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0. Introduction

The "restricted" Chow variety $C_\lambda(X)$ of a projective algebraic variety X is the space consisting of all effective cycles with a given homology class λ. This space is a subvariety of the usual Chow variety and has the advantage of not depending on the projective embedding of X. The main purpose of this paper is to calculate the Euler characteristic of $C_\lambda(X)$, when X admits a linear action of an algebraic torus, for which there are only finitely many irreducible invariant subvarieties. An important class of such varieties are the projective toric varieties.

When $X = \mathbf{P}^n$ and $\lambda = d[\mathbf{P}^p]$ then

$$\mathscr{C}_\lambda(\mathbf{P}^n) = \mathscr{C}_{p,d}(\mathbf{P}^n)$$

where $\mathscr{C}_{p,d}(\mathbf{P}^n)$ is the Chow variety of effective cycles of dimension p and degree d in $\mathbf{P}^n$. Very little is known about $\mathscr{C}_{p,d}(X)$. In [LY87] H. B. Lawson and S. S. Yau posed the problem of computing the Euler characteristic $\chi(\mathscr{C}_\lambda)$ of $\mathscr{C}_\lambda$. They introduced a formal power series which satisfies, for $X = \mathbf{P}^n$, the following equality

$$\sum_{d=0}^{\infty} \chi(\mathscr{C}_{p,d}(\mathbf{P}^n))t^d = \left(\frac{1}{1-t}\right)^{\binom{n+1}{p+1}}. \tag{1}$$

This shows that the formal power series on the left is rational and solves the problem of computing the Euler characteristic of $\mathscr{C}_{p,d}(\mathbf{P}^n)$. They also computed the case $X = \mathbf{P}^n \times \mathbf{P}^m$.

When we try to generalize the above series in order to compute $\chi(\mathscr{C}_\lambda)$ we face some problems. If a basis for the integral homology group (modulo torsion) of X is fixed, and we write it as a multiplicative group in the canonical way, we arrive at a series where some of the powers could be negative integers. In others words, we do not obtain a formal power series,

and the concept of rationality is not defined a priori. In the author's thesis [Eli92] an "ad hoc" definition of rationality was given and it was proved that for any smooth projective toric variety the corresponding series is rational. In this paper we follow an intrinsic formulation suggested by E. Bifet [Bif92] in order to reinterpret the results in [Eli92] and prove them for any projective algebraic variety with a finite number of invariant subvarieties under an algebraic torus action. The definition of the monoid C, rationality and Lemma 1.2 are taken from him.

The main results can be stated in the following way. Let C be the monoid of homology classes of effective p-cycles, and let $\mathbf{Z}[[C]]$ be the ring of $\mathbf{Z}$-valued functions (with respect to the convolution product) over C. Denote by $\mathbf{Z}[C]$ the ring of functions with finite support on C. We say that an element of $\mathbf{Z}[[C]]$ is rational if it is the quotient of two elements in $\mathbf{Z}[C]$. We define the Euler series of X by

$$E_p = \sum_{\lambda \in C} \chi(\mathscr{C}_\lambda)\lambda \in \mathbf{Z}[[C]]. \tag{2}$$

This generalizes the series in Eq. (1). Let X be a projective algebraic variety and $V_1, \ldots, V_N$ be its p-dimensional invariant irreducible subvarieties under the action of the algebraic torus. Denote by $e_{[V_i]} \in \mathbf{Z}[C]$ the characteristic function of the set $\{[V_i]\}$. Theorem 2.1 says,

$$E_p = \prod_{i=1}^{N} \left(\frac{1}{1 - e_{[V_i]}} \right). \tag{3}$$

If X is a smooth projective toric variety we define C_T to be the monoid of equivariant cohomology classes of invariant effective p-cycles. Denote by $\mathbf{Z}[[C_T]]$ and by $\mathbf{Z}[C_T]$ the ring of functions and the ring of functions with finite support on C_T respectively. An element of $\mathbf{Z}[[C_T]]$ is rational if it is the quotient of two elements of $\mathbf{Z}[C_T]$. In section three the equivariant Euler series E_p^T is defined and it is proved in Theorem 3.4 that E_p^T is rational. In fact an explicit formula for E_p^T is obtained. Furthermore, a ring homomorphism

