COMPOSITIO MATHEMATICA

DMITRII I. PANYUSHEV

The jacobian modules of a representation of a Lie algebra and geometry of commuting varieties

Compositio Mathematica, tome 94, nº 2 (1994), p. 181-199

<http://www.numdam.org/item?id=CM_1994__94_2_181_0>

Compositio Mathematica **94**: 181–199, 1994.

The Jacobian modules of a representation of a Lie algebra and geometry of commuting varieties

DMITRII I. PANYUSHEV
ul. akad. Anokhina, d.30, kor.1, kv.7 117602, Moscow, Russia

Received 23 March 1993; accepted in final form 25 October 1993

Introduction

(0.1) Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field k of characteristic 0. Consider a finite-dimensional representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$. By V^* we denote the dual $\mathfrak{g}$-module. Two bilinear mappings and two (generalized) commuting varieties are naturally assigned to ρ. The first variety is the zero-fiber of the *moment mapping* φ: $V \times V^* \to \mathfrak{g}^*$ and the second one is the zero-fiber of the map $\psi: \mathfrak{g} \times V \to V$, $\psi(g, v) = \rho(g)v$. (Details see in 1.1).

EXAMPLES. (1) Consider the adjoint representation of $\mathfrak{g}$. Then $\psi^{-1}(0)$ is the obvious commuting variety, i.e. the set of pairs of commuting elements in $\mathfrak{g}$.

(2) For the coadjoint representation (i.e. $V = \mathfrak{g}^*$) we have $\varphi = \psi$. Therefore these varieties coincide.

(3) Let S_n be the set of symmetric $n \times n$-matrices and let $\mathfrak{so}_n$ be the Lie algebra of skew-symmetric matrices. Then the moment mapping for the natural representation of $\mathfrak{so}_n$ in S_n is nothing else but the obvious matrix commutator $[\,,]: S_n \times S_n \to \mathfrak{so}_n$. Thus the variety of pairs of commuting symmetric matrices is the zero-fiber of a moment mapping.

(0.2) It has been shown in [BPV] that for the adjoint representation of a Lie algebra $\mathfrak{g}$ there exists a module (the *Jacobian module*) over the ring of the regular functions $k[\mathfrak{g}]$, such that $\psi^{-1}(0)_{\text{red}}$ is isomorphic to the spectrum of the symmetric algebra of it. The main idea of [BPV] is to apply known results on symmetric algebras to investigation of geometry of the (obvious) commuting variety and vice versa.

In this paper we shall show that for any linear representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ one can define two modules E and E' over $R = k[V]$ in such a way, that the subschemes $\varphi^{-1}(0)$ and $\psi^{-1}(0)$ are isomorphic to the spectra of the corresponding symmetric algebras. Thus, our E' may be regarded as a generalization of the Jacobian module from [BPV]. Therefore both E and E' will be referred to as the Jacobian modules of a representation. The constructions of E and E' are dual to each other and for the coadjoint representation they are glued together. We obtain in Section 1 simple estimations of the rank and the projective dimension of E and E', as well as the description of their dual modules. However, having a representation of a Lie algebra, it is rather natural to think that this one is a differential of a representation of a connected group G such that $\mathfrak{g} = \operatorname{Lie} G$. This assumption provides a more geometric framework for our considerations. For instance, we get an ability to introduce the *Jacobian sheaf* on a smooth G-variety.

In many papers (see e.g. [HSV], [SV]) a series of conditions $(\mathscr{F}_p)$ on a presentation of an R-module E has been treated. They are closely related with properties of the symmetric algebra $S_R(E)$. In Section 2 we shall give an interpretation of this condition for the Jacobian modules of a representation of an algebraic group G in terms of sheets of the corresponding G-action.

Most of the results of Section 1, 2 grew out of the attempts to understand and present the construction from [BPV] in a coordinate-free form. Our approach to the Jacobian modules allow us to prove a number of assertions from [loc.cit] in a more general form. For instance, our Theorem 1.9 gives sufficient conditions for $\operatorname{pd}_R E' = 2$. Since these conditions hold for the adjoint representations of semisimple Lie algebras, we obtain a unified proof of Theorem 5.1 in [BPV], as well as the description of the "generic Cartan subalgebra".

