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Period invariants of Hilbert modular forms, II

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Introduction

Let E be a totally real field of degree d over $\mathbb{Q}$, $\Sigma = \Sigma_E = \{\sigma_1, \ldots, \sigma_d\}$ the set of real embeddings of $E, G = R_{E/\mathbb{Q}}\mathrm{GL}(2, \mathrm{E})$. Let π be an irreducible automorphic representation of G, attached to a holomorphic Hilbert modular cusp form F. We assume the central character ξ_π of π to be *motivic* in the sense that ξ_π equals an integral power of the idele norm multiplied by a character of finite order. In the article [H3], to which we refer henceforth as Part I, we have shown how to attach invariants $v^I(\pi) \in \mathbb{C}^\times$ to π, for every subset $I \subset \Sigma$. These invariants are well-defined up to scalars in $\mathbb{Q}(\pi, I)$, where $\mathbb{Q}(\pi)$ is the field of definition of the non-archimedean component π_f of π and $\mathbb{Q}(\pi, I)$ is a certain subfield of the composite of $\mathbb{Q}(\pi)$ and the Galois closure of E over $\mathbb{Q}$. When $I = \varnothing$ we can take $v^{\varnothing}(\pi) = 1$, whereas $v^{\Sigma}(\pi)$ is the normalized Petersson inner product of F with itself, if F is taken to be an arithmetic new fofm in π. The definition of $v^I(\pi)$ in general is recalled briefly in (1.2.5).

The main result of Part I is the expression of values of Rankin-Selberg convolutions $L(s, \pi \otimes \pi')$, up to scalars in $\mathbb{Q}(\pi, I) \cdot \mathbb{Q}(\pi', I')$, in terms of the invariants $v^I(\pi)$ and $v^{I'}(\pi')$, when $s = m$ is a *critical value* of $L(s, \pi \otimes \pi')$, in the sense of Deligne (and Shimura). Here π' is another irreducible cuspidal automorphic representation, again attached to a holomorphic Hilbert modular form. The exact formula is recalled in (4.1.2); here we simply mention that the existence of critical values of $L(s, \pi \otimes \pi')$ implies the existence of a unique partition $\Sigma = I \amalg I'$, associated to the pair $\{\pi, \pi'\}$, and the critical values are expressible as elementary multiples of $v^I(\pi) \cdot v^{I'}(\pi')$.

On the other hand, some years earlier Shimura had found another expression for these critical values, valid in most cases. Suppose D and D' are quaternion algebras over E. Let $\Sigma(D)$ and $\Sigma(D')$ denote the subsets of Σ at which D and D' are unramified. In the situation of the preceding paragraph, we suppose $\Sigma(D) = I$, $\Sigma(D') = I'$, and we assume that the local constituent π_v

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of π (resp. π'_v of π') belongs to the discrete series of $\mathrm{GL}(2, E_v)$ at any place v of E at which D (resp. D') ramifies. Then the Jacquet-Langlands correspondence [JL] asserts the existence of automorphic representations π^D and $\pi'^{,D'}$ of $D^\times$ and $D'^{,\times}$, respectively, such that $\pi_v^D \cong \pi_v$ (resp. $\pi_v'^{,D'} \cong \pi'_v$) for every place v at which $D_v^\times$ (resp. $D_v'^{,\times}$) $\cong \mathrm{GL}(2, E_v)$. In this setting, Shimura has associated invariants $q^D(\pi)$, $q^{D'}(\pi') \in \mathbb{C}^\times/\bar{\mathbb{Q}}^\times$ to π and π', respectively. (Shimura denotes them $Q(\chi, D)$, where χ is the family of Hecke eigenvalues attached to places at which π is unramified; our notation is taken from [H5]). Roughly speaking, $q^D(\pi)$ is the normalized Petersson inner product with itself of an arithmetic holomorphic form in π^D; a more precise definition can be found in [H5], where it is shown how to define $q^D(\pi)$ up to $\mathbb{Q}(\pi, \Sigma(D))^\times$. Shimura's formula [S3; cf. Theorem 4.1.4 below] expresses the critical value $L(m, \pi \otimes \pi')$, up to algebraic factors, as an elementary multiple of $q^D(\pi) \cdot q^{D'}(\pi')$.

Let λ, $\mu \in \mathbb{C}$, and let L be a subfield of $\mathbb{C}$. We write $\lambda \sim_L \mu$ if either $\lambda \cdot \mu = 0$ or if $\lambda/\mu \in L^\times$. Assume there is a critical value $L(m, \pi \otimes \pi')$ which does not vanish. With our normalization, we then obtain the relation:

$$v^I(\pi) \cdot v^{I'}(\pi') \sim_{\bar{\mathbb{Q}}} q^D(\pi) \cdot q^{D'}(\pi'). \tag{0.1}$$

It is natural to conjecture that in fact

$$v^I(\pi) \sim_{\bar{\mathbb{Q}}} q^D(\pi). \tag{0.2}$$

Indeed, the invariants $v^I(\pi)$ were introduced as generalizations of Shimura's invariants $q^D(\pi)$ when no pair (D, π^D) as above exists, in response to a conjecture of Shimura [S3, Conjecture 5.10].

Suppose π and π' are associated to algebraic Hecke characters of a quadratic CM extension $\mathscr{K}$ of E. In other words, suppose ω and ω' are Hecke characters of $\mathscr{K}_{\mathbb{A}}^\times/\mathscr{K}^\times$, and let $\pi = \pi(\omega, \mathscr{K})$, $\pi' = \pi(\omega', \mathscr{K})$ be the corresponding automorphic representations of G [JL, Prop. 12.1], normalized as in Part I, §4. We denote by ρ the non-trivial element of $\mathrm{Gal}(\mathscr{K}/E)$. Then ρ acts on $\mathscr{K}_{\mathbb{A}}^\times/\mathscr{K}^\times$, and we let $\omega^\rho(x) = \omega(x^\rho)$, as in Part I. Then (cf. (4.3.7), below) we can write

$$L(s, \pi \otimes \pi') = L_{\mathscr{K}}(s, \omega \cdot \omega') \cdot L_{\mathscr{K}}(s, \omega \cdot \omega'^{,\rho})$$

as a product of Hecke L-functions on $\mathrm{GL}(1)_{\mathscr{K}}$. In this case, we have a third expression for the critical values of $L(s, \pi \otimes \pi')$, due to Shimura and Blasius [B]. For our purposes, it is convenient to use the formulation of [H5, §1]. For any Hecke character η of $\mathscr{K}^\times$ we let $\tilde{\omega} = \omega/\omega^\rho$. Then one can define invariants $p_{\mathscr{K}}(\tilde{\omega}, I)$, $p_{\mathscr{K}}(\tilde{\omega}', I')$ ([H5, §1]; cf. §4, below) such that the critical values $L(m, \pi \otimes \pi')$ are elementary multiples of $p_{\mathscr{K}}(\tilde{\omega}, I) \cdot p_{\mathscr{K}}(\tilde{\omega}', I')$. The invariants $p_{\mathscr{K}}(\tilde{\omega}, I)$ can be expressed as periods of motives of CM type; i.e., in the tensor category generated by abelian varieties with complex multiplication over number fields. Again, this leads to the conjecture that

$$v^I(\pi(\omega, \mathscr{K})) \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}, I). \tag{0.3}$$

Up to $\bar{\mathbb{Q}}^\times$, the CM period $p_{\mathscr{K}}(\tilde{\omega}, I)$ depends only on the infinity-type of $\tilde{\omega}$, and then $p_{\mathscr{K}}(\tilde{\omega}, I)$ coincides with the invariants defined by Shimura in [S1]. Thus (0.3) is also a translation of part of Shimura's Conjecture 5.10 in [S3].

Of course, when there is a quaternion algebra D over E such that the Jacquet-Langlands transfer π^D exists (briefly: π^D exists), with $\pi = \pi(\omega, \mathscr{K})$, (0.2) and (0.3) together imply that $q^D(\pi(\omega, \mathscr{K})) \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}, I)$. This was in fact proved by Shimura in [S3]. However, $p_{\mathscr{K}}(\tilde{\omega}, I)$ can be defined whenever the *weights* (in the sense of holomorphic Hilbert modular forms, cf. §1) of $\pi(\omega, \mathscr{K})$ at places in I are $\geqslant 2$; the definition of $q^D(\pi(\omega, \mathscr{K}))$ requires that the weights at places in $\Sigma - I$ be $\geqslant 2$. Thus (0.2) and (0.3) are logically independent of one another, even for binary theta functions.

The main theorem of the present paper is

THEOREM 1. *Let π be an irreducible automorphic representation of G, attached to a holomorphic Hilbert modular cusp form, with motivic central character. Let $I \subset \Sigma$, and suppose the weights of π at places in I are $\geqslant 2$.*

(a) *Let D be a quaternion algebra over E such that π^D exists. Then $v^I(\pi) \sim_{\mathbb{Q}} q^D(\pi)$.*
(b) *Suppose $\pi = \pi(\omega, \mathscr{K})$ for some algebraic Hecke character ω of a CM quadratic extension $\mathscr{K}$ of E. Then $i^{\eta(I,\pi)} \cdot v^I(\pi(\omega, \mathscr{K})) \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}, I)$, where the invariants $\eta(I, \pi)$ are defined as in Part I, 3.5.*

In (b) the term $i^{\eta(I,\pi)}$ is a fourth root of unity, and its presence in the formula may seem superfluous. It should rather be taken as an indication of what is to be expected when $\bar{\mathbb{Q}}$ is replaced by $\mathbb{Q}$ in the relations above. Indeed, most of the steps in the proof give results in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant form, and the power of i is an artifact of the normalization of the Petersson inner product, which does not appear in the definition of $p_{\mathscr{K}}(\tilde{\omega}, I)$.

Theorem 1 has applications to special values of triple product L-functions; a typical one is indicated in Corollary 6.5.

The main theorem of [H5] is a factorization of the quaternionic invariants $q^D(\pi)$, under the hypothesis that the local component π_v of π is special or supercuspidal at some finite prime v. This theorem, together with the results of Part I and Theorem 1 of the present paper, complete the proof of Shimura's Conjecture 5.10 [S3] in almost all cases.

The proof of Theorem 1 has three main steps. We present these steps in logical order, which is slightly different from the order followed in the text. First, using the relation (0.1) and slight generalizations thereof, one reduces

Theorem 1 to the special case of binary theta functions; essentially, to (b). In order to obtain results for forms of low weight, one has to demonstrate that sufficiently many critical L-values do not vanish. The main result of [H4] provides the necessary non-vanishing; alternatively, we could have quoted a theorem of Rohrlich.

The second step is provided by a somewhat mysterious theorem of Shimura [S2, Theorem 3.7]. Let $|I|$ denote the cardinality of I. One can interpret Shimura's theorem as the special case of Theorem 1 in which $|I| = 1$, provided it can be shown that certain cohomological cup products do not vanish. Here the non-vanishing is provided by the functional analytic methods of [H2, §7].

The first step allows us to restrict our attention to binary theta functions; the second step demonstrates the theorem when $|I| = 1$. The final step is an induction on $|I|$. The induction step, in the case of binary theta functions, is provided by Theorem 4.7 of Part I, which in turn is based on my joint work with Kudla on the central critical values of triple product L-functions [HK]. The induction step shows that, in certain cases, a partition $I = I_1 \amalg I_2$ defines a factorization:

$$v^I(\pi(\omega, \mathscr{K}))^2 \sim_{\bar{\mathbb{Q}}} v^{I_1}(\pi(\omega, \mathscr{K}))^2 \cdot v^{I_2}(\pi(\omega, \mathscr{K}))^2. \tag{0.4}$$

Since the invariants $p_\chi(\tilde{\omega}, I)$ are already known to have analogous factorizations, this suffices for the induction step.

