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VALERY LUNTS

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Equivariant sheaves on toric varieties

VALERY LUNTS
Indiana University, Bloomington, IN 47405

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0. Introduction

(0.1) Let X be a normal toric variety for a torus T. Our goal is to give an algebraic description of the category $D^b_{T,c}(X)$ – the (constructible, bounded) derived category of T-equivariant sheaves on X defined in [BL]. Let $\{L_1, \dots, L_k\}$ be the total collection of simple equivariant perverse sheaves on X. Put $L = \oplus L_i$ and let $A^o = \mathrm{Ext}_{D_T(X)}(L, L)$ be the corresponding graded algebra. Let A be the opposite algebra. Let $e_i \colon L \to L_i$ be the projection and denote by $Q_i = Ae_i$ the corresponding projective A-module. Consider the DG-algebra $\mathcal{A} = (A, d = 0)$ with the zero differential. Consider the derived category $D_{\mathcal{A}}$ of DG-modules over $\mathcal{A}$ (see Section 1 below). Let $D^f_{\mathcal{A}} \subset D_{\mathcal{A}}$ be the ("finite") full triangulated subcategory generated by DG-modules Q_i (with the zero differential). Our main result is the following

(0.1.1) THEOREM. *Assume that X is affine or projective. Then there exists a natural equivalence of triangulated categories*

$$D^b_{T,c}(X) \simeq D^f_{\mathcal{A}}.$$

The main point of the theorem is the formality of some "geometric" DG-algebra $\mathcal{B}$ with the cohomology $H(\mathcal{B}) = A$. That is, we prove the quasiisomorphism $\mathcal{B} \simeq \mathcal{A}$. See 0.3 below for more details.

(0.1.2) REMARK. Consider the category $D^b_{G,c}(pt)$ for a connected Lie group G. In the paper [BL] this category was described in a similar way. So the above theorem may be considered as a natural extension of this result to the case when X has finitely many orbits and $G =$ torus.

We hope that the analogue of the above theorem holds in many cases of algebraic actions. In particular, let us formulate the following

(0.1.3) CONJECTURE. *Let G be a complex reductive group acting on a projective variety X with finitely many orbits. Then the analogue of the above theorem holds for the category $D^b_{G,c}(X)$.*

(0.2) Suppose we have an equivalence of categories as above

$$D^b_{G,c}(X) \simeq D^f_{\mathcal{A}}. \qquad (*)$$

Consider the derived category $D(A\text{-mod})$ of graded A-modules. Notice that there is a natural "forgetful" functor

$$g\colon D(A\text{-mod}) \to D_{\mathcal{A}},$$

which sends a complex of graded A-modules to its total complex. That is g makes a bigraded object into a graded one. In this sense $D(A\text{-mod})$ is a natural "mixed" version of $D_{\mathcal{A}}$. Notice, that the category $D(A\text{-mod})$ is simpler (!) than $D_{\mathcal{A}}$.

Let $D_g \subset D(A\text{-mod})$ be the full subcategory generated by Q_i's. Then D_g is a natural "mixed" version of $D^f_{\mathcal{A}}$ and, granted the equivalence $(*)$, of $D^b_{G,c}(X)$.

This point of view is implicit in the paper of W. Soergel [S], where he tries to relate representation theory to geometry on the level of categories extending the work of Adams–Barbasch–Vogan [ABV]. In particular our conjecture is essentially an explicit version of Soergel's Conjecture 2 ([S], 5.2).

Furthermore, let $D(A\text{-Mod})$ be the derived category of nongraded A-modules. We have the natural second forgetful functor

$$r\colon D(A\text{-mod}) \to D(A\text{-Mod}),$$

which forgets the grading of modules. Following Soergel, denote by $D_r \subset D(A\text{-}$ $\text{Mod})$ the full subcategory generated by simple A-modules. Let $D_{\text{rep}} \subset D(A\text{-Mod})$ be the image of D_r under the functor r.

Let us summarize the above functors in a diagram

$$\begin{array}{ccccc} D_r & \hookrightarrow & D(A\text{-mod}) & \hookleftarrow & D_g \\ \Big\downarrow{\scriptstyle r} & & & & \Big\downarrow{\scriptstyle g} \\ D_{\text{rep}} & & & & D^f_{\mathcal{A}} \end{array}.$$

Soergel considers categories D_{rep} and $D^f_{\mathcal{A}}$ (or their mixed versions D_r and D_g) as (Kozsul) dual to each other. And his point of view is that $D^f_{\mathcal{A}}$ is related to geometry (as explained above) and D_{rep} is related to representations (see [S] for details). So our present work is related to the geometric part of Soergel's conjectures.

(0.3) Let us explain how the most general form of our conjecture is "almost" true. Let Y be a topological space. Let $D^+(Y)$ be the bounded below derived category of sheaves on Y. Let $D \subset D^+(Y)$ be a triangulated category generated by a finite collection of objects $\{F_1, \ldots, F_k\} \subset D^+(Y)$. We may (and will) assume that F_i's consist of injective sheaves. Put $F = \oplus F_i$ and $B^o = \text{Ext}_{D(Y)}(F, F)$. Then B^o is the cohomology ring of the DG-algebra

$$\mathcal{B}^o := \text{Hom}^{\cdot}(F, F) = \Gamma(\mathit{Hom}^{\cdot}(F, F)).$$

Let B and $\mathcal{B}$ denote the corresponding opposite algebras. Let $e_i: F \to F_i$ be the projection and $\mathcal{P}_i = \mathcal{B}e_i$ be the corresponding DG-module over $\mathcal{B}$ (i.e. a $\mathcal{B}$-module). Let $D_{\mathcal{B}}$ be the derived category of $\mathcal{B}$-modules and $D^f_{\mathcal{B}} \subset D_{\mathcal{B}}$ be the triangulated subcategory generated by $\mathcal{P}_i$'s. The following proposition is easy to prove

(0.3.1) PROPOSITION. *There exists a natural equivalence of triangulated categories*

$$D \simeq D^f_{\mathcal{B}}.$$

Now we would like to replace the DG-algebra $\mathcal{B}$ by the DG-algebra $(B, d = 0)$. However, we do not know that $\mathcal{B}$ is formal, i.e. quasiisomorphic to its cohomology B, and hence cannot do this. So our theorem and conjecture essentially claim that some geometric DG-algebra (like $\mathcal{B}$ above) is formal.

(0.4) Let us briefly describe the method to prove the theorem. Let $\bar{X} = T\backslash X$ be the quotient space. It is a topological space with finitely many points. There is a natural sheaf $\mathcal{A}_{\bar{X}}$ of DG-algebras on $\bar{X}$. For each point $p \in \bar{X}$, the cohomology of the stalk $\mathcal{A}_{\bar{X},p}$ is the cohomology ring of the classifying space of the stabilizer of the orbit $o \subset X$ corresponding to p.

Consider the derived category $D_{\mathcal{A}_{\bar{X}}}$ of sheaves of DG-modules over $\mathcal{A}_{\bar{X}}$ (see Section 1). It has a natural full "constructible" subcategory $D_{\mathcal{A}_{\bar{X}},c} \subset D_{\mathcal{A}_{\bar{X}}}$.

Step 1. We establish a natural equivalence (Theorem 2.6).

$$D^b_{T,c}(X) \simeq D_{\mathcal{A}_{\bar{X}},c}.$$

Step 2. We prove that the sheaf of DG-algebras $\mathcal{A}_{\bar{X}}$ is formal (Theorem 3.1), hence we may replace $\mathcal{A}_{\bar{X}}$ by its cohomology $\mathcal{H}$.

After the above two steps we get a nice description of the category $D^b_{T,c}(X)$ which is interesting by itself (Theorem 3.3).

$$D^b_{T,c}(X) \simeq D_{\mathcal{H},c}.$$

Step 3. Finally we use results of [BL] (which in turn use the decomposition theorem for perverse sheaves) to finish the proof of the theorem. In particular we use the injectivity of the natural map

$$\mathrm{Ext}^{\cdot}(L, L) \to \mathrm{Hom}(H_T(L), H_T(L)),$$

(Theorem 4.0.3) where $H_T(L) = H_T(X, L)$ is the T-equivariant cohomology of L.

(0.4.1) REMARK. Although the same method does not apply directly to prove our conjecture, we believe that some variation of the method will.

