mR$, $m_0 = 0$. Let $f \in R_1$. There is a dependence relation $f^s + a_1 f^{s-1} + \dots + a_s = 0$ with $a_i \in m^i$. mR radical implies $f \in mR$.

Hence $mR = \bigoplus_{n>0} R_n$, and $m^n R_0 = R_n$ for all $n > 0$. $R_0 \cong k^s$ for some s . The surjection $R_0 \otimes_k (A/I) \rightarrow R$ is an isomorphism since the map is flat SGA1 I 4.8. Hence $Y \cong \amalg C_X$.

Let $\mathcal{N} = \text{spec}(\mathcal{A}) \in \text{Et}(X)$.

$$\mathcal{N} \cong \text{spec}(\pi_*(\pi^*\mathcal{A})^G) \cong \text{spec} \left(\pi_* \left(\bigoplus \left(\bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n) \right) \right)^G \right) \cong \amalg X.$$

Hence $\pi_1(X) = 0$.

If k is not algebraically closed, we get $\pi_1(X \times_k \bar{k}) = 0$, and then $\pi_1(X) \cong \text{Gal}(\bar{k}/k)$ by SGA1 IX 6.1.

As a corollary of Theorem 2 we have

THEOREM 8 (Grothendieck SGA2 XII 3.5). *Suppose that k is an algebraically closed field and $X \subset \mathbb{P}_k^n$ is a complete intersection of dimension ≥ 2 . Then $\pi_1(X) = 0$.*

LEMMA 9. *Suppose that $X \subset \mathbb{P}_k^n$ is as in Lemma 7 and k is algebraically closed. Then $\text{Pic}(X)$ has no torsion of order prime to $p = \text{char}(k)$.*

Proof. Given $\mathcal{L} \in \text{Pic}(X)$ of order s prime to p , choose a \mathcal{O}_X module isomorphism $\sigma : \mathcal{L}^{\otimes s} \cong \mathcal{O}_X$, which induces an algebra structure on $\mathcal{A} = \bigoplus_{n=0}^{s-1} \mathcal{L}^{\otimes n}$ making $W = \text{spec}(\mathcal{A})$ into an irreducible étale cover of X . $s = 1$ by Lemma 7.

2. Lefschetz conditions

The following two definitions are from SGA2.

DEFINITION 1. Let X be a scheme and let Z be a closed subscheme. Let $U = X - Z$. (X, Z) is pure if for all open subsets V of X , the map $\text{Et}(V) \rightarrow \text{Et}(V \cap U)$, $V' \mapsto V' \times_V (V \cap U)$ is an equivalence of categories. If (A, \mathfrak{m}) is a noetherian local ring, then A is pure if $(\text{spec}(A), \mathfrak{m})$ is pure.

DEFINITION 2 (Lefschetz conditions). Let X be a locally noetherian scheme, and Y a closed subscheme. Let \hat{X} be the formal completion of X along Y .

- (1) The condition $\text{Lef}(X, Y)$ is true if for all open subsets U of X containing Y , and for all coherent locally free sheaves E on U , $\Gamma(U, E) = \Gamma(\hat{X}, \hat{E})$.
- (2) The condition $\text{Leff}(X, Y)$ is true if $\text{Lef}(X, Y)$ and if for all coherent locally free sheaves \mathcal{E} on \hat{X} , there exists an open neighborhood U of Y and a coherent locally free sheaf E on U such that $\hat{E} \cong \mathcal{E}$.

Consider the natural map $\Phi : \{\text{Germs of étale covers around } Y\} \rightarrow \text{Et}(\hat{X})$. If $\text{Lef}(X, Y)$ holds then Φ is fully faithful. If $\text{Leff}(X, Y)$ holds then Φ is an equivalence.

Lemma 10 isolates an argument of Grothendieck in SGA2 X.

LEMMA 10. *Suppose that (A, m) is noetherian, local. Suppose that $I \subset A$ is an ideal such that A is I -adically complete. Let $X = \text{spec}(A)$, $Y = V(I)$, $U = \text{spec}(A) - m$. Suppose that*

- (1) $\text{Leff}(U, U \cap Y)$
- (2) $(X, X - V)$ is pure for any open $V \subset X$ with $U \cap Y \subset V$

Then (A/I) is pure.

Proof. We must show that $\text{Et}(Y) \rightarrow \text{Et}(U \cap Y)$ is an equivalence. Consider the natural diagram of restriction maps

$$\begin{array}{ccc}
 \text{Et}(X) & \xrightarrow{a} & \text{Et}(U) \\
 \downarrow c & & \downarrow b \\
 \text{Et}(Y) & \xrightarrow{d} & \text{Et}(U \cap Y)
 \end{array}$$

a is an equivalence by assumption. c is an equivalence since

$$\text{Et}(X) \cong \text{Et}(\hat{X}) \cong \text{Et}(Y)$$

by EGA III 5.1.6 and SGA2 X 1.1. There is an equivalence $\text{Et}(\hat{U}) \rightarrow \text{Et}(U \cap Y)$ by SGA2 X 1.1. It suffices to show that the restriction map $e : \text{Et}(U) \rightarrow \text{Et}(\hat{U})$ is an equivalence. e is fully faithful by SGA2 X 2.3. Suppose that $\mathcal{E} \in \text{Et}(\hat{U})$. By SGA2 X 2.3 there exists a neighborhood V of $Y \cap U$ in U and $\mathcal{G} \in \text{Et}(V)$ such that $\hat{\mathcal{G}} \cong \mathcal{E}$. Since $(X, X - V)$ is pure, there exists $\mathcal{F} \in \text{Et}(U)$ such that $\hat{\mathcal{F}} \cong \mathcal{E}$.

LEMMA 11. *Suppose that (A, m) is noetherian, local. Suppose that $I \subset A$ is an ideal such that A is I -adically complete. Let $X = \text{spec}(A)$, $Y = V(I)$, $U = \text{spec}(A) - m$. Suppose that*

- 1. A is pure
- 2. $\text{Leff}(U, U \cap Y)$

Then $Y - m$ is connected.

Proof. Consider the natural diagram of restriction maps

$$\begin{array}{ccc}
 \text{Et}(X) & \xrightarrow{a} & \text{Et}(U) \\
 \downarrow c & & \downarrow b \\
 \text{Et}(Y) & \xrightarrow{d} & \text{Et}(U \cap Y)
 \end{array}$$

a is an equivalence by assumption. c is an equivalence by EGA III 5.1.6 and SGA2 X 1.1. There is an equivalence $\text{Et}(\hat{U}) \rightarrow \text{Et}(U \cap Y)$ by SGA2 X 1.1. The restriction map $\text{Et}(U) \rightarrow \text{Et}(\hat{U})$ is fully faithful by SGA2 X 2.3. Hence d is fully faithful which shows that $\Gamma(Y, \mathcal{O}_Y) = \Gamma(U \cap Y, \mathcal{O}_Y)$. We conclude that $U \cap Y$ is connected.