It deserves to be stressed that the key ingredients in the first and third steps are period relations, which derive from the possibility of expressing special values of certain L-functions in several different ways, and from results which guarantee that these special values do not vanish. As a byproduct of the construction used in the third step, we include in Section 6 a theorem on the non-vanishing of certain triple product L-functions.

The text indicates what needs to be done to make each step $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant. Briefly: a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant version of the first step should appear in a forthcoming joint paper with Garrett. The methods of [HK] should, in principle, provide $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariance in the second step. Finally, Theorem 4.7 of Part I, on which (0.4) is based, is already $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant. However, this only suffices to prove a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant version of Theorem 1 when the period invariants are replaced by their *squares*. The presence of uncontrollable square roots in (0.4) seems to be an inevitable consequence of the appeal to triple product L-functions.

At various points in Part I, reference is made to the future contents of Part II. However, in the interim I found a proof of Shimura's conjecture on the factorization of the quaternionic invariants, using methods largely unrelated to those of Part I [H5]. Most of the material on quaternionic modular forms, originally intended for Part II, was incorporated into [H5], with the result that the anticipatory references in Part I are no longer correct. I apologize for any confusion this may cause.

Most of the results of this paper were announced in [H1]. Some refinements, mainly related to forms of low weight, were worked out when I was visiting the Steklov Mathematical Institute in Moscow during the academic year 1989–90, in the context of an exchange program sponsored by the National Academy of Sciences and the Academy of Sciences of the USSR. I thank the Steklov Institute for their hospitality.

Notation and conventions. We retain the notation of Part I, some of which is recalled in Section 1.1.

1. Review of notation

1.1. We recall the conventions of Part I, to which we refer for details. Let Y be the group $GL(2, \mathbb{R})$, $\mathfrak{y}$ its complexified Lie algebra, $K = O(2)$, Z the center of Y, which we identify with $\mathbb{R}^\times$, embedded diagonally in Y. Let $Y^+ \subset Y$ denote the subgroup of elements of Y with positive determinant, $K^+ = K \cap Y^+$. For $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$, we denote by $r(\vartheta)$ the matrix

$$\begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \in K^+;$$

let X_+ (resp. X_-) be the matrix

$$\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \left(\text{resp. } \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}\right)$$

in $\mathfrak{y}$. A holomorphic (resp. anti-holomorphic) vector in a $(\mathfrak{y}, K^+)$-module V is a vector annihilated by $X-$ (resp. X_+); an irreducible $(\mathfrak{y}, K^+)$-module is in the discrete series (or limit of discrete series) if it is generated by a holomorphic or antiholomorphic vector. For k, $r \in \mathbb{Z}$, $k > 0$, let $\pi(k, r)^+$ denote the discrete series (or limit of discrete series) $(\mathfrak{y}, K^+)$-module containing a holomorphic vector v_k satisfying $v_k(zgr(\vartheta)) = z^r e^{ki\vartheta} \cdot v_k(g)$, $g \in Y$, $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$, $z \in Z$. Let $\pi(k, r)$ denote the $(\mathfrak{y}, K)$-module induced from $\pi(k, r)^+$. Then

$$\pi(k, r)|_{(\mathfrak{y}, K^+)} \cong \pi(k, r)^+ \oplus \pi(k, r)^-$$

where $\pi(k, r)^-$ is generated by an antiholomorphic vector v_{-k} satisfying $v_{-k}(\mathrm{zgr}(\vartheta)) = z^r e^{-ki\vartheta} \cdot v_{-k}(g)$, g, ϑ, z as above.

Define E, $\Sigma = \Sigma_E$, and G as in the introduction. We identify $G(\mathbb{R}) \cong \mathrm{GL}(2, \mathbb{R})^d$ via Σ, $G(\mathbb{R})^+$ its identity component, and let $K_\infty = O(2)^d \subset G(\mathbb{R})$, $K_\infty^+ = G(\mathbb{R})^+ \cap K_\infty$. Thus $Z_G(\mathbb{R}) \cdot K_\infty^+$ is the stabilizer in $G(\mathbb{R})^+$ of the point $(i, \ldots, i)$ under the usual action of $G(\mathbb{R})^+$ on the d-fold product $\mathfrak{H}^d$ of upper half-planes. If $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$, $j = 1, 2, \ldots, d$, we let $r_j(\vartheta) \in K_\infty^+$ be the element $r(\vartheta)$ in the jth factor of $K_\infty^+ \cong \mathrm{SO}(2)^d$. Let

$$\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R})),\ \mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(Z_G)_{\mathbb{C}} \oplus \mathfrak{k}_{\infty,\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-,\ \mathfrak{p}^+ = \prod_{j=1}^{d} \mathfrak{p}_j^+,\ \mathfrak{p}^- = \prod_{j=1}^{d} \mathfrak{p}_j^-,$$

as in Part I, where $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) maps naturally to the holomorphic (resp. antiholomorphic) tangent space to $\mathfrak{H}^d$ at $(i, \ldots, i)$. Each $\mathfrak{p}_j^+$ (resp. $\mathfrak{p}_j^-$) is generated by its respective copy of X_+ (resp. X_-).

We have $G(\mathbf{A}) \cong \Pi' G_v$ (restricted direct product), where v runs through the places of E and $G_v \cong \mathrm{GL}(2, E_v)$. If v is non-archimedean, let $\mathcal{O}_v$ be the maximal order in F_v, and let $K_v = \mathrm{GL}(2, \mathcal{O}_v) \subset G_v$; let $K_f = \Pi_v K_v$, as v runs over non-archimedean places. If v is an archimedean place, we let $\mathfrak{g}_v = \mathrm{Lie}(G_v)$, $K_v = O(2) \subset G_v$, $K_v^+ = \mathrm{SO}(2)$. The Haar measure $dg = \Pi\, dg_v$ is normalized as in Part I, 1.6.

Let $\underline{k} = (k_1, \ldots, k_d)$ be a d-tuple of positive integers, $|\underline{k}| = \Sigma_{j=1}^d k_j$. Let $r \in \mathbb{Z}$ be an integer such that

$$k_i \equiv r \pmod 2 \quad (i = 1, \ldots, d). \tag{1.1.1}$$

A (motivic) *Hilbert modular form of weight* $(\underline{k}, r)$ *for* E is an automorphic form f on $G(\mathbb{Q})\backslash G(\mathbf{A})$ such that

$$f(z_\infty g) = N_{E/\mathbb{Q}}(z_\infty)^r f(g),\ g \in G(\mathbf{A}),\ z_\infty \in Z_G(\mathbb{R}); \tag{1.1.2}$$

$$f(g \cdot r_j(\vartheta)) = e^{ik_j\vartheta} \cdot f(g),\ g \in G(\mathbf{A}); \tag{1.1.3}$$

$$R(\mathfrak{p}^-) f = 0. \tag{1.1.4}$$

Here $N_{E/\mathbb{Q}}$: $R_{E/\mathbb{Q}}\mathbb{G}_{m,E} \to \mathbb{G}_{m,\mathbb{Q}}$ is the norm map, viewed as a homomorphism of algebraic tori over $\mathbb{Q}$, and $R(\cdot)$ is the right regular action on functions of the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Denote by $\mathscr{A}(\underline{k}, r) = \mathscr{A}(\underline{k}, r, E)$ the space of Hilbert modular forms of weight $(\underline{k}, r)$ for E; let $\mathscr{A}_0(\underline{k}, r)$ be the space of cusp forms.

Let $\mathscr{M} = \mathscr{M}_E$ be the Hilbert modular Shimura variety associated to G. Over the complex numbers, there is an isomorphism

$$\mathscr{M}(\mathbb{C}) = \varprojlim_{U} G(\mathbb{Q})\backslash G(\mathbf{A})/K_\infty^+ \times U,$$

where U runs through the set of open compact subgroups of $G(\mathbf{A}^f)$. Then if $(\underline{k}, r)$ satisfies (1.1.1), there is a $G(\mathbf{A}^f)$-homogeneous line bundle $\mathscr{E}_{(\underline{k},r)}$ over $\mathscr{M}(\mathbb{C})$ which admits a canonical isomorphism

$$\underline{\mathrm{Lift}} = \underline{\mathrm{Lift}}_{(\underline{k},r)} : \Gamma(\mathscr{M}(\mathbb{C}), \mathscr{E}_{(\underline{k},r)}) \xrightarrow{\sim} \mathscr{A}(\underline{k}, r), \tag{1.1.5}$$

with the addition of the usual condition at the cusps when $E = \mathbb{Q}$. The definition is given in Part I, Section 1.2, where we also define $\mathscr{E}_{(\underline{k},r)}$ for negative k_j; there are natural isomorphisms

$$\mathscr{E}_{(\underline{k},r)} \otimes \mathscr{E}_{(\underline{k}',r')} \xrightarrow{\sim} \mathscr{E}_{(\underline{k}+\underline{k}',r+r')} \tag{1.1.6}$$

where the $(d + 1)$-tuples are added as vectors. When it is necessary to specify the field E, we write $\mathscr{E}^E_{(\underline{k},r)} = \mathscr{E}_{(\underline{k},r)}$.

The Shimura variety $\mathscr{M}_E(\mathbb{C})$ has a canonical model over $\mathbb{Q}$, which we denote $\mathscr{M}$ or $\mathscr{M}_E$. If $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, let $\underline{k}^\sigma = (k^\sigma_1, \ldots, k^\sigma_d)$, with $k^\sigma_j =$ the $\sigma^{-1} \circ \sigma_j$ index in $\underline{k}$, let $\Gamma(\underline{k}) = \{\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \mid \underline{k}^\sigma = \underline{k}\}$, and let $E(\underline{k}) \subset \bar{\mathbb{Q}}$ be the fixed field of $\Gamma(\underline{k})$ (N.B.: the definition of $\underline{k}^\sigma$ in Part I contains a misprint). Then the $G(\mathbf{A}^f)$-equivariant line bundle $\mathscr{E}_{(\underline{k},r)}$ is rational over $E(\underline{k})$; moreover, $\sigma(\mathscr{E}_{(\underline{k},r)}) = \mathscr{E}_{(\underline{k}^\sigma,r)}$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

The space $\Gamma(\mathscr{M}, \mathscr{E}_{(\underline{k},r)})(E(\underline{k}))$ of $E(\underline{k})$-rational sections of $\mathscr{E}_{(\underline{k},r)}$ is determined, via (1.1.5), by the Fourier coefficients of their lifts to $\mathscr{A}(\underline{k}, r)$; see Part I, Prop. 1.3.3, for details.

For any subset $I \subset \Sigma$, let $\underline{k}(I)$ be the d-tuple $(k(I)_1, \ldots, k(I)_d)$, where $k(I)_j = k_j$ if $j \in \Sigma - I$, $k(I)_j = 2 - k_j$ if $j \in I$.