$$J \colon \mathbf{Z}[[C_T]] \to \mathbf{Z}[[C]]$$

is defined and we recover formula (3) from the following equality

$$J(E_p^T) = E_p.$$

1. The Euler series

In [LY87] B. Lawson and S. S. Yau introduced a series associated to the Chow monoid of a projective variety that becomes a formal power series when a basis for homology is fixed. They proved it is a rational function for the cases of $\mathbf{P}^n$ and $\mathbf{P}^n \times \mathbf{P}^m$. In this section we define a more general series and state the problem of its rationality in intrinsic terms for any projective algebraic variety. We follow an approach suggested by E. Bifet [Bif92]. The definition of the monoid C, rationality and Lemma 1.2 are taken from him. We start with some basic definitions and some of their properties.

Throughout this section any projective algebraic variety X comes with a fixed embedding $X \overset{j}{\hookrightarrow} \mathbf{P}^n$. An effective p-cycle c on X is a finite (formal) sum $c = \Sigma n_s V_s$ where each n_s is a positive integer and each V_s is an irreducible p-dimensional subvariety of X. From now on, we shall use the term *cycle* for *effective cycle*. For any projective algebraic variety $X \overset{j}{\hookrightarrow} \mathbf{P}^n$ we denote by $\mathscr{C}_{p,d}(X)$ the Chow variety of X of all cycles of dimension p and degree d in $\mathbf{P}^n$ with support on X. By convention, we write $\mathscr{C}_{p,0}(X) = \{\varnothing\}$. Let λ be an element in $H_{2p}(X, \mathbf{Z})$ and denote by $\mathscr{C}_\lambda(X)$ the space of all cycles on X whose homology class is λ. Note that $\mathscr{C}_\lambda(X)$ is contained in $\mathscr{C}_{p,d}(X)$, where $j_* \lambda = d[\mathbf{P}^p]$.

LEMMA 1.1. *Let λ be an element of $H_{2p}(X, \mathbf{Z})$, then $\mathscr{C}_\lambda(X)$ is a projective algebraic variety.*

Proof. Since $\mathscr{C}_{p,d}(X)$ is a projective variety (see [Sam55], [Cv37]) we can write $\mathscr{C}_{p,d}(X) = \bigcup_{i=1}^M \mathscr{C}_{p,d}^i(X)$, where $\mathscr{C}_{p,d}^i(X)$ are its irreducible components. Suppose $\mathscr{C}_\lambda(X) \cap \mathscr{C}_{p,d}^{i_0}(X) \neq \varnothing$. Any two cycles in $\mathscr{C}_{p,d}^{i_0}$ are algebraically equivalent, hence they represent the same element in homology. Therefore $\mathscr{C}_{p,d}^{i_0}(X) \subset \mathscr{C}_\lambda(X)$ for some λ. Consequently $\mathscr{C}_\lambda(X) = \bigcup_{j=1}^l \mathscr{C}_{p,d}^{i_j}(X)$, where $\mathscr{C}_\lambda(X) \cap \mathscr{C}_{p,d}^{i_j}(X) \neq \varnothing$ for $j = 1, \ldots, l$. $\qquad\square$

For an effective p-cycle $c = \Sigma n_i V_i$, we denote by $[c]$ its homology class in $H_{2p}(X, \mathbf{Z})$. Now, let C be the **monoid of homology classes of effective p-cycles** in $H_{2p}(X, \mathbf{Z})$, and let $\mathbf{Z}[[C]]$ be the **set of all integer valued functions on C**. We shall write the elements of $\mathbf{Z}[[C]]$ as

$$\sum_{\lambda \in C} a_\lambda \lambda \quad \text{where } a_\lambda \in \mathbf{Z}.$$

The following lemma allows us to prove that $\mathbf{Z}[[C]]$ is a ring,

LEMMA 1.2. *Let C be the monoid of homology classes of effective p-cycles on X. Then*

$$+ : C \times C \to C$$

has finite fibers.