3. Excellent rings

The following notations will be in use in this section. Let A be a noetherian normal domain which is a quotient of a regular ring. Let $I \subset a \subset A$ be ideals. Given an appropriate object M , \hat{M} will denote the I -adic completion of M .

Let X_1 be the S_2 -ification of $\text{Proj}(\bigoplus_{n \geq 0} I^n)$, with natural projection $\sigma : X_1 \rightarrow \text{spec}(A)$. $X_1 \rightarrow \text{Proj}(\bigoplus_{n \geq 0} I^n)$ is finite by EGA IV 5.11.2. $\bigoplus_{n \geq 0} \sigma_*(I^n \mathcal{O}_{X_1})$ is a module of finite type over $\bigoplus_{n \geq 0} I^n$ by EGA III 3.3.1. Hence there exists $m > 0$ such that if $J = \sigma_*(I^m \mathcal{O}_{X_1})$, then $J^n = \sigma_*(J^n \mathcal{O}_{X_1})$ for all $n \geq 0$ by EGA II 2.1.6 v. J is an ideal since $\text{spec}(A)$ is normal. We may also choose m sufficiently large so that $R^1 \sigma_*(J^n \mathcal{O}_{X_1}) = 0$ for all $n > 0$.

Let $X = \text{spec}(\hat{A})$, $Y = V(I)$, $Z = V(a)$, $U = X - Z$. Let $\tilde{X} = X_1 \otimes \hat{A}$, $\gamma = \sigma \otimes 1 : \tilde{X} \rightarrow X$.

LEMMA 12. *Suppose that A is excellent, $f \in A$. Then $(A_f)^\wedge$ is normal and a quotient of a regular ring. $X_1 \otimes_A (A_f)^\wedge$ is S_2 .*

Proof. We will first show that $(A_f)^\wedge$ is normal. It suffices to show that $((A_f)^\wedge)_p$ is normal for all $p \in V(I) \cap D(f) \subset \text{spec}(A)$. Let $'$ denote p -adic completion. We have a flat map $((A_f)^\wedge)_p \rightarrow (A_p)' \cdot (A_p)'$ is normal by EGA IV 7.8.3. Hence $((A_f)^\wedge)_p$ is normal by EGA IV 6.4.1, IV 6.5.3.

The fact that $X_1 \otimes_A (A_f)^\wedge$ is S_2 follows from similar arguments applied to the map $(\mathcal{O}_{X_1} \otimes_A (A_f)^\wedge)_x \rightarrow (\mathcal{O}_{X_1, x})'$ for $x \in \sigma^{-1}(V(I) \cap D(f))$, where $'$ denotes completion with respect to the maximal ideal of $\mathcal{O}_{X_1, x}$.

It remains to show that $(A_f)^\wedge$ is the quotient of a regular ring. There is a surjection $\phi : B \rightarrow A_f$ where B is regular. Let $b = \phi^{-1}(I)$, and let \hat{B} be the completion of B along b . Given $p \in V(b)$, let $'$ denote p -adic completion. We have a flat map $(\hat{B})_p \rightarrow (B_p)'$ where $(B_p)'$ is regular. Hence $(\hat{B})_p$ is regular by EGA IV 6.5.3.

THEOREM 13. *Let S be a noetherian scheme which can locally be embedded in a regular scheme, T a closed subset of S . Let $W = S - T$, $f : W \rightarrow S$ be inclusion. Suppose that \mathcal{F} is a coherent \mathcal{O}_W module, \mathcal{F} is S_n , and $\text{codim}(T, S) \geq n + 1$. Then $R^i f_*(\mathcal{F})$ is coherent for $i < n$.*

Proof. By EGA I 9.4.7 there exists a coherent extension $\bar{\mathcal{F}}$ of \mathcal{F} to S . By the long exact sequence of local cohomology, we have an exact sequence

$$0 \rightarrow \underline{H}_T^0(\bar{\mathcal{F}}) \rightarrow \bar{\mathcal{F}} \rightarrow f_*(\mathcal{F}) \rightarrow \underline{H}_T^1(\bar{\mathcal{F}}) \rightarrow 0,$$

and isomorphisms $R^i f_*(\mathcal{F}) \cong \underline{H}_T^{i+1}(\bar{\mathcal{F}})$ for $i > 0$.

For $x \in W$, let $\{\bar{x}\}$ denote the closure of x in S . Suppose that $\text{codim}(\{\bar{x}\} \cap T, \{\bar{x}\}) = 1$. Then $\dim(\{\bar{x}\}) \leq \dim(T) + 1 \leq \dim(S) - n$. Hence $\dim(\mathcal{O}_S, x) \geq n$, and $\text{depth}(\mathcal{F}_x) \geq n$.

By (ii) \Rightarrow (iii) of the finiteness theorem of SGA2 VIII 2.3, we have that the local cohomology sheaves $H_T^i(\bar{\mathcal{F}})$ are coherent for $i \leq n$, and the Theorem follows.

REMARK 14. We will use Theorem 13 to show that $f_*(\mathcal{F})$ is coherent. Then $\text{depth } f_*(\mathcal{F})_x \geq 2$ for all $x \in T$ by SGA2 III 3.5.

THEOREM 15. Suppose that \tilde{X} is an S_2 domain and $\dim(\gamma^{-1}(V(a))) \leq \dim(X) - 3$. Then $\text{Leff}(U, U \cap Y)$.

Proof. Let $\tilde{Y} = \gamma^{-1}(Y)$ and $\tilde{U} = \gamma^{-1}(U)$. Let $\beta = \gamma|_{\tilde{U}}$.

Suppose that V is an open subset of U containing $Y \cap U$, and that E is a locally free coherent \mathcal{O}_V module. We must show that the natural map $\Gamma(V, E) \rightarrow \Gamma(\hat{U}, \hat{E})$ is an isomorphism.

Let $\tilde{V} = \beta^{-1}(V)$. Let $\alpha = \beta|_{\tilde{V}}$. There is a commutative diagram of maps where $f, \tilde{f}, u, \tilde{u}$ are the natural inclusions.

$$\begin{array}{ccccc}
 \tilde{V} & \xrightarrow{\tilde{u}} & \tilde{U} & \xrightarrow{\tilde{f}} & \tilde{X} \\
 \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow \\
 V & \xrightarrow{u} & U & \xrightarrow{f} & X
 \end{array}$$

Let $\tilde{Z} = \tilde{U} - \tilde{V}$, Z' be the closure of \tilde{Z} in \tilde{X} . $Z' \cap \tilde{Y} \subset \gamma^{-1}(V(a))$ implies that $\dim(Z' \cap \tilde{Y}) \leq \dim \gamma^{-1}(V(a)) \leq \dim X - 3$. Since \tilde{Y} is a Cartier divisor, the principal ideal theorem implies that $\dim \tilde{Z} \leq \dim X - 2$.