(0.5) As is clear from the context, our main object of study is the derived category of equivarient sheaves $D^b_{T,c}(X)$. The theory of such categories and functors between them was worked out in detail in [BL]. Actually, the reader does not need to know anything about the category $D^b_{T,c}(X)$, except its definition which will be stated when needed.

Our second main ingredient is the language of DG-modules (also worked out in [BL] in the appropriate generality). This is reviewed and extended to sheaves of DG-modules in Section 1 below.

So this paper is self-contained for the most part, with the exception of the final step in the proof of the theorem which relies heavily on some results of [BL].

1. Sheaves of DG-modules

The general theory of DG-modules over a DG-algebra was worked out in [BL]. This includes the definition of the corresponding derived category and of various derived functors. Here we will need a slight extension of this theory to sheaves of DG-modules over a topological space with finitely many points. Actually the finiteness is needed only for the existence of enough $\mathcal{K}$-projectives (1.7.4). Most of the constructions and the definitions are identical to the original case of a single DG-algebra.

(1.0) Let Y be a space with finitely many points.

(1.0.1) Let $\mathcal{A} = \mathcal{A}_Y$ be a *sheaf of DG-algebras* on Y. The sheaf $\mathcal{A}$ consits of a sheaf $A = A_Y = \oplus_{i=-\infty}^{\infty} A^i$ of unitary graded associative $\mathbb{C}$-algebras with a $\mathbb{C}$-linear differential d of degree 1 satisfying

$$d^2 = 0,$$
$$d(a \cdot b) = da \cdot b + (-1)^{\deg(a)} a \cdot db$$

and

$$d(1) = 0.$$

(1.0.2) A *left DG-module over* $\mathcal{A}$, or simply an $\mathcal{A}$*-module*, is a sheaf $M = \oplus M^i$ of graded left A-modules with a differential d_M of degree 1 such that

$$d_M^2 = 0,$$
$$d_M(a \cdot m) = da \cdot m + (-1)^{\deg(a)} a \cdot d_M m.$$

We will usually denote an $\mathcal{A}$-module (M, d_M) simply by M.

A morphism of $\mathcal{A}$-modules is a morphism of degree zero of graded A-modules, which commutes with the differential. The *abelian category* of $\mathcal{A}$*-modules* is denoted by $\mathcal{M} = \mathcal{M}_{\mathcal{A}}$.

(1.0.3) Given $M \in \mathcal{M}$ and $U \subset Y$ we may consider the *cohomology* $H(M(U)) =$

$\operatorname{Ker} d_M(U)/\operatorname{Im} d_M(U)$.

(1.0.4) The *translation functor* $[1]\colon \mathcal{M} \to \mathcal{M}$ is an automorphism of $\mathcal{M}$ s.t.

$$(M[1])^i = M^{i+1}, \quad d_{M[1]} = -d_M,$$

and the A-module structure on $M[1]$ is twisted, that is

$$a \circ m = (-1)^{\deg(a)} am,$$

where $a \circ m$ is the multiplication in $M[1]$ and am is the multiplication in M.

(1.0.5) Two morphisms $f, g\colon M \to N$ are *homotopic* if there exists a morphism of A-modules (possibly not of $\mathcal{A}$-modules) $M \xrightarrow{s} N[-1]$ s.t.

$$f - g = sd_M + d_N s.$$

Null homotopic morphisms $\operatorname{Hot}(M, N)$ form a 2-sided ideal in $\operatorname{Hom}_{\mathcal{M}}(M, N)$ and we define the *homotopy category* $\mathcal{K} = \mathcal{K}_{\mathcal{A}}$ to have the same objects as $\mathcal{M}$ and morphisms

$$\operatorname{Hom}_{\mathcal{K}}(M, N) = \operatorname{Hom}_{\mathcal{M}}(M, N)/\operatorname{Hot}(M, N).$$

(1.0.6) The *cone* $C(u)$ of a morphism $M \xrightarrow{u} N$ is defined in the usual way. Namely, $C(u) = N \oplus M[1]$ with the differential $d_{N \oplus M[1]} = (d_N + u, -d_M)$. We have the obvious diagram

$$M \xrightarrow{u} N \to C(u) \to M[1],$$

in $\mathcal{M}$ which is called a *standard triangle.*

(1.0.7) An *exact triangle* in $\mathcal{K}$ is a diagram isomorphic (in $\mathcal{K}$) to a standard triangle above.

(1.0.8) PROPOSITION. *The homotopy category $\mathcal{K}$ with the translation functor [1] and the exact triangles defined above forms a triangulated category* (see *[Ve]).* □

(1.0.9) A morphism $M \xrightarrow{u} N$ in $\mathcal{M}$ is called a *quasiisomorphism* if for each point $p \in Y$ it induces an isomorphism on the stalk cohomology

$$u\colon H(M_p) \xrightarrow{\sim} H(N_p).$$

(1.0.10) The *derived category* $D = D_{\mathcal{A}}$ is the localization of $\mathcal{K}$ with respect to quasi-isomorphisms (see [Ve]).

(1.0.11) PROPOSITION. *The derived category D inherits a natural triangulation from $\mathcal{K}$.* □

(1.0.12) REMARK. One can check that a short exact sequence

$$0 \to M \to N \to K \to 0,$$

in $\mathcal{M}$ defines an exact triangle in D.

(1.0.13) REMARK. If $A^i = 0$, $i \neq 0$, then $D_{\mathcal{A}} = D(\dot{A}\text{-mod})$ – the derived category of complexes of sheaves of A-modules.

(1.1) As for any triangulated category, the functors $\mathrm{Hom}_{\mathcal{K}}(M, \cdot)$, $\mathrm{Hom}_{\mathcal{K}}(\cdot, N)$, $\mathrm{Hom}_D(M, \cdot)$, $\mathrm{Hom}_D(\cdot, N)$ from $\mathcal{K}$ or D to the category of abelian groups are *cohomological*. That is they take exact triangles into long exact sequences. Fix a point $p \in Y$. The functor $H(\cdot)_p$ – cohomology of the stalk at p – is cohomological on $\mathcal{K}$ or D. The functor $H(Y, \cdot)$ – global cohomology – is cohomological on $\mathcal{K}$.

(1.2) *Hom*$^{\cdot}$. Let $M, N \in \mathcal{M}$. Define a sheaf of complexes of $\mathbb{C}$-modules as follows

$$\mathit{Hom}^n(M, N) := \{\text{morphisms of } A\text{-modules } M \to N[n]\};$$

if $f \in \mathit{Hom}^n(M, N)$, then

$$df = d_N f - (-1)^n f d_M.$$

Put $\mathrm{Hom}^{\cdot}(M, N) = \Gamma(Y, \mathit{Hom}^{\cdot}(M, N))$ – the complex of global sections.

Note that by definition $\mathrm{Hom}_{\mathcal{M}}(M, N)$ = zero cycles in $\mathrm{Hom}^{\cdot}(M, N)$ and $\mathrm{Hom}_{\mathcal{K}}(M, N) = H^0(\mathrm{Hom}^{\cdot}(M, N))$.

The bifunctor $\mathit{Hom}^{\cdot}(\cdot, \cdot)$ preserves homotopies and defines an exact bifunctor

$$\mathit{Hom}^{\cdot}(\cdot, \cdot)\colon \mathcal{K}^0_{\mathcal{A}} \times \mathcal{K}_{\mathcal{A}} \to \mathcal{K}_{\mathbb{C}}.$$

(1.3) *Right modules*. One can develop a similar theory for right DG-modules.

(1.3.1) DEFINITION. A *right DG-module* (M, d_M) over $\mathcal{A} = (A, d)$ is a sheaf of right graded A-modules $M = \oplus M^i$ with a differential $d_M\colon M \to M$ of degree 1, s.t. $d_M^2 = 0$ and

$$d_M(ma) = d_M m \cdot a + (-1)^{\deg(m)} m \cdot da.$$

Denote the abelian category of right $\mathcal{A}$-modules by $\mathcal{M}^r_{\mathcal{A}}$.

One can either proceed to define the homotopy category $\mathcal{K}^r_{\mathcal{A}}$ and the derived category $D^r_{\mathcal{A}}$ in a way similar to left $\mathcal{A}$-modules, or simply reduce the study of right modules to that of left modules using the following Remark (1.3.3) (the two approaches yield the same result).