Proof. If $X = \mathbf{P}^n$ the result is obvious since C is isomorphic to $\mathbf{N}$. Let

$$j_* : H_{2p}(X, \mathbf{Z}) \to H_{2p}(\mathbf{P}^n, \mathbf{Z})$$

be the homomorphism induced by the embedding $X \overset{j}{\hookrightarrow} \mathbf{P}^n$. Denote by $C' \cong \mathbf{N}$ the monoid of homology classes of p-cycles on $\mathbf{P}^n$. It follows from the proof of Lemma 1.1 that

$$j_*|_C : C \to C'$$

has finite fibers. Finally, the lemma follows from the commutative diagram below

$$
\begin{array}{ccc}
 & + & \\
C \times C & \longrightarrow & C \\
\Big\downarrow{\scriptstyle j_* \times j_*|_{C \times C}} & & \Big\downarrow{\scriptstyle j_*|_C} \\
C' \times C' & \longrightarrow & C' \\
 & + &
\end{array}
\qquad \square
$$

Directly from this Lemma we obtain:

PROPOSITION 1.3. *Let C and $\mathbf{Z}[[C]]$ be defined as above. Then $\mathbf{Z}[[C]]$ is a ring under the convolution product, i.e.*

$$(f \cdot g)(\lambda) = \sum_{\lambda = \mu_1 + \mu_2} f(\mu_1)g(\mu_2).$$

We are ready for the following definition.

DEFINITION 1.4. *Let X be a projective algebraic variety. The* **Euler series** *of X, in dimension p, is the element*

$$E_p = \sum_\lambda \chi(\mathscr{C}_\lambda)\lambda \in \mathbf{Z}[[C]]$$

where $\mathscr{C}_\lambda(X)$ is the space of all effective cycles on X with homology class λ, and $\chi(\mathscr{C}_\lambda(X))$ is the Euler characteristic of $\mathscr{C}_\lambda$.

By convention, if $\mathscr{C}_\lambda$ is the empty set then its Euler characteristic is zero.

Let $\mathbf{Z}[C]$ be the monoid-ring of C over $\mathbf{Z}$. This ring consists of all elements

of $Z[[C]]$ with finite support. Observe that the multiplicative identity is the function which is 1 on the class $0 \in C \subset H_{2p}(X, Z)$ and 0 elsewhere. We arrive at the following definition,

DEFINITION 1.5. An element of $Z[[C]]$ is rational if it is the quotient of two elements of $Z[C]$.

REMARK 1.6. Denote by H the homology group $H_{2p}(X, Z)$ together with a fixed basis $\mathscr{A}$. Consider H as a multiplicative group in the standard way and suppose that C is isomorphic to the monoid N^k where N is the natural numbers and k is a positive integer. Then it is easy to see that $Z[[C]]$ is isomorphic to the ring of formal power series in k variables. Therefore $Z[C]$ is the ring of polynomials in k variables and the last definition just says when a formal power series is a rational function.
    We are interested in the following problem.

PROBLEM. When is the **Euler series** rational in the sense of the last definition?
    Let X be a path-connected projective algebraic variety. We know that, for any basis, the homology group $H = H_0(X, Z)$ is isomorphic to the integers Z, and C to the monoid of natural numbers N. Let us consider H as a multiplicative group. Then $Z[[C]]$ is isomorphic to the ring of formal power series in the variable t. We obtain directly from the computation in [Mac62] that

$$E_0(X) = \frac{1}{(1-t)^{\chi(X)}}.$$

The present article, in particular, recovers the results for both cases $X = \mathbf{P}^n$ and $X = \mathbf{P}^n \times \mathbf{P}^m$ which were worked out in [LY87].

