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On the zeta function of monodromy of a polynomial map*

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1. Introduction

In a classic work [We], Weil connected, at least conjecturally, the topology of the complex projective hypersurface defined say by a polynomial $f(X)$ rational over a number field K with the study of the number of solutions of the reduced polynomial $\tilde{f}(X)$ in the various residue class fields of K and their finite extensions. This connection, proved in full generality by Deligne, is now seen to be a basic example in the theory of motives. The present work, in a similar spirit, ties together some properties of the topology of the holomorphic map defined by a polynomial $f(X)$ which is nondegenerate with respect to its Newton polyhedron and rational over a number field K and certain properties of the exponential sums defined by the reduction of f taken over the various residue class fields $k(\mathfrak{P})$ of K (and their finite extensions). The motivic framework of this work is not clear at the present time.

To be more precise, consider a polynomial or Laurent polynomial $f(X) = \Sigma A(\mu)X^{\mu} \in K[X_1, \ldots, X_n, (X_1 \cdots X_n)^{-1}]$ defined over a field K. As usual, $\mathrm{Supp}(f) = \{\mu \in \mathbb{Z}^n \mid A(\mu) \neq 0\}$. Following Kushnirenko [Ku], we define the Newton polyhedron of f at ∞,

$$\Delta_\infty(f) = \text{convex closure in } \mathbb{R}^n \text{ of } \mathrm{Supp}(f) \cup \{0\}.$$

For any set $\Sigma \subseteq \mathbb{R}^n$, let

$$f_\Sigma = \sum_{\mu \in \mathrm{Supp}(f) \cap \Sigma} A(\mu)X^{\mu}.$$

Then f is nondegenerate [Ku] over K with respect to $\Delta_\infty(f)$ provided

$$\frac{\partial f_\sigma}{\partial X_1} = 0, \ldots, \frac{\partial f_\sigma}{\partial X_n} = 0$$

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have no simultaneous solutions in $(\bar{K}^*)^n$ for every (closed) face σ not containing 0. (Here $\bar{K}$ denotes an algebraic closure of K). If $f(X)$ has coefficients in a number field K then for almost all primes $\mathfrak{P}$ of K it makes sense to reduce f obtaining $\tilde{f}$ mod $\mathfrak{P}$ and $\Delta_\infty(f) = \Delta_\infty(\tilde{f})$. Recall that if f is nondegenerate with respect to $\Delta = \Delta_\infty(f)$ over K then for almost all primes $\mathfrak{P}$ of K, $\tilde{f}$ is nondegenerate with respect to Δ over $k(\mathfrak{P})$, the residue class field of $\mathfrak{P}$.

Assume f is defined over $\mathbb{C}$ and is nondegenerate with respect to $\Delta_\infty(f)$. In Section 2, we consider a locally trivial fibration defined by $f\colon(\mathbb{C}^*)^n \to \mathbb{C}$ outside a disk in $\mathbb{C}$ of sufficiently large radius. If $\eta \in \mathbb{C}$ traverses a sufficiently large circle, then one obtains an automorphism M of $H_i(f^{-1}(\eta), \mathbb{Q})$ which we call the monodromy at ∞. One may then define the zeta function of monodromy at ∞ by

$$\zeta(s) = \prod_{i=0}^{2n} \det(I - sM \mid H_i(f^{-1}(\eta), \mathbb{Q}))^{(-1)^{i+1}}.$$

Our main result in Section 2 is the following.

THEOREM 1. *Assume $f \in \mathbb{C}[X_1, \ldots, X_n, (X_1 \cdots X_n)^{-1}]$ is non-degenerate with respect to $\Delta_\infty(f)$, and $\dim \Delta_\infty(f) = n$. Then the zeta function of monodromy at ∞ is given by*

$$\prod (1 - s^{m_\sigma})^{(-1)^{n-1}(n-1)!\ \mathrm{Vol}(\sigma)}$$

where the product is taken over all faces σ of $\Delta_\infty(f)$ of codimension one not containing the origin, and where the affine hyperplane L_σ spanned by σ has normalized equation $\Sigma_{i=1}^n a_i^\sigma x_i = m_\sigma$ with $\{\{a_i^\sigma\}_{i=1}^n, m_\sigma\} \subseteq \mathbb{Z}$, relatively prime, and $m_\sigma > 0$. $\mathrm{Vol}(\sigma)$ is computed in L_σ relative to the measure for which a fundamental domain for $\mathbb{Z}^n \cap L_\sigma$ has measure 1.

In Theorem 2 (Section 2) this result is generalized to the case $f\colon(\mathbb{C}^*)^{n_1} \times \mathbb{C}^{n_2} \to \mathbb{C}$. We recall that this result is entirely analogous to the result of Varchenko [V] computing the zeta function of monodromy at an isolated singular point say $P \in \mathbb{C}^n$ of a holomorphic map $f\colon N_P \to \mathbb{C}$ (in a neighborhood N_P of P) in terms of analogous invariants associated with $\Delta_p(f)$, the Newton polyhedron of f at P.

The referee has kindly informed us that F. Loeser has also calculated the monodromy in an unpublished paper "Déterminants et Faisceaux de Kummer".

When V is a variety defined over the finite field $\mathbb{F}_q$ (of characteristic p) and $\tilde{f} \in \Gamma(V, \mathcal{O}_V)$ is a regular function on V we may consider the exponential sum

$$S_l(V, \tilde{f}, \Psi) = \sum \Psi_l(\tilde{f}(X))$$

where X runs over the $\mathbb{F}_q$-rational points of V, Ψ is an additive character of $\mathbb{F}_q$, and Ψ_l is the additive character of $\mathbb{F}_{q^l}$ defined by composing Ψ with the trace map from $\mathbb{F}_{q^l}$ to $\mathbb{F}_q$. The associated L-function

$$L(V, \tilde{f}, \Psi, T) = \exp\left(\sum_{l=1}^{\infty} S_l(V, \tilde{f}, \Psi)T^l/l\right)$$

is well-known to be a rational function of T with coefficients in the cyclotomic field $\mathbb{Q}(\zeta_p)$. In the case when $V = \mathbb{G}_m^n$ and $\tilde{f}$ is nondegenerate with respect to $\Delta_\infty(\tilde{f})$, then $L(\mathbb{G}_m^n, \tilde{f}, \Psi, T)^{(-1)^{n+1}}$ is a polynomial of degree equal to $n!\operatorname{Vol}\Delta_\infty(\tilde{f})$ [A-S]. Furthermore, the Newton polygon of $L(\mathbb{G}_m^n, \tilde{f}, \Psi, T)^{(-1)^{n+1}}$ lies over a "Hodge-type polygon", i.e. the Newton polygon of a polynomial $H(\tilde{f}, T)$ determined from $\Delta_\infty(\tilde{f})$. In Section 3, we prove that for every $r \in \mathbb{Q}$, the multiplicity of $\exp(-2\pi i r)$ as an exponent of monodromy at ∞ is the same as the multiplicity of the reciprocal zeros γ of $H(\tilde{f}, T)$ satisfying $\operatorname{ord}_q(\gamma) \equiv r \pmod{\mathbb{Z}}$. Conjecturally [A-S] the Newton polygons of $L(\mathbb{G}_m^n, \tilde{f}, \Psi, T)^{(-1)^{n+1}}$ and $H(\tilde{f}, T)$ agree in certain cases. In fact, in a recent work [Wa], Wan has shown that for p in a certain arithmetic progression the Newton polygon of $L(\mathbb{G}_m^n, \tilde{f}, \Psi, T)^{(-1)^{n+1}}$ and $H(\tilde{f}, T)$ agree for generic $\tilde{f}$. If so then in these cases, for every $r \in \mathbb{Q}$ the multiplicity of $\exp(-2\pi i r)$ as an exponent of monodromy is the same as the multiplicity of reciprocal zeros γ of $L(\mathbb{G}_m^n, \tilde{f}, \Psi, T)^{(-1)^{n+1}}$ satisfying $\operatorname{ord}_q(\gamma) \equiv r \pmod{\mathbb{Z}}$.