(1.3.2) For a DG-algebra $\mathcal{A} = (A, d)$ we define its *opposite* $\mathcal{A}^o = (A^o, d)$ to have the same elements and the same differential d, but a new multiplication $a \cdot b$ defined by

$$a \cdot b := (-1)^{\deg(a)\cdot\deg(b)} ba,$$

where ba denotes the multiplication in A.

(1.3.3) REMARK. The categories $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}^r_{\mathcal{A}^o}$ are naturally isomorphic. Namely, let $M \in \mathcal{M}_{\mathcal{A}}$ be a *left* $\mathcal{A}$-module. Define on M the structure of a *right* $\mathcal{A}^o$-module as follows:

$$m \circ a := (-1)^{\deg(a)\cdot\deg(m)} am.$$

(1.3.4) A DG-algebra is called *supercommutative* if $ab = (-1)^{\deg(a)\cdot\deg(b)} ba$. In other words, $\mathcal{A}$ is supercommutative if $\mathcal{A} = \mathcal{A}^o$.

(1.4) $\otimes_{\mathcal{A}}$.

Let $M \in \mathcal{M}^r_{\mathcal{A}}$, $N \in \mathcal{M}_{\mathcal{A}}$ be a right and a left $\mathcal{A}$-module. Consider the complex of sheaves $M \otimes_A N$ with the differential

$$d(m \otimes n) = d_M m \otimes n + (-1)^{\deg(m)} m \otimes d_N n.$$

Denote this complex of sheaves by $M \otimes_{\mathcal{A}} N$.

The bifunctor $\otimes_{\mathcal{A}}$ preserves homotopies and descends to an exact bifunctor

$$\otimes_{\mathcal{A}}: \mathcal{K}^r_{\mathcal{A}} \times \mathcal{K}_{\mathcal{A}} \to \mathcal{K}_{\mathbb{C}}.$$

(1.5) In case $\mathcal{A}$ is supercommutative the sheaves of complexes $Hom^{\cdot}(M, N)$ and $M \otimes_{\mathcal{A}} N$ are in fact $\mathcal{A}$-modules. Namely, for $f \in Hom^{\cdot}(M, N)$ put $(af)(m) = af(m)$; and $a(m \otimes n) = (-1)^{\deg(a)\cdot\deg(m)} ma \otimes n$.

(1.6.1) Given an open subset $U \subset Y$ and $M \in \mathcal{M}_{\mathcal{A}}$ denote by $M_U \in \mathcal{M}_{\mathcal{A}}$ the extension by zero of $M|_U$ to Y.

(1.6.2) Define the *constructible subcategory* $D_{\mathcal{A},c} \subset D_{\mathcal{A}}$ to be the full subcategory generated by $\mathcal{A}$-modules $\{\mathcal{A}_U\}$, $U \subset Y$ open. Notice that since the space Y is finite it suffices to take $\mathcal{A}_U$'s for only irreducible open subsets $U \subset Y$.

(1.7) DERIVED FUNCTORS

We want to define the derived functors of $Hom^{\cdot}$ and $\otimes_{\mathcal{A}}$ in the sense of Deligne ([D]). We will use the notion of a $\mathcal{K}$-projective object introduced in [Sp]. The main fact is that $D_{\mathcal{A}}$ has enough $\mathcal{K}$-projectives. This is the only place, where we use that the space Y is finite.

(1.7.1) Let $P \in \mathcal{M}_{\mathcal{A}}$. We say that P is *$\mathcal{K}$-projective* if one of the following equivalent properties holds:

(i) For each $M \in \mathcal{M}_{\mathcal{A}}$

$$\mathrm{Hom}_{\mathcal{K}_{\mathcal{A}}}(P, M) = \mathrm{Hom}_{D_{\mathcal{A}}}(P, M).$$

(ii) For each $M \in \mathcal{M}_{\mathcal{A}}$, such that $H(M_p) = 0$ for every point $p \in Y$

$$H(\mathrm{Hom}^{\cdot}(P, M)) = 0.$$

(1.7.2) EXAMPLE. Let $U \subset Y$ be an irreducible open subset. Then the $\mathcal{A}$-module $\mathcal{A}_U$ is $\mathcal{K}$-projective. Indeed, let $p \in U$ be a point so that U is the smallest open subset containing p. Then for any $M \in \mathcal{M}_{\mathcal{A}}$ we have $M(U) = M_p$. Hence $\mathrm{Hom}_{\mathcal{K}}(\mathcal{A}_U, M) = H(M_p)$.

(1.7.3) Let $P \in \mathcal{M}_{\mathcal{A}}$ be $\mathcal{K}$-projective. A quasiisomorphism $P \xrightarrow{\sim} M$ is called a *$\mathcal{K}$-projective resolution* of M.

(1.7.4) PROPOSITION. *Every $M \in \mathcal{M}_{\mathcal{A}}$ has a $\mathcal{K}$-projective resolution.*

Sketch of proof:

Step 1. Construct a complex of $\mathcal{A}$-modules

$$\cdots \xrightarrow{\delta^{-2}} P^{-1} \xrightarrow{\delta^{-1}} P^0 \xrightarrow{\delta^0} M \longrightarrow 0,$$

such that

(a) For each point $p \in Y$ the complex

$$\to H(P_p^{-1}) \to H(P_p^0) \to H(M_p) \to 0,$$

is exact.

(b) Each P^{-i} is a direct sum $P^{-i} = \oplus \mathcal{A}_U[?]$ of shifted $\mathcal{K}$-projective modules $\mathcal{A}_U$ for irreducible open subsets $U \subset Y$.

Step 2. Define a new $\mathcal{A}$-module $P = \oplus P^{-i}[i]$, where the $\mathcal{A}$-module structure on $P^{-i}[i]$ is the *same* as on P^{-i} and the differential $d: P^{-i}[i] \to P^{-i}[i] \oplus P^{-i+1}[i-1]$ is

$$d(p) = (d_{P^{-i}}(p), \quad (-1)^{\deg(p)}\delta^{-i}(p)).$$

By the construction the map

$$\delta^0 : P \to M,$$

is a quasiisomorphism.

Step 3. Prove that P is $\mathcal{K}$-projective by verifying the property (ii) of 1.7.1. □

(1.8) Now we can define the derived functors *RHom*$^{\cdot}$ and RHom$^{\cdot}$ as follows. Let $P \xrightarrow{\sim} M$ be a $\mathcal{K}$-projective resolution. Then put

$$\mathit{RHom}^{\cdot}(M, N) := \mathit{Hom}^{\cdot}(P, N)$$

$$\mathrm{RHom}^{\cdot}(M, N) := \mathrm{Hom}^{\cdot}(P, N).$$

We have $\mathrm{Hom}_{D_{\mathcal{A}}}(M, N) = H^0(\mathrm{Hom}^{\cdot}(P, N))$.

Thus *RHom*˙ and RHom˙ become exact bifunctors

$$RHom^{\cdot}: D^{o}_{\mathcal{A}} \times D_{\mathcal{A}} \to D_{\mathbb{C}}$$

$$\mathrm{RHom}^{\cdot}: D^{o}_{\mathcal{A}} \times D_{\mathcal{A}} \to D(\mathbb{C}\text{-vect}).$$

(1.9) An $\mathcal{A}$-module $Q \in \mathcal{M}_{\mathcal{A}}$ is called *$\mathcal{K}$-flat* if, given a right module $N \in \mathcal{M}^{r}_{\mathcal{A}}$ such that $H(N_p) = 0$ for every $p \in Y$, the same is true for $N \otimes_{\mathcal{A}} Q$, i.e. $H(N \otimes_{\mathcal{A}} Q)_p = 0$ for all $p \in Y$.

It is easy to check that the $\mathcal{K}$-projective module P constructed in the proof of Proposition (1.7.4) is $\mathcal{K}$-flat. Hence *every* $\mathcal{K}$-projective is also $\mathcal{K}$-flat.

(1.10) The previous discussion allows us to define the derived functor $\overset{L}{\otimes}_{\mathcal{A}}$ of the tensor product $\otimes_{\mathcal{A}}$. Namely, given $N \in \mathcal{M}^{r}_{\mathcal{A}}$, $M \in \mathcal{M}_{\mathcal{A}}$ and a $\mathcal{K}$-projective resolution $P \to M$ we put

$$N \overset{L}{\otimes}_{\mathcal{A}} M := N \otimes_{\mathcal{A}} P.$$

Thus we obtain an exact bifunctor

$$\overset{L}{\otimes}_{\mathcal{A}}: D^{r}_{\mathcal{A}} \times D_{\mathcal{A}} \to D_{\mathbb{C}}.$$

(1.11.1) Let $\mathcal{B} = (B, d)$ be another sheaf of DG-algebras over Y and let $\phi: \mathcal{A} \to \mathcal{B}$ be a homomorphism of DG-algebras. That is, ϕ is a unitary homomorphism of sheaves of algebras $A \to B$ that commutes with the differentials.