(0.3) In Section 3, 4 commuting varieties for representation of *reductive* algebraic groups are being considered. We find sufficient conditions for $(\mathscr{F}_0)$ and $(\mathscr{F}_1)$. As an application we describe a class of representations of reductive groups such that all fibers of the moment mapping φ are irreducible reduced complete intersections (3.2). This class contains, for instance, stable locally free θ-groups of E. Vinberg [Vi1]. In Section 4 we prove normality of fibers of the moment mapping for isotropy representations of symmetric spaces of the maximal rank. In particular this is the case in the situation of Example 3.

(0.4) Our basic references for invariant theory are [VP] and [K]. We follow mainly the terminology and notations of [VP].

1. The Jacobian modules of a representation

(1.1) Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field k of characteristic 0. Consider a finite-dimensional representation

$\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. We can assign two natural bilinear maps to ρ. First, the representation mapping

$$\psi\colon \mathfrak{g} \times V \to V,\ \psi(g, v) = \rho(g)v$$

and second, the *moment mapping*

$$\varphi\colon V \times V^* \to \mathfrak{g}^*,$$

where $\langle\varphi(v, \xi), g\rangle := \langle\rho(g)v, \xi\rangle$, $g \in \mathfrak{g}$, $v \in V$, $\xi \in V^*$ and $\langle , \rangle$ is the pairing of dual modules. Further we shall write $g * v$ instead of $\rho(g)v$. As usual, $\mathfrak{g}v = \{g * v \mid g \in \mathfrak{g}\}$ and $\mathfrak{g}_v = \{g \in \mathfrak{g} \mid g * v = 0\}$. By $\mathrm{Ann} M$ we denote the annihilator subspace in V^* of a subset $M \subset V$. By definition put $\mathfrak{E} = \varphi^{-1}(0)_{\mathrm{red}}$, $\mathfrak{E}' = \psi^{-1}(0)_{\mathrm{red}}$. Then

$$\mathfrak{E} = \{(v, \xi) \in V \times V^* \mid \xi \in \mathrm{Ann}(\mathfrak{g}v)\}$$
$$\mathfrak{E}' = \{(g, v) \in \mathfrak{g} \times V \mid g \in \mathfrak{g}_v\}. \tag{1}$$

Following Examples 0.1 we shall say that $\mathfrak{E}$ and $\mathfrak{E}'$ are the *commuting varities.*

(1.2) If Y is any affine variety and L is a linear space, then the set $\mathrm{Mor}(Y, L)$ of all morphisms from Y into L is a free $k[Y]$-module of rank $\dim_k L$. Actually,

$$\mathrm{Mor}(Y, L) \cong \mathrm{Hom}(L^*, k[Y]) \cong L \otimes k[Y]. \tag{2}$$

More explicitly, if $l \otimes f \in L \otimes k[Y]$, then the corresponding map $\gamma \in \mathrm{Mor}(Y, L)$ is defined by $\gamma(y) = f(y)l$.

Henceforth we use the following notation: $R := k[V] = S_k(V^*)$ is the algebra of the regular functions on V, $n = \dim_k V$, $m = \dim_k \mathfrak{g}$. Thus, $\mathrm{Mor}(V, \mathfrak{g})$ and $\mathrm{Mor}(V, V)$ are free R-modules of ranks m and n respectively. Consider a homorphism of R-modules

$$\hat{\varphi}\colon \mathrm{Mor}(V, \mathfrak{g}) \to \mathrm{Mor}(V, V),\ \gamma \mapsto \tilde{\gamma}, \tag{3}$$

where $\tilde{\gamma}(v) = \gamma(v) * (v)$, $v \in V$. By definition put $E = \operatorname{coker} \hat{\varphi}$. This is the (first) Jacobian module of a representation. By $S_R(E)$ we denote the symmetric algebra of R-module E.

(1.3) THEOREM. $k[\varphi^{-1}(0)] \cong S_R(E)$. *In particular, going down to the reduced varieties, we have* $\mathrm{Spec}(S_R(E))_{\mathrm{red}} \cong \mathfrak{E}$.