2. Varieties with a torus action

Throughout this section X is a projective algebraic variety, on which an algebraic torus T acts linearly having only a finite number of invariant irreducible subvarieties of dimension p. In particular, we will see that the result is true for any projective toric variety. Let us denote by H the homology group $H_{2p}(X, Z)$.
    The action of T on X induces an action on the Chow variety $\mathscr{C}_{p,d}(X)$. Let λ be an element in H and denote by $\mathscr{C}_\lambda^T$ the fixed point set of $\mathscr{C}_\lambda(X)$ under the action of T. Then its Euler characteristic $\chi(\mathscr{C}_\lambda^T)$ is equal to the number of invariant subvarieties of X with homology class λ. We have

$$E_p(\lambda) = \chi(\mathscr{C}_\lambda) = \chi(\mathscr{C}_\lambda^T) \tag{4}$$

where the first equality is just the definition of E_p and the last one is proved in [LY87]. The following theorem tells us that E_p is rational.

THEOREM 2.1. *Let E_p be the Euler series of X. Denote by $V_1, \ldots, V_N$ the p-dimensional invariant irreducible subvarities of X. Let $e_{[V_i]} \in Z[C]$ be the characteristic function of the subset $\{[V_i]\}$ of C. Then*

$$E_p = \prod_{1 \leqslant i \leqslant N} \left(\frac{1}{1 - e_{[V_i]}} \right) \tag{5}$$

Proof. For each V_i define f_i in $Z[[C]]$ by

$$f_i(\lambda) = \begin{cases} 1 & \text{if } \lambda = n \cdot [V_i], \ n \geqslant 0 \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

It is easy to see from Eqs. (4) and (6) that E_p can be written as

$$E_p = \prod_{1 \leqslant i \leqslant N} f_i. \tag{7}$$

The theorem follows because of the equality

$$1 = (1 - e_{[V_i]}) \cdot f_i \qquad\qquad \square$$

Observe that if we fix a basis for H modulo torsion and consider H multiplicatively as in Remark 1.6, then the elements of $Z[C]$ can be identified with Laurent polynomials. Under this identification any rational element of $Z[[C]]$ is a rational function.

The next lemma tells us that the result is true for any projective toric variety.

LEMMA 2.2. *Let X be a projective (perhaps singular) toric variety. Then any irreducible subvariety V of X which is invariant under the torus action is the closure of an orbit. Therefore, any invariant cycle has the form*

$$c = \Sigma \, n_i \overline{\mathcal{O}}_i$$

where each n_i is a nonnegative integer and each $\overline{\mathcal{O}}_i$ is the closure of the orbit $\mathcal{O}_i$.

Proof. The fan Δ associated to X is finite because X is compact. Hence there is a finite number of cones, therefore a finite number of orbits. Let V be an invariant irreducible subvariety of X. We can express V as the closure

of the union of orbits. Since there is a finite number of them we must have that

$$V = \bar{\mathcal{O}}_1 \cup \bar{\mathcal{O}}_2 \cup \cdots \cup \bar{\mathcal{O}}_N$$

where $\bar{\mathcal{O}}_i$ is the closure of an orbit. Finally since V is irreducible, there must be i_0 such that $V = \bar{\mathcal{O}}_{i_0}$. $\square$

3. Smooth toric varieties

In this section we give an equivariant version of the Euler series and find a relation between the equivariant and non-equivariant Euler series. The use of equivariant cohomology allows us to analize the Euler series from a geometrical point of view. This approach might help to understand other cases. Throughout this section, unless otherwise stated, X is a smooth projective toric variety of dimension n, and we use cohomology instead of homology by applying Poincaré duality.