Then ϕ induces a functor of restriction of scalars

$$\phi_*: D_{\mathcal{B}} \to D_{\mathcal{A}}.$$

Consider $\mathcal{B}$ as a right $\mathcal{A}$-module via ϕ. We get a functor of extension of scalars

$$\phi^*: D_{\mathcal{A}} \to D_{\mathcal{B}}$$

$$\phi^*(M) = \mathcal{B} \overset{L}{\otimes}_{\mathcal{A}} M.$$

(1.11.2) PROPOSITION. *Assume that $\phi: \mathcal{A} \to \mathcal{B}$ is a quasiisomorphism, i.e. $\phi: H(\mathcal{A}_p) \xrightarrow{\sim} H(\mathcal{B}_p)$ for every $p \in Y$. Then the above functors ϕ_* and ϕ^* are mutually inverse equivalences of categories $D_{\mathcal{A}} \simeq D_{\mathcal{B}}$. They also induce the equivalence of constructible subcategories $D_{\mathcal{A},c} \simeq D_{\mathcal{B},c}$.*

Proof. In case $Y = pt$ this is proved in [BL], Theorem (10.12.5.1). The same proof works here, since in [BL] we only used the fact that a $\mathcal{K}$-projective module is $\mathcal{K}$-flat (see (1.9) above).

(1.11.3) REMARK. The above equivalence of categories preserves stalk cohomology, i.e. $H(\phi^*(M)_p) = H(M_p)$ and $H(\phi_*(N)_p) = H(N_p)$ for $M \in D_{\mathcal{A}}$, $N \in D_{\mathcal{B}}$, $p \in Y$.

2. Localization and global sections: an equivalence of categories

(2.1) Let $T = (\mathbb{C}^*)^n$ be a torus. Let us recall the category $D^b_{T,c}(X)$ for a toric variety X. We will state the definition of $D^b_{T,c}(X)$ which is the most convenient for our purposes here.

Let $E \to BT$ be the classifying bundle for T. For any T-space Z put $Z_T = Z \times_T E$. Then $D^b_{T,c}(X)$ may be viewed as the full subcategory of $D^b(X_T)$ consisting of complexes C with the following property: for any orbit $o \subset X$ and any cohomology sheaf $\mathcal{H}^i(C)$ the restriction of $\mathcal{H}^i(C)$ to o_T is a constant sheaf of finite rank. For different choices of the classifying bundle $E \to BT$ the corresponding categories are naturally equivalent.

(2.2) Assume that X is quasiprojective. Then we can find a T-equivariant embedding $X \hookrightarrow \mathbb{P}^N$ for a linear action of T on $\mathbb{P}^N$. By choosing the space E appropriately we may (and will) assume that the space $\mathbb{P}^N_T$ is paracompact and is an inductive limit of manifolds. In particular, it is an ∞-dimensional manifold according to the definition in [BL], 12.2. Let $\Omega^{\cdot}_{\mathbb{P}^N_T}$ be its deRham complex (as defined in [BL], 12.2.2.). We know that $\Omega^{\cdot}_{\mathbb{P}^N_T}$ is a resolution of the constant sheaf $\mathbb{C}_{\mathbb{P}^N_T}$ and consists of soft sheaves. Since $\mathbb{P}^N_T$ is paracompact, the sheaves in $\Omega^{\cdot}_{\mathbb{P}^N_T}$ are acyclic. Moreover, for any sheaf S on $\mathbb{P}^N_T$ and any $p \geqslant 0$ the sheaf $\Omega^p_{\mathbb{P}^N_T} \otimes S$ is also soft (as a module over the soft sheaf of rings $\Omega^0_{\mathbb{P}^N_T}$) and hence is acyclic. The complex $\Omega^{\cdot}_{\mathbb{P}^N_T}$ has a natural multiplicative structure (the wedge product), so for an open subset $U \subset \mathbb{P}^N_T$, the global sections $\Omega^{\cdot}_{\mathbb{P}^N_T}(U)$ form a (supercommutative) DG-algebra.

Finally we put

$$\mathcal{F}^{\cdot} = \mathcal{F}^{\cdot}_{X_T} := \Omega^{\cdot}_{\mathbb{P}^N_T}|X_T.$$

(2.3) Consider the space $\bar{X} = T\backslash X$. It has finitely many points and the quotient topology. We have a natural continuous map

$$q: X_T \to \bar{X},$$

which sends o_T to the point $T\backslash o$ for any orbit $o \subset X$. Consider the sheaf of DG-algebras

$$\mathcal{A} = \mathcal{A}_{\bar{X}} := q_* \mathcal{F}^{\cdot},$$

on $\bar{X}$. We consider the corresponding derived category $D_{\mathcal{A}}$ of DG-modules over $\mathcal{A}$ and its constructible subcategory $D_{\mathcal{A},c}$ as defined in Section 1 above.

(2.4) Let us define the functor of "gobal sections" $\gamma: D^+(X_T) \to D_{\mathcal{A}}$ as follows. Let $F \in D^+(X_T)$ be a bounded below complex. Consider the complex $\mathcal{F}^{\cdot} \otimes_{\mathbb{C}} F$. It is bounded below complex consisting of soft sheaves, which is quasiisomorphic to F (see (2.2) above). Moreover $\mathcal{F}^{\cdot} \otimes F$ is naturally a sheaf of DG-modules over $\mathcal{F}^{\cdot}$. We put

$$\gamma(F) := q_*(\mathcal{F}^{\cdot} \otimes F),$$

which is an $\mathcal{A}$-module, hence an element of $D_{\mathcal{A}}$. Notice that γ preserves quasi-isomorphisms, hence is a well defined functor

$$\gamma: D^+(X_T) \to D_{\mathcal{A}}.$$

(2.5) Let us define the localization functor $\mathcal{L}: D_{\mathcal{A}} \to D(X_T)$. Let $M \in D_{\mathcal{A}}$ be an $\mathcal{A}$-module. Choose a $\mathcal{K}$-projective resolution $P \to M$ (see (1.7.3.4)). Consider $q^*(P)$ as a sheaf of $q^*\mathcal{A}$-modules. Finally, put

$$\mathcal{L}(M) := \mathcal{F}^{\cdot} \otimes_{q^*\mathcal{A}} q^*(P).$$

(2.6) THEOREM. *The above functors γ and $\mathcal{L}$ preserve the subcategories $D^b_{T,c}(X) \subset D^+(X_T)$ and $D_{\mathcal{A},c} \subset D_{\mathcal{A}}$ and induce mutually inverse equivalences*

$$D^b_{T,c}(X) \underset{\mathcal{L}}{\overset{\gamma}{\rightleftarrows}} D_{\mathcal{A},c}.$$

Proof. Recall that the category $D_{\mathcal{A},c}$ is generated by the objects $\mathcal{A}_W$, where $W \subset \bar{X}$ is an irreducible open subset (1.6.2). Notice that such $\mathcal{A}_W$ is $\mathcal{K}$-projective, hence $\mathcal{L}(\mathcal{A}_W) = \mathcal{F}^{\cdot} \otimes_{q^*\mathcal{A}} q^*(\mathcal{A}_W)$ lies in $D^+(X_T)$. So the composition $\gamma \cdot \mathcal{L}$ is a well defined functor from $D_{\mathcal{A},c}$ to $D_{\mathcal{A}}$.

(2.6.1) Let us define morphisms of functors

$$\alpha: Id_{D_{\mathcal{A},c}} \to \gamma \cdot \mathcal{L}$$

$$\beta: \mathcal{L} \cdot \gamma \to Id_{D^b_{T,c}(X)},$$

as follows.