Finally in §4, we extend our earlier results from the case of the to the torus $\mathbb{G}_m^n$ to the case of affine space or a product of affine space and a torus $\mathbb{G}_m^{n_1} \times \mathbb{A}^{n_2}$.

2. Calculation of the monodromy at ∞

The purpose of this section is to prove Theorems 1 and 2 calculating explicitly the zeta function of monodromy at ∞. In what follows, we will view f as a combination of characters

$$f = \sum A_i \chi_i, \quad A_i \in \mathbb{C} \tag{2.1}$$

We seek a resolution of the base points of the pencil of hypersurfaces which are compactifications of the hypersurfaces

$$F_t\colon f - t = 0 \tag{2.1$'$}$$

in $\mathbb{G}_m^n$. In particular, we are seeking $X \supseteq \mathbb{G}_m^n$ such that the projective closures of F_t in X form a base-point free pencil. In fact construction of X

will be done in two steps. In the first step (the major one), we define a toric variety $X_F \supseteq \mathbb{G}_m^n$ such that the pencil defined by F_t in X_F has reduced base locus with non-singular components having normal crossings. The second step is just the blowing up of X_F along this base locus of the pencil F_t in X_F. The use of toroidal compactifications occur as well in related contexts in the works of Varchenko [V] and Denef–Loeser [DL].

To construct X_F we recall that a toric variety of dimension n is an algebraic variety, with an action of the torus $\mathbb{G}_m^n$, which contains $\mathbb{G}_m^n$ as a dense set. The action of $\mathbb{G}_m^n$ on the toric variety is induced from the action of $\mathbb{G}_m^n$ on itself via translation. To fix notations we write $(\mathbb{Z}^n)_{\mathrm{ch}} = \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m)$ for the lattice of characters, and $(\mathbb{R}^n)_{\mathrm{ch}} = (\mathbb{Z}^n)_{\mathrm{ch}} \otimes_{\mathbb{Z}} \mathbb{R}$. The dual group to $(\mathbb{Z}^n)_{\mathrm{ch}}$ is the group of one-parameter subgroups $(\mathbb{Z}^n)_{\mathrm{sg}}$. We recall that toric varieties are in one-one correspondence with the "fans" of $(\mathbb{R}^n)_{\mathrm{sg}}$. A fan is a collection of rational polyhedral cones in $(\mathbb{R}^n)_{\mathrm{sg}}$ satisfying certain axioms; the union of the cones in the collection is the underlying set of the fan. Given a fan v, the corresponding toric variety will be denoted T_v. The essential fact is that $v \mapsto T_v$ is a functor from the category of fans in $(\mathbb{R}^n)_{\mathrm{sg}}$ into the category of toric varieties. In particular, this means that if v_1 is a refinement of v_2 then one has an equivariant surjective morphism $T_{v_1} \to T_{v_2}$. A toric variety T_v is complete if and only if the fan v is complete (i.e. the underlying set $|v|$ of v is all of $(\mathbb{R}^n)_{\mathrm{sg}}$). A toric variety is non-singular if and only if the fan is simple (i.e., every cone σ in the fan v has the property that the set of minimal lattice vectors in its one-dimensional edges may be extended to form a basis of $(\mathbb{Z}^n)_{\mathrm{sg}}$).

Consider the Newton polyhedron $\Delta_\infty(f) \subseteq (\mathbb{R}^n)_{\mathrm{ch}}$ of the Laurent polynomial $f(X) \in \mathbb{C}[X_1, \ldots, X_n, (X_1 \cdots X_n)^{-1}]$ which we assume to have dimension n. Recall that there is a dual (complete) fan $v(f)$ of $\Delta_\infty(f)$ in which each i-dimensional cone in $v(f)$ corresponds to an $(n - i)$-dimensional face of $\Delta_\infty(f)$. In fact the cone corresponding to a given face γ consists of the subset of $(\mathbb{R}^n)_{\mathrm{sg}}$ formed of linear functions for which the minimal value on $\Delta_\infty(f)$ is attained on the face γ. Alternatively, the closures of the cones of maximal dimenson of $v(f)$ can be described as the cones in $(\mathbb{R}^n)_{\mathrm{sg}}$ such that the function $h_f(\tau) = \min\{\langle \tau, x\rangle \mid x \in \Delta_\infty(f)\}$ is linear on each cone. h_f is called the support function of the Newton polyhedron of f.

Let us make an additional subdivision of the fan $v(f)$ to obtain the fan $\bar{v}(f)$ such that each cone is generated by a set of vectors in $(\mathbb{Z}^n)_{\mathrm{sg}}$ which can be extended to a basis of $(\mathbb{Z}^n)_{\mathrm{sg}}$ (i.e., each cone in the fan is simple), to assure non-singularity of the corresponding variety which we shall denote X_F. The fact that such a subdivision is always possible is a standard fact (cf. [O]).

We will say an orbit of X_F is at ∞ if it corresponds to a cone in $\bar{v}(f)$ which is contained in a cone of $v(f)$ itself corresponding to a face not containing the origin.

LEMMA 1. *The closures of the hypersurfaces F_t say $\bar{F}_t$ in X_F have transverse intersections with the orbits at ∞.*

Recall that on a toric variety complete linear systems correspond to the functions on $(\mathbb{R}^n)_{sg}$, linear on each cone of the fan (cf. [D], [O]). In the case of the toric variety corresponding to the fan $\bar{v}(f)$, the linear system corresponding to the pullback of the support function of $\Delta_\infty(f)$ from $v(f)$ to $\bar{v}(f)$ is the linear system containing the closures of the hypersurfaces F_t. We will denote this system by D_F. It is base point free as a consequence of the upper convexity of the support function of an n-dimensional polyhedron (cf. [O], p. 76). Hence a generic divisor G from D_F has as support a non-singular manifold transverse to all orbits of X_F (Bertini's theorem). The restriction of the linear system G to $\mathbb{G}_m^n$ produces a Laurent polynomial (having the same Newton polyhedron as f but with "generic" coefficients). This Laurent polynomial defines a pencil of hypersurfaces in X_F whose base locus is the intersection of G with certain orbits at ∞ of X_F. Therefore the lemma follows immediately for generic G. It remains to show however that Kouchnirenko's non-degeneracy condition implies this genericity, and this is well-known.

Lemma 1 implies that the resolution of the base locus of the pencil given by $\bar{F}_t$ in X_F can be achieved by simple blowing-ups of X_F along the components of the base locus. Indeed the blow-up along an irreducible component of the $\bar{F}_{t_1} \cap \bar{F}_{t_2}$ (where $t_1 \neq t_2$) produces a pencil which has as base locus the union of the proper preimages of the remaining components (i.e., such a blow-up produces a pencil with base locus having strictly fewer irreducible components). The resulting manifold we will denote by $\tilde{X}_F$.

Let $\bar{F}_\infty$ be the divisor corresponding to $t = \infty$. Our goal now is to find the multiplicities of the components of $\bar{F}_\infty$. This divisor is clearly supported on the union of the closures of the codimension one orbits of X_F.