1.2. Let (π, H_π) be an irreducible cuspidal automorphic representation of $G(\mathbf{A})$ which is generated by a Hilbert modular form F of weight $(\underline{k}, r)$. We say that such π is *of type* $(\underline{k}, r)$. Then π factors as a (restricted) tensor product $\otimes_v \pi_v$ of representations of G_v (or $(\mathrm{Lie}(G_v), K_v)$-modules if v is archimedean). If v corresponds to the real embedding σ_j then $\pi_v \cong \pi(k_j, r)$. Denote by $(\pi_f, H_{\pi,f})$ (resp. $(\pi_\infty, H_{\pi,\infty})$) the restriction of (π, H_π) to $G(\mathbf{A}^f)$ (resp. to $(\mathfrak{g}_\mathbb{C}, K_\infty)$). Then π_f can be realized over a finite extension $\mathbb{Q}(\pi)$ of $\mathbb{Q}$, which is either totally real or a CM field, and which contains $E(\underline{k})$. Let $H^{\mathrm{hol}}_\pi \subset H_\pi$ denote the subspace of holomorphic vectors; then the $G(\mathbf{A}^f)$-action on H^{hol}_π is isomorphic to π_f. Moreover, $H^{\mathrm{hol}}_\pi \subset \mathscr{A}_0(\underline{k}, r)$, hence is isomorphic to a $G(\mathbf{A}^f)$-submodule $\Gamma(\mathscr{M}, \mathscr{E}_{(\underline{k},r)})^\pi \subset \Gamma(\mathscr{M}, \mathscr{E}_{\underline{k},r)})$. Now $\Gamma(\mathscr{M}, \mathscr{E}_{(\underline{k},r)})$ has a natural $E(\underline{k})$-rational structure, defined by the sections rational over $E(\underline{k})$. Thus $\Gamma(\mathscr{M}, \mathscr{E}_{(\underline{k},r)})^\pi$ has a natural $\mathbb{Q}(\pi) \cdot E(\underline{k})$-rational structure. We let $\Gamma(\mathscr{M}, \mathscr{E}_{(\underline{k},r)})^\pi(\mathbb{Q}(\pi))$ denote the $\mathbb{Q}(\pi)$-rational elements, and let $H^{\mathrm{hol}}_\pi(\mathbb{Q}(\pi))$ denote the corresponding $\mathbb{Q}(\pi)$-subspace of H^{hol}_π.

For any $I \subset \Sigma$, let

$$H_\pi^I = \{\varphi \in H_\pi \mid R(\mathfrak{p}_j^+)\varphi = 0 \text{ if } \sigma_j \in I,\ R(\mathfrak{p}_j^-)\varphi = 0 \text{ if } \sigma_j \in \Sigma - I\} \tag{1.2.1}$$

Thus $H_\pi^{\mathrm{hol}} = H_\pi^{\varnothing}$. Then for all I, the $G(\mathbf{A}^f)$-action on H_π^I is isomorphic to π_f. Let $\pi(\underline{k}, r)_j^I = \pi(k_j, r)^-$ if $\sigma_j \in I$, $\pi(\underline{k}, r)_j^I = \pi(k_j, r)^+$ if $\sigma_j \in \Sigma - I$ (there is a misprint in Part I). Then we may decompose $H_\pi = \oplus_{I \subset \Sigma} H_\pi(I) \otimes H_{\pi,f}$, where

$$H_\pi(I) \cong \bigotimes_{j=1}^{d} \pi(\underline{k}, r)_j^I, \tag{1.2.2}$$

H_π^I is the lowest K_∞^+-type subspace of $H_\pi(I) \otimes H_{\pi,f}$.

There is a natural embedding

$$H_\pi^I \hookrightarrow \bar{H}^{|I|}(\mathscr{E}_{(\underline{k}(I),r)}) \tag{1.2.3}$$

of $G(\mathbf{A}^f)$-modules, where $\bar{H}^{\cdot}(\mathscr{E}_{(\underline{k}(I),r)})$ is the "interior cohomology" studied in Part I, 1.4. Recall that $\bar{H}^{\cdot}(\mathscr{E}_{(\underline{k}(I),r)})$ has a natural $G(\mathbf{A}^f)$-equivariant $E(\underline{k}(I))$-rational structure, and the image $\bar{H}^{|I|}(\mathscr{E}_{(\underline{k}(I),r)})^\pi$ of H_π^I via (1.2.2) is a $\mathbb{Q}(\pi, I) := E(\underline{k}(I)) \cdot \mathbb{Q}(\pi)$-rational subspace of $\bar{H}^{|I|}(\mathscr{E}_{(\underline{k}(I),r)})$ (ibid., Prop. 1.4.3). The corresponding subspace of $\mathbb{Q}(\pi, I)$-rational vectors in H_π^I is denoted $H_\pi^I(\mathbb{Q}(\pi, I))$. On the other hand, the Whittaker model determines a second $\mathbb{Q}(\pi, I)$-rational subspace ${}^L H_\pi^I(\mathbb{Q}(\pi, I)) \subset H_\pi^I$, which is also $G(\mathbf{A}^f)$-equivariant. Indeed, there is a map

$$H_\pi^{\mathrm{hol}}(\mathbb{Q}(\pi)) \xrightarrow{\sim} {}^L H_\pi^I(\mathbb{Q}(\pi, I)) \text{ denoted } F \mapsto F^I. \tag{1.2.4}$$

These two rational structures are related by the formula

$$v^I(\pi) \cdot H_\pi^I(\mathbb{Q}(\pi, I)) = {}^L H_\pi^I(\mathbb{Q}(\pi, I)) \tag{1.2.5}$$

where $v^I(\pi) \in \mathbb{C}^\times$ is well-defined up to multiplication by $\mathbb{Q}(\pi, I)^\times$. (For all this, cf. Part I, Lemma 1.4.5).

1.3. REMARK. Everything we have defined up to this point makes sense when E is replaced by a product $\Pi_{j=1}^a E_i$ of totally real number fields. The only difference is that r is replaced by an a-tuple $\underline{r} = (r_1, \ldots, r_a)$, and that $\underline{k}$ can be decomposed as $(k_{j_i} \mid i = 1, \ldots, a, \sigma_{j_i} \in \Sigma_{E_i})$, such that

$$k_{j_i} \equiv r_i (\mathrm{mod}\, 2) \text{ for all } i. \tag{1.3.1}$$

2. Existence of certain cusp forms

2.1. Now let $E' = \Pi_{i=1}^{a} E_i$ be a product of totally real extensions of E, and let $\Delta: E \to E'$ denote the diagonal map. Let $G' = R_{E'/\mathbb{Q}}\mathrm{GL}(2)_{E'}$, and let $j: G \to G'$ be the natural embedding; this defines a morphism of Shimura varieties

$$\mathcal{M}_E \to \mathcal{M}_{E'}. \tag{2.1.1}$$

Let $d' = \dim_{\mathbb{Q}} E'$, $\delta = d'/d = \dim_E E'$. Let $\Sigma' = \{\sigma'_1, \ldots, \sigma'_{d'}\}$ denote the set of homomorphisms from E' to $\bar{\mathbb{Q}}$; then restriction defines a map of sets $\mu: \Sigma' \to \Sigma$. For $\sigma \in \Sigma$, let $\Sigma'(\sigma) = \mu^{-1}(\sigma)$; then each $\Sigma'(\sigma)$ has cardinality δ. Let $\mu^*: \Sigma \to \Sigma'$ be any section of μ, i.e. a choice, for each σ, of an element $\mu^*(\sigma) \in \Sigma'(\sigma)$. Let $J_1 \subset \Sigma$, $J_2 = \Sigma - J_1$; let $I_\alpha = \mu^*(J_\alpha)$, $\alpha = 1, 2$. For each $\sigma \in J_1$, let $\Sigma'(\sigma/I_1) = \Sigma'(\sigma) - \mu^*(\sigma)$. Let $(\underline{k}, r)$ be a (d', a)-tuple as in 1.3, satisfying (1.3.1). We consider automorphic forms of type $(\underline{k}, r)$ on G', antiholomorphic at places in I_1 and holomorphic at other real places, and their restrictions to G. The element of $\underline{k}$ corresponding to $\sigma' \in \Sigma'$ is denoted $k(\sigma')$.

2.2. LEMMA. *Let $\pi' \subset \mathcal{A}_0(G')$ be a cuspidal automorphic representation of G' corresponding to a holomorphic automorphic form of type $(\underline{k}, r)$. Define $\lambda(\sigma) \in \mathbb{Z}$ by the following formulas*:

$$\lambda(\sigma) := k(\mu^*(\sigma)) - \sum_{\sigma' \in \Sigma'(\sigma/I_1)} k(\sigma'), \quad \textit{for } \sigma \in J_1;$$

$$\lambda(\sigma) := \sum_{\sigma' \in \Sigma'(\sigma)} k(\sigma'), \quad \textit{for } \sigma \in J_2. \tag{2.2.1}$$

Assume $\lambda(\sigma) \geqslant 3$ for all $\sigma \in \Sigma$. Let $r^\mu = -\Sigma_{i=1}^{a} r_i$, and let $\underline{\lambda}$ be the d-tuple with $\lambda_j = \lambda(\sigma_j)$. Then there exist $f \in H_{\pi'}^{I_1}$ and a holomorphic cusp form F on G of type $(\underline{\lambda}, r^\mu)$ such that

$$\int_{Z_G(\mathbf{A}) \cdot G(\mathbb{Q}) \backslash G(\mathbf{A})} f(g) F^{I_2}(g) dg \neq 0, \quad F \mapsto F^{I_2} \text{ as in (1.2.4).} \tag{2.2.2}$$

REMARK. Here and in what follows, it is implicit in the notation that the central character of F is the inverse of the restriction to $Z_G(\mathbf{A})$ of the central character of π', so that the integral (2.2.2) is well-defined.

Proof. We want to apply Theorem 7.4 of [H2]. Let $\mathfrak{g}' = \mathrm{Lie}(G'(\mathbb{R}))_{\mathbb{C}}$, and define $K'^{+}_{\infty} \subset G'(\mathbb{R})^+$ as in Section 1. Define $H_{\pi'}(I_1)$ as in (1.2.2); it is the archimedean component of the $(\mathfrak{g}', K'^{+}_{\infty})$-module generated by f; let $\mathcal{H}_{\pi'}(I_1)$ denote the corresponding Hilbert space representation of $G'(\mathbb{R})^+$.

The function f belongs to the lowest K'^{+}_{∞}-type subspace $W_{\pi'}(I_1) \subset \mathscr{H}_{\pi'}(I_1)$. Let $R_G(\mathscr{H}_{\pi'}(I_1))$ denote the restriction of $\mathscr{H}_{\pi'}(I_1)$ to $G(\mathbb{R})^+$. Let

$$H = H(\underline{\lambda}, r^\mu) \cong \bigotimes_{j=1}^{d} \pi(\lambda_J, r^\mu)^+, \quad \text{as } (\mathfrak{g}, K_\infty^+)\text{-module}$$

and let $\mathscr{H}$ be the corresponding Hilbert space representation of $G(\mathbb{R})^+$. Our hypothesis (2.2.1) is that $\mathscr{H}$ belongs to the integrable discrete series of $G(\mathbb{R})^+$ (more precisely, its restriction to $G(\mathbb{R})^{+,\mathrm{der}}$ is integrable and square integrable). Then it has to be verified (cf. 7.2–7.4 of [H2]) that

$R_G(\mathscr{H}_{\pi'}(I_1))$ contains $\bar{\mathscr{H}} = \mathscr{H}^*$, the contragredient of $\mathscr{H}$, as a discrete direct summand; (2.2.3)

The orthogonal projection of $R_G(\mathscr{H}_{\pi'}(I_1))$ onto $\bar{\mathscr{H}}$ is injective on $W_{\pi'}(I_1)$. (2.2.4)

For any pair (k, r) of integers of the same parity, $k > 0$, let $\mathscr{H}(k, r)^\pm$ be the Hilbert space representation of $Y^+ = \mathrm{GL}(2, \mathbb{R})^+$ associated to $\pi(k, r)^\pm$. Now (2.2.3–4) can be checked separately at each place $\sigma \in \Sigma$. Thus let $(k_1, \ldots, k_\delta)$ be a δ-tuple of positive integers, $(r_1, \ldots, r_\delta)$ a δ-tuple of integers; let

$$\lambda^1 = k_1 - \sum_{i=2}^{\delta} k_i, \lambda^2 = -\sum_{i=1}^{\delta} k_i, r^\mu = \sum_{i=1}^{\delta} r_i.$$

Let

$$R^1_{Y^+} = \mathscr{H}(k_1, r_1)^- \otimes \bigotimes_{i=2}^{\delta} \mathscr{H}(k_i, r_i)^+ \,|_{Y^+} \text{ (completed tensor product)},$$

as a Hilbert space representation of Y^+; let $W \subset R^1_{Y^+}$ be the lowest $\mathrm{SO}(2)^\delta$-type subspace. Similarly, let $R^2_{Y^+} = \otimes_{i=1}^{\delta} \mathscr{H}(k_i, r_i)^+ \,|_{Y^+}$. It suffices to show that

$\mathscr{H}(\lambda^\alpha, r^\mu)^-$ is a discrete direct summand of $R^\alpha_{Y^+}$, $\alpha = 1, 2$. (2.2.5)

The orthogonal projection of $R^\alpha_{Y^+}$ onto $\mathscr{H}(\lambda^\alpha, r^\mu)^-$ is injective on W. (2.2.6)

Evidently, the assertions with regard to $R^\alpha_{Y^+}$ are the local conditions corresponding to places in J_α, $\alpha = 1, 2$.