Let $P \in D_{\mathcal{A},c}$ be $\mathcal{K}$-projective. Then

$$\gamma \cdot \mathcal{L}(P) = q_*(\mathcal{F}^{\cdot} \otimes_{\mathbb{C}} (\mathcal{F}^{\cdot} \otimes_{q^*\mathcal{A}} q^*P)).$$

Consider the quasiisomorphism

$$\theta: \mathcal{F}^{\cdot} \otimes_{\mathbb{C}} (\mathcal{F}^{\cdot} \otimes_{q^*\mathcal{A}} q^*P) \to \mathcal{F}^{\cdot} \otimes_{q^*\mathcal{A}} q^*P$$

$$\theta: \omega \otimes \omega' \otimes p \mapsto \omega\omega' \otimes p,$$

and the induced quasiisomorphism of $\mathcal{A}$-modules

$$q_*(\theta): \gamma \cdot \mathcal{L}(P) \to q_*(\mathcal{F}^{\cdot} \otimes_{q^*\mathcal{A}} q^*P).$$

Define a morphism of $\mathcal{A}$-modules

$$\eta: P \to q_*(\mathcal{F}^\cdot \otimes_{q^*\mathcal{A}} q^*P)$$

$$\eta: p \mapsto 1 \otimes p.$$

Finally, put

$$\alpha := q_*(\theta)^{-1} \cdot \eta.$$

Let $F \in D^b_{T,c}(X)$. Let $\varepsilon: P \to q_*(\mathcal{F}^\cdot \otimes F)$ be a $\mathcal{K}$-projective resolution. Then

$$\mathcal{L} \cdot \gamma(F) = \mathcal{F}^\cdot \otimes_{q^*\mathcal{A}} q^*P.$$

We define the morphism $\beta: \mathcal{L} \cdot \gamma(F) \to \mathcal{F}^\cdot \otimes F \simeq F$ as the composition

$$\mathcal{F}^\cdot \otimes_{q^*\mathcal{A}} q^*P \xrightarrow{1\otimes q^*\varepsilon} \mathcal{F}^\cdot \otimes_{q^*\mathcal{A}} q^*q_*(\mathcal{F}^\cdot \otimes_{\mathbb{C}} F) \xrightarrow{\delta} \mathcal{F}^\cdot \otimes F,$$

where $\delta: \omega \otimes \omega' \otimes f \mapsto \omega\omega' \otimes f$.

(2.6.2) CLAIM. *The morphisms α and β are isomorphisms.*

Let $V \subset X$ be the star of some orbit and $j: V_T \hookrightarrow X_T$ be the corresponding open embedding. Put $W = q(V_T)$ – an irreducible open subset in $\bar{X}$. To prove the claim (and hence the Theorem (2.6)) it suffices to show that $\alpha(\mathcal{A}_W)$ and $\beta(j_!\mathbb{C}_{V_T})$ are isomorphisms.

(2.6.3) LEMMA. *There exist natural morphisms which are quasiisomorphisms in $D^b_{T,c}(X)$ (in (a)) in $D_{\mathcal{A},c}$ (in (b)):*

(a) $j_!\mathbb{C}_{V_T} \to \mathcal{F}^\cdot \otimes_{q^*\mathcal{A}} q^*(\mathcal{A}_W)$

(b) $\mathcal{A}_W \to q_*(\mathcal{F}^\cdot \otimes j_!\mathbb{C}_{V_T})$.

(c) *The morphism $\eta(\mathcal{A}_W): \mathcal{A}_W \to q_*(\mathcal{F}^\cdot \otimes_{q^*\mathcal{A}} q^*(\mathcal{A}_W))$ as defined above is a quasiisomorphism.*

Proof of lemma.

(a) Note that $j_!\mathbb{C}_{V_T}$ is naturally a subsheaf of $q^*(\mathcal{A}_W)$ and define a map

$$j_!\mathbb{C}_{V_T} \to \mathcal{F}^\cdot \otimes_{q^*\mathcal{A}} q^*(\mathcal{A}_W) \qquad (*)$$

by $s \mapsto 1 \otimes s$. We will check that this map induces a quasiisomorphism at each point $x \in X_T$.

Let $x \in V_T$. Then the RHS in $(*)$ is isomorphic to the stalk $(\mathcal{F}^\cdot)_x$ which is quasiisomorphic to $\mathbb{C}$.

Let $x \notin V_T$. Then the stalk $q^*(\mathcal{A}_W)_x = 0$ and hence also $(\mathcal{F}^\cdot \otimes_{q^*\mathcal{A}} q^*(\mathcal{A}_W))_x = 0$.

So $(*)$ is a quasiisomorphism.

(b) The map

$$\mathcal{A}_W \to q_*(\mathcal{F}^{\cdot} \otimes j_!\mathbb{C}_{V_T})$$

is the obvious one since the restriction of RHS to W is by definition equal to $\mathcal{A}|_W$. It remains to prove that the stalks of RHS outside of W are acyclic. Fix a point $y \in \bar{X}$, $y \notin W$. Let $\tau_T = q^{-1}(y)$ for an orbit τ. Let $U \subset X$ be the star of the orbit τ. It amounts to show the vanishing of the cohomology

$$H^i(U_T, j_!\mathbb{C}_{V_T}) = 0, \quad \forall i.$$

But this is clear, since the space τ_T is a homotopy retract of U_T by the action of a subgroup $\mathbb{C}^* \subset T$, and $\tau_T \cap V_T = \emptyset$.

(c) As in (b) it is clear that η is an isomorphism over W. So again it remains to show that stalks of RHS outside of W are acyclic. But this is already done in the proof of (b) above, since by (a) we have $\mathcal{F}^{\cdot} \otimes_{q^*\mathcal{A}} q^*(\mathcal{A}_W) \simeq j_!\mathbb{C}_{V_T}$.

(2.6.4) COROLLARY. *The morphism α is an isomorphism.*

Indeed, $\alpha = q_*(\theta)^{-1} \cdot \eta$ and η is an isomorphism by (c) in the above lemma. □

It remains to show that $\beta(j_!\mathbb{C}_{V_T})$ is a quasiisomorphism. By (b) in Lemma (2.6.3) the map $\varepsilon: \mathcal{A}_W \to q_*(\mathcal{F}^{\cdot} \otimes j_!\mathbb{C}_{V_T})$ is a $\mathcal{K}$-projective resolution. So $\beta(j_!\mathbb{C}_{V_T})$ is the following composition

$$\mathcal{F}^{\cdot} \otimes_{q^*_{\mathcal{A}}} q^*(\mathcal{A}_W) \xrightarrow{1 \otimes q^*\varepsilon} \mathcal{F}^{\cdot} \otimes_{q^*\mathcal{A}} q^*q_*(\mathcal{F}^{\cdot} \otimes j_!\mathbb{C}_{V_T}) \xrightarrow{\delta} \mathcal{F}^{\cdot} \otimes j_!\mathbb{C}_{V_T}.$$

Consider the quasiisomorphism

$$\mu: j_!\mathbb{C}_{V_T} \to \mathcal{F}^{\cdot} \otimes_{q^*\mathcal{A}} q^*(\mathcal{A}_W)$$

from part (a) in Lemma (2.6.3). Observe that the composition $\beta(j_!\mathbb{C}_{V_T}) \cdot \mu$ is the inclusion

$$j_!\mathbb{C}_{V_T} \to \mathcal{F}^{\cdot} \otimes j_!\mathbb{C}_{V_T}, \quad s \mapsto 1 \otimes s.$$

Hence $\beta(j_!\mathbb{C}_{V_T})$ is a quasiisomorphism, which finishes the proof of Claim (2.6.2) and of Theorem 2.6.

(2.7) REMARK. Let $F \in D^b_{T,c}(X)$ and put $M = \gamma(F) \in D_{\mathcal{A},c}$. Fix an orbit $o \subset X$ and let $V = \mathrm{St}(o) \subset X$ be its star. Let $p = T\backslash o \in \bar{X}$ be the corresponding point. Then by the definition of γ we have

$$H(M_p) = H_T(V, F) := H(V_T, F).$$

3. Formality of the sheaf $\mathcal{A}_{\bar{X}}$

(3.0) Our next goal is to establish the formality of the sheaf of DG-algebras $\mathcal{A} = \mathcal{A}_{\bar{X}}$ (see 2.3).