LEMMA 2. *The multiplicity of the closure of a codimension one orbit corresponding to a generator e_σ, of a one-dimensional cone σ of the fan $\bar{v}(f)$ is equal to $-h_f(e_\sigma)$ where e_σ is a primitive lattice vector in the cone σ.*

Proof. This follows from [O, p. 69]. One can see it directly as follows. Let us calculate the limit of F_t in a chart $\mathbb{C}^n$ corresponding to a cone $\tilde{\sigma}$ of $\bar{v}(f)$ spanned over $\mathbb{R}_+$ by $\{e_1, \dots, e_n\}$ and chosen so that $e_\sigma = e_1$. We define new coordinates via $x_i = \Pi_{j=1}^n u_j^{b_i(e_j)}$ where $b_i(e)$ denotes the ith coordinate of the vector e. The equation of F_t, $\Sigma A_\alpha X^\alpha - t$, becomes in the given chart

$$\sum A_\alpha u_1^{\langle \alpha, e_1 \rangle} \cdots u_n^{\langle \alpha, e_n \rangle} - t = 0$$

We see that the equation in the given chart of that part of the hypersurface moving in the pencil is

$$\sum A_\alpha \prod_{j=1}^{n} u_j^{\langle \alpha, e_j \rangle - h_f(e_j)} - t \prod_{j=1}^{n} u_j^{-h_f(e_j)} = 0$$

in which the sum on the left is taken over all monomials $\alpha \in \mathrm{Supp}(f)$. When $t \to \infty$ we obtain that the equation for $\bar{F}_\infty$ in the given chart is

$$\prod_{j=1}^{n} u_j^{-h_f(e_j)} = 0$$

and Lemma 2 follows. □

Before proceeding to the calculation of the zeta function of monodromy, we prove a result describing the restriction of the linear system on X_F corresponding to the support function h_f to the closure of a codimension one orbit.

LEMMA 3. *Let L_F be the linear system on X_F corresponding to $\Delta_\infty(f)$. Let σ be a one-dimensional cone of $\bar{v}(f)$ generated by the primitive vector e_σ and corresponding to a codimension one orbit T_σ of X_F. Then the restriction of L_F to the closure $\bar{T}_\sigma$ of T_σ in X_F is the linear system on the toric variety $\bar{T}_\sigma$ corresponding to the support function of the polyhedron which is the intersection of the hyperplane $\langle \sigma, x \rangle = h_f(e_\sigma)$ in $(\mathbb{R}^n)_{\mathrm{ch}}$ with $\Delta_\infty(f)$.*

Proof. L_F has as representative the divisor $-\Sigma h_f(e_\tau)\bar{T}_\tau$ (Lemma 2, [O], p. 69) where e_τ runs through the primitive vectors in the various one-dimensional cones τ of the fan $\bar{v}(f)$. On the other hand $\Delta_\infty(f)$ is the convex hull of the points in $(\mathbb{R}^n)_{\mathrm{ch}}$ which we may view as linear functions on $(\mathbb{R}^n)_{\mathrm{sg}}$. The linear functions on the n-dimensional cones of the fan $v(f)$ (or $\bar{v}(f)$) corresponding to the restrictions of the support function h_f to these cones are precisely the vertices of $\Delta_\infty(f)$. If l_σ is a linear function on $(\mathbb{R}^n)_{\mathrm{sg}}$ such that $h_f(e_\sigma) = l_\sigma(e_\sigma)$, then

$$(L_F)_\sigma = \sum_\tau (h_f(e_\tau) - l_\sigma(e_\tau))\bar{T}_\sigma$$

is another divisor in L_F. Because the $\bar{T}_\tau$ are transversal to one another and $\bar{T}_\sigma$ does not belong to the support of $(L_F)_\sigma$, the pullback of L_F on $\bar{T}_\sigma$ is given by

$$-\sum (h_f(e_\tau) - l_\sigma(e_\tau))(\bar{T}_\tau \cap \bar{T}_\sigma)$$

where the summation runs over all rays τ for which there is a cone in the fan having both τ and σ in their closure. Therefore the support function corresponding to the restriction of L_F on $\bar{T}_\sigma$ is given on $(\mathbb{R}^n)_{\mathrm{sg}}/(\sigma)$ (which is the natural domain for the fan of the toric variety $\bar{T}_\sigma$ (cf. [O], p. 11, cor.

1.7)) by the push-forward of $h_f(e_\tau) - l_\sigma(e_\tau)$ where σ and τ are in the closure of a cone. But the convex hull of those points corresponding to the linear functions of which this piecewise linear function is composed is equivalent to the face of $\Delta_\infty(f)$ on which the linear function defined by σ, i.e. $\langle \sigma, - \rangle$, achieves its minimum. $\square$

Before we proceed to the calculation of the ζ-function of the monodromy of the pencil (2.1) at ∞ we shall state a version of A'Campo's formula for the ζ-function of the monodromy on the cohomology of a fibre of a morphism $\phi: X \to S$ ($S = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$) about the special fibre which is a divisor with normal crossings. The proof in [AC] is given for proper morphisms and is based on the spectral sequence $E_2^{p,q} = \mathscr{H}^p(\phi^{-1}(0), \psi^q) \Rightarrow \mathscr{H}^{p+q}(\phi^{-1}(1))$ (ψ^q is the sheaf of vanishing q-cycles) which is not valid in general in the non-proper case (cf. SGA2, p. 8). We have however the following lemma.

LEMMA 4. *Let $\phi: X \to S$ be a proper holomorphic map of a non-singular variety onto a disk S which is a locally trivial fibration outside of the center of S. Assume that the central fiber $\phi^{-1}(0) = \bigcup E_i$ is a divisor with normal crossings. Let D be a divisor on X such that $D \cap \mathrm{Supp}(\phi^{-1}(0))$ is a divisor with normal crossings as well. Assume that $\phi|_D: D \to S$ is a locally trivial fibration outside the center of S. Then the ζ-function of the monodromy of $\phi|_{X-D}: X - D \to S$ is given by $\Pi(1 - s^{m_i})^{\chi(S_i - S_i \cap D)}$ where m_i is the multiplicity of the ith component E_i of $\phi^{-1}(0)$, S_i is the open set of E_i which is non-singular on $\phi^{-1}(0)$, χ is the Euler characteristic and the product is taken over all irreducible components of the central fiber.*

Proof. This result may be derived from the case D empty which is contained in [AC] using induction on the number of components of D. Assume that one already has removed i components, that one has the formula from this lemma, and that one takes away an additional component D_{i+1}. Then

$$\phi|_{D_{i+1}}: D_{i+1} \to S$$

is a locally trivial fibration outside zero and $(\phi|_{D_{i+1}})^{-1}(0)$ is a divisor with normal crossings. A point of it is singular if and only if it is singular on $\phi^{-1}(0)$ and the multiplicity of a component S_l of $\phi^{-1}(0)$ is the same as the multiplicity of $S_l \cap D_{i+1}$ in $(\phi|_{D_{i+1}})^{-1}(0)$ as follows from the assumptions on D in particular, the fact that $D \cap \mathrm{Supp}\, \phi^{-1}(0)$ is a divisor with normal crossings. Hence the addivity of the Euler characteristic and the ζ-function (which is, for example, a consequence of its interpretation via the numbers of fixed points of geometric monodromy ([AC])) implies the inductive step. $\square$

Proof of Theorem 1

Now we are in a position to give the proof of Theorem 1. Let X_F be the compactification of $\mathbb{G}_m^n$ constructed above and let D_σ be the closure of the codimension one orbit in X_F corresponding to a one-dimensional cone σ in the fan $\bar{v}(f)$. Let $\mathscr{D}$ be the union of those D_σ's for which the codimension one orbit T_σ does not belong to the support of $\bar{F}_\sigma$. (These are the orbits corresponding to one-dimensional cones σ of $\bar{v}(f)$ for which $h_f(e_\sigma) = 0$). Let B be the base locus of the pencil F_t on X_F. Then the pencil defines the map

$$\phi: X_F - \mathscr{D} - B \to \mathbb{P}^1.$$

The fibres of ϕ over points $t \in \mathbb{P}^1 - \{\infty\}$ are the hypersurfaces $F = t$ in the torus $\mathbb{G}_m^n$. Let S be the complement in $\mathbb{C}$ of a disk of sufficiently large radius containing all the critical values of f and those values t for which F_t is not transversal to a "finite" orbit of X_F (i.e. an orbit corresponding to a cone of $\bar{v}(f)$ contained in a cone of $v(f)$ corresponding to a face of $\Delta_\infty(f)$ containing the origin). Then by Lemma 1, the closure of F_t in X_F for t in S form a locally trivial fibration.