Now it is well-known (cf. [M]) that $\otimes_{i=2}^{\delta} \mathscr{H}(k_i, r_i)^+ \,|_{Y^+}$, where Y^+ is embedded diagonally in $(Y^+)^{\delta-1}$, is a direct sum of holomorphic representations of Y^+ (even of Harish-Chandra modules). Moreover, the decomposition of the tensor product is given in [R] for $\delta - 1 = 2$, and it follows easily by induction on $\delta - 1$ that the module $\mathscr{H}(\kappa, \rho)^+$ occurs with multiplicity one, where $\kappa = \Sigma_{i=2}^{\delta} k_i$, $\rho = \Sigma_{i=2}^{\delta} r_i$. Furthermore, the orthogonal projection of the holomorphic (minimal $SO(2)^{\delta-1}$-type) subspace $W_h \subset \otimes_{i=2}^{\delta} \mathscr{H}(k_i, r_i)^+$ onto $\mathscr{H}(\kappa, \rho)^+$ is injective, and maps to the lowest SO(2)-type subspace of $\mathscr{H}(\kappa, \rho)^+$. Indeed, under the diagonal action of $SO(2) \subset SO(2)^{\delta-1}$, W_h is of SO(2)-type (κ, ρ), and this SO(2)-type occurs with multiplicity one in both $\otimes_{i=2}^{\delta} \mathscr{H}(k_i, r_i)^+$ and $\mathscr{H}(\kappa, \rho)^+$.

We are thus reduced to the case $\delta = 2$; we have to show that $\mathscr{H}(\lambda^1, r^\mu)$ (resp. $\mathscr{H}(\lambda^2, r^\mu)$) is a discrete direct summand of $\mathscr{H}(k_1, r_1)^- \otimes \mathscr{H}(\kappa, \rho)^+$ (resp. $\mathscr{H}(k_1, r_1)^+ \otimes \mathscr{H}(\kappa, \rho)^+$), and that the analogue of (2.2.6) holds. But the assertion regarding $\mathscr{H}(\lambda^1, r^\mu)$ has already been treated in [H2, §8.5], and the assertion regarding $\mathscr{H}(\lambda^2, r^\mu)$ is just a special case of the preceding paragraph.

2.3. EXAMPLE. In this example E is arbitrary, $E' = E \times E$; we identify $\Sigma' = \Sigma^{(1)} \times \Sigma^{(2)}$, where $\Sigma^{(\alpha)} = \Sigma$ is the set of embeddings of the αth copy of E in $E \times E$. $\alpha = 1, 2$. We assume we are given a partition $\Sigma = \mathscr{I}_1 \amalg \mathscr{I}_2 \amalg \mathscr{I}_3$ as a disjoint union of places, and define $\mu^* : \Sigma \to \Sigma'$ so that $\mu^*(\mathscr{I}_\alpha) \subset \Sigma^{(\alpha)}$, $\alpha = 1, 2$; $\mu^*(\mathscr{I}_3)$ is irrelevant. In the lemma, we let $J_1 = \mathscr{I}_1 \amalg \mathscr{I}_2$, $J_2 = \mathscr{I}_3$. Let π_1 and π_2 be irreducible automorphic representations of G associated to holomorphic cusp forms of weights $(\underline{k}, r)$ and $(\underline{l}, r')$, respectively. Let $r'' = -r - r'$, and define $\underline{\lambda}$ by the formula

$$\lambda_j = k_j - l_j \text{ (resp. } l_j - k_j\text{, resp. } k_j + l_j) \quad \text{if } \sigma_j \in \mathscr{I}_1 \text{(resp. } \mathscr{I}_2\text{, resp. } \mathscr{I}_3).$$

For π' in the lemma we take the irreducible automorphic representation $\pi_1 \otimes \pi_2$ of G'. Then the lemma asserts the following: there exist elements $f_\alpha \in H_{\pi_\alpha}^{I_\alpha}$, $\alpha = 1, 2$, and a holomorphic cusp form F on G of type $(\underline{\lambda}, r'')$, such that

$$\int_{Z_G(\mathbf{A}) \cdot G(\mathbb{Q}) \backslash G(\mathbf{A})} f_1(g) f_2(g) F^{\mathscr{I}_3}(g) dg \neq 0, \tag{2.3.1}$$

provided $\lambda_j \geqslant 3$ for all j.

We may assume F belongs to an irreducible automorphic representation. Now (2.3.1) is just the special case of (4.2.2) of Part I when Δ^d is the trivial differential operator (the I_α of Part I are replaced by $\mathscr{I}_\alpha$ here). Thus

2.3.2. COROLLARY. *There always exists a cusp form F'' satisfying the hypothesis of Theorem* 4.7 *of Part I, provided $\lambda_j \geqslant 3$ for all j. In particular, the period relations asserted in that theorem are valid for such λ.*

We recall these period relations in Section 5, below. An additional application of the lemma to the central critical value of triple product L-functions is given in Section 6.

2.4. EXAMPLE. Here in Lemma 2.2 we take $E = \mathbb{Q}$, and for E' we take an arbitrary totally real field, which we denote E. We write $\Sigma_{\mathbb{Q}}$ instead of Σ, Σ_E instead of Σ'; we also change notation and let $d = [E:\mathbb{Q}]$. Thus $\Sigma_{\mathbb{Q}}$ is a singleton, and we let $J_1 = \Sigma_{\mathbb{Q}}$, $J_2 = \emptyset$; $I_1 = \sigma_1 \in \Sigma_E$ is an arbitrary embedding. Let π be an irreducible automorphic representation of $G = \mathrm{GL}(2)_E$, associated to a holomorphic cusp form of weight $(\underline{k}, r)$. Let

$$m = k_1 - \sum_{j=2}^{d} k_j, \; r^{\mu} = -dr.$$

In this case, Lemma 2.2 asserts that

2.4.1. COROLLARY. *With the above notation, suppose* $m \geqslant 3$. *Then there exists* $f \in I_{\pi}^{\{\sigma_1\}}$ *and a holomorphic cusp form* F *of weight* $(m, -dr)$ *for* $\mathrm{GL}(2)_{\mathbb{Q}}$, *such that*

$$I(f, F) := (2\pi i)^{-1} \cdot \int_{\mathbf{A}^{\times} \cdot \mathrm{GL}(2,\mathbb{Q}) \backslash \mathrm{GL}(2,\mathbf{A})} f(g) \cdot F(g) dg \neq 0. \tag{2.4.2}$$

3. Applications of a theorem of Shimura

3.1. The integral $I(f, F)$ defined by (2.4.2) can be interpreted as a pairing in coherent cohomology. Recall (1.2.3) that f defines an element of $\bar{H}^1(\mathcal{M}_E, \mathscr{E}^E_{(\underline{k}(\{\sigma_1\}),r)})$, which we continue to denote f. Similarly, (1.1.5) identifies F with an element of $\bar{H}^0(\mathcal{M}_{\mathbb{Q}}, \mathscr{E}^{\mathbb{Q}}_{(m,-dr)})$. Let j: $\mathcal{M}_{\mathbb{Q}} \to \mathcal{M}_E$ be the natural morphism (2.1.1) of Shimura varieties. Then it follows from the construction (Part I, (1.2.6)) of the automorphic line bundles $\mathscr{E}_{(\cdot,\cdot)}$ that

$$j^*(\mathscr{E}^E_{(\underline{k}(\{\sigma_1\}),r)}) \cong \mathscr{E}^{\mathbb{Q}}_{(2-m,dr)}.$$

Thus there is a natural restriction map:

$$j^*: \bar{H}^1(\mathcal{M}_E, \mathscr{E}^E_{(\underline{k}(\{\sigma_1\}),r)}) \to \bar{H}^1(\mathcal{M}_{\mathbb{Q}}, \mathscr{E}^{\mathbb{Q}}_{(2-m,dr)})$$

which, composed with the cup product pairing induced by the isomorphism

$$\mathscr{E}^{\mathbb{Q}}_{(2-m,dr)} \otimes \mathscr{E}^{\mathbb{Q}}_{(m,-dr)} \xrightarrow{\sim} \mathscr{E}^{\mathbb{Q}}_{(2,0)} \quad \text{(cf. (1.1.6))},$$

defines a pairing

$$\bar{H}^1(\mathcal{M}_E, \mathscr{E}^E_{(\underline{k}(\{\sigma_1\}),r)}) \otimes \bar{H}^0(\mathcal{M}_{\mathbb{Q}}, \mathscr{E}^{\mathbb{Q}}_{(m,-dr)}) \xrightarrow{\cup} \bar{H}^1(\mathcal{M}_{\mathbb{Q}}, \mathscr{E}^{\mathbb{Q}}_{(2,0)}). \tag{3.1.1}$$

All of these maps, including (3.1.1), are rational over the field of definition of $\mathscr{E}^E_{(\underline{k}(\{\sigma_1\}),r)}$, namely $E(\underline{k}(\{\sigma_1\}))$. Denote by $\mathscr{M}^*_{\mathbb{Q}}$ the compactification of $\mathscr{M}_{\mathbb{Q}}$ by adding cusps; it is a projective limit of smooth curves. As in Part I, 1.3.6, we can identify $\mathscr{E}^{\mathbb{Q}}_{(2,0)} \cong \Omega^1_{\mathscr{M}_{\mathbb{Q}}}$; we then obtain (cf. [H2, 2.2.6, 2.3]) an isomorphism

$$\bar{H}^1(\mathscr{M}_{\mathbb{Q}}, \mathscr{E}^{\mathbb{Q}}_{(2,0)}) \tilde{\to} H^1(\mathscr{M}^*_{\mathbb{Q}}, \Omega^1_{\mathscr{M}^*_{\mathbb{Q}}}). \tag{3.1.2}$$

Letting Tr: $H^1(\mathscr{M}^*_{\mathbb{Q}}, \Omega^1_{\mathscr{M}^*_{\mathbb{Q}}}) \to \mathbb{C}$ be the GL(2, $\mathbf{A}^f$)-equivariant trace map of Serre duality theory, we find (cf. [H2, 7.7.1; Part I, (222.4.3)]):

$$I(f, F) = \mathrm{Tr}(f \cup F); \; f \in I^{\{\sigma_1\}}_\pi, \; F \in \mathscr{A}_0(m, -dr, \mathbb{Q}). \tag{3.1.3}$$

Recall that Tr is rational over $\mathbb{Q}$. It follows in particular that the pairing $I(f, F)$ of (2.4.2) is rational over $\mathbb{Q}(\pi, \{\sigma_1\})$. From this it is immediate that,