Let $\tilde{E}_1 = \tilde{u}_*(\alpha^*(E))$. By Theorem 13, \tilde{E}_1 is coherent, and $\text{depth}((\tilde{E}_1)_p) \geq 2$ at all points p of \tilde{Z} by Remark 14. In particular, \tilde{E}_1 is S_2 .

Let $E_1 = u_*(E)$. E_1 is coherent since $E_1 = \beta_*(\tilde{E}_1)$.

Let $\mathcal{J}_1 = J\mathcal{O}_U$. Let D, D_1 be the respective effective cartier divisors on \tilde{X}, \tilde{U} , such that $J\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-D)$ and $\mathcal{J}_1\mathcal{O}_{\tilde{U}} = \mathcal{O}_{\tilde{U}}(-D_1)$.

We will show that $\Gamma(U, E_1) = \Gamma(\hat{U}, \hat{E}_1) = \Gamma(\hat{U}, \hat{E})$, from which it follows that $\Gamma(V, E) = \Gamma(\hat{U}, \hat{E})$.

By the comparison theorem of SGA2 IX, $\Gamma(U, E_1) = \Gamma(\hat{U}, \hat{E}_1)$ if

$$\bigoplus_{n \geq 0} R^i f_*(E_1 \mathcal{J}_1^n) \text{ is a module of finite type over}$$

$$\bigoplus_{n \geq 0} (\hat{J}^n)^\sim \text{ for } i = 0 \text{ and } 1. \tag{1}$$

By the fact that β is an isomorphism over $U - V$, the projection formula and our choice of J , we have

$$\beta_*(\tilde{E}_1 \otimes \mathcal{O}_{\tilde{U}}(-nD_1)) \cong E_1 \otimes (\beta_*\mathcal{O}_{\tilde{U}}(-nD_1)) \cong E_1\mathcal{J}_1^n \quad \text{for all } n \geq 0$$

In order to verify (1), we will need that $\tilde{f}_*(\tilde{E}_1)$ and $R^1\tilde{f}_*(\tilde{E}_1)$ are coherent. This follows from Theorem 13 since \tilde{E}_1 is S_2 , and $\dim \gamma^{-1}(V(a)) \leq \dim X - 3$.

Now we will verify (1) for $i = 0$.

$$\begin{aligned} f_*(E_1\mathcal{J}_1^n) &\cong f_*\beta_*(\tilde{E}_1 \otimes \mathcal{O}_{\tilde{U}}(-nD_1)) \\ &\cong \gamma_*\tilde{f}_*(\tilde{E}_1 \otimes \mathcal{O}_{\tilde{U}}(-nD_1)) \\ &\cong \gamma_*(\tilde{f}_*\tilde{E}_1 \otimes \mathcal{O}_{\tilde{X}}(-nD)) \end{aligned}$$

by the projection formula. Since $J\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-D)$, $\bigoplus_{n \geq 0} f_*(E_1\mathcal{J}_1^n)$ is a $\bigoplus_{n \geq 0} (\hat{J}^n)^\sim$ module of finite type by EGA III 3.3.1.

Finally, we will verify (1) for $i = 1$. For $n \geq 0$ set $\mathcal{M}_n = \tilde{E}_1 \otimes \mathcal{O}_{\tilde{U}}(-nD_1)$. There are exact sequences

$$\begin{aligned} 0 \rightarrow R^1\gamma_*\tilde{f}_*\mathcal{M}_n &\rightarrow R^1(f \circ \beta)_*\mathcal{M}_n \rightarrow \gamma_*R^1\tilde{f}_*\mathcal{M}_n, \\ 0 \rightarrow R^1f_*(\beta_*\mathcal{M}_n) &\rightarrow R^1(f \circ \beta)_*\mathcal{M}_n. \end{aligned}$$

$\bigoplus_{n \geq 0} R^1f_*(E_1\mathcal{J}_1^n)$ is a finite type $\bigoplus_{n \geq 0} (\hat{J}^n)^\sim$ module since

$$\begin{aligned} \bigoplus_{n \geq 0} R^1\gamma_*\tilde{f}_*\mathcal{M}_n &\cong \bigoplus_{n \geq 0} R^1\gamma_*(\tilde{f}_*\tilde{E}_1 \otimes \mathcal{O}_{\tilde{X}}(-nD)) \quad \text{and} \\ \bigoplus_{n \geq 0} \gamma_*R^1\tilde{f}_*\mathcal{M}_n &\cong \bigoplus_{n \geq 0} \gamma_*((R^1(\tilde{f}_*\tilde{E}_1) \otimes \mathcal{O}_{\tilde{X}}(-nD))) \end{aligned}$$

are finite $\bigoplus_{n \geq 0} (\hat{J}^n)^\sim$ modules by EGA III 3.3.1.

COROLLARY 16 (Faltings Corollary 3 [F]). *Suppose that A is local and I -adically complete, I is generated by n elements, and $\dim A \geq n + 2 + \dim A/a$. Let $\hat{\mathcal{F}}$ be a locally free sheaf on \hat{U} which is defined by the germ of a vector bundle on U along $U \cap Y$. Then $M = \Gamma(\hat{U}, \hat{\mathcal{F}})$ is a finitely generated torsion free S_2 A -module. $\hat{\mathcal{F}}$ is isomorphic to the coherent sheaf on \hat{U} defined by M .*

Proof. There is a finite A -morphism from X_1 to a closed subscheme of \mathbf{P}_A^{n-1} . Hence $\text{Leff}(U, U \cap Y)$ by Theorem 15.

Let \mathcal{F} be a vector bundle on an open subset V of U containing $U \cap Y$ such that $\mathcal{F} \otimes \hat{\mathcal{O}}_U \cong \hat{\mathcal{F}}$. Let $i : V \rightarrow X$ be the inclusion. As in the first part of the proof of Theorem 15 we have $\text{codim}(X - V, X) \geq 2$. By Theorem 13 and Remark 14 we

have $i_*(\mathcal{F})$ is coherent and S_2 . $M = \Gamma(V, \mathcal{F}) = \Gamma(\hat{U}, \hat{\mathcal{F}})$ then has the desired properties.

THEOREM 17. *Suppose that A is excellent and there exists an ideal b of A with $I \subset b \subset a$ such that $\gamma^{-1}(X - V(b))$ is S_3 , and $\dim \gamma^{-1}(V(b)) \leq \dim(X) - 4$. Then $\text{Leff}(U, U \cap Y)$.*

Proof. The hypothesis of Theorem 15 hold, so that $\text{Leff}(U, U \cap Y)$ holds. Suppose that \mathcal{E} is coherent and locally free on \hat{U} . We must find an open neighborhood V of $U \cap Y$ and a locally free coherent sheaf E on V such that $\hat{E} \cong \mathcal{E}$.