The monodromy of $F = t$ about infinity is the monodromy of ϕ. Now we shall apply Lemma 4 to $\tilde{X}_F$ (blow-up of X_F along B) with D equal to the union of $\mathscr{D}$ and the exceptional locus $\tilde{B}$ of the blow-up of X_F. A point on a component with positive multiplicity is non-singular if and only if it either belongs to a codimension one orbit properly or in the case of higher codimension if it is in the closure of exactly one orbit of maximal dimension having positive multiplicity. In the latter case, it also belongs to the closure of an orbit having multiplicity zero. Hence, in the notation of Lemma 4, $S_i - S_i \cap D$ is the complement to the base locus B in a codimension one orbit, the closure of which has positive multiplicity. Thus, the Euler characteristic of $S_i - S_i \cap D$ is minus the Euler characteristic of the intersection of the base locus with the codimension one orbit. The part of the base locus of the pencil inside this orbit is just an element in the restriction of the linear system defined by f to this codimension one orbit.

By Lemma 3, this linear system will correspond to the polyhedron which is a face of $\Delta_\infty(f)$ not containing the origin. The Euler characteristic of a hypersurface in a torus with given Newton polyhedron can be calculated using the formula of [BKH] (cf. also [A]). In particular, the base locus will have non-zero Euler characteristic only in those orbits corresponding to vectors $\sigma \in (\mathbb{R}^n)_{sg}$ which are generators such that $\langle \sigma, x \rangle = -h_f(\sigma)$ is the hyperplane spanned by a codimension one face of the polyhedron $\Delta_\infty(f)$. The multiplicity of the corresponding orbit is given in Lemma 2. Hence we obtain Theorem 1.

Theorem 1 can be generalized as follows. Let K be an arbitrary field and $f(X) \in K[X_1, \dots, X_{n_1}, \; X_{n_1+1}, \dots X_{n_1+n_2}, (X_1 \cdots X_{n_1})^{-1}]$. Set $S_1 = \{1, \dots, n_1\}$, $S_2 = \{n_1 + 1, \dots, n\}$, where $n = n_1 + n_2$. For any subset $A \subseteq S_2$, define f_A to be the Laurent polynomial in $n - |A|$ variables obtained from f by setting $X_i \to 0$ for every $i \in A$. We say $f(X)$ is convenient with respect to the variables $\{X_i\}_{i \in S_2}$ provided $\dim \Delta_\infty(f_A) = n - |A|$ for every $A \subseteq S_2$.

THEOREM 2. *Assume $K = \mathbb{C}$ and let f as above be non-degenerate and convenient with respect to $\{X_i\}_{i \in S_2}$. Then the zeta function of monodromy at ∞ of the map $f\colon \mathbb{G}_m^{n_1} \times \mathbb{C}^{n_2} \to \mathbb{C}$ is given by*

$$\prod (1 - s^{m(\sigma)})^{(-1)^{\dim \sigma} (\dim \sigma)! \operatorname{Vol}(\sigma)}$$

where the product is taken over all faces σ of codimension one in $\Delta_\infty(f_A)$ not containing the origin for all $A \subseteq S_2$. For any such σ, Vol(σ) *is the volume in the affine hyperplane L_σ spanned by σ in the space $\mathbb{R}^{n-|A|}$ having coordinates $\{X_i\}_{i \in S_1 \cup (S_2 - A)}$ with respect to the measure for which a fundamental domain in $\mathbb{Z}^n \cap H_\sigma$ has measure equal to* 1.

Theorem 2 can be derived from Theorem 1 as follows. $\mathbb{G}_m^{n_1} \times \mathbb{C}^{n_2}$ can be compactified in a way entirely similar to the compactification of $\mathbb{G}_m^n$. For each $i \in S_2$, the convenience hypothesis implies that $\dim \Delta_\infty(f_{\{i\}}) = n - 1$,so that $\Delta_\infty(f_{\{i\}})$ is a face of $\Delta_\infty(f)$ of codimension 1. As a consequence, the rays along the last n_2 coordinates in $(\mathbb{R}^n)_{sg}$ belong to $v(f)$ which assures that the compactification of the torus is at the same time a compactification of $\mathbb{G}_m^{n_1} \times \mathbb{C}^{n_2}$. Then $F_t = f - t$ induces the pencil on each of the tori of $(\mathbb{G}_m^{n_1} \times \mathbb{C}^{n_2}) - \mathbb{G}_m^n$. The additivity of ζ-functions allows us to express the ζ-function of monodromy on $\mathbb{G}_m^{n_1} \times \mathbb{C}^{n_2}$ as the product of ζ-functions corresponding to each of these added tori. The added tori correspond to the cones which are subsets of all possible coordinate hyperplanes with non-zero coordinates among the various subsets of the last n_2 coordinates and for which these non-zero entries are positive real. In particular, for each $A \subseteq S_2$ let W_A be the torus in $V = \mathbb{G}_m^{n_1} \times \mathbb{C}^{n_2}$ having coordinates $\{X_i\}_{i \in S_1 \cup (S_2 - A)}$. Then

$$V = \bigcup_{A \subseteq S_2} W_A \quad \text{is the decomposition}$$

of V into orbits. The additivity of the zeta function implies

$$\zeta_{V,f} = \prod_{A \subseteq S_2} \zeta_{W_A, f_A}$$

which then yields Theorem 2 as a direct consequence of Theorem 1.

For example if $f(X, Y) = X^3 + X^2Y^2 + Y^4 + XY$ then $\Delta_\infty(f) =$ convex closure $\{(0,4), (2,2), (3,0), (0,0)\}$ and viewing f on $(\mathbb{C}^*)^2$, the zeta function of monodromy is $\zeta(s) = (1-s^4)^{-2}(1-s^6)^{-1}$ and viewing f on $\mathbb{C}^2$, the zeta function of monodromy is $\zeta(s) = (1-s^4)^{-2}(1-s^6)^{-1}(1-s^3)(1-s^4)$.