In (2.4.2), we may assume $f \in H^{\{\sigma_1\}}_\pi(\mathbb{Q}(\pi, I))$, $F \in \bar{H}^0(\mathscr{M}_{\mathbb{Q}}, \mathscr{E}^{\mathbb{Q}}_{(m,-dr)})(\mathbb{Q})$, and in this case $I(f, F) \in \mathbb{Q}(\pi, I)^\times$. (3.1.4)

By (1.2.5), we may replace $H^{\{\sigma_1\}}_\pi(\mathbb{Q}(\pi, I))$ by ${}^L H^{\{\sigma_1\}}_\pi(\mathbb{Q}(\pi, I))$ in (3.1.4). Again, since $\mathrm{Tr}(\cdot \cup \cdot)$ is rational over $\mathbb{Q}$, it follows easily from 2.4.1 that

3.1.5. COROLLARY. *Under the hypotheses of* 2.4.1, *we can find*

$$f \in {}^L H^{\{\sigma_1\}}_\pi(\mathbb{Q}(\pi, I)) \quad \textit{and} \quad F \in \bar{H}^0(\mathscr{M}_{\mathbb{Q}}, \mathscr{E}^{\mathbb{Q}}_{(m,-dr)})(\mathbb{Q})$$

such that

$$I(f, F) = \alpha \cdot v^{\{\sigma_1\}}(\pi) \quad \textit{for some } \alpha \in \mathbb{Q}(\pi, I)^\times.$$

3.2. In [S2], Shimura has given an alternative characterization of the integrals $I(f, G)$, for $f \in {}^L H^{\{\sigma_1\}}_\pi(\mathbb{Q}(\pi, I))$, $F \in \bar{H}^0(\mathscr{M}_{\mathbb{Q}}, \mathscr{E}^{\mathbb{Q}}_{(m,-dr)})(\mathbb{Q})$, valid in most cases. In order to describe it, we assume $k_i \geqslant 2$ for all i. Suppose there exists a quaternion algebra $D = D(\{\sigma_1\})$ over E, unramified at σ_1 and ramified at all other real places, and an automorphic representation π^D of $D^\times$ which corresponds to π under the Jacquet-Langlands correspondence. This is true, for example, if d is odd, or if d is even and π_v belongs to the discrete series for some finite place v. We define the quadratic period $q^D(\pi)$ as in 1.4. The following theorem is a paraphrase of a special case of [S2, 3.7(iv)], with allowances for the different normalizations of arithmetic automorphic forms:

3.2.1. THEOREM (Shimura). *For any* $f \in {}^L H^{\{\sigma_1\}}_\pi(\bar{\mathbb{Q}})$, $F \in \bar{H}^0(\mathscr{M}_{\mathbb{Q}}, \mathscr{E}^{\mathbb{Q}}_{(m,-dr)})(\bar{\mathbb{Q}})$, *we have* $I(f, F) \sim_{\bar{\mathbb{Q}}} q^D(\pi)$.

Combining this result with Corollary 3.1.5, we find:

3.2.2. COROLLARY. *Suppose* $m = k_1 - \Sigma_{j=2}^{d} k_j \geqslant 3$, $k_j \geqslant 2$ *for all j, and there exist D and* π^D *as above. Then* $v^{\{\sigma_1\}}(\pi) \sim_{\bar{\mathbb{Q}}} q^D(\pi)$.

This is the special case of our main theorem 1(a) in which $|I| = 1$ and the weights k_j satisfy the indicated inequalities. In the next section we show how to remove the inequalities.

3.3. REMARK. The proof of Shimura's theorem, in this special case, is based on the theta correspondence between automorphic forms on $GL(2)_{\mathbb{Q}}$ and automorphic forms on the orthogonal similitude group $GO(R_{E/\mathbb{Q}}D)$. Here the reduced norm makes D into a quadratic space over E; the signature of $R_{E/\mathbb{Q}}D$ at the (unique) real place of $\mathbb{Q}$ is $(4d\text{-}2,2)$. In particular, the symmetric space atached to $GO(R_{E/\mathbb{Q}}D)(\mathbb{R})$, or rather its identity component $GO(R_{E/\mathbb{Q}}D)(\mathbb{R})^0$, is of hermitian type. There is thus a Shimura variety Sh_D attached to $GO(R_{E/\mathbb{Q}}D)$, of complex dimension $4d$-2, and holomorphic automorphic forms on $GL(2)_{\mathbb{Q}}$ can be lifted to holomorphic automorphic forms on Sh_D. The articles [S1, S2, S3, O] of Shimura and Oda investigate the arithmeticity of the lifting of automorphic forms in this situation. Shimura actually treats a much more general situation, in which the orthogonal group is anisotropic. At least when D is unramified at finite primes, a careful adaptation of Oda's arguments, combined with the methods of [HK], should easily permit replacement of Corollary 3.2.2 by a $Gal(\bar{\mathbb{Q}}/\mathbb{Q}^{ab})$-equivariant version. More generally, it should be a routine (but lengthy) exercise to extend the methods of [HK, §§13–15], which treat the theta lifting from GL(2) to $GO(D)$ when D is a quaternion algebra over $\mathbb{Q}$, to the case considered by Shimura; one would then be able to replace $\mathbb{Q}$ by $\mathbb{Q}(\pi, \{\sigma_1\})$ in the statement of Corollary 3.2.2, and the relation there should satisfy the natural transformation law under $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$.

4. Reducing to the case of binary theta functions

4.1. Let π and π' be two irreducible automorphic representations of $GL(2, E)$, attached to holomorphic Hilbert modular forms of weights $(\underline{k}, r)$ and $(\underline{l}, r')$, respectively. The Rankin-Selberg convolution $L(s, \pi \otimes \pi')$ is given the motivic normalization, as in Part I. We suppose that $\Sigma = I \amalg I'$ is a partition of the set of real places of E such that

$$k_j > l_j \quad \text{if } \sigma_j \in I,\ k_j < l_j \quad \text{if } \sigma_j \in I'. \tag{4.1.1}$$

Theorem 3.5.1 of Part I then asserts that, for certain integers m,

$$L(m, \pi \otimes \pi') \sim_{\mathbb{Q}(\pi,\pi')\cdot E(\underline{k}(I))} (2\pi i)^{d\lambda(m)} \cdot i^{\eta(I,\pi)+\eta(I',\pi')} \cdot G(\xi_0 \cdot \xi_0') \cdot v^I(\pi) \cdot v^{I'}(\pi'). \tag{4.1.2}$$

Here $\lambda(m) = 2m - 2 + r + r'$, $G(\xi_0 \cdot \xi_0')$ is a certain Gauss sum, and $\eta(I, \pi)$,

$\eta(I', \pi')$ are integers. Furthermore, we must have

$$m_{\min} := \tfrac{1}{2}(3 - r - r') \leqslant m \leqslant m_{\max} := \min_{1 \leqslant j \leqslant d}\{\tfrac{1}{2}((2 - r - r') + |k_j - l_j|)\} \tag{4.1.3}$$

Finally, (4.1.2) is naturally $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant (cf. Part I, (3.5.3)).

Let D and D' be the quaternion algebras over E. As in the introduction, we assume $\Sigma(D) = I$, $\Sigma(D') = I'$; we assume that the local constituent π_v of π (resp. π'_v of π') belongs to the discrete series of $\mathrm{GL}(2, E_v)$ at any place v of E at which D (resp. D') ramifies. Thus we have automorphic representations π^D and $\pi'^{,D'}$ of $D^\times$ and $D'^{,\times}$, respectively, such that $\pi_v^D \cong \pi_v$ (resp. $\pi_v'^{,D'} \cong \pi'_v$) for every place v at which $D_v^\times$ (resp. $D_v'^{,\times}$) $\cong \mathrm{GL}(2, E_v)$. Shimura's theorem, to which we alluded in the introduction, is the following:

4.1.4. THEOREM (Shimura, [S3, Theorem 5.3]): *Let m be an integer satisfying* (4.1.3). *Then*

$$L(m, \pi \otimes \pi') \sim_{\bar{\mathbb{Q}}} (2\pi i)^{d\lambda(m)} \cdot i^{\eta(I,\pi)+\eta(I',\pi')} \cdot G(\xi_0 \cdot \xi'_0) \cdot q^D(\pi) \cdot q^{D'}(\pi').$$

The algebraic factors have been added to Shimura's formula in anticipation of the proof of a $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant version, which should appear in forthcoming work with P. Garrett. The following corollary (cf. (0.1) of the introduction) should likewise be $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant.

4.1.5. COROLLARY. *Under the hypotheses of* 4.1.4, *suppose* $m_{\max} > m_{\min} + \frac{1}{2}$. *Then*

$$v^I(\pi) \cdot v^{I'}(\pi') \sim_{\bar{\mathbb{Q}}} q^D(\pi) \cdot q^{D'}(\pi').$$

Proof. It suffices to show that, when $m_{\max} > m_{\min} + \frac{1}{2}$, there is some integer m in the range (4.1.3) for which $L(m, \pi \otimes \pi') \neq 0$. Certainly under the hypotheses there is an integer m such that $m_{\max} \geqslant m \geqslant m_{\min} + \frac{1}{2}$. But $m_{\min}$ is the center of symmetry for the functional equation of $L(s, \pi \otimes \pi')$. By a theorem of Jacquet-Shalika and Shahidi,

$$L(s, \pi \otimes \pi') \neq 0 \quad \text{for } \mathrm{Re}(s) \geqslant m_{\min} + \tfrac{1}{2} \text{ [GS, p. 69]}.$$

4.2. In order to prove Theorem 1 for forms of low weight, we need a slight generalization of Theorem 4.1.4, adapted to the case in which π' is a space of binary theta functions. Let $\mathscr{K}$ be a quadratic CM extension of E,$(\bar{\omega}:(\mathscr{K}_{\mathbb{A}}^\times/\mathscr{K}^\times) \to \mathbb{C}^\times$ a motivic Hecke character (i.e., a Hecke character of type A^0, in Weil's terminology). Let $\pi(\omega, \mathscr{K})$ be the associated automorphic representation of $\mathrm{GL}(2, E)$, normalized so that $L(s, \pi(\omega, \mathscr{K})) = L_{\mathscr{K}}(s + \frac{1}{2}, \omega)$, as in Part I, 4.6.1; here the left-hand side is the standard (Hecke) L-function of $\pi(\omega, \mathcal{K})$ and the right-hand side is the L-function of the Hecke character ω. Suppose $\pi' = \pi(\omega, \mathcal{K})$ in the preceding paragraphs. Thus $(\underline{l}, r') = (\underline{k}(\omega) + 1, r(\omega))$ in the notation of Part I. We write $\underline{l}(\omega) = \underline{k}(\omega) + 1$. (Warning: in the notation of [H5], $(\underline{l}, r') = (\underline{k}(\omega), 1 - w(\omega))$; note that the definitions of $\underline{k}(\omega)$ in Part I and in [H5] do not coincide! This is why we write $\underline{l}(\omega)$ instead of $\underline{k}(\omega)$.)