Recall the definition of $\mathcal{A}$. We considered (2.2) a T-equivariant embedding $X \hookrightarrow \mathbb{P}^N$ and defined $\mathcal{F}^{\cdot} = \mathcal{F}^{\cdot}_{X_T}$ to be the restriction to X_T of the deRham complex $\Omega^{\cdot}_{\mathbb{P}^N_T}$. Then in (2.3) we defined

$$\mathcal{A} := q_* \mathcal{F}^{\cdot}$$

for the natural map

$$q: X_T \to \bar{X}.$$

(3.1) THEOREM. *The sheaf of DG-algebras $\mathcal{A}$ is formal. More precisely there exist sheaves of DG-algebras $\mathcal{B}$, $\mathcal{H}$ on $\bar{X}$, where $\mathcal{H}$ has zero differential, and quasiisomorphisms*

$$\mathcal{A} \leftarrow \mathcal{B} \rightarrow \mathcal{H}.$$

(3.2) COROLLARY. *The categories $D_{\mathcal{A}}$ and $D_{\mathcal{H}}$* (resp. *$D_{\mathcal{A},c}$ and $D_{\mathcal{H},c}$*) *are naturally equivalent* (see *1.11.2*).

Summarizing Theorem (2.6) and Corollary (3.2) we get

(3.3) THEOREM. *The categories $D^b_{T,c}(X)$ and $D_{\mathcal{H},c}$ are naturally equivalent.*

(3.4) REMARK. Let $F \in D^b_{T,c}(X)$ and $M \in D_{\mathcal{H},c}$ be objects corresponding to each other under the above equivalence. Let $o \subset X$ be an orbit and $p \in \bar{X}$ be the corresponding point. Let $V = \mathrm{St}(o)$ be the star of the orbit o. Then

$$H(M_p) = H_T(V, F) := H(V_T, F).$$

Using Lemma (5.2) below this is also equal to $H_T(F|_o)$.

Proof of Theorem 3.1. Let $o_0 = T, o_1, \ldots, o_r$ be all T-orbits in X. As usual $\mathrm{St}(o_i)$ denotes the star of o_i and we put $V_i := \mathrm{St}(o_i)_T \subset X_T$. The open subsets $V_i \subset X_T$ correspond to irreducible open subsets of $\bar{X}$ via the map $q: X_T \to \bar{X}$. Hence, in order to define a sheaf on $\bar{X}$ it suffices to specify its value for each V_i and the restriction morphisms.

Let us construct the sheaf $\mathcal{B}$.

Consider the classifying map $\pi: X_T \to BT$. Let $\omega_1, \ldots, \omega_n \in \Omega^2(BT)$ be generators of the cohomology ring $H(BT)$. Put $\Omega = \Sigma \mathbb{C} \pi^* \omega_k \subset \mathcal{F}^2(X_T)$, where π^* means the composition of the pullback of smooth forms from BT under the smooth map $\mathbb{P}^N_T \to BT$ with the restriction to $X_T \subset \mathbb{P}^N_T$ (2.2). Then for all i the restriction of Ω to $\mathcal{F}^2(V_i)$ generates the cohomology ring $H(V_i)$. Denote this restriction again by Ω.

Put $K_i := d^{-1}(\Omega) \subset \mathcal{F}^1(V_i)$, where $d\colon \mathcal{F}^1(V_i) \to \mathcal{F}^2(V_i)$ is the differential, and let $N_i := \operatorname{Ker} d\colon K_i \to \Omega$. Notice that $N_i \subset d\mathcal{F}^0(V_i)$ since $H^1(V_i) = 0$.

(3.5) LEMMA. *There exist subspaces $S_i \subset \mathcal{F}^0(V_i)$ with the following properties*

(i) $d\colon S_i \xrightarrow{\sim} N_i$,

(ii) *if* $V_i \subset V_j$ *then* $S_j|_{V_i} \subset S_i$.

Assume the lemma. Then we define $\mathcal{B}(V_i)$ to be the free supercommutative algebra on the graded vector space $S_i \oplus K_i \oplus \Omega$ ($\deg S_i = 0, \deg K_i = 1, \deg \Omega = 2$) with the differential induced by $d\colon S_i \to N_i \subset K_i$, $d\colon K_i \to \Omega$. The restrictions $\mathcal{B}(V_i) \to \mathcal{B}(V_j)$ are obvious. We have the obvious morphism of sheaves

$$\mathcal{B} \to \mathcal{A},$$

$$S_i \to S_i, \quad K_i \to K_i, \quad \Omega \to \Omega,$$

which is a quasiisomorphism since $H(\mathcal{B}(V_i)) = H(V_i)$. Indeed, it is known that the cohomology of a free superalgebra on a complex of vector spaces $C^\cdot$ is the free superalgebra on the cohomology $H(C^\cdot)$ (see for example [GM], V.3.6, Lemma (7)). Moreover, let $I(V_i) \subset \mathcal{B}(V_i)$ be the ideal generated by S_i, K_i, $d(K_i)$. Then $\mathcal{B}(V_i)/I(V_i) \simeq H(V_i)$ and so we get the second quasiisomorphism $\mathcal{B} \to \mathcal{H}$, where $\mathcal{H}(V_i) = (H(V_i), d = 0)$. This proves the theorem. So it remains to prove the lemma.

Proof of lemma. Let $C(V_i) \subset \mathcal{F}^0(V_i)$ denote the subspace of constant functions. On $V_0 = T_T$ choose a linear complement $\tilde{\mathcal{F}}^0(V_0) \subset \mathcal{F}^0(V_0)$ to $C(V_0)$. Note that the restriction map $\mathcal{F}^0(V_j) \to \mathcal{F}^0(V_i)$ is injective, hence $\tilde{\mathcal{F}}^0(V_i) := \tilde{\mathcal{F}}^0(V_0) \cap \mathcal{F}^0(V_i)$ is a complement to $C(V_i)$ in $\mathcal{F}^0(V_i)$. The differential d is an isomorphism

$$d\colon \tilde{F}^0(V_i) \xrightarrow{\sim} d\mathcal{F}^0(V_i) \subset \mathcal{F}^1(V_i).$$

Now put $S_i := d^{-1}(N_i) \subset \tilde{\mathcal{F}}^0(V_i)$. This proves the lemma.

4. Proof of Theorem (0.1.1)

(4.0) In this section we will finish the proof of the main Theorem (0.1.1).

(4.0.1) LEMMA. *Suppose that in the Theorem (0.1.1) the toric variety X is affine. Then we may assume that X has a fixed point.*

Proof. Let $o \subset X$ be the orbit of the minimal dimension and let T_1 be its stabilizer. Then $X = T \times_{T_1} X_1$, where X_1 is an affine toric variety (with a fixed point) for the torus T_1. The categories $D^b_{T,c}(X)$ and $D^b_{T_1,c}(X_1)$ are naturally equivalent (the induction equivalence ([BL])). This equivalence preserves simple equivariant perverse sheaves, so we may replace X by X_1.

For the rest of this paper we assume that X is a normal toric variety, which is either projective or affine with a fixed point.

Let BT be the classifying space for $T = (\mathbb{C}^*)^n$, $A_T := H(BT, \mathbb{C})$. It is known that A_T is a polynomial ring on n variables $A_T = \mathbb{C}[x_1, \dots, x_n]$ with $\deg(x_i) = 2$. Given $F \in D^b_{T,c}(X)$ its equivariant cohomology $H_T(F)$ is naturally an A_T-module. In particular, $\mathcal{H}$ is a sheaf of A_T-algebras on $\bar{X}$ and any $M \in D_{\mathcal{H}}$ is an A_T-module.

Let $M_1, \dots, M_k \in D_{\mathcal{H},c}$ be the objects corresponding to $L_1, \dots, L_k \in D^b_{T,c}(X)$ under the equivalence of Theorem (3.3) above. Put $M = \oplus M_i$. We will use the following

(4.0.2) THEOREM. *The A_T-module* $\mathrm{Ext}^{\cdot}(M, M)$ *is torsion free.*

Since $\mathrm{Ext}^{\cdot}(M, M) = \mathrm{Ext}^{\cdot}(L, L)$ this theorem follows from the following two theorems.

(4.0.3) THEOREM. *The natural map*

$$\mathrm{Ext}^{\cdot}(L_i, L_j) \to \mathrm{Hom}_{A_T}(H_T(L_i), H_T(L_j)),$$

is injective.

(4.0.4) THEOREM. *The equivariant cohomology $H_T(L_i)$ is a free A_T-module.*

These two theorems will be proved in the next Section 5.