3. Results from number theory

In this section, we recall some results from number theory, and relate the formula above for the zeta function of monodromy at ∞ to a formula estimating the p-divisibility of an exponential sum defined over a field of characteristic p. We begin as in the previous section with the case of the n-dimensional split torus $V = \mathbb{G}_m^n$ defined over an arbitrary field K, $f \in \Gamma(V, \mathcal{O}_V)$ a regular function f defined on V represented by a Laurent polynomial.

$$f(X) = \sum A(\alpha)X^\alpha \in K[X_1, \ldots, X_n, (X_1 \cdots X_n)^{-1}].$$

Let cone(f) be the union of all rays in $\mathbb{R}^n$ beginning at O and passing through points of $\Delta_\infty(f)$ (distinct from O); let $M(f) = \text{cone}(f) \cap \mathbb{Z}^n$, an additive monoid. Then $R = K[M(f)]$ is the monoid-algebra consisting of Laurent-polynomials with support in cone(f). R is a filtered ring. For each $\alpha \in M(f)$, we define $w(\alpha)$ to be the smallest non-negative rational number β such that $\alpha \in \beta \cdot \Delta_\infty(f)$ where the set on the right is the image of $\Delta_\infty(f)$ under the homothetie centered at O, of magnitude β. Then the Newton weight w has the following properties:

(i) $w(M(f)) \subseteq 1/(M)\mathbb{Z}_+$, for some positive integer M;
(ii) $w(\alpha) = 0$ if any only if $\alpha = 0$;
(iii) $w(c\alpha) = cw(\alpha)$, for $c \in \mathbb{Z}_+$;
(iv) $w(\alpha_1 + \alpha_2) \leqslant w(\alpha_1) + w(\alpha_2)$ and equality holds if and only if the rays from O to α_1 and to α_2 intersect the same closed face of $\Delta_\infty(f)$.

For $i \in 1/(M)\mathbb{Z}_+$, $\text{Fil}_i(R)$ consists of the elements of R all of whose terms have weight less than or equal to i. If $\bar{R} = \text{gr}(R) = \oplus \bar{R}_i$ is the associated graded ring where $\bar{R}_i = \text{Fil}_i(R)/\text{Fil}_{i-1/M}(R)$, then we denote $h_i = \dim_K \bar{R}_i$. By definition, $f(X) \in \text{Fil}_1(R)$, and its image in $\bar{R}_1$ will be denoted by $F(X)$. Consider the Koszul complex on $\bar{R}$ defined by $\{X_i \partial F/\partial X_i\}_{i=1}^n$. If f is nondegenerate with respect to $\Delta_\infty(f)$ and $\dim \Delta_\infty(f) = n$ then [Ku] the Koszul complex is acyclic except in degree 0, and

$$H_0(\bar{R}) \left(= \bar{R} \Big/ \sum_{j=1}^n X_j \frac{\partial F}{\partial X_j} \bar{R} \right)$$

is a graded K-algebra of dimension equal to $n!\,\mathrm{Vol}(\Delta_\infty(f))$. Furthermore if $\bar{h}_i$ is the dimension of the ith graded piece,

$$\bar{R}_i \Big/ \sum_{j=1}^{n} X_j \frac{\partial F}{\partial X_j} \bar{R}_{i-1},$$

of $H_0(\bar{R})$ then the acyclicity of the Koszul complex

$$0 \to \bar{R} \to \bar{R}^n \to \cdots \to \bar{R}^n \to \bar{R} \to H_0(\bar{R}) \to 0$$

yields

$$\bar{h}_i = \sum_{j=0}^{n} (-1)^j \binom{n}{j} h_{i-j}. \tag{3.1}$$

The purpose of this section is to relate $\bar{h}_i$ and the invariants of $\Delta_\infty(f)$ appearing in the formula in Theorem 1 for the monodromy about ∞. Recall that for $l \in 1/(M)\mathbb{Z}_+$, the multiplicity of $e^{-2\pi i l}$ as a zero of $\zeta(s)$ is given by

$$N_l = (n-1)! \sum \mathrm{Vol}(\sigma)$$

where the sum runs over all faces σ of codimension 1 not passing through O such that $m_\sigma \cdot l \in \mathbb{Z}$. Let

$$H_l = \sum \bar{h}_i$$

where the sum runs over all $i \in 1/(M)\mathbb{Z}_+$, $i \equiv l \pmod{\mathbb{Z}}$.

THEOREM 3. *For* $l \in 1/(M)\mathbb{Z}_+$, $H_l = N_l$.

Before proceeding with the proof we recall the number-theoretic significance of the integers $\bar{h}_i$. The following result appears in [A-S].

Let $f(X) \in \mathbb{F}_q[X_1, \ldots, X_n, (X_1 \cdots X_n)^{-1}]$ be non-degenerate with respect to $\Delta_\infty(f)$ and $\dim \Delta_\infty(f) = n$. Then $L(\mathbb{G}_m^n/\mathbb{F}_q, f, \Psi, T)^{(-1)^{n+1}}$ is a polynomial in $\mathbb{Q}(\zeta_p)[T]$ of degree equal to $n!\,\mathrm{Vol}(\Delta_\infty(f))$. Furthermore, the Newton polygon of $L(\mathbb{G}_m^n, f, \Psi, T)^{(-1)^{n+1}}$ lies over the Newton polygon of $H(\bar{f}, T) = \Pi(1 - q^i T)^{\bar{h}_i}$ where the index of the product runs over $i \in 1/(M)\mathbb{Z}_+$. In fact this is a polynomial since $\bar{h}_i = 0$ for $i > n$ (in the case O is not an interior point of $\Delta_\infty(f)$, $\bar{h}_i = 0$ for $i \geqslant n$).

We now proceed with the proof of Theorem 3. For each σ a face of $\Delta_\infty(f)$ of codimension one not containing the origin we set

$$N_{l,\sigma} = \begin{cases} (n-1)!V(\sigma), & \text{if } m_\sigma l \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $N_l = \Sigma N_{l,\sigma}$ where the sum runs over all faces σ of $\Delta_\infty(f)$ of codimension 1 not containing the origin. Recall [Ku] there is an exact sequence

$$O \to \bar{R} \to \bigoplus_{\substack{\mathrm{cod}(\sigma)=1 \\ 0 \notin \sigma}} R_\sigma \to \bigoplus_{\substack{\mathrm{cod}(\tau)=2 \\ 0 \notin \tau}} R_\tau \to \cdots \to \bigoplus_{\substack{\mathrm{cod}(\mu)=n \\ 0 \notin \mu}} R_\mu \to R^{(\varepsilon)} \to 0$$

where

$$R^{(\varepsilon)} = \begin{cases} O, & \text{if } O \notin \text{interior } \Delta_\infty(f); \\ K, & \text{if } O \in \text{interior } \Delta_\infty(f). \end{cases}$$

Thus $h_i^{(\varepsilon)} = 0$, unless $O \in \text{interior}(\Delta_\infty(f))$ and $i = 0$, in which case $h_0^{(\varepsilon)} = 1$. More generally, for any face σ of $\Delta_\infty(f)$ not containing O we will use the notation

$$f_\sigma = \sum_{\alpha \in \mathrm{Supp}(f) \cap \sigma} A(\alpha) X^\alpha.$$

to denote the "part" of f having support in σ. (Of course $\Delta_\infty(f_\sigma)$ has only one face (namely σ) not containing O of dimension equal to $\dim \sigma$. As a consequence, R_σ is not only filtered but graded and $R_\sigma = \bar{R}_\sigma$. In this case, we denote $\Delta_\infty(f_\sigma) = \hat{\sigma}$.)