Define $\rho \in \mathrm{Gal}(\mathcal{K}/E)$ and ω^ρ as in the introduction. Let $\Sigma_{\mathcal{K}}$ be the set of complex embeddings of $\mathcal{K}$, and let Ψ be a subset of $\Sigma_{\mathcal{K}}$ such that $\Psi \cap \Psi\rho = \emptyset$; we call such Ψ a *partial CM-type*. In [H5], we have defined period invariants $p_{\mathcal{K}}(\omega, \Psi)$ for any such Ψ. We recall (very briefly) one definition. Let $I \subset \Sigma$ be the set of restrictions to E of Ψ and let D be any quaternion algebra over E which splits over $\mathcal{K}$, with $\Sigma(D) = I$. Then the identity component $D^\times(\mathbb{R})^0$ of $D^\times(\mathbb{R})$ acts on $\mathfrak{H}^{|I|}$. Choose a $\mathbb{Q}$-rational embedding $j: \mathcal{K}^\times \to D^\times$ such that $\mathcal{K}^\times(\mathbb{R})$ fixes a point $p \in \mathfrak{H}^{|I|}$ of type Ψ—e.g., the σth coordinate of p, for $\sigma \in I$, is given by $\tau(\alpha)$, for some $\alpha \in \mathcal{K}^\times$, where $\tau \in \Psi$ restricts to σ on E. Let f be any holomorphic automorphic form on $D^\times$ and let $j^*(f)$ denote its pullback to a function on $\mathcal{K}_{\mathbb{A}}^\times/\mathcal{K}^\times$. We suppose f to be rational over $\bar{\mathbb{Q}}$. Write $j^*(f)$ as a Fourier series $\Sigma\, c(\eta, f)\eta$, where the sum runs over Hecke characters of $\mathcal{K}_{\mathbb{A}}^\times/\mathcal{K}^\times$, with coefficients $c(\eta) \in \mathbb{C}$. We can find f (of some weight) such that $c(\omega, f) \neq 0$; then $p_{\mathcal{K}}(\omega, \Psi)$ is an algebraic multiple of $c(\omega, f)$. This determines $p_{\mathcal{K}}(\omega, \Psi)$ up to factors in $\bar{\mathbb{Q}}^\times$. We can descend to the field $\mathbb{Q}(\omega)$ generated by the values of ω on finite adeles by being more careful.

This definition is strictly a posteriori, because the definition of $\bar{\mathbb{Q}}$-rational holomorphic automorphic forms on quaternion algebras is based on the construction of rational structures on spaces of automorphic forms on CM tori, which in turn is based on the theory of periods of abelian varieties with complex multiplication. This is all explained in [H5] and in the references cited there. Here we note merely that, up to factors in $\bar{\mathbb{Q}}^\times$, $p_{\mathcal{K}}(\omega, \Psi)$ depends only on Ψ and on the archimedean component ω_∞ of ω. Then, up to factors in $\bar{\mathbb{Q}}^\times$, $p_{\mathcal{K}}(\omega_\infty, \Psi)$ is *multiplicative* in ω_∞, in the obvious sense. It is also multiplicative in Ψ, in the sense that, if $\Psi = \Psi_1 \amalg \Psi_2$, then

$$p_{\mathcal{K}}(\omega_\infty, \Psi) \sim_{\bar{\mathbb{Q}}} p_{\mathcal{K}}(\omega_\infty, \Psi_1) \cdot p_{\mathcal{K}}(\omega_\infty, \Psi_2)$$

(The multiplicative relations can be descended to optimal number fields, as in [H5, §1]). These properties figure essentially in the proof of Lemma 5.3, below.

The invariants defined by Shimura in [S1, §1] depend on archimedean components of Hecke characters and on elements of the free group $I_{\mathcal{K}}$ on $\Sigma_{\mathcal{K}}$. In particular, if $|\Psi| = \Sigma_{\sigma \in \Psi}\, \sigma \in I_{\mathcal{K}}$, the $p_{\mathcal{K}}(\omega_\infty, |\Psi|)$ defined by Shimura (in a different notation) coincides with our $[p_{\mathcal{K}}(\omega_\infty, \Psi)/p_{\mathcal{K}}(\omega_\infty, \rho\Psi)]^{1/2}$.

Let $I(\Psi) \subset \Sigma$ be the set of restrictions of elements of Ψ to E. Let $\sigma_j \in \Sigma$ and let $\mathscr{K}_j = \mathscr{K} \otimes_{E,\sigma_j} \mathbb{R}$. If $\sigma_j \in I(\Psi)$, let τ_j be the element of Ψ which restricts to σ_j; then τ_j defines an isomorphism $\mathscr{K}_j \cong \mathbb{C}$. In terms of this identification, the local component ω_j of ω at σ_j can be written

$$\omega_j(z) = z^{a_j} \cdot \bar{z}^{b_j},$$

and then $l_j(\omega) = |a_j - b_j| + 1$. We say Ψ is *strictly compatible with* ω if $a_j < b_j$ for all $\sigma_j \in I(\Psi)$. It is obvious that, if Ψ and Ψ' are partial CM types, both strictly compatible with ω, such that $I(\Psi') \subset I(\Psi)$, then $\Psi' \subset \Psi$. In particular,

4.2.1 *If $I \subset \Sigma$ is a subset such that $l_j(\omega) \geqslant 2$ for all $\sigma_j \in I$, then there is a unique partial CM type $\Psi = \Psi(I, \omega)$, strictly compatible with ω, such that $I(\Psi) = I$. In that case, we can write $p_{\mathscr{K}}(\omega, I) = p_{\mathscr{K}}(\omega, \Psi)$ without fear of ambiguity.*

Let π be as in the first paragraph, $\pi' = \pi(\omega, \mathscr{K})$. Let $\xi_\omega = \omega|_{E_{\mathbb{A}}^\times}$, and let ξ_π denote the central character of π. We define $\tilde{\omega} = \omega/\omega^\rho$. Assume $\xi_\omega \cdot \xi_\pi = 1$; then $r + r(\omega) = 1$. Assume further that (4.1.1) holds. Then $m = 1$ is the smallest element of the set (4.1.3), and $\lambda(m) = 1$. Assume finally that there exists a quaternion algebra D over E, with $\Sigma(D) = I$, such that the automorphic representation π^D exists. Then Proposition 7.1.8 of [H5], slightly extending previous results of Shimura, asserts that

$$L(1, \pi \otimes \pi') \sim_{\bar{\mathbb{Q}}} (2\pi i)^d q^D(\pi) \cdot p_{\mathscr{K}}(\tilde{\omega}, I'). \tag{4.2.2}$$

Note that $l_j(\tilde{\omega}) = 2l_j(\omega) - 1 \geqslant 2k_j - 1$, for all $\sigma_j \in I'$, by (4.1.1). Now $k_j \geqslant 1$ for all j, so $p_{\mathscr{K}}(\tilde{\omega}, I')$ is well-defined. (In fact, the existence of π^D implies that $k_j \geqslant 2$ for $\sigma_j \in I'$.) We write $v^J(\omega, \mathscr{K})$ instead of $v^J(\pi(\omega, \mathscr{K}))$ for $J \subset \Sigma$. Then (4.1.2) and (4.2.2) together imply:

4.2.3. COROLLARY. *Let π and $\pi' = \pi(\omega, \mathscr{K})$ be as above; in particular, assume* (4.1.1) *holds and* $\xi_\omega \cdot \xi_\eta = 1$. *Finally, assume* $L(1, \pi \otimes \pi') \neq 0$. *Then*

$$v^I(\pi) \cdot (i^{\eta(I',\pi')} \cdot v^{I'}(\omega, \mathscr{K})) \sim_{\bar{\mathbb{Q}}} q^D(\pi) \cdot p_{\mathscr{K}}(\tilde{\omega}, I').$$

4.2.4. REMARK. The power of i on the left-hand side is explained in the introduction.

4.3. COROLLARY 4.2.3. allows us to reduce the proof of Theorem 1 to the special case of binary theta functions. More precisely:

4.3.1. LEMMA. *Let I be a subset of Σ, $I' = \Sigma - I$. Let D be a quaternion algebra over E with $\Sigma(D) = I$. Suppose there exists an integer $M > 0$ such that, for every*

CM quadratic extension $\mathscr{K}$ of E and every motivic Hecke character ω of $\mathscr{K}^{\times}$ such that $l_j(\omega) \geqslant M$ for all $j \in I'$, the relation

$$i^{\eta(I'\pi')} \cdot v^{I'}(\omega, \mathscr{K}) \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}, I') \tag{4.3.2}$$

is valid. Then for every π of type $(\underline{k}, r)$, with $k_j \geqslant 2$ for all j, such that π^D exists, we have

$$v^I(\pi) \sim_{\bar{\mathbb{Q}}} q^D(\pi). \tag{4.3.3}$$

Proof. By Corollary 4.2.3, it suffices to show the existence of some ω satisfying the hypotheses of the corollary, such that $l_j(\omega) \geqslant M$ for all $j \in I'$. As in [H5], the condition $\xi_\omega \cdot \xi_\pi = 1$ implies that $l_j(\omega) \equiv k_j + 1 \pmod 2$ for all j. It will be enough to find ω such that $\xi_\omega \cdot \xi_\pi = 1$, $L(1, \pi \otimes \pi(\omega, \mathscr{K})) \neq 0$,

$$l_j(\omega) = k_j - 1 \quad \text{for } j \in I,\ l_j(\omega) \geqslant \max\{M, k_j + 1) \quad \text{for } j \in I'.$$

But the existence of such ω follows from the main theorem of [H4].

4.4. Suppose D' is a quaternion algebra over E such that $\pi(\omega, \mathscr{K})^{D'}$ exists. We write $q^{D'}(\omega, \mathscr{K})$ instead of $q^{D'}(\pi(\omega, \mathscr{K}))$. It has been proved by Shimura [S3, cf. also H5, §7.2] that (with $\pi' = \pi(\omega, \mathscr{K})$)

$$p_{\mathscr{K}}(\tilde{\omega}, \Sigma(D')) \sim_{\bar{\mathbb{Q}}} i^{\eta(I', \pi')} \cdot q^{D'}(\omega, \mathscr{K}) \text{ if } l_j(\omega) \geqslant 2 \quad \text{for all } j. \tag{4.4.1}$$

Again, the hypothesis on $l_j(\omega)$ allows us to define $p_{\mathscr{K}}(\tilde{\omega}, \Sigma(D'))$ unambiguously. It then follows from Corollary 3.2.2 that

4.4.2. LEMMA. *Let $M = 3 + 2(d - 1)$. Let $I' \subset \Sigma$ be a subset consisting of one element, say $I' = \{\sigma_1\}$. For every CM quadratic extension $\mathscr{K}$ of E and every motivic Hecke character ω of $\mathscr{K}^{\times}$ such that* (i) $l_1(\omega) \geqslant M$; (ii) $l_j(\omega) \geqslant 2$ *for all* $j > 1$; *and* (iii) *there exists a quaternion algebra D′ over E, unramified at σ_1 and ramified at all other archimedean primes of E, such that $\pi(\omega, \mathscr{K})^{D'}$ exists, we have*

$$i^{\eta(I', \pi')} \cdot v^{I'}(\omega, \mathscr{K}) \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}, I') \sim_{\bar{\mathbb{Q}}} i^{\eta(I', \pi')} \cdot q^{D'}(\omega, \mathscr{K})\ (\pi' = \pi(\omega, \mathscr{K})) \tag{4.4.3}$$

In order to apply this to the situation of Lemma 4.3.1, we need to remove conditions (i)–(iii); then, of course, we will only have the first relation of (4.4.3). Thus, we assume only

$$l_1(\omega) \geqslant 2,$$

and let $\pi' = \pi(\omega, \mathscr{K})$. Fix a second motivic Hecke character η of $\mathscr{K}^{\times}$ satisfying (i), (ii), and (iii), and such that $k_1(\eta) \gg k_1(\omega)$. Let ω'' be an auxiliary motivic

Hecke character of $\mathscr{K}$ such that $\xi_{\pi'} \cdot \xi_{\omega''} = 1$, $L(1, \pi(\omega, \mathscr{K}) \otimes \pi(\omega'', \mathscr{K})) \neq 0$, and

$$l_1(\omega'') < l_1(\omega),\ l_j(\omega'') \gg \max\{l_j(\eta),\ l_j(\omega)\}. \tag{4.4.4}$$

The existence of such ω' is again guaranteed by [H4]. Let $I'' = \Sigma - I'$, $\pi'' = \pi(\omega'', \mathscr{K})$. Then the proof of [H5], Lemma 1.9.5, yields

$$[i^{\eta(I',\pi')} \cdot v^{I'}(\omega, \mathscr{K})] \cdot [i^{\eta(I'',\pi'')} \cdot v^{I''}(\omega'', \mathscr{K})] \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}, I') \cdot p_{\mathscr{K}}(\tilde{\omega}'', I''). \tag{4.4.5}$$

In loc. cit., we assumed $l_1(\omega'') = l_1(\omega) - 1$, $l_j(\omega'') = l_j(\omega) + 1$, for $j > 1$, but this hypothesis is only introduced to simplify keeping track of the powers of $2\pi i$, which eventually disappear; the crucial hypothesis is that $L(1, \pi(\omega, \mathscr{K}) \otimes \pi(\omega'', \mathscr{K})) \neq 0$.