Let $W = X - V(b)$. Let u and f be the natural inclusions $u : W \rightarrow U$, $f : U \rightarrow X$ and let $h = f \circ u$.

Let $\mathcal{F} = \mathcal{E}|_{\hat{W}}$. We will first prove that $\hat{h}_*(\mathcal{F})$ is coherent. Let $\mathcal{J}_1 = J\mathcal{O}_W$. By the existence theorem of SGA2 IX it suffices to show that

$$\bigoplus_{n \geq 0} R^i \hat{h}_*(\hat{\mathcal{J}}_1^n \hat{\mathcal{F}} / \hat{\mathcal{J}}_1^{n+1} \hat{\mathcal{F}}) \text{ is a finite type } \bigoplus_{n \geq 0} (J^n / J^{n+1})^\Delta \text{ module}$$

$$\text{for } i = 0 \text{ and } 1. \tag{2}$$

There is a natural diagram of maps

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{g} & \tilde{X} \\ \alpha \downarrow & & \downarrow \gamma \\ W & \xrightarrow{h} & X \end{array}$$

where $\tilde{W} = \gamma^{-1}(W)$, $\alpha = \gamma|_{\tilde{W}}$, and g is the natural inclusion. Let D, D_1 be effective Cartier divisors defined by $\mathcal{O}_{\tilde{X}}(-D) = J\mathcal{O}_{\tilde{X}}$ and $\mathcal{O}_{\tilde{W}}(-D_1) = J\mathcal{O}_{\tilde{W}}$. Taking I -adic completion, we get a diagram

$$\begin{array}{ccc} \hat{\tilde{W}} & \xrightarrow{\hat{g}} & \hat{\tilde{X}} \\ \hat{\alpha} \downarrow & & \downarrow \hat{\gamma} \\ \hat{W} & \xrightarrow{\hat{h}} & \hat{X} \end{array}$$

$\hat{\alpha}^*(\mathcal{F}) \otimes \mathcal{O}_{D_1}$ is a coherent $\mathcal{O}_{\hat{W}}$ module. \mathcal{O}_{D_1} is S_2 since $\mathcal{O}_{\hat{W}}$ is S_3 and $\mathcal{O}_{\hat{W}}(-D_1)$ is locally principal. $R^i \hat{g}_*(\hat{\alpha}^*(\mathcal{F}) \otimes \mathcal{O}_{D_1})$ is then a coherent $\mathcal{O}_{\hat{D}}$ module for $i = 0$ and 1 by Theorem 13, since $\text{codim}(D - D_1, D) \geq 3$, and $\hat{\alpha}^*(\mathcal{F}) \otimes \mathcal{O}_{D_1}$ is an S_2 \mathcal{O}_{D_1} module.

We will verify (2) for $i = 0$. By EGA III 4.1.5, for all $n \geq 0$

$$\begin{aligned} \hat{\alpha}_*(\hat{\alpha}^*\mathcal{F} \otimes \hat{\mathcal{O}}_{\tilde{W}}(-nD_1)) &\cong \mathcal{F} \otimes \hat{\alpha}_*\hat{\mathcal{O}}_{\tilde{W}}(-nD_1) \\ &\cong \mathcal{F} \otimes (\alpha_*\mathcal{O}_{\tilde{W}}(-nD_1))^\wedge \cong \mathcal{F}\hat{J}_1^n. \end{aligned}$$

By our choice of J , $R^1\hat{\alpha}_*(\hat{\mathcal{O}}_{\tilde{W}}(-nD_1)) \cong R^1\alpha_*\mathcal{O}_{\tilde{W}}(-nD_1)^\wedge = 0$ for all $n > 0$. There is an exact sequence

$$\begin{aligned} 0 \rightarrow \hat{\alpha}^*\mathcal{F} \otimes \hat{\mathcal{O}}_{\tilde{W}}(-(n+1)D_1) &\rightarrow \hat{\alpha}^*\mathcal{F} \otimes \hat{\mathcal{O}}_{\tilde{W}}(-nD_1) \\ &\rightarrow \hat{\alpha}^*\mathcal{F} \otimes \hat{\mathcal{O}}_{D_1}(-nD_1) \rightarrow 0 \end{aligned}$$

apply α_* to get

$$\begin{aligned} 0 \rightarrow \mathcal{F}\hat{J}_1^{n+1} \rightarrow \mathcal{F}\hat{J}_1^n &\rightarrow \mathcal{F} \otimes \hat{\alpha}_*(\hat{\mathcal{O}}_{D_1}(-nD_1)) \\ &\rightarrow \mathcal{F} \otimes (R^1\hat{\alpha}_*\hat{\mathcal{O}}_{\tilde{W}}(-(n+1)D_1)) = 0. \end{aligned}$$

Hence $\mathcal{F} \otimes \hat{\alpha}_*(\hat{\mathcal{O}}_{D_1}(-nD_1)) \cong \mathcal{F}\hat{J}_1^n/\mathcal{F}\hat{J}_1^{n+1}$ for all $n \geq 0$.

$$\begin{aligned} \hat{h}_*(\mathcal{F}\hat{J}_1^n/\mathcal{F}\hat{J}_1^{n+1}) &\cong \hat{\gamma}_*\hat{g}_*(\hat{\alpha}^*(\mathcal{F}) \otimes \mathcal{O}_{D_1}(-nD_1)) \\ &\cong \gamma_*(g_*(\hat{\alpha}^*(\mathcal{F}) \otimes \mathcal{O}_{D_1}) \otimes \mathcal{O}_{\tilde{X}}(-nD)) \\ \bigoplus_{n \geq 0} \hat{h}_*(\mathcal{F}\hat{J}_1^n/\mathcal{F}\hat{J}_1^{n+1}) &\cong \bigoplus_{n \geq 0} \gamma_*(g_*(\hat{\alpha}^*(\mathcal{F}) \otimes \mathcal{O}_{D_1}) \otimes \mathcal{O}_{\tilde{X}}(-nD)) \end{aligned}$$

is then a $\bigoplus_{n \geq 0} (\hat{J}^n)^\sim$ module of finite type by EGA III 3.3.1, and hence a $\bigoplus_{n \geq 0} (J^n/J^{n+1})^\Delta$ module of finite type.