Since the maps in the above exact sequence are homogeneous of degree zero,

$$h_i = \sum_\sigma (-1)^{\mathrm{cod}(\sigma)+1} h_{i,\sigma} + (-1)^n h_i^{(\varepsilon)} \tag{3.2}$$

in which the sum runs over all closed faces σ of $\Delta_\infty(f)$ not containing the origin. Consider the following Poincaré series

$$\begin{aligned}
P(T) &= \sum h_i T^{iM} && \text{for the graded ring } \bar{R}, \\
P_\sigma(T) &= \sum h_{i,\sigma} T^{iM} && \text{for the graded ring } R_\sigma, \\
Q(T) &= \sum \bar{h}_i T^{iM} && \text{for the graded ring } H_0(\bar{R}) \\
Q_\sigma(T) &= \sum \bar{h}_{i,\sigma} T^{iM} && \text{for the graded ring } H_0(R_\sigma)
\end{aligned}$$

in which all sums run over $i \in 1/(M)\mathbb{Z}_+$, and $H_0(R_\sigma)$ is the zero-dimensional homology of the Koszul complex defined on

$$R_\sigma = \bigoplus_{i \in 1/(M)\mathbb{Z}_+} R_{i,\sigma} \quad \text{by} \left\{ X_i \frac{\partial f_\sigma}{\partial X_i} \right\}_{i=1}^n$$

where f_σ in fact lies in $R_{1,\sigma}$. As in the case of $\bar{R}$, the acyclicity of this Koszul complex yields

$$\bar{h}_{i,\sigma} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} h_{i-j,\sigma} \tag{3.3}$$

Equation (3.2) implies that

$$P(T) = \sum (-1)^{\operatorname{cod}(\sigma)+1} P_\sigma(T) + h_0^{(\varepsilon)}, \tag{3.4}$$

the sum running over faces σ of $\Delta_\infty(f)$ not containing the origin. Each $P_\sigma(T)$ is a rational function of T having a pole at $T = 1$ of order $\dim(\sigma) + 1$ [Ku]. Equations (3.1) and (3.3) imply

$$Q(T) = (1 - T^M)^n P(T),$$
$$Q_\sigma(T) = (1 - T^M)^n P_\sigma(T).$$

so that multiplying (3.4) by $(1 - T^M)^n$ we obtain

$$Q(T) = \sum_{\substack{\operatorname{cod}(\sigma)=1 \\ 0 \notin \sigma}} Q_\sigma(T) + (1 - T^M)U(T),$$

for some polynomial $U(T)$. For any given

$$l, l \in \frac{1}{M}\mathbb{Z}_+, 0 \leqslant l < 1, \quad H_l = \sum_{j \equiv l (\operatorname{mod} \mathbb{Z})} \bar{h}_j$$

is the coefficient of $T^{l'M}$ in $(1 - T^M)^{-1}Q(T)$ for any l' such that $l' \equiv l (\operatorname{mod} \mathbb{Z})$ and l' large enough (in fact, $l' > n-1$ will suffice). Thus if $H_{l,\sigma} = \Sigma_{j \equiv l(\operatorname{mod}\mathbb{Z})} \bar{h}_{j,\sigma}$, then

$$H_l = \sum H_{l,\sigma}$$

where σ runs over all codimension one faces of $\Delta_\infty(f)$ not passing through the origin. Thus the theorem is implied by the assertion $H_{l,\sigma} = N_{l,\sigma}$. Note that trivially $H_{l,\sigma} = 0$ if $l \cdot m_\sigma \notin \mathbb{Z}$ (since $w(M(f_\sigma)) \subseteq 1/(m_\sigma)\mathbb{Z}_+$).

We claim for $l, l' \in (1/m_\sigma)\mathbb{Z}_+$, $0 \leqslant l, l' < 1$, that $H_{l,\sigma} = H_{l',\sigma}$. If so, using

$$\sum_{\substack{0 \leqslant l < 1 \\ l \in 1/(M_\sigma)\mathbb{Z}_+}} H_{l,\sigma} = \sum_{j \in (1/M_\sigma)\mathbb{Z}_+} \bar{h}_{j,\sigma} = n!\,\mathrm{Vol}(\hat{\sigma})$$

then

$$H_{l,\sigma} = \frac{n!\,\mathrm{Vol}(\hat{\sigma})}{m_\sigma} = (n-1)!\,\mathrm{Vol}(\sigma)$$

as desired. However we know by [Ehr] (also [A-S (4.5)]) that

$$h_{i,\sigma} = V(\sigma) i^{n-1} + \sum_{r=0}^{n-2} a_r(l) i^r \tag{3.5}$$

for all $i \equiv l \pmod{\mathbb{Z}}$. Fix $l, l \in (1/m_\sigma)\mathbb{Z}_+$, $0 \leqslant l < 1$. We know by (3.3) that

$$\bar{h}_{i,\sigma} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} h_{i-j,\sigma},$$

it being understood that $h_{r,\sigma} = 0$ for $r < 0$. The right-hand side then is a polynomial in i (for $i \geqslant n$) and is zero (by the finite dimensionality of $H_0(R_\sigma)$) for i sufficiently large. Hence it is identically zero. Therefore

$$\begin{aligned} H_{l,\sigma} &= \sum_{\substack{i \equiv l \pmod{\mathbb{Z}} \\ i < n}} \bar{h}_{i,\sigma} \\ &= \sum_{i=0}^{n-1} \bar{h}_{l+i,\sigma} \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{i} (-1)^j \binom{n}{j} h_{l+i-j,\sigma} \\ &= \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} h_{l+k,\sigma}. \end{aligned}$$

Trivially, if

$$f(x) = \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} x^k = (x-1)^{n-1} \quad \text{and} \quad \delta = x\frac{\mathrm{d}}{\mathrm{d}x},$$

then

$$(\delta^j f)(1) = \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} k^j = 0 \quad \text{for } j < n-1,$$

and

$$(\delta^{n-1} f)(1) = \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{n-1}{k} k^{n-1} = (n-1)!$$

Hence substituting (3.5) into this last formula, we obtain

$$H_{l,\sigma} = (n-1)!\, V(\sigma)$$

as desired.

4. Generalization to $V = \mathbb{G}_m^{n_1} \times \mathbb{A}^{n_2}$

The purpose of this section is to extend the results above (especially, Theorem 3) to the case $V = \mathbb{G}_m^{n_1} \times \mathbb{A}^{n_2}$. Let

$$f(X) \in K[X_1, \ldots, X_{n_1}, X_{n_1+1}, \ldots, X_n, (X_1 \cdots X_{n_1})^{-1}].$$

Assume f is non-degenerate and convenient with respect to $\{X_i\}_{i \in S_2}$. For $l \in \mathbb{Q}$, let $\tilde{N}_l$ denote the multiplicity of $\exp(-2\pi i l)$ as a zero of $\zeta_{V,f}(s)$, and $N_{l,A}$ denote the multiplicity of $\exp(-2\pi i l)$ as a zero of $\zeta_{W_A, f_A}(s)$. Then the additivity of zeta functions implies

$$\tilde{N}_l = \sum_{A \subseteq S_2} (-1)^{|A|} N_{l,A}$$

It is our goal now to relate $\tilde{N}_l$ with a quantity arising in the estimation of the p-adic size of the roots of the L-function associated with a certain exponential sum. We will proceed algebraically and we will recall the precise relevant result from number theory subsequently. Let $A \subseteq S_2$. Denote by R_A the subring $K[M(f_A)]$ of $K[\{X_i\}_{i \in S_1 \cup (S_2 - A)}, (X_1 \cdots X_{n_1})^{-1}]$. R_A is filtered by $\Delta_\infty(f)$; denote the associated graded ring by $\bar{R}_A = \oplus_{i \in M_A^{-1}\mathbb{Z}} \bar{R}_{A,i}$ and the image of

$$X_i \frac{\partial f_A}{\partial X_i} \quad \text{in } \bar{R}_{A,1} = \mathrm{Fil}_1(R_A)/\mathrm{Fil}_{1-1/M_A}(R_A)$$

by $F_{A,i}$. Denote by $K(A)_{\cdot}$ the Koszul complex defined on $\bar{R}_A$ by $\{F_{A,i}\}_{i \in S_1 \cup (S_2 - A)}$.