Now we let $\pi = \pi(\eta, \mathscr{K})$, and apply (4.1.2) with π' replaced by π''. Note that the lower boundary $m_{\min}$ in (4.1.3) corresponds to the center of symmetry of the functional equation of $L(s, \pi \otimes \pi')$. Our hypotheses imply that $m_{\max} \gg m_{\min}$, so letting $m = [m_{\max}]$ (greatest integer), it follows that

$$0 \neq L(m, \pi \otimes \pi'') \sim_{\bar{\mathbb{Q}}} (2\pi i)^{d\lambda(m)} \cdot i^{\eta(I',\pi) + \eta(I'',\pi'')} \cdot v^{I'}(\eta, \mathscr{K}) \cdot v^{I''}(\omega'', \mathscr{K})). \tag{4.4.6}$$

But (cf. [H5, (1.8.4)]):

$$L(s, \pi \otimes \pi'') = L_{\mathscr{K}}(s, \eta \cdot \omega'') \cdot L_{\mathscr{K}}(s, \eta \cdot \omega''^{,\rho}), \tag{4.4.7}$$

where $L_{\mathscr{K}}(s, \chi)$ is Hecke's L-function attached to the Hecke character χ. As in the proof of [H5, Lemma 1.9.5], the right-hand side of (4.4.7) can be interpreted, via Blasius' theorem, in terms of the period invariants $p_{\mathscr{K}}(\cdot, \cdot)$. Arguing as in loc. cit., we easily obtain

$$L_{\mathscr{K}}(m, \eta \cdot \omega'') \cdot L_{\mathscr{K}}(m, \eta \cdot \omega''^{,\rho}) \sim_{\bar{\mathbb{Q}}} (2\pi i)^{d\lambda(m)} \cdot p_{\mathscr{K}}(\tilde{\eta}, I') \cdot p_{\mathscr{K}}(\tilde{\omega}'', I'')$$

which, combined with (4.4.6) and (4.4.7), yields

$$[i^{\eta(I',\pi)} \cdot v^{I'}(\eta, \mathscr{K})] \cdot [i^{\eta(I'',\pi'')} \cdot v^{I''}(\omega'', \mathscr{K})] \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\eta}, I') \cdot p_{\mathscr{K}}(\tilde{\omega}'', I''). \tag{4.4.8}$$

Dividing (4.4.5) by (4.4.8) we find that

$$[i^{\eta(I',\pi')} \cdot v^{I'}(\omega, \mathscr{K})]/[i^{\eta(I',\pi)} \cdot v^{I'}(\eta, \mathscr{K})] \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}, I')/p_{\mathscr{K}}(\tilde{\eta}, I'). \tag{4.4.9}$$

But η satisfies the hypotheses of Lemma 4.4.2, so we can cancel the denominators in (4.4.9). Thus:

4.4.10. COROLLARY. *Suppose* ω *is a motivic Hecke character of* $\mathscr{K}^\times$ *such that* $l_1(\omega) \geqslant 2$. *Then*

$$i^{\eta(I',\pi')} \cdot v^{I'}(\omega, \mathscr{K}) \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}, I').$$

Applying Lemma 4.2.1, we obtain:

4.5. COROLLARY. *Theorem* 1 *holds whenever* $|I| = 1$.

5. Proof of the theorem for binary theta functions

5.1. We now let $I \subset \Sigma$ be any subset with $|I| > 1$, and suppose Theorem 1 to be known for all proper subsets of I. Write I as a disjoint union of proper subsets $I = I_1 \amalg I_2$. We verify the hypothesis of Lemma 4.2.1, with the roles of I' and I exchanged. Thus let M be a large number, soon to be determined, and let ω_0 be any motivic Hecke character of $\mathscr{K}^\times$ which satisfies

$$l_j(\omega_0) \geqslant M \quad \text{for all } \sigma_j \in I. \tag{5.1.1}$$

We will show that

$$i^{\eta(I,\pi)} \cdot v^I(\omega_0, \mathscr{K}) \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}_0, I). \tag{5.1.2}$$

We first choose a partial CM type Ψ of $\mathscr{K}$, with $I(\Psi) = I$, strictly compatible with ω_0, and partition $\Psi = \Psi_1 \amalg \Psi_2$ compatibly with the partition of I. To begin with, we lose nothing by making the additional hypotheses:

$$l_j(\omega_0) \geqslant 3 \text{ for all } \sigma_j;\ \omega_{0,w} \neq \omega^c_{0,w} \text{ for some finite prime } w \text{ of } \mathscr{K} \text{ which is fixed by } \mathrm{Gal}(\mathscr{K}/E). \tag{5.1.3}$$

Indeed, these can be removed just as in the case $|I| = 1$, by the argument used to prove Corollary 4.4.10. The second hypothesis guarantees that $\pi(\omega_0, \mathscr{K})_v$ is supercuspidal (cf. [JL, Theorem 4.6]), where v is the restriction of w to E. Now we factor $\omega_0 = \omega_1 \cdot \omega_2$ in such a way that

$$\Psi \text{ is strictly compatible with } \omega_1 \text{ and } \omega_2. \tag{5.1.4}$$

This implies that

$$l_j(\omega_0) = l_j(\omega_1) + l_j(\omega_2) - 1 \quad \text{for } \sigma_j \in I,$$

whereas we always have

$$l_j(\omega_1)+l_j(\omega_2) \geqslant l_j(\omega_0)+1,\ l_j(\omega_1)+l_j(\omega_2) \equiv l_j(\omega_0)+1 \pmod 2$$

(cf. Part I, Theorem 4.6.1(d) for these and subsequent computations of weights of binary theta functions). It also implies that $l_j(\omega_\alpha) \geqslant 2$ for all σ_j, $\alpha = 1, 2$. We make the additional hypothesis that

$$l_j(\omega_1) > l_j(\omega_2)+2 \quad (\sigma_j \in I_1); \qquad l_j(\omega_2) > l_j(\omega_1)+4 \quad (\sigma_j \in I_2). \tag{5.1.5}$$

By the inequalities recalled above, this is possible provided we take $M \geqslant 11$. We are thus in the situation of Corollary 2.3.2, and we can apply Theorem 4.7 of Part I:

$$\begin{aligned} v^I(\omega_0, \mathscr{K}) \cdot v^I(\omega_1\omega_2^\varrho, \mathscr{K}) &\sim_{\bar{\mathbb{Q}}} v^{I_1}(\omega_1, \mathscr{K})^2 \cdot v^{I_2}(\omega_2, \mathscr{K})^2 \\ &\sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}_1, I_1)^2 \cdot p_{\mathscr{K}}(\tilde{\omega}_2, I_2)^2. \end{aligned} \tag{5.1.6}$$

by the induction hypothesis.

Let $\omega^* = \omega_1\omega_2^\varrho$; then

$$\omega^* \text{ is strictly compatible with } \Psi_1 \amalg \rho\Psi_2 \tag{5.1.7}$$

and we have

$$\begin{aligned} &l_j(\omega^*) > l_j(\omega_1) \geqslant 5 \quad (\sigma_j \in I_1); \\ &l_j(\omega^*) \geqslant l_j(\omega_2) - l_j(\omega_1) \geqslant 5 \quad (\sigma_j \in I_2). \end{aligned} \tag{5.1.8}$$

Finaly, if we initially assume that $\omega_{2,v} \neq \omega_{2,v}^\varrho$, we find that $\pi(\omega^*, \mathscr{K})_v \cong \pi(\omega, \mathscr{K})_v$ is supercuspidal. Thus

5.1.9. There is a quaternion algebra D over E, unramified at primes in I, ramified at all primes in $\Sigma - I$, and possibly also at v, such that the automorphic representations $\pi(\omega, \mathscr{K})^D$ and $\pi(\omega^*, \mathscr{K})^D$ of $D^\times$ exist.

5.2. We now choose an auxiliary motivic Hecke character η of $\mathscr{K}^\times$, with the property that

$$\begin{aligned} &\min\{l_j(\omega_0), l_j(\omega^*)\} > l_j(\eta) + 2 \quad (\sigma_j \in I) \\ &l_j(\eta) > \max\{l_j(\omega_0), l_j(\omega^*)\} + 2 \quad (\sigma_j \in \Sigma - I); \end{aligned} \tag{5.2.1}$$

this is possible by (5.1.8) and (5.1.1). We apply the discussion of 4.1 to the pair

$$\pi = \pi(\eta, \mathscr{K}),\ \pi' = \text{either } \pi(\omega_0, \mathscr{K}) \text{ or } \pi(\omega^*, \mathscr{K}).$$

In particular, we let $\underline{k} = \underline{l}(\eta)$ and define $\underline{l}$, r, and r' accordingly. The hypothesis (5.2.1) then guarantees that, in (4.1.3),

$$m_{\max} - m_{\min} \geqslant 1.$$

By choosing the local component of η appropriately at some finite place, we may also assume, as in 5.1.9, that the automorphic representation $\pi^{D'}$ of $D'^{,\times}$ exists, for some quaternion algebra D' over E which ramifies at I and is unramified at $\Sigma - I$. Applying Corollary 4.1.5, we conclude:

$$v^I(\omega_0, \mathscr{K}) \cdot v^{\Sigma-I}(\eta, \mathscr{K}) \sim_{\bar{\mathbb{Q}}} q^D(\omega_0, \mathscr{K}) \cdot q^{D'}(\eta, \mathscr{K}); \tag{5.2.2}$$

$$v^I(\omega^*, \mathscr{K}) \cdot v^{\Sigma-I}(\eta, \mathscr{K}) \sim_{\bar{\mathbb{Q}}} q^D(\omega^*, \mathscr{K}) \cdot q^{D'}(\eta, \mathscr{K}). \tag{5.2.3}$$

Multiplying the two sides of (5.2.3) and (5.2.2), and appealing to (4.4.1), we find:

$$v^I(\omega_0, \mathscr{K}) \cdot v^I(\omega^*, \mathscr{K}) \cdot v^{\Sigma-I}(\eta, \mathscr{K})^2 \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}_0, I) \cdot p_{\mathscr{K}}(\tilde{\omega}^*, I) \cdot p_{\mathscr{K}}(\tilde{\eta}, \Sigma - I)^2 \tag{5.2.4}$$

(one checks that the powers of i disappear). Now (5.1.6) yields

$$p_{\mathscr{K}}(\tilde{\omega}_1, I_1)^2 \cdot p_{\mathscr{K}}(\tilde{\omega}_2, I_2)^2, \cdot v^{\Sigma-I}(\eta, \mathscr{K})^2 \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}_0, I) \cdot p_{\mathscr{K}}(\tilde{\omega}^*, I) \cdot p_{\mathscr{K}}(\tilde{\eta}, \Sigma - I)^2. \tag{5.2.5}$$