Now we will verify (2) for $i = 1$. Set $\mathcal{M}_n = \hat{\alpha}^*(\mathcal{F}) \otimes \mathcal{O}_{D_1}(-nD_1)$ for $n \geq 0$. \mathcal{M}_n are coherent $\mathcal{O}_{\tilde{X}}$ modules. There are exact sequences

$$\begin{aligned} 0 \rightarrow R^1\gamma_*(g_*\mathcal{M}_n) \rightarrow R^1(h \circ \alpha)_*(\mathcal{M}_n) &\rightarrow \gamma_*R^1g_*(\mathcal{M}_n), \\ 0 \rightarrow R^1h_*(\alpha_*\mathcal{M}_n) \rightarrow R^1(h \circ \alpha)_*(\mathcal{M}_n), \end{aligned}$$

(2) follows since

$$\begin{aligned} \bigoplus_{n \geq 0} \gamma_*(R^1g_*\mathcal{M}_n) &\cong \bigoplus_{n \geq 0} \gamma_*(R^1g_*\mathcal{M}_0 \otimes \mathcal{O}_{\tilde{X}}(-nD)) \quad \text{and} \\ \bigoplus_{n \geq 0} R^1\gamma_*g_*\mathcal{M}_n &\cong \bigoplus_{n \geq 0} R^1\gamma_*(g_*\mathcal{M}_0 \otimes \mathcal{O}_{\tilde{X}}(-nD)) \end{aligned}$$

are $\bigoplus_{n \geq 0} (\hat{J}^n)^\sim$ modules of finite type by EGA III 3.3.1.

The next step is to show that $\hat{u}_*(\mathcal{F}) = \mathcal{E}$. Then $\hat{f}_*(\mathcal{E}) = \hat{f}_*(\hat{u}_*(\mathcal{F})) = \hat{h}_*(\mathcal{F})$ is coherent.

Let $Z_1 = V(b)$. We must show that $\Gamma(V - Z_1, \mathcal{E}) = \Gamma(V, \mathcal{E})$ for all open $V \subset \hat{U}$. Write $V = \cup \mathcal{D}(f_i)$ with $f_i \in A$ so that $\mathcal{E}|_{\mathcal{D}(f_i)}$ is free for all i . There is a commutative diagram of cech complexes, with exact rows.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Gamma(V, \mathcal{E}) & \longrightarrow & \bigoplus_i \Gamma(\mathcal{D}(f_i), \mathcal{E}) & \longrightarrow & \bigoplus_{ij} \Gamma(\mathcal{D}(f_i f_j), \mathcal{E}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Gamma(V - Z_1, \mathcal{E}) & \longrightarrow & \bigoplus_i \Gamma(\mathcal{D}(f_i) - Z_1, \mathcal{E}) & \longrightarrow & \bigoplus_{ij} \Gamma(\mathcal{D}(f_i f_j) - Z_1, \mathcal{E})
 \end{array} \tag{3}$$

Suppose that f is an f_i or an $f_i f_j$. Then $\mathcal{D}(f) = \text{spf}((A_f)^\wedge)$. Let $S = \text{spec}((A_f)^\wedge)$. By Lemma 12 and Theorem 15 we have $\text{Lef}(S - Z_1, (S - Z_1) \cap Y)$. Applying EGA III 4.1.5 and SGA2 III 3.5 gives $\Gamma(\hat{S}, \hat{\mathcal{O}}_S) = \Gamma(S, \mathcal{O}_S) = \Gamma(S - Z_1, \mathcal{O}_S) = \Gamma(\hat{S} - Z_1, \hat{\mathcal{O}}_S)$. Hence the second and third vertical arrows in (3) are isomorphisms and $\Gamma(V, \mathcal{E}) = \Gamma(V - Z_1, \mathcal{E})$.

We can now complete the proof of the theorem. There exists a coherent \hat{A} module E such that $\hat{f}_*(\mathcal{E}) \cong E^\Delta$ by EGA III 5.1.6. Hence there exists a neighborhood V of $U \cap Y$ in U such that $E|_V$ is locally free, and $(E|_V)^\wedge \cong \mathcal{E}$.

Suppose that A is local with maximal ideal m . The analytic deviation of I is

$$\text{ad}(I) = \dim \sigma^{-1}(m) + 1 - \text{ht}(I) = \dim \left(\bigoplus_{n \geq 0} I^n / mI^n \right) - \text{ht}(I).$$

We have $\text{ad}(I) \leq \delta(I)$.

COROLLARY 18. *Suppose that A is local Cohen–Macaulay and excellent with maximal ideal \mathfrak{a} and that I is equidimensional. Set $B = A/I$. Suppose that*

1. $\dim B \geq 3$
2. I_q is a complete intersection in A_q for $q \in \text{spec } A$ with $\dim B_q < 3 + \text{ad}(I)$.

Then $\text{Lef}(U, U \cap Y)$.

Proof. \hat{A} is Cohen–Macaulay and normal and \hat{I} is equidimensional since A is excellent by EGA IV 7.8.3. There is an exact sequence

$$A^s \xrightarrow{M} A^r \rightarrow I \rightarrow 0,$$

$I_{r-\text{ht}(I)}(M)$ is the ideal of the locus where I is not a complete intersection. Hence $I_{r-\text{ht}(I)}(M)\hat{A}$ is the ideal of the locus where \hat{I} is not a complete intersection. We may thus assume that A is I -adically complete.

If B is a complete intersection in A , then $\dim \gamma^{-1}(a) = \dim A = \dim B - 1 \leq \dim A - 4$ and the assumptions of Theorem 17 are satisfied with $b = a$.

If B is not a complete intersection in A , let b be an ideal defining the locus of points in A where B is not a complete intersection. If q is an associated prime of b then

$$\begin{aligned} 3 + \text{ad}(I) &\leq \dim B_q \\ &= \dim A_q - \text{ht } I_q \\ &= \dim A - \dim V(q) - \text{ht } I. \end{aligned}$$

Hence $\dim V(q) \leq \dim A - 4 - \dim \gamma^{-1}(a)$

$$\dim \gamma^{-1}(V(b)) \leq \dim V(b) + \dim(\gamma^{-1}(a)) \leq \dim A - 4.$$

The assumptions of Theorem 17 are then satisfied.

THEOREM 19. *Suppose that (A, m) is excellent, a quotient of a regular local ring and equidimensional of dimension ≥ 3 . If A is a complete intersection in codimension $2 + \delta(A)$, then A is pure.*

Proof. Let $A = R/I$ where R is regular. Let \hat{R} be the I -adic completion of R . Let $X = \text{spec}(\hat{R})$, $Y = V(I)$, $U = \text{spec}(\hat{R}) - m$. Suppose that V is an open subset of X such that $Y \cap U \subset V$. Then $\dim(X - V) \leq \dim R - 2$ as in the first part of the proof of Theorem 15. $(X, X - V)$ is then pure by SGA2 X 3.3 and purity for regular rings (Theorem 1). $A = \hat{A}$ is pure by Corollary 18 and Lemma 10.

REMARK. The proof shows the stronger statement that if A is a complete intersection in codimension $2 + \text{ad}(I)$ then A is pure. Theorems 19–22 are then true with deviation δ weakened to analytic deviation ad .

THEOREM 20. *Suppose that (A, m) is excellent, a quotient of a regular local ring and equidimensional. If A is a complete intersection in codimension $2 + \delta(A)$, then A is pure in codimension ≥ 3 .*

Proof. Suppose $p, q \in \text{spec}(A)$ with $p \subset q$ and $\text{ht}(p) \leq 2 + \delta(A_q)$. Then $\text{ht}(p) \leq 2 + \delta(A)$ so that A_p is a complete intersection. In particular, A_q is a complete intersection in codimension $2 + \delta(A_q)$. By Theorem 19 A is pure in codimension ≥ 3 .