Let H and H_T be the cohomology group $H^{2(n-p)}(X, \mathbf{Z})$ and the equivariant cohomology group $H_T^{2(n-p)}(X, \mathbf{Z})$ of X, respectively. Denote by Δ the fan associated with X and by $\bar{\mathcal{O}}_1, \ldots, \bar{\mathcal{O}}_N$ the p-codimensional orbits closures. Let $\mathscr{C}_\lambda^T$ and $\mathscr{C}_\lambda$ be the spaces of all p-dimensional effective invariant cycles and p-dimensional effective cycles on X with cohomology class λ. It is proved in [LY87] that

$$\chi(\mathscr{C}_\lambda^T) = \chi(\mathscr{C}_\lambda). \tag{8}$$

The next lemma is crucial for the following results,

LEMMA 3.1. *Let λ be an element in H. Then $\mathscr{C}_\lambda^T$ is a finite set.*
    Proof. By Lemma 2.2 we know that any invariant effective cycle c in $\mathscr{C}_\lambda^T$ has the form $c = \Sigma_{i=1}^N \beta_i \bar{\mathcal{O}}_i$ with $\beta_i \in \mathbf{N}$. Hence, we obtain that $\mathscr{C}_\lambda^T$ has a countable number of elements. We know that $\mathscr{C}_\lambda$ is a projective algebraic variety (see Lemma 1.1), and since $\mathscr{C}_\lambda^T$ is Zariski closed in $\mathscr{C}_\lambda$, we have that $\mathscr{C}_\lambda^T$ is a finite set. $\square$

Our next step is to define the equivariant Euler series for X.

Let $\bar{\mathcal{O}}$ be an irreducible invariant cycle in a smooth toric variety (Lemma 2.2). Since $\bar{\mathcal{O}} \subset X$ is smooth, we have an equivariant Thom-Gysin sequence

$$\cdots \to H_T^{i-2\operatorname{cod}\bar{\mathcal{O}}}(\bar{\mathcal{O}}) \to H_T^i(X) \to H_T^i(X - \bar{\mathcal{O}}) \to \cdots$$

and we define $[\bar{\mathcal{O}}]_T$ as the image of 1 under

$$H_T^0(\bar{\mathcal{O}}) \to H_T^{2\,\mathrm{cod}\,\bar{\mathcal{O}}}(X).$$

Let $\{D_1, \ldots, D_K\}$ be the set of T-invariant divisors on X. To each D_i we associate the variable t_i in the polynomial ring $\mathbf{Z}[t_1, \ldots, t_K]$. Let $\mathscr{I}$ be the ideal generated by the (square free) monomials $\{t_{i_1} \cdots t_{i_l} | D_{i_1} + \cdots + D_{i_l} \notin \Delta\}$. It is proved in [BDP90] that

$$\mathbf{Z}[t_1, \ldots, t_K]/\mathscr{I} \cong H_T^*(X, \mathbf{Z}). \tag{9}$$

The arguments given there also prove the following.

PROPOSITION 3.2. *For any T-orbit $\mathcal{O}$ in a smooth projective toric variety X, one has*

$$[\bar{\mathcal{O}}]_T = \prod_{\bar{\mathcal{O}} \subset D_i} [D_i]_T.$$

Furthermore if $\mathcal{O}$ and $\mathcal{O}'$ are distinct orbits, then

$$[\bar{\mathcal{O}}]_T \neq [\bar{\mathcal{O}}']_T.$$

It is natural to define the cohomology class for any effective invariant cycle $V = \Sigma m_i \mathcal{O}_i$ as $[V]_T = \Sigma_i m_i [\bar{\mathcal{O}}_i]_T$ where $\mathcal{O}_i \neq \mathcal{O}_j$ if $i \neq j$. In a similar form as we define C, $\mathbf{Z}[[C]]$ and $\mathbf{Z}[C]$, we denote by C_T the monoid of equivariant cohomology classes of invariant effective cycles of dimension p, by $\mathbf{Z}[[C_T]]$ the set of functions on C_T, and by $\mathbf{Z}[C_T]$ the set of functions with finite support on C_T. Since $C_T \simeq \mathbf{N}^N$ where N is the number of orbits of dimension p, we obtain that

$$+ : C_T \times C_T \to C_T$$

has finite fibers. Observe that if $\pi : H_T \to H$ denotes the standard surjection, we obtain from Lemma 3.1 that

$$\pi : C_T \to C$$

is onto with finite fibers. We arrive at the following definition:

DEFINITION 3.3. Let X be a smooth projective toric variety and let $H_T(X)$ be the equivariant cohomology of X. Let us denote by $\mathscr{C}_\xi^T$ the space of all invariant effective cycles on X whose equivariant cohomology class is ξ. The **equivariant Euler series** of X is the element

$$E_p^T = \prod_\xi \chi(\mathscr{C}_\xi^T)\xi \in \mathbf{Z}[[C_T]]$$

where the sum is over C_T.

Let us define the ring homomorphism

$$J: \mathbf{Z}[[C_T]] \to \mathbf{Z}[[C]]$$

by

$$J(\xi) = \sum_\lambda \left(\sum_{\pi(\beta) = \lambda} a_\beta \right) \lambda$$

where $\xi = \Sigma_\beta a_\beta \beta$. This is well defined since π has finite fibers.

THEOREM 3.4. *Let X be a smooth projective toric variety. Denote by E_p, E_p^T and J the Euler series, the equivariant Euler series and the ring homomorphism defined above. Then $J(E_p^T) = E_p$. Furthermore,*

$$E_p^T = \prod_{1 \leqslant i \leqslant N} \left(\frac{1}{1 - e_{[\bar{\mathcal{O}}_i]_r}} \right)$$

and therefore

$$E_p = \prod_{1 \leqslant i \leqslant N} \left(\frac{1}{1 - e_{[\bar{\mathcal{O}}_i]}} \right).$$

Proof. We define for each orbit $\mathcal{O}_i$ an element $f_i^T \in \mathbf{Z}[[C_T]]$ by

$$f_i^T(\xi) = \begin{cases} 1 & \text{if } \xi = n \cdot [\bar{\mathcal{O}}_i]_T, \ n \geqslant 0 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

and denote by e_ξ the characteristic function of $\{\xi\}$. It follows from both Definition (3.3) and Eq. (10) that

$$E_p^T = \prod_{1 \leqslant i \leqslant N} f_i^T$$

and

$$(1 - e_{[\bar{\mathcal{O}}_i]_r}) \cdot f_i^T = 1.$$

Therefore the equivariant Euler series is rational and

$$E_p^T = \prod_{1 \leqslant i \leqslant N} \left(\frac{1}{1 - e_{[\bar{\mathcal{O}}_i]_T}} \right). \tag{11}$$

For each V_i we defined (see Eq. 6) a function f_i in $\mathbf{Z}[[C]]$ by

$$f_i(\lambda) = \begin{cases} 1 & \text{if } \lambda = n \cdot [V_i], \ n \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

And we know from Theorem 2.1 that

$$E_p = \prod_{i=1}^{N} f_i \quad \text{with} \quad f_i \cdot (1 - e_{[V_i]}) = 1.$$

Now, the result follows since $\pi([\bar{\mathcal{O}}_i]_T) = [\bar{\mathcal{O}}_i]$ and J is a ring homomorphism satisfying

$$J(f_i^T) = f_i \quad \text{and} \quad J(e_\sigma) = e_{\pi(\sigma)}.$$

4. Some examples

(I) *The projective space* $\mathbf{P}^n$

Let $X = \mathbf{P}^n$ be the complex projective space of dimension n. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbf{R}^n$. Consider $A = \{e_1, \ldots, e_{n+1}\}$ a set of generators of the fan Δ where $e_{n+1} = -\Sigma_{i=1}^n e_i$. We have the following equality

$$H^*(X, \mathbf{Z}) \cong \mathbf{Z}[t_1, \ldots, t_{n+1}]/I$$

where I is the ideal generated by

(i) $t_1 \cdots t_{n+1}$

and

(ii) $\displaystyle\sum_{j=1}^{n+1} e_i^*(e_j) t_j \quad \text{for } i = 1, \ldots, n,$

where $e_i^* \in (\mathbf{R}^n)^*$ is the element dual to e_i.