Proof. It follows from (2), that we get the exact sequence of R-modules

$$\mathfrak{g} \otimes R \xrightarrow{\hat{\varphi}} V \otimes R \xrightarrow{\nu} E \to 0.$$

Functorial properties of symmetric algebras yield the surjective homomorphism of R-algebras

$$S(\nu)\colon S_R(V \otimes R) \to S_R(E).$$

Moreover, the kernel of $S(\nu)$ is the ideal, generated by the image of $\hat{\varphi}$ [Bo, Ch. 3]. Clearly, $S_R(V \otimes R) \cong k[V^* \times V]$ and $\operatorname{Ker} S(\nu)$ is generated by $\hat{\varphi}(\mathfrak{g} \otimes 1)$. A simple consequence of the definition of $\hat{\varphi}$ is the fact, that $\hat{\varphi}(\mathfrak{g} \otimes 1) = \varphi^*(\mathfrak{g}) \subset V \otimes V^*$, where $\varphi^*\colon k[\mathfrak{g}^*] \to k[V^* \times V]$ is the comorphism, associated with the moment mapping $\varphi, \mathfrak{g} = k[\mathfrak{g}^*]_1$, and $V \otimes V^* \subset k[V^* \times V]_2$. Since the subspace $k[\mathfrak{g}^*]_1$ generates the ideal of the point $0 \in \mathfrak{g}^*$, we obtain the assertion of the theorem. □

Let $\rho^*\colon \mathfrak{g} \to \mathfrak{gl}(V^*)$ be the dual representation. We can also carry the construction of the Jacobian module for ρ^*. We shall get a module $\tilde{E}$ over $\tilde{R} = k[V^*] = S_k(V)$. Obviously, the definition of the moment mapping is symmetric with respect of (V, ρ) and (V^*, ρ^*). Therefore we get

COROLLARY. *$S_R(E) \cong S_{\tilde{R}}(\tilde{E})$ as k-algebras.* □

(1.4) Let us remark the $V^* \times V$ is the cotangent bundle of V and $\operatorname{Mor}(V, V)$ is nothing else but the set of algebraic vector fields on V. This observation leads to a global version of Theorem 1.3.

Let G be an algebraic group, $\mathfrak{g} = \operatorname{Lie} G$, and let Y be a smooth irreducible G-variety. It is well-known, that the cotangent bundle T^*Y is a symplectic variety with the Hamiltonian G-acton. Therefore, the moment mapping $\varphi\colon T^*Y \to \mathfrak{g}^*$ is well-defined. Let $\mathcal{O}_Y$ be the structure sheaf of Y and $\mathcal{T}_Y$ be the tangent sheaf of Y. The action of G on Y induces the homomorphism $\tau\colon \mathfrak{g} \to H^0(Y, \mathcal{T}_Y)$. Define a homomorphism of sheaves $\hat{\varphi}\colon \mathcal{O}_Y^m \cong \mathfrak{g} \otimes \mathcal{O}_Y \to \mathcal{T}_Y$ as follows. If $U \subset Y$ is an open subset, $g \in \mathfrak{g}$, $f \in H^0(U, \mathcal{T}_Y)$, then the section $\sigma = \hat{\varphi}(g \otimes f)$ is determined by the formula $\sigma(v) = f(v)[\tau(g)(v)]$ for $v \in U \subset Y$. Here $\mathfrak{g}$ is regarded as a constant sheaf on Y. Put $\mathcal{E} = \operatorname{coker} \hat{\varphi}$. This sheaf of $\mathcal{O}_Y$-algebras is said to be the Jacobian sheaf of the action of G on Y.

THEOREM. *The subscheme $\varphi^{-1}(0) \subset T^*Y$ is isomorphic to the spectrum of the sheaf of $\mathcal{O}_Y$-algebras $S_{\mathcal{O}_Y}(\mathcal{E})$.*

Proof. Taking a suitable affine open covering $Y = \bigcup_i Y_i$ we may assume Y is affine and the tangent bundle is trivial. (This covering is not required to be G-invariant, it is enough to have the homomorphism $\tau\colon \mathfrak{g} \to H^0(Y_i, \mathcal{T}_{Y_i})$.) Then we can argue as in (1.3). □

(1.5) For any R-module by $(-)^* = \mathrm{Hom}_R(-, R)$ we denote the dual R-module. Clearly, $\mathrm{Mor}(V, L)^* = \mathrm{Mor}(V, L^*)$. Now, dualizing the construction of $\hat{\varphi}$ (3) we get the homomorphism

$$\hat{\psi} = \hat{\varphi}^*\colon \mathrm{Mor}(V, V^*) \to \mathrm{Mor}(V, \mathfrak{g}^*),\ \delta \mapsto \tilde{\delta},$$

where $\langle\tilde{\delta}(v), g\rangle := \langle\delta(v), g * v\rangle$, $v \in V$, $g \in \mathfrak{g}$. By definition put $E' = \operatorname{coker}\hat{\psi}$. This is the (second) Jacobian module of the representation ρ.