Two applications of Theorem 20 are:

THEOREM 21. *Suppose that (A, m) is excellent, a quotient of a regular local ring, normal and a complete intersection in codimension $2 + \delta(A)$. If B/A is a finite extension with B normal, then $\text{codim}(\text{Ramlocus}(B/A)) \leq 2$.*

Proof. This is immediate from Theorem 20 and Lemma 3.

THEOREM 22. *Suppose that (A, m) is excellent, equicharacteristic, a quotient of a regular local ring, normal and R_2 . If A is a complete intersection in codimension*

$2 + \delta(A)$, then the divisor class group of A , $Cl(A)$, has no torsion of order prime to the characteristic of A .

Proof. This is immediate from Theorem 20, Theorem 1 and Lemma 4.

We will now show that the affine cones over some standard examples in projective geometry (which are not complete intersections) satisfy the conditions of Theorem 19.

EXAMPLE 1. (The affine cone over the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^2$ in \mathbf{P}^5). Let k be a field, $A = k[\{x_{ij}\}_{1 \leq i \leq 2, 1 \leq j \leq 3}]$, X be the 2×3 generic matrix with entries x_{ij} . Let $I_2(X)$ be the ideal in R generated by the 2×2 minors of X . $I_2(X)$ has height 2, and deviation 1. $A/I_2(X)$ is a normal domain, and has an isolated singularity at the vertex, so that the codimension of the singular locus is 4. $R/I_2(X)$ is then pure at the vertex.

EXAMPLE 2. (The affine cones over the Plucker embeddings of some Grassmanians). Let G_m^n be the Grassmanian of m -planes in n -space. Let k be a field, $A = k[x_1, \dots, x_N]$, with $N = \binom{n}{m}$. Let $I \subset A$ be the ideal of the $\binom{n}{m+1}$ independent Plucker relations determining the embedding of G_k^n in $\mathbf{P}^{\binom{n}{m}-1}$. I has height $\binom{n}{m} - (m(n - m) + 1)$, and deviation $\delta(I) = \binom{n}{m+1} - ((\binom{n}{m}) - (m(n - m) + 1))$. R/I is a normal singularity with an isolated singularity at the vertex, so that the codimension of the singular locus is $1 + m(n - m)$. R/I is pure at the vertex for $\binom{n}{m} - \binom{n}{m+1} \geq 3$. Some cases where this condition holds are G_3^5 and G_3^6 with deviations 2 and 5.

EXAMPLE 3. (The affine cones over the varieties of some Pfaffians). A classical discussion of Varieties of Pfaffians is in [R]. Let k be a field, $A = k[\{x_{ij}\}_{1 \leq i \leq j \leq 2n+1}]$, $G_n(k)$ is generic alternating $(2n + 1) \times (2n + 1)$ matrix. Let $Pf_{2n}(G_n(k))$ be the ideal in A generated by the $2n$ th order Pfaffians of $G_n(k)$. $\delta(Pf_{2n}(G_n(k))) = 2n - 2$. It is shown in Proposition 6.1 of [BE] that $R_n(k)/Pf_{2n}(G_n(k))$ is a normal domain, nonsingular in codimension 6. Purity at the vertex thus holds for $n = 2$ and 3. These examples have deviation 2 and 4 respectively.

4. Purity in equicharacteristic zero

Suppose that $p \in \mathbf{C}^n$. Then $\partial B_\delta(p)$ will denote the boundary of the ball $B_\delta(p)$ of radius δ about p . The following theorem is a special case of theorems of Hamm [H] and Goresky–MacPherson [GM].

THEOREM 23. *Suppose that U is a nonsingular complex algebraic affine variety of dimension n and $X \subset U$ is an algebraic subvariety defined by the vanishing of r functions on U . Suppose $p \in X$ and $r < n - 2$. Then for δ sufficiently small*

$$\pi_1^{\text{top}}(X \cap \partial B_\delta(p)) = 0.$$

Proof. Suppose that X is defined by the vanishing of f_1, \dots, f_r in U . Consider \mathbf{C}^r with coordinate functions z_1, \dots, z_r . Let $X' \subset U \times \mathbf{C}^r$ be the variety defined by

$$z_1 = f_1, \dots, z_r = f_r.$$

Let H be the linear subspace of codimension r in $U \times \mathbf{C}^r$ defined by the vanishing of z_1, \dots, z_r . There are maps

$$\begin{aligned} \pi_1^{\text{top}}(X \cap \partial B_\delta(p)) &= \pi_1^{\text{top}}(X' \cap H \cap \partial B_\delta(p)) \xrightarrow{\alpha} \\ &\pi_1^{\text{top}}(X' \cap \partial B_\delta(p)) \xrightarrow{\beta} \pi_1^{\text{top}}((U \times \mathbf{C}^r) \cap \partial B_\delta(p)). \end{aligned}$$

We will show that α is an isomorphism.

Let $\phi(k)$ be the dimension of the set of points $x \in X' - H$ such that a neighbourhood of $x \in X'$ can be defined in $U \times \mathbf{C}^r$ by k equations and no fewer. $\phi(k) = -\infty$ if this set is empty. It follows from Theorem 2 of Part II, Section 5.3 [GM], making use of the comment in “further” following the statement of the theorem, that

$$\pi_i^{\text{top}}(X' \cap H \cap \partial B_\delta(p)) \cong \pi_i^{\text{top}}(X' \cap \partial B_\delta(p))$$

for δ sufficiently small if

$$i < \inf_k(\dim_{\mathbf{C}} U \times \mathbf{C}^r - k - \inf(\phi(k), \text{codim}(H) - 1) - 2).$$

For $k > r$, $\phi(k) = -\infty$. For $k \leq r$,

$$\begin{aligned} &\dim_{\mathbf{C}} U \times \mathbf{C}^r - k - \inf(\phi(k), \text{codim}(H) - 1) - 2. \\ &\geq n + r - k - \inf(n + r, r - 1) - 2 \\ &= n - 1 - k \geq n - 1 - r > 1. \end{aligned}$$

Hence α is an isomorphism.

β is an isomorphism since X' is a complete intersection of dimension ≥ 3 . This follows, for instance, from another application of the above cited theorem of [GM]. Finally, $\pi_1^{\text{top}}((U \times \mathbf{C}^r) \cap \partial B_\delta(p)) = 0$ since $U \times \mathbf{C}^r$ is nonsingular of dimension ≥ 2 .

THEOREM 24. *Suppose that (A, m) is the analytic local ring of a point p of a normal complex variety X . Then for δ sufficiently small there is an isomorphism*

$$\pi_1^{\text{top}}(X \cap \partial B_\delta(p))^\wedge \cong \pi_1(\text{spec}(A) - m)$$

where $\hat{\pi}$ denotes pro-finite completion of π with respect to subgroups of finite index.