(4.1) We will proceed in two steps. Choose a $\mathcal{K}$-projective resolution $P_i \xrightarrow{\sim} M_i$ in $D_{\mathcal{H},c}$. Put $P = \oplus P_i$. Let $\mathcal{B}^o$ be the DG-algebra $\mathrm{Hom}^{\cdot}(P, P)$. Let $\mathcal{B}$ be the opposite DG-algebra (1.3.2). Let $D_{\mathcal{B}}$ be the derived category of $\mathcal{B}$-modules. Consider the full subcategory $D^f_{\mathcal{B}} \subset D_{\mathcal{B}}$ generated by the modules $\mathcal{P}_i = \mathcal{B}e_i$, where $e_i \colon P \to P_i$ is the projection.

Consider the functor $\theta \colon D_{\mathcal{H}} \to D_{\mathcal{B}}$ defined by

$$\theta(M) \colon \mathrm{Hom}^{\cdot}(P, M).$$

(4.1.1) PROPOSITION. *The functor θ above induces an equivalence of full subcategories*

$$\theta \colon D_{\mathcal{H},c} \to D^f_{\mathcal{B}}.$$

Notice that $H(\mathcal{B}) = (\mathrm{Ext}^{\cdot}(M, M))^o = (\mathrm{Ext}^{\cdot}(L, L))^o = A$ as defined in the introduction. Consider the DG-algebra $\mathcal{A} = (A, d = 0)$ with the zero differential. Let $D_{\mathcal{A}}$ be the derived category of DG-modules over $\mathcal{A}$. Consider the full subcategory $D^f_{\mathcal{A}} \subset D_{\mathcal{A}}$ generated by projective A-modules $Q_i = Ae_i$, where $e_i \colon L \to L_i$ is the projection.

(4.1.2) PROPOSITION. *The DG-algebra $\mathcal{B}$ is formal, i.e. there exists a quasiisomorphism of DG-algebras $\mathcal{B} \simeq \mathcal{A}$. Hence there is an equivalence of categories $D_{\mathcal{B}} \simeq D_{\mathcal{A}}$. This equivalence induces an equivalence $D^f_{\mathcal{B}} \simeq D^f_{\mathcal{A}}$.*

The Theorem (0.1.1) follows from Proposition (4.1.1), (4.1.2) and from Theorem (3.3). Proposition (4.1.1) is of a very general nature and the proof is easy. The proof of Proposition (4.1.2) uses the Theorem (4.0.2).

(4.2) PROOFS.

(4.2.1) *Proof of Proposition (4.1.1).*
The following statements are easy to check:

(1) $\theta(P_i) = \mathcal{P}_i$

(2) $\mathrm{Ext}_{D_{\mathcal{H}}}(P_i, P_i) = \mathrm{Ext}_{D_{\mathcal{B}}}(\mathcal{P}_i, \mathcal{P}_i)$ (use that P_i, $\mathcal{P}_i$ and $\mathcal{K}$-projective).

This proves Proposition 4.1.1.

(4.2.2) *Proof of Proposition (4.1.2).*
To prove the formality of $\mathcal{B}$ we need to choose $\mathcal{K}$-projective resolutions $P_i \xrightarrow{\sim} M_i$ carefully. Let $\{p_1, \dots, p_s\} \subset \bar{X}$ be the image of the fixed point set of X.

We know that the cohomology of the stalk $H(M_j)_{p_i}$ is a free $\mathcal{H}_{p_i} = A_T$-module (use Theorem (4.0.4) applied to the star of the corresponding fixed point in X and Remark (2.7)). Hence we may find a direct sum $P_j^0 = \oplus \mathcal{H}_U[?]$ of (shifted) modules $\mathcal{H}_U$ for irreducible open U's and a morphism $\varepsilon\colon P_j^0 \to M_j$ which is a quasiisomorphism at each point p_i and induces a surjection on the stalk cohomology at every point.

We proceed to construct a complex

$$\to P_j^{-2} \xrightarrow{\delta^{-1}} P_j^{-1} \xrightarrow{\delta^0} P_j^0$$

(actually finite) as in (1.7.4) such that

(i) Each P_j^{-m}, $m > 0$ is a direct sum of (shifted) sheaves $\mathcal{H}_U$, where U is an irreducible open such that $U \cap \{p_1, \dots, p_s\} = \emptyset$.

(ii) For each point $p \in \bar{X}$ this sequence is a resolution of the stalk cohomology $H(M_j)_p$. Hence

$$\varepsilon\colon P_j = \oplus P_j^{-m}[m] \xrightarrow{\sim} M_j$$

is a $\mathcal{K}$-projective resolution (1.7.3). We fix one such resolution for every M_j and will use them to compute $\mathrm{Ext}^{\cdot}(M_i, M_j)$.

(4.2.2.1) REMARKS. 1. The $\mathcal{H}$-module P_j is "complex-like", since the differential in P_j^{-m} is zero (and hence the differential in P_j is $\pm\delta^{-m}$). We will use this fact shortly to define a new grading on the complex $\mathrm{Hom}^{\cdot}(P_i\, P_j)$.

2. Note that the A_T-modules P_j^{-m} are torsion for $m > 0$.

Fix $1 \leqslant i, j \leqslant k$. The complex $\mathrm{Hom}^{\cdot}(P_i, P_j)$ has a natural grading (besides its usual one) which we denote by a lower index:

$$\mathrm{Hom}_{\cdot} = \mathrm{Hom}_{\cdot}(P_i, P_j) = \oplus_m \mathrm{Hom}_m(P_i, P_j)$$

such that $f_m: P_i^{-s} \to P_j^{-s+m}$, if $f_m \in \mathrm{Hom}_m$. Clearly, the differential in $\mathrm{Hom}^{\cdot}$ preserves this grading, i.e. $d: \mathrm{Hom}_m \to \mathrm{Hom}_{m+1}$. In particular we may consider the cohomology $H^i(\mathrm{Hom}_{.})$ and the truncation $(\mathrm{Hom}_{.})_{\tau\leqslant i}$.

CLAIM. *The complex* $\mathrm{Hom}_{.}$ *above is acyclic except at 0, i.e.*

$$\mathrm{Hom}_{.} \simeq H^0(\mathrm{Hom}_{.}).$$

Proof. The cohomology $H(\mathrm{Hom}_{.}) = \mathrm{Ext}^{\cdot}(M_i, M_j)$ is torsion free as an A_T-module (Theorem (4.0.2)). On the other hand the modules P_i^{-m}, P_j^{-m} are torsion if $m > 0$. So the only nonzero contribution to $H(\mathrm{Hom}_{.})$ comes from $\mathrm{Hom}(P_i^0, P_j^0) \subset \mathrm{Hom}_0(P_i, P_j)$. This proves the claim.

Now it follows that the DG-algebra $\mathcal{B}^o = \mathrm{Hom}(\oplus P_i, \oplus P_i)$ is formal. Hence also $\mathcal{B}$ is formal. Indeed, by the above claim the obvious morphisms of DG-algebras are quasiisomorphisms

$$H^0(\mathcal{B})^o \leftarrow \mathcal{B}^o_{\tau\leqslant 0} \hookrightarrow \mathcal{B}^o.$$

This induces an equivalence of categories $D_{\mathcal{A}} \simeq D_{\mathcal{B}}$ and proves the first part of Proposition (4.1.2). Under this equivalence $\mathcal{P}_i$ corresponds to Q_i so we have $D^f_{\mathcal{A}} \simeq D^f_{\mathcal{B}}$, which proves Proposition (4.1.2) and Theorem (0.1.1).

5. Proof of Theorems (4.0.3) and (4.0.4)

(5.0) Let X be a normal toric variety for the torus $T = (\mathbb{C}^*)^n$. We keep our assumption of Section 4 that X is either projective or an affine with a fixed point.

(5.1) *Proof of Theorem (4.0.4).* Let $\mathrm{Supp}(L_i) = Z \subset X$. Then $L_i|_Z = IC_T(Z)$ – the T-equivariant intersection cohomology complex on Z and $L_i = j_* IC_T(Z)$ for a closed embedding $j: Z \to X$. It was proved in [BL] (13.4, 14.3(ii)) that the equivariant intersection cohomology $IH_T(Z)$ is a free A_T-module. But the equivariant cohomology commutes with the direct image j_*, so $H_T(L_i) = IH_T(Z)$ is a free A_T-module. This proves Theorem (4.0.4).