For every B, $B \subseteq S_2 - A$ we define as well a subcomplex $K(A, B)_.$ of $K(A)_.$ (so that $K(A, \phi)_. = K(A)_.$). In particular, we set

$$K(A, B)_i = \bigoplus \bar{R}_A^{C \cap B} e_{c_1} \wedge \cdots \wedge e_{c_i}$$

where the sum runs over subsets $C = \{c_1, \ldots, c_i\}$ of $S_1 \cup (S_2 - A)$ of cardinality i and where for $B \subseteq S_2 - A$, $\bar{R}_A^B$ denotes the sub K-vector space of $\bar{R}_A$ generated by monomials $\Pi_{i \in S_1 \cup (S_2 - A)} X_i^{\mu_i}$ with $\mu_i > 0$ for $i \in B$. Since multiplication by $F_{A,i}$ takes $\bar{R}_A^B$ to $\bar{R}_A^{B \cup \{i\}}$ the restriction of the maps of $K(A)_.$ to $K(A, B)_.$ define a subcomplex. Recall that in [A-S, Appendix] it is shown that if f is nondegenerate with respect to $\Delta_\infty(f)$ and f is convenient then $K(\phi, S_2)_.$ is acyclic except in dimension zero, that

$$H_0(K(\phi, S_2)_.) = \bar{R}^{S_2} \Big/ \sum_{i \in S_1 \cup S_2} F_i \bar{R}^{S_2 - \{i\}}$$

and that

$$\dim_K H_0(K(\phi, S_2)_.) = \sum_{j=0}^{|S_2|} (-1)^j (N - j)! \, V_{N-j}$$

where $V_N = \operatorname{Vol} \Delta_\infty(f)$ and $V_{N-j} = \Sigma_{|A| = j} \operatorname{Vol} \Delta_\infty(f_A)$ where $\operatorname{Vol} \Delta_\infty(f_A)$ is computed with respect to Haar measure in $\oplus_{i \in S_1 \cup (S_2 - A)} \mathbb{R} e_i$ normalized so that a fundamental domain for the lattice $\mathbb{Z}^N \cap \oplus_{i \in S_1 \cup (S_2 - A)} \mathbb{R} e_i$ has measure 1. For $j \in (1/M)\mathbb{Z}_+$, let

$$\tilde{h}_j = \dim_K \bar{R}_j^{S_2} \Big/ \sum_{i \in S_1 \cup S_2} F_i \bar{R}_{j-1}^{S_2 - \{i\}}$$

Set $\tilde{H}_l = \Sigma_{j \equiv l (\operatorname{mod} \mathbb{Z})} \tilde{h}_j$.

THEOREM 4. *For* $l \in 1/(M)\mathbb{Z}_+$,

$$\tilde{H}_l = \tilde{N}_l.$$

Before proceeding with the proof we indicate the number-theoretic significance of the quantities $\tilde{h}_j$. Let $K = \mathbb{F}_q$ be the finite field with q elements. Let $\tilde{f}(X) \in \mathbb{F}_q[X_1, \ldots, X_N, (X_1 \cdots X_{n_1})^{-1}]$. Consider the family of exponential sums

$$S_l(\mathbb{G}_m^{n_1} \times \mathbb{A}^{n_2}, \bar{f}, \Psi) = \sum \Psi_l(\bar{f}(X))$$

where the sum runs over $X \in (\mathbb{F}_{q^l}^*)^{n_1} \times (\mathbb{F}_{q^l})^{n_2}$, Ψ is an additive character of $\mathbb{F}_q$, and $\Psi_l = \Psi \circ \mathrm{Tr}_{\mathbb{F}_{q^l}/\mathbb{F}_q}$. Associated with this exponential sum is the L-function

$$L(\mathbb{G}_m^{n_1} \times \mathbb{A}^{n_2}, \bar{f}, \Psi, T) = \exp\left(\sum_{l=1}^{\infty} S_l(\mathbb{G}_m^{n_1} \times \mathbb{A}^{n_2}, \bar{f}, \psi) T^l/l\right)$$

We recall the following result. [A-S, Corollary 3.11].

Let $\bar{f}(X)$ be nondegenerate with respect to $\Delta_\infty(\bar{f})$ and convenient with respect to S_2. Assume $\dim \Delta_\infty(\bar{f}) = n$. Then $L(\mathbb{G}_m^{n_1} \times \mathbb{A}^{n_2}, \bar{f}, \Psi, T)^{(-1)^{N+1}}$ is a polynomial and the Newton polygon of $L(\mathbb{G}_m^{n_1} \times \mathbb{A}^{n_2}, \bar{f}, \Psi, T)^{(-1)^{N+1}}$ lies over the Newton polygon of the polynomial $\Pi_{i \in (1/M)\mathbb{Z}_+}(1 - q^i T)^{\tilde{h}_i}$.

We proceed now with the proof of Theorem 4.

Proof. Since $\tilde{N}_l = \Sigma_{A \subseteq S_2}(-1)^{|A|} N_{l,A}$ and by the previous section, $N_{l,A} = H_{l,A}$ for all $A \subseteq S_2$, it suffices to show

$$\tilde{H}_l = \sum_{A \subseteq S_2} (-1)^{|A|} H_{l,A}$$

We claim the existence of an exact sequence of complexes

$$\begin{aligned} 0 \longrightarrow K(\phi, S_2)_{.} \xrightarrow{\varepsilon} K(\phi, \phi)_{.} \xrightarrow{\delta_0} \bigoplus_{i \in S_2} K(\{i\}, \phi)_{.} \xrightarrow{\delta_1} \bigoplus_{\substack{i,j \in S_2 \\ i \neq j}} K(\{i,j\}, \phi)_{.} \cdots \\ \longrightarrow K(S_2, \phi)_{.} \longrightarrow 0 \end{aligned} \tag{4.1}$$

the maps of which are all homogeneous of degree zero. Let us assume this claim and prove the theorem. Since f is nondegenerate and convenient therefore f_A is nondegenerate with respect to $\Delta_\infty(f_A)$ and $\dim \Delta_\infty(f_A) = N - |A|$. As a consequence $K(A, \phi)_{.}$ is acyclic except in dimension zero. But then (for example, by breaking the exact sequence into a system of short exact sequences), we obtain an exact sequence of homology:

$$\begin{aligned} 0 \to H_0(K(\phi, S_2)_{.}) \to H_0(K(\phi, \phi)_{.}) \to \bigoplus_{i \in S_2} H_0(K(\{i\}, \phi)_{.}) \\ \to \bigoplus_{\substack{i,j \in S_2 \\ i \neq j}} H_0(K(\{i, j\}, \phi)_{.}) \to \cdots \to H_0(K(S_2, \phi)_{.}) \to 0 \end{aligned} \tag{4.2}$$

in which the maps are homogeneous of degree zero. But then for $j \in (1/M)\mathbb{Z}_+$,

$$\tilde{h}_j = \sum_{A \subseteq S_2} (-1)^{|A|} \bar{h}_{j,A},$$

where

$$\bar{h}_{j,A} = \dim_K \left(\bar{R}_{A,j} \Big/ \sum_{i \in S_1 \cup (S_2 - A)} F_{A,i} \bar{R}_{A,j-1} \right)$$

and

$$\tilde{H}_l = \sum_{A \subseteq S_2} (-1)^{|A|} H_{l,A}.$$

It remains then to prove the above claim. Whenever $B' \subseteq B \subseteq S_2 - A$, there is an obvious inclusion of complexes

$$i(A; B, B')_. : K(A, B)_. \to K(A, B')_. .$$

Whenever $A' \subseteq A \subseteq S_2 - B$, there is an obvious surjection of complexes

$$\pi(A', A; B) : K(A', B)_. \to K(A, B)_. .$$

Furthermore [A-S, Appendix]