However (ii) says that $t_i \sim t_j$ for all i and j. Therefore

$$H^*(X, \mathbf{Z}) = \mathbf{Z}[t]/t^{n+1}.$$

Consequently, any two cones of dimension $n - p$ represent the same

element in cohomology, and Theorem 3.4 implies that

$$\prod_{i=1}^{\binom{n+1}{n-p}} \left(\frac{1}{1-t}\right) = \left(\frac{1}{1-t}\right)^{\binom{n+1}{n-p}} = \left(\frac{1}{1-t}\right)^{\binom{n+1}{p+1}} = E_p.$$

(II) $\mathbf{P}^n \times \mathbf{P}^m$

Denote by $X(\Delta)$ the toric variety associated to the fan Δ, and recall that $X(\Delta \times \Delta') \cong X(\Delta) \times X(\Delta')$. Using the same notation as in Example I, we have that a set of generators of $\Delta \times \Delta'$ is given by

$$\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}, e_{n+m+1}, e_{n+m+2}\}$$

where $e_{n+m+1} = -\Sigma_{i=1}^n e_i$, $e_{n+m+2} = -\Sigma_{i=n+1}^{n+m} e_i$ and $\{e_1, \ldots, e_{n+m}\}$ a basis for $\mathbf{R}^{n+m}$. Then

$$H^*(X, \mathbf{Z}) = \mathbf{Z}[t_1, \ldots, t_{n+m+2}]/I$$

where I is the ideal generated by

(i) $\left\{ t_1 \cdots t_n t_{n+m+1}, \ t_{n+1} \cdots t_{n+m} t_{n+m+2}, \ \prod_{i=1}^{n+m+2} t_i \right\}$

and

(ii) $\displaystyle\sum_{j=1}^{n+m+2} e_i^*(e_j)t_j \qquad i = 1, \ldots, n+m.$

From (ii) we obtain,

$$t_i \sim t_{n+m+1} \quad \text{if} \quad 1 \leqslant i \leqslant n$$

$$t_j \sim t_{n+m+2} \quad \text{if} \quad n+1 \leqslant j \leqslant n+m$$

The number of cones of dimensions $(n+m) - p$ is equal to $\Sigma_{k+l=p}$ $\binom{n+1}{n-k}\binom{m+1}{m-1} = \Sigma_{k+l=p}\binom{n+1}{k+1}\binom{m+1}{l+1}$.

Denote by $t_{k,l} = t_{n+m+1}^k t_{n+m+2}^l$. Then

$$\prod_{k+l=p} \left(\frac{1}{1-t_{k,l}}\right)^{\binom{n+1}{k+1}\binom{m+1}{l+1}} = E_p.$$

(III) *Blow up of* $\mathbf{P}^n$ *at a point*

The fan $\tilde{\Delta}$ associated to the blow up $\tilde{\mathbf{P}}^n$ of the projective space at the fixed point given by the cone $\mathbf{R}^+e_2 + \cdots + \mathbf{R}^+e_{n+1}$ is generated by $\{e_1,\ldots,e_{n+1},e_{n+2}\}$ where $e_{n+2} = -e_1$. Denote by D_i the 1-dimensional cone $\mathbf{R}^+e_i$ and by s_i its class in cohomology where

$$H^*(X,\mathbf{Z}) = \mathbf{Z}[s_1,\ldots,s_{n+2}]/I$$

and I is the ideal generated by

(i) $\{s_{i_1} \cdots s_{i_k} \mid D_{i_1} + \cdots + D_{i_k} \text{ is not in } \tilde{\Delta}\}$

and

(ii) $\displaystyle\sum_{j=1}^{n+2} e_i^*(e_j)s_j \qquad i = 1,\ldots,n.$

However (ii) is equivalent to

(ii) $s_2 \sim \cdots \sim s_3 \sim s_{n+1}$ and $s_1 \sim s_{n+1} + s_{n+2}.$

Note that a p-dimensional cone cannot contain both D_{n+2} and D_1. The reason is that D_{n+2} is generated by $-e_1$ and D_1 by e_1, but by definition, a cone does not contain a subspace of dimension greater than 0. We would like to find a basis for $H^*(\tilde{\mathbf{P}}^n)$ and write any monomial of degree p in terms of it. Consider the monomial $s_{i_1} \cdots s_{i_p}$. There are three possible situations:

(1) s_{i_j} is different from both s_{n+2} and s_{n+1}. In this situation we have from (ii) that $s_{i_1} \cdots s_{i_p} = s_{n+1}^p.$

(2) s_{n+2} is equal to s_{i_j} for some $j = 1,\ldots,p$. Then from (ii) we obtain that $s_{i_1} \cdots s_{i_p} = s_{n+1}^{p-1}s_{n+2}.$

(3) s_1 is equal to s_{i_j} for some $j = 1,\ldots,p$. Then from (ii) we obtain $s_{i_1} \cdots s_{i_p} = (s_{n+1} + s_{n+2})s_{n+1}^{p+1} = s_{n+1}^p + s_{n+2}s_{n+1}^{p-1}$ which is the sum of (1) and (2).

We conclude that s_{n+1}^p and $s_{n+2}s_{n+1}^{p-1}$ form a basis for H^{2p} if $p < n$. If $p = n$ then $s_{n+1}^p = 0$ and the only generator is $s_{n+2}s_{n+1}^{p-1}$. Let us call s_{n+1} by t_1 and $s_{n+2}s_{n+1}^{p-1}$ by t_2. The Euler series for $\tilde{\mathbf{P}}^n$ is:

$$E_{n-p} = \left(\frac{1}{1-t_1}\right)^{\binom{n}{p}} \left(\frac{1}{1-t_1t_2}\right)^{\binom{n}{p-1}} \left(\frac{1}{1-t_2}\right)^{\binom{n}{p-1}} \quad \text{if } p < n.$$

$$E_0 = \left(\frac{1}{1-t_2}\right)^{\binom{n+2}{n}}.$$

(IV) *Hirzebruch surfaces*

A set of generators for the fan Δ that represents the Hirzebruch surface $X(\Delta)$ is given by $\{e_1, \ldots, e_4\}$ with $\{e_1, e_2\}$ the standard basis for $\mathbf{R}^2$, and $e_3 = -e_1 + ae_2$, and $e_4 = -e_2$. With the same notation as in the last examples, we have

$$H^*(X(\Delta)) = \mathbf{Z}[t_1, \ldots, t_4]/I$$

where I is generated by

(i) $\{t_1 t_3, t_2 t_4\}$

and

(ii) $\{t_1 - t_3, \, t_2 + at_3 - t_4\}$

from (ii) we have the following conditions for the t_i's in $H^*(X)$

$$t_1 \sim t_3 \quad \text{and} \quad t \sim (t_4 - at_3). \tag{12}$$

A basis for $H^*(X)$ is given by $\{\{0\}, t_3, t_4, t_4 t_1\}$ (see [Dan78], [Ful93]). The Euler series for each dimension is:

(1) Dimension 0. There are four orbits (four cones of Dimension 2), and all of them are equivalent in homology. From Theorem 3.4 we obtain

$$E_0 = \left(\frac{1}{1 - t}\right)^4.$$

(2) Dimension 1. Again, there are four orbits (four cones of Dimension 1), and the relation among them, in homology, is given by 12. From Theorem 3.4 we obtain

$$E_1 = \left(\frac{1}{1 - t_3}\right)^2 \left(\frac{1}{1 - t_4}\right) \left(\frac{1}{1 - t_3^{-a} t_4}\right).$$

(3) Dimension 2. The only orbit is the torus itself so

$$E_2 = \left(\frac{1}{1 - t}\right).$$

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