(5.2) LEMMA. *Let $o \subset X$ be an orbit, and let $W = \mathrm{St}(o)$ be its star. Let $S \in D^b_{T,c}(W)$. Then the restriction $S \to S|_o$ induces an isomorphism $H_T(W, S) \xrightarrow{\sim} H_T(S|_o)$.*

Proof. Let $j: V = W - o \hookrightarrow W$ be the open embedding. It suffices to prove that $H_T(j_! j^* S) = 0$. By devissage (on V and on S) it suffices to prove that $H_T(W, \mathbb{C}) = H_T(o, \mathbb{C})$, which is clear since o_T is a homotopy retract of W_T by the action of some $\mathbb{C}^* \subset T$.

For the rest of this section we fix L_i, L_j and put $F = L_i$, $F' = L_j$.

(5.3) LEMMA. *Let $0 \neq f \in \mathrm{Ext}^{\cdot}(F, F')$. Then there exists an orbit $o \subset X$ such that $f|_o \neq 0$.*

Proof. Let us make some preliminary remarks.

For an A_T-module M its dimension $d(M)$ is the dimension of $\mathrm{Supp}(M) \subset \mathrm{Spec}\, A_T$. Let $i: o \hookrightarrow X$ be the inclusion of an orbit of codimension k, and $V = \mathrm{St}(o)$. Let $T_1 = \mathrm{Stab}(o)$ be the stabilizer of o. Then $V = T \times_{T_1} Y$, where Y is an affine toric variety for T_1.

(5.3.1) REMARK. By the Corollary 14.3 in [BL] (and using the induction equivalence $D_T(V) \simeq D_{T_1}(Y)$) the restriction i^*F(resp. i^*F') is a direct sum of constant equivariant sheaves on o. Similarly for the corestrictions $i^!F$, $i^!F'$.

Hence,

(1) Every nonzero A_T-submodule of $\mathrm{Ext}^{\cdot}(F|_o, F'|_o)$ has dimension k.
(2) Every nonzero A_T-submodule of $\mathrm{Ext}^{\cdot}(i_!i^*F, F')$ has dimension k.

Let $j: W \hookrightarrow X$ be an open embedding, where W is a union of some orbits of codimension $< k$.

(3) The A_T-modules $\mathrm{Ext}^{\cdot}(\,\cdot\,, j_!j^*F')$ and $\mathrm{Ext}^{\cdot}(j_!j^*F, \,\cdot\,)$ have dimension less than k.

Define a filtration of X by open subsets

$$T = U_0 \subset U_1 \subset \cdots \subset U_n = X,$$

where

$$U_k = \coprod_{\mathrm{codim}\, o \leqslant k} o.$$

Put

$$Z_k = U_k - U_{k-1} = \coprod_{\mathrm{codim}\, o = k} o.$$

Fix $1 \leqslant k \leqslant n$ and let

$$U_{k-1} \overset{j}{\hookrightarrow} U_k \overset{i}{\hookleftarrow} Z_k$$

be the open and closed embeddings. Denote $F_k = F|_{U_k}$, $F^k = F|_{Z_k}$ and similarly for F'. For every k the morphism $f \in \mathrm{Ext}(F, F')$ induces a morphism of exact triangles on U_k:

$$\begin{array}{ccccc} j_!F_{k-1} & \xrightarrow{a} & F_k & \xrightarrow{b} & i_*F^k \\ \downarrow & & \downarrow f & & \downarrow \\ j_!F'_{k-1} & \xrightarrow{a'} & F'_k & \xrightarrow{b'} & i_*F'^k \end{array} \qquad (*)$$

Let d be the dimension of the A_T-module $A_T \cdot f$. Using descending induction on k it suffices to prove the following claim.

CLAIM. *Assume that* $f|_{U_k} \neq 0$.

(a) *if* $d = k$, *then* $f|_{Z_k} \neq 0$.
(b) *if* $d < k$, *then* $f|_{U_{k-1}} \neq 0$.

Proof of Claim.

(a) Assume by contradiction that $f|_{Z_k} = 0$. Then $b'f = 0$. Hence $f = a'g$ for some $g \in \operatorname{Ext}(F_k, j_!F'_{k-1})$. But $A_T \cdot g$ has dimension less than k by (3) above. So $f|_{Z_k} \neq 0$.
(b) In this case $f|_{Z_k} = 0$ by (1) above.

Assume by contradiction that $f|_{U_{k-1}} = 0$. That is, both left and right vertical arrows in $(*)$ are zero. By diagram chasing we find $\alpha \in \operatorname{Ext}(i_*F^k, j_!F'_{k-1})$ such that $f = a'\alpha b$. But the module $A_T \cdot \alpha$ has dimension less than k by (3) and $A_T \cdot a'\alpha$ has dimension k by (2). This contradiction proves the claim and the lemma.

(5.4) LEMMA. *Let* $\{q_1, \ldots, q_s\} = X^T$ *be the fixed point set. Let* $S \in D^b_{T,c}(X)$. *The natural map of* A_T*-modules*

$$H_T(X, S) \to \oplus H_T(S_{q_\iota})$$

is an isomorphism at the generic point of Spec A_T.

Proof. This is clear, since the A_T-module $H_T(X - X^T, S)$ is torsion.

(5.5) Fix $0 \neq f \in \operatorname{Ext}^{\cdot}(F, F')$. Let $o \in X$ be an orbit such that $f|_o \neq 0$ (Lemma 5.3). Let $k = \operatorname{codim}(o)$ and $W = \operatorname{St}(o)$. By Remark 5.3.1 the restrictions $F|_o$, $F'|_o$ are isomorphic to shifted direct sums of the constant equivariant sheaf on o. Hence f induces a nonzero map $f\colon H_T(F|_o) \to H_T(F'|_o)$. Then by Lemma 5.2 the map $f\colon H_T(W, F) \to H_T(W, F')$ is also nonzero. Let $\tau \subset \bar{o}$ be another orbit of codimension $k + 1$. Put $V = \operatorname{St}(\tau)$.

(5.5.1) LEMMA. *In the above notations the map of* A_T*-modules*

$$f\colon H_T(V, F) \to H_T(V, F')$$

is not zero.

(5.5.2) REMARK. This lemma finishes the proof of Theorem (4.0.3). Indeed, using repeatedly the last lemma we arrive at the case $\tau = pt = q$. So f induces a nonzero map of free A_T-modules

$$f\colon H_T(\operatorname{St}(q), F) \to H_T(\operatorname{St}(q), F').$$

Now apply Lemmas (5.2), (5.4) to conclude that f induces a nonzero map

$$f\colon H_T(X, F) \to H_T(X, F'),$$

which proves Theorem (4.0.3).

Proof of Lemma (5.5.1). By the same argument as in 4.0.1 we may (and will) assume that $\tau = q$ is a fixed point and hence o is an orbit of dimension 1. Put $V^0 = V - \{q\}$. Let $\lambda \simeq \mathbb{C}^* \subset T$ be a 1-parameter subgroup that contracts V to q. This action of λ defines on V the structure of a (quasi)-homogeneous cone over a projective toric variety $\bar{V} = \lambda\backslash V^0$ for the torus $\bar{T} = T/\lambda$. Let $\varphi: T \to \bar{T}$ be the factor map and $g: V^0 \to \bar{V}$ be the corresponding (quotient) φ-map. Consider the direct image functor $Q_{g*} = Q_*: D^b_{T,c}(X) \to D^b_{\bar{T},c}(\bar{X})$ (see [BL]). Then $Q_*F = \bar{F}$ (resp. $Q_*F' = \overline{F'}$) is a simple $\bar{T}$-equivariant perverse sheaf on $\bar{V}$ ([BL], 9.1). Put $r = \lambda\backslash o \in \bar{V}$. Then $\lambda\backslash W = \mathrm{St}(r)$. The direct image Q_* commutes with the equivariant cohomology (up to restriction of scalars). Hence the map

$$Q_*(f): H_{\bar{T}}(\mathrm{St}(r), \bar{F}) \to H_{\bar{T}}(\mathrm{St}(r), \overline{F'})$$

is not zero. Now by the Remark (5.5.2) applied to the variety $\bar{V}$ (with $q = r$) we conclude that the map

$$Q_*(f): H_{\bar{T}}(\bar{V}, \bar{F}) \to H_{\bar{T}}(\bar{V}, \overline{F'})$$

is not zero. So it suffices to show that the natural map

$$H_T(V, F) \to H_T(V^o, F) = H_{\bar{T}}(\bar{V}, \bar{F})$$

is surjective (and similar for F'). But this follows from a more precise result 14.6, [BL]. This proves Lemma (5.5.1).

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