(i) if $B'' \subseteq B' \subseteq B \subseteq S_2 - A$, then

$$i(A; B', B'') \circ i(A; B, B') = i(A; B, B'')$$

(ii) if $A'' \subseteq A' \subseteq A \subseteq S_2 - B$, then

$$\pi(A', A; B) \circ \pi(A'', A'; B) = \pi(A'', A; B),$$

(iii) if $A' \subseteq A$ and $B' \subseteq B \subseteq S_2 - A$ then

$$i(A; B, B') \circ \pi(A', A; B) = \pi(A', A; B') \circ i(A'; B, B')$$

(iv) if $B \subseteq S_2 - A$ and $i \in B$ then the following sequence is exact

$$0 \longrightarrow K(A, B)_. \xrightarrow{i(A;B,B-\{1\})} K(A, B-\{i\})_. \xrightarrow{\pi(A,A\cup\{i\};B-\{i\})} K(A\cup\{i\}, B-\{i\})_. \longrightarrow 0.$$

We will show how these properties enable us to define the sequence (4.1), prove it is a complex, and then show it is exact. (In fact, the proof applies not only in the category of complexes but in any abelian category. For this reason, in the rest of the proof, we will write $K(A, B)$ for $K(A, B)_{\cdot}$.) First, we define the sequence. The map ε on the left is given by $\varepsilon = i(\phi; S_2, \phi)$ and is clearly injective. The rest of the sequence is essentially a Koszul complex using the projections $\pi(A, A \cup \{i\}; \phi)$. More precisely, let $\{i_1, \ldots, i_r\} \subseteq S_2$ and $i_1 < i_2 < \cdots < i_r$, and let

$$p(i_1, \ldots, i_r)\colon \bigoplus_{\substack{j_1 < j_2 < \cdots < j_r \\ \{j_1, \ldots, j_r\} \subseteq S_2}} K(\{j_1, \ldots, j_r\}, \phi) \to K(\{i_1, \ldots, i_r\}, \phi)$$

be the obvious projection. Then δ_{r-1} is defined by

$$p(i_1, \ldots, i_r) \circ \delta_{r-1}|_{K(\{i_1, \ldots, \hat{i}_l, \ldots, i_r\}, \phi)} = (-1)^{l+1} \pi(\{i_1, \ldots, \hat{i}_l, \ldots, i_r\}, \{i_1, \ldots, i_r\}; \phi)$$

for each l, $1 \leqslant l \leqslant r$. Since

$$\varepsilon = \prod_{i \in S_2} i(\phi; \{i\}, \phi),$$

clearly $\delta_0 \circ \varepsilon = 0$. Furthermore, by usual combinatorial business $\delta_r \circ \delta_{r-1} = 0$ for $1 \leqslant r \leqslant |S_2|$, so the sequence (4.1) is a complex.

We are indebted to Joel Roberts for the following argument proving the exactness of the complex (4.1). We will prove instead the more general assertion that for any $A \subseteq S_2$ the following complex is exact

$$0 \to K(A, S_2 - A) \to K(A, \phi) \to \bigoplus_{i \in S_2 - A} K(A \cup \{i\}, \phi) \to \bigoplus_{\substack{\{i,j\} \subseteq S_2 - A \\ i \neq j}} K(A \cup \{i, j\}, \phi) \to \cdots$$

$$\to K(S_2, \phi) \to 0. \tag{4.3}$$

If so, then $A = \phi$ is the claim we are trying to prove. We proceed first by induction on $|S_2| = n_2$. The assertion is trivial if $S_2 = \phi$ and follows directly from (iv) above in the case $|S_2| = 1$. Now assume $S_2 = \{1, \ldots, n_2\}$ and (4.3) is exact for all proper subsets S_2' of S_2 and all $A' \subseteq S_2'$. We proceed to prove the exactness of (4.3) for S_2. We proceed by induction on $|A|$. The case $A = S_2$ is trivial:

$$0 \to K(S_2, \phi) \to K(S_2, \phi) \to 0.$$

Assume $n_2 \in A$ and we will assume (4.3) is exact for S_2 and A. We now prove it is exact for S_2 and $A' = A - \{n_2\}$. Set $S_2' = S_2 - \{n_2\}$. Consider the map of complexes:

$$\begin{array}{ccccccccc}
0 & & 0 & & & & 0 & & \\
\uparrow & & \uparrow & & & & \uparrow & & \\
K(A',\phi) & \to & \bigoplus_{i\in S_2'-A'} K(A'\cup\{i\},\phi) & \to & & \cdots\to & K(S_2',\phi) & \to & 0 \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
K(A',\phi) & \to & \bigoplus_{i\in S_2-A'} K(A'\cup\{i\},\phi) & \to & & \cdots\to & \bigoplus_{i\in S_2-A} K(S_2-\{i\},\phi) & \to & K(S_2,\phi)\to 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & K(A,\phi) & \to & \bigoplus_{i\in S_2-A} K(A\cup\{i\},\phi) & \cdots\to & \bigoplus_{i\in S_2-A} K(S_2-\{i\},\phi) & \to & K(S_2,\phi)\to 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

Let $\mathbb{A}$ be the complex in the top row, and $\mathbb{B}_\cdot$ the complex in the middle row. Since $0 \to K(A', S_2') \to \mathbb{A}_\cdot$ is the complex (4.3) for the pair (A', S_2'), we have that the top row is exact by induction hypothesis on $|S_2|$. The bottom row has the form $\mathbb{D}_\cdot^+ = (0 \to \mathbb{D}_\cdot)$ where $0 \to K(A, S_2 - A) \to \mathbb{D}_\cdot$ is the complex (4.3) for the pair (A, S_2) and hence $\mathbb{D}_\cdot$ is exact by induction hypothesis on $|A|$. The short exact sequence of complexes

$$0 \to \mathbb{D}_\cdot^+ \to \mathbb{B}_\cdot \to \mathbb{A}_\cdot \to 0$$

gives rise to a long exact sequence of homology from which we conclude $H_i(\mathbb{B}_\cdot) = 0$, for $i \geqslant 2$, and

$$0 \to H_0(\mathbb{B}_\cdot) \to H_0(\mathbb{A}_\cdot) \xrightarrow{\gamma} H_1(\mathbb{D}_\cdot^+) \to H_1(\mathbb{B}_\cdot) \to 0 \tag{4.4}$$

The exactness of (4.3) for the pair (A', S_2') yields $H_0(\mathbb{A}_\cdot) = K(A', S_2' - A')_\cdot$ and the exactness of the complex (4.3) for the pair (A, S_2) yields $H_1(\mathbb{D}_\cdot^+) = H_0(\mathbb{D}_\cdot) = K(A, S_2 - A)$. Since the map $\gamma = \pi(A', A; S_2 - A)$ is surjective it follows that $H_1(\mathbb{B}_\cdot) = 0$. Finally the beginning part of (4.4) may now be identified as

$$0 \to H_0(\mathbb{B}_\cdot) \to K(A', S_2 - A' - \{n_2\}) \to K(A, S_2 - A' - \{n_2\})$$

which identifies $H_0(\mathbb{B}_\cdot)$ as $K(A', S_2)$ by (iv) above.

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