5.3. LEMMA. *Under the above hypotheses,*

$$p_{\mathscr{K}}(\tilde{\omega}_1, I_1)^2 \cdot p_{\mathscr{K}}(\tilde{\omega}_2, I_2)^2 \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}_0, I) \cdot p_{\mathscr{K}}(\tilde{\omega}^*, I).$$

Proof. The proof is identical to the argument preceding Lemma 1.8.9 of [H5]. It follows from (5.1.4) and (5.1.7) that

$$p_{\mathscr{K}}(\tilde{\omega}_\alpha, I) = p_{\mathscr{K}}(\tilde{\omega}_\alpha, \Psi),\ \alpha = 0, 1, 2;\ p_{\mathscr{K}}(\tilde{\omega}^*, I) = p_{\mathscr{K}}(\tilde{\omega}^*, \Psi_1 \amalg \rho\Psi_2) \tag{5.3.2}$$

(cf. (4.1.4)). Then the right-hand side of (5.3.1) can be written

$$\begin{aligned} &p_{\mathscr{K}}(\tilde{\omega}_0, \Psi_1 \amalg \Psi_2) \cdot p_{\mathscr{K}}(\tilde{\omega}^*, \Psi_1 \amalg \Psi_2) \\ &\quad \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}_0, \Psi_1) \cdot p_{\mathscr{K}}(\tilde{\omega}_0, \Psi_2) \cdot p_{\mathscr{K}}(\tilde{\omega}^*, \Psi_1) \cdot p_{\mathscr{K}}(\tilde{\omega}^*, \rho\Psi_2) \end{aligned} \tag{5.3.3}$$

by [H5, Corollary 1.5]. Factoring $\tilde{\omega}_0 = \tilde{\omega}_1 \cdot \tilde{\omega}_2$; $\tilde{\omega}^* = \tilde{\omega}_1 \cdot \tilde{\omega}_2^{\varrho} = \tilde{\omega}_1 / \tilde{\omega}_2$, and applying [H5, Prop. 1.4] several times, we find that

$$p_{\mathscr{K}}(\tilde{\omega}_0, \Psi_1 \amalg \Psi_2) \cdot p_{\mathscr{K}}(\tilde{\omega}^*, \Psi_1 \amalg \rho\Psi_2) \sim_{\bar{\mathbb{Q}}} P_1 \cdot P_2, \tag{5.3.4}$$

where

$$P_1 = p_{\mathscr{K}}(\tilde{\omega}_1, \Psi_1)^2 \cdot p_{\mathscr{K}}(\tilde{\omega}_1, \Psi_2) \cdot p_{\mathscr{K}}(\tilde{\omega}_1, \rho\Psi_2).$$
$$P_2 = p_{\mathscr{K}}(\tilde{\omega}_2, \Psi_1) \cdot p_{\mathscr{K}}(\tilde{\omega}_2, \Psi_2) \cdot p_{\mathscr{K}}(\tilde{\omega}_2^{-1}, \Psi_1) p_{\mathscr{K}}(\tilde{\omega}_2^{-1}, \rho\Psi_2)$$

Now we apply [H5, Lemma 1.6] several times (in loc. cit. ρ is denoted ι); we find

$$\begin{aligned} P_1 &\sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}_1, \Psi_1)^2 \cdot p_{\mathscr{K}}(\tilde{\omega}_1, \Psi_2) \cdot p_{\mathscr{K}}(\tilde{\omega}_1^{\rho}, \Psi_2) \\ &= p_{\mathscr{K}}(\tilde{\omega}_1, \Psi_1)^2 \cdot p_{\mathscr{K}}(\tilde{\omega}_1, \Psi_2) \cdot p_{\mathscr{K}}(\tilde{\omega}_1^{-1}, \Psi_2); \\ P_2 &\sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}_2, \Psi_1) \cdot p_{\mathscr{K}}(\tilde{\omega}_2, \Psi_2) \cdot p_{\mathscr{K}}(\tilde{\omega}_2^{-1}, \Psi_1) p_{\mathscr{K}}((\tilde{\omega}_2^{-1})^{\rho}, \Psi_2) \\ &= p_{\mathscr{K}}(\tilde{\omega}_2, \Psi_2)^2 \cdot p_{\mathscr{K}}(\tilde{\omega}_2, \Psi_1) \cdot p_{\mathscr{K}}(\tilde{\omega}_2^{-1}, \Psi_1). \end{aligned} \tag{5.3.5}$$

Finally [loc. cit., Prop. 1.4] implies:

$$p_{\mathscr{K}}(\tilde{\omega}_1, \Psi_2) \cdot p_{\mathscr{K}}(\tilde{\omega}_1^{-1}, \Psi_2) \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\omega}_2, \Psi_1) \cdot p_{\mathscr{K}}(\tilde{\omega}_2^{-1}, \Psi_1) \sim_{\bar{\mathbb{Q}}} 1. \tag{5.3.6}$$

The lemma is a consequence of (5.3.2–6).

5.3.7. REMARK. As in [loc. cit.], this lemma can be proved in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-equivariant form.

5.4. Combining the preceding lemma with (5.2.5) and (4.4.1), we obtain

$$v^{\Sigma - I}(\eta, \mathscr{K})^2 \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\tilde{\eta}, \Sigma - I)^2 \sim_{\bar{\mathbb{Q}}} q^{D'}(\eta, \mathscr{K})^2. \tag{5.4.1}$$

Now we square both sides of (5.2.2) and cancel with (5.4.1). We obtain:

5.4.2. LEMMA. *Let ω_0 be any motivic Hecke character of $\mathscr{K}^\times$ satisfying* (5.1.1) *and* (5.1.3), *and let D be a quaternion algebra over E as in* 5.1.9. *Then*

$$[i^{\eta(I,\pi)} \cdot v^I(\omega_0, \mathscr{K})]^2 \sim_{\bar{\mathbb{Q}}} q^D(\omega_0, \mathscr{K})^2 \sim_{\bar{\mathbb{Q}}} p_{\mathscr{K}}(\omega_0, I)^2.$$

At this point, the power of i is completely irrelevant, since the method of proof necessarily produces relations between squares.

As remarked above, the hypothesis (5.1.3) can be removed by arguing as in 4.4. Thus the hypothesis of Lemma 4.3.1 is valid for I. This completes the induction step, and thus the proof of Theorem 1(a). The derivation of Theorem 1(b) from Lemma 5.4.2 is identical to the proof of Corollary 4.4.10.

6. Non-vanishing of certain L-functions

Let $G = \mathrm{GL}(2)_E$. We return to the situation of Example 2.3. The discussion in Section 4 of Part I shows that Corollary 2.3.2 is actually equivalent to the following assertion:

6.1. PROPOSITION. *Let π_1 and π_2 be irreducible automorphic representations of G associated to holomorphic cusp forms of weights $(\underline{k}, r)$ and $(\underline{l}, r')$, respectively. Define r'', $\underline{\lambda}$, and the partition $\Sigma = \mathscr{I}_1 \amalg \mathscr{I}_2 \amalg \mathscr{I}_3$ as in 2.3. Suppose $\lambda_j \geqslant 3$ for all j. Then there exists an irreducible automorphic representation π_3 of G, associated to holomorphic cusp forms of weight $(\underline{\lambda}, r'')$, such that the central characters ξ_i of π_i $(i = 1, 2, 3)$, satisfy $\xi_1 \cdot \xi_2 \cdot \xi_3 = 1$, and such that the triple product L-function $L(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ does not vanish at its central critical point $s = 2$.*

By generalizing the argument of Part I, Section 4 in various ways, we can modify the hypotheses:

6.2. By applying non-trivial differential operators in the variables z_j with $\sigma_j \in \mathscr{I}_3$, we can let

$$\lambda_j = k_j - l_j - 2a_j,\ \sigma_j \in \mathscr{I}_1;$$
$$\lambda_j = l_j - k_j - 2a_j,\ \sigma_j \in \mathscr{I}_2;$$
$$\lambda_j = k_j + l_j + 2a_j,\ \sigma_j \in \mathscr{I}_3, \tag{6.2.1}$$

for any collection of non-negative integers a_j such that $\lambda_j \geqslant 3$ for all j.

6.3. Let T be a finite set of non-archimedean places of E, of *even* cardinality. By replacing GL(2) by the totally indefinite quaternion algebra D_T over E ramified exactly at places in T, we can insist that the local root numbers satisfy

$$\varepsilon(2, \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v}) = -1 \quad \text{if and only if } v \in T. \tag{6.3.1}$$

6.4. More generally, let T be any finite set of places of E of even cardinality, and let $\Sigma = \mathscr{I}_1 \amalg \mathscr{I}_2 \amalg \mathscr{I}_3 \amalg \mathscr{I}_4$, where $\mathscr{I}_4 = T \cap \Sigma$. By replacing GL(2) by the totally indefinite quaternion algebra D_T over E ramified exactly at places in T, we can insist that λ_j satisfy (6.2.1) at $\mathscr{I}_1 \amalg \mathscr{I}_2 \amalg \mathscr{I}_3$, and

$$\min(k_j + l_j - \lambda_j, k_j + \lambda_j - l_j, l_j + \lambda_j - k_j) > 0 \text{ for all } \sigma_j \in \mathscr{I}_4. \tag{6.4.1}$$

Again, we have to assume $\lambda_j \geqslant 3$ for all j. We can also assume (6.3.1); as shown in [HK], conditions (6.3.1) and (6.4.1) are consistent.

The extensions 6.2–6.4 of Proposition 6.1 follow easily from the arguments used to prove Lemma 2.2, and from the results in [HK]. Roughly, the critical values $L(2, \pi_1 \otimes \pi_2 \otimes \pi_3)$ can be expressed, as in [loc. cit., 9.2], as the square of an integral similar to (2.3.1), with G replaced by $D_T^\times$. The proof of 2.2 can be extended to show that not all such integrals vanish, and this implies the non-vanishing of the critical value. Details are left to the reader.

The non-vanishing results described here are admittedly esoteric, but I know of no simpler way to prove them.

Finally, Theorem 1, whose proof is based on knowledge of special values of certain L-functions, can be applied to special values of other L-functions. Here is a typical example.

6.5. COROLLARY. *Let π_1, π_2, and π_3 be irreducible automorphic representations of G attached to motivic holomorphic cusp forms of weights $(\underline{k}, r)$, $(\underline{l}, r')$, and $(\underline{\lambda}, -r'-r'')$. Assume the product of the central characters ξ_i of π_i satisfy $\xi_1 \cdot \xi_2 \cdot \xi_3 = 1$. Suppose $\underline{\lambda}$ satisfies the relations (6.2.1). Suppose finally that there are quaternion algebras D_i over E with $\Sigma(D_i) = \mathcal{I}_i$, such that $\pi_i^{D_i}$ exists for $i = 1, 2, 3$. Then the central critical value of the triple product L-function satisfies*

$$L(2, \pi_1 \otimes \pi_2 \otimes \pi_3) \sim_{\bar{\mathbb{Q}}} \pi^{2d} \cdot [q^{D_1}(\pi_1) \cdot q^{D_2}(\pi_2) \cdot q^{D_3}(\pi_3)]^2.$$

Proof. This is an immediate corollary of Theorem 4.5 of Part I and Theorem 1 of this paper. In Part I the Euler factors of $L(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ at bad finite primes were taken to be those whose existence was proved by Piatetski-Shapiro and Rallis, whereas here we are implicitly assuming that the local factors are those defined by the corresponding representation of the local Weil groups. These are known to be the same in almost all cases, thanks to work of Ikeda. However, as far as the rationality assertion of the corollary is concerned, the choice of local factors is immaterial.

By generalizing the methods of [HK, §§12–15], one can obtain similar results under the hypotheses of 6.3, and probably under those of 6.4 as well.

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