Proof. For all δ sufficiently small, $B_\delta(p) \cap X$ is homeomorphic to the cone over $X \cap \partial B_\delta(p)$ (c.f. Part I, Section 1.4 [GM]). Hence there exists $\delta > 0$ such that $\pi_1^{\text{top}}(X \cap \partial B_\delta(p)) \cong \pi_1^{\text{top}}((X - p) \cap B_\delta(p))$ and $\pi_1^{\text{top}}((X - p) \cap \partial B_{\delta_1}(p)) \cong \pi_1^{\text{top}}((X - p) \cap \partial B_{\delta_2}(p))$ for all $\delta_1, \delta_2 \leq \delta$.

Let $\mathcal{C}^{an}(Y)$ be the category of finite unramified analytic covers of a normal complex space Y . The natural map of $\mathcal{C}^{an}(Y)$ to the category of finite topological covers of Y is an equivalence.

For $\delta_1 < \delta_2 \leq \delta$ the restriction map

$$\mathcal{C}^{an}((X - p) \cap B_{\delta_2}(p)) \rightarrow \mathcal{C}^{an}((X - p) \cap B_{\delta_1}(p))$$

is an equivalence, since we have an isomorphism of the corresponding fundamental groups. Hence there is a natural map

$$\text{Et}(\text{spec}(A) - m) \rightarrow \mathcal{C}^{an}((X - p) \cap B_\delta(p))$$

defined by extension of a representative of a germ over $(X - p) \cap B_{\delta_1}(p)$ for δ_1 small. This map is an equivalence by [GR].

LEMMA 25. *Suppose that S is a domain essentially of finite type over a field k of characteristic zero. Then there exists a ring A of finite type over a field K containing k , and a maximal ideal m of $\text{spec}(A)$ such that $S \cong A_m$.*

Proof. There is an ideal $I = (f_1, \dots, f_r) \subset k[x_1, \dots, x_n]$ such that if $R = k[x_1, \dots, x_n]/I$ $S = R_P$. Let $\bar{u}_1, \dots, \bar{u}_s$ be a transcendence basis of R_P/P_P over k . Choose $u_i \in R_P$ which map to \bar{u}_i . Let $K = k(u_1, \dots, u_n) \subset R_P$. Let A be the k -algebra generated by K and R in S . A is of finite type over K . Let $m = PR_P \cap A \in \text{spec}(A)$. We have inclusions $K \subset A/m \subset R_P/P_P$. R_P/P_P is finite over K so that A/m is a field and m is a maximal ideal. $P = m \cap R$ implies $R_P \subset A_m$ so that $A_m = R_P$.

THEOREM 26. *Suppose that*

- (1) (A, m) is normal, local, essentially of finite type over a field k of characteristic zero, $p \in \text{spec}(A)$ is such that $\dim(A_p) - g\delta(A_p) \geq 3$.
- (2) B is a finite extension ring of A such that B is normal.
- (3) $q \in \text{spec}(B)$ is such that $q \cap A = p$ and B_q is unramified over A_p away from q .

Then B is unramified at q .

Proof. By Lemma 25 there exists a regular ring R of finite type over a field K containing k ; $f_1, \dots, f_t \in R$ and a maximal ideal p' of R such that $A_p = (R/I)_{p'}$, where $I = \sqrt{(f_1, \dots, f_t)}$, $g\delta(A_p) = t - (\dim R_{p'} - \dim A_p)$ and R/I is normal. Let $n = \dim R_{p'}$.

Let S be the normalization of R/I in the quotient field of B . Then S is of finite type over K and there exists $q' \in \text{spec}(S)$ such that $S_{q'} = B_q$. R, I, S , the f_i and the map $R/I \rightarrow S$ are defined over a finitely generated extension field of \mathbf{Q} , which can be embedded in \mathbf{C} . Hence there exists an extension field L containing K and \mathbf{C} , rings V, T, U of finite type over \mathbf{C} such that $T \rightarrow U$ is finite

$$R \otimes_K L \rightarrow (R/I) \otimes_K L \rightarrow S \otimes_K L$$

is obtained from

$$V \rightarrow T \rightarrow U$$

by base change with L , and $T = V/J$ where J is set theoretically defined by t equations.

T, U are normal and V is regular, by EGA IV 6.7.4. There exist maximal ideals a of T and b of U such that

$$a(T \otimes_{\mathbf{C}} L) \cap R/I = p', \quad b(U \otimes_{\mathbf{C}} L) \cap S = q', \quad b \cap T = a.$$

It suffices to show that $T \rightarrow U$ is unramified at b since $T \rightarrow U$ unramified at $b \Rightarrow (\Omega_{T/U}^1)_b = 0 \Rightarrow (\Omega_{S/(R/I)}^1)_{q'} = 0$, since base change by a field is faithfully flat, $\Rightarrow R/I \rightarrow S$ is unramified at q' .

Let E be the analytic local ring of $\text{specan}(T)$ at a , F be the analytic local ring of $\text{specan}(U)$ at b , with respective maximal ideals a' and b' . T_a and E have the same completions, as do U_b and F , so it suffices to show that $E \rightarrow F$ is unramified. Let

$$i : \text{spec}(E) - a' \rightarrow \text{spec}(E)$$

be inclusion.

By construction, each E is the analytic local ring of a point on a normal complex variety, which is set theoretically defined by t functions on an n dimensional nonsingular variety, with $n - t \geq 3$. F is a finite extension of E , by Theorem 4.2 [M] since E is Henselian, which is unramified away from b' . By Theorems 23 and 24 $\pi_1(\text{spec}(E) - a') = 0$. Hence $\mathcal{O}_F|_{\text{spec}(E) - a'}$ is a direct sum of copies of $\mathcal{O}_E|_{\text{spec}(E) - a'}$ as a sheaf of $\text{spec}(E) - a'$ algebras. $\mathcal{O}_F \cong i_*(\mathcal{O}_F|_{\text{spec}(E) - a'})$ is then a direct sum of copies of \mathcal{O}_E as an \mathcal{O}_E algebra since F and E are S_2 . F is then unramified over E .

COROLLARY. *Suppose that (A, m) is normal, local, essentially of finite type over a field k of characteristic zero and $\dim(A) - g\delta(A) \geq 3$. Then A is pure.*

Proof. Suppose that $X \rightarrow \text{spec}(A) - m$ is an etale cover. Let $i : \text{spec}(A) - m \rightarrow \text{spec}(A)$ be inclusion. Write $X = \text{spec}(\mathcal{A})$, where \mathcal{A} is a sheaf of $\text{spec}(A) - m$ algebras. $i_*\mathcal{A}$ is a normal finite A algebra. It suffices to show that $\text{spec}(i_*\mathcal{A}) \rightarrow \text{spec}(A)$ is unramified. This follows from Theorem 26.

THEOREM 27. *Suppose that A is normal, essentially of finite type over a field of characteristic zero, and that B is a finite extension of A such that B is normal. Suppose that r is a positive integer such that for every prime P of A of height $> r$*

$$\dim(A_P) - g\delta(A_P) \geq 3.$$

Then

$$\text{codim}(\text{Ramlocus}(B/A)) \leq r.$$

Proof. Suppose that p is a minimal prime of the branch locus. Let $q = p \cap A$. $A_q \rightarrow B_p$ is ramified only at q . By Theorem 26, $\dim(A_q) - g\delta(A_q) \leq 2$. Hence $\dim(A_q) \leq r$.

COROLLARY. *Suppose that A is normal, essentially of finite type over a field of characteristic zero, and that B is a finite extension of A such that B is normal. Suppose that A is a complete intersection in codimension $2 + g\delta(A)$. Then*

$$\text{codim}(\text{Ramlocus}(B/A)) \leq 2.$$

THEOREM 28. *Suppose that A is normal, essentially of finite type over a field of characteristic zero, and that B is a finite extension of A such that B is normal. Then*

$$\text{codim}(\text{Ramlocus}(B/A)) \leq 2 + g\delta(A).$$

Proof. Suppose that p is a minimal prime of the branch locus. Let $q = p \cap A$. $A_q \rightarrow B_p$ is ramified only at q . By Theorem 26

$$\dim(A_q) - g\delta(A_q) \leq 2.$$

Hence

$$\dim(A_q) \leq 2 + g\delta(A_q) \leq 2 + g\delta(A).$$

5. Factoriality in local rings of analytic deviation two

LEMMA 28. *Let (R, m) be a universally catenary noetherian local domain. Let $I \subset R$ be an equidimensional ideal. Suppose that $S = \bigoplus_{n \geq 0} I^n / I^{n+1}$ is S_r . Then*

$$\text{depth}(I^n / I^{n+1})_p \geq \inf(r, \dim(R/I)_p - \text{ad}(I))$$

for all $n \geq 0$, $p \in V(I)$.

Proof. We will use the criteria of SGA2 III 3.1, III 3.3. Let $\pi : \text{spec}(S) \rightarrow \text{spec}(R/I)$. S is equidimensional, since $\text{spec}(S)$ is the cone over $\text{proj}(S)$, which is equidimensional by the principal ideal theorem.

Let $p \in V(I)$. Then

$$\begin{aligned} \dim V(pS) &\leq \dim R/p + \dim V(mS) \\ &= \dim R/p + \text{ad}(I) + \text{ht}(I). \end{aligned}$$

If $q \in V(pS)$

$$\begin{aligned} \dim S_q &= \dim S - \dim V(q) \\ &\geq \dim S - \dim R/p - \text{ad}(I) - \text{ht}(I) \\ &= \dim R/I - \dim R/p - \text{ad}(I) \\ &= \dim(R/I)_p - \text{ad}(I). \end{aligned}$$

Hence $\text{depth } S_q \geq \inf(r, \dim(R/I)_p - \text{ad}(I))$.

Let $U = \text{spec}((R/I)_p) - p$, V be an open subset of $\text{spec}(R/I)_p$. Then

$$\begin{aligned} \bigoplus_{n \geq 0} H^i(V, (I^n/I^{n+1})_p) &\cong H^i(\pi^{-1}(V), \mathcal{O}_S \otimes R_p) \\ &\rightarrow H^i(\pi^{-1}(V \cap U), \mathcal{O}_S \otimes R_p) \\ &\cong \bigoplus_{n \geq 0} H^i(V \cap U, (I^n/I^{n+1})_p) \end{aligned}$$

is a bijection if $i < \inf(r, \dim(R/I)_p - \text{ad}(I)) - 1$, an injection if $i = \inf(r, \dim(R/I)_p - \text{ad}(I)) - 1$.

THEOREM 29. *Suppose that (A, m) is a normal factorial Cohen–Macaulay excellent local ring which is a quotient of a regular ring. Let $I \subset A$ be an ideal and $B = A/I$. Assume that*

- (1) I_q is a complete intersection in A_q for $q \in \text{spec}(A)$ with $\dim B_q < 3 + \text{ad}(I)$.
- (2) I^n/I^{n+1} is S_4 for all $n \geq 0$.
- (3) $\dim B_p \leq 3$ implies B_p factorial.

Then B is factorial.

Proof. Given $p \in \text{spec}(A)$ such that $\dim B_p \geq 4$, $\text{ad}(I_p) \leq \text{ad}(I)$ implies that A_p satisfies 1, 2 and 3. Hence by induction, it suffices to prove the theorem when B_p is factorial if $p \neq m$, and $\dim B \geq 4$.

Let $X = \text{spec}(\hat{A})$, $Y = V(I)$, $U = X - m$. We have $\text{Leff}(U, U \cap Y)$ by Corollary 18. By (2), $H^i(U, \tilde{I}^n/I^{n+1}) = 0$ for $i = 1, 2$ and $n \geq 0$. Suppose that V is an open subset of U containing $U \cap Y$. If $p \in U - V$, then (1) implies that $\dim V(p) \leq \dim A - 2$ by an argument as in the first part of the proof of Theorem 15. Hence A_p is parafactorial by SGA2 XI 3.10. The assumptions of SGA2 XI 3.12 are satisfied, so that $0 = \text{Pic}(U) \cong \text{Pic}(U \cap Y)$. Hence B is factorial.

COROLLARY 30. *Let A be a factorial Cohen–Macaulay excellent local ring which is a quotient of a regular ring. Let $P \in \text{spec}(A)$ and $B = A/P$. Assume that*

- (1) $\text{ad}(P) = 2$.
- (2) B is Cohen–Macaulay.
- (3) P_q is a complete intersection in A_q for $q \in \text{spec}(A)$ with $\dim B_q \leq 3 + \text{ad}(P)$.
- (4) $\dim B_q \leq 3$ implies that B_q is factorial.

Then B is factorial

Proof. A is Gorenstein since it is Cohen–Macaulay and factorial. $\bigoplus_{n \geq 0} P^n$ is Cohen–Macaulay by Corollaries 2.21 and 4.20 of [HH]. $\bigoplus_{n \geq 0} P^n / P^{n+1}$ is then Cohen–Macaulay as explained in the introduction of [HH].

Given $q \in V(P)$, if P_q is not a complete intersection

$$\text{depth}(P^n / P^{n+1})_q \geq \dim B_q - \text{ad}(P) \geq 4$$

by Lemma 28.

REMARKS. Our condition 3 is similar to the deviation condition of Hoobler’s factoriality Theorem [H]. It follows from Corollary 4.4 of [V] that there does not exist P as in corollary 30 with analytic deviation 1 (instead of analytic deviation 2).

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