THEOREM. $k[\psi^{-1}(0)] \cong S_R(E')$. *In particular*, $\mathrm{Spec}(S_R(E'))_{\mathrm{red}} \cong \mathfrak{E}'$.

Proof. It goes in the same way as in 1.3. □

Let us give the global version of this theorem. Keep the notations of 1.4. Suppose TY is the tangent bundle of Y and Ω_Y is the sheaf of differentials. Define a homomorphism of sheaves of $\mathcal{O}_Y$-modules $\hat{\psi}\colon \Omega_Y \to \mathfrak{g}^* \otimes \mathcal{O}_Y \cong \mathrm{Hom}(\mathfrak{g}, \mathcal{O}_Y)$ as follows. If $\sigma \in H^0(U, \Omega_Y)$, $\sigma(y) := \xi_\sigma$, then $[\hat{\psi}(\sigma)(g)](y) = \xi_\sigma(\tau(g)(y))$. Put $\mathcal{E}' = \operatorname{coker}\hat{\psi}$. Consider a morphism $\bar{\varphi}\colon \mathfrak{g} \times Y \to TY$, such that $\bar{\varphi}(g, y) = \tau(g)(y)$.

THEOREM. *The spectrum of the sheaf of $\mathcal{O}_Y$-algebras $S_{\mathcal{O}_Y}(\mathcal{E}')$ is isomorphic to the subscheme $\bar{\varphi}^{-1}(Y)$, where Y is being considered as the zero section of TY.*

Proof. By taking a suitable open covering of Y it is reduced to the previous Theorem (cf. 1.4). □

(1.6) Thus we have constructed two exact sequences of R-modules:

$$0 \to \operatorname{Ker}\hat{\varphi} \to \mathrm{Mor}(V, \mathfrak{g}) \xrightarrow{\hat{\varphi}} \mathrm{Mor}(V, V) \to E \to 0$$
$$0 \to \operatorname{Ker}\hat{\psi} \to \mathrm{Mor}(V, V^*) \xrightarrow{\hat{\psi}} \mathrm{Mor}(V, \mathfrak{g}^*) \to E' \to 0. \tag{4}$$

Let us begin to get some information about E and E'. It follows from the definitions that

$$\operatorname{Ker}\hat{\varphi} = \{\gamma \in \mathrm{Mor}(V, \mathfrak{g}) \mid \gamma(v) \in \mathfrak{g}_v \text{ for any } v \in V\}$$
$$\operatorname{Ker}\hat{\psi} = \{\delta \in \mathrm{Mor}(V, V^*) \mid \delta(v) \in \mathrm{Ann}(\mathfrak{g}v) \text{ for any } v \in V\}. \tag{5}$$

Since R is a domain, one can define the rank of R-modules by $\operatorname{rank} M = \dim_{Q(R)}(M \otimes_R Q(R))$, where $Q(R)$ is the fraction field of R.

(1.7) PROPOSITION.

(i) $\operatorname{rk}\hat{\varphi} = \operatorname{rk}\hat{\psi} = \max_{v\in V} \dim \mathfrak{g}v$;

(ii) $\operatorname{rank} E = n - \max_{v\in V} \dim \mathfrak{g}v$, $\operatorname{rank} E' = m - \max_{v\in V} \dim \mathfrak{g}v$;

(iii) $\operatorname{Ker}\hat{\varphi} \cong (E')^*$, $\operatorname{Ker}\hat{\psi} \cong E^*$.

Proof. (i) Consider a homorphism $\hat{\varphi}_v: \mathfrak{g} \to V$, $g \mapsto g * v$. This is a specialization of $\hat{\varphi}$. Therefore $\operatorname{rk}\hat{\varphi} = \max_{v \in V} \operatorname{rk}\hat{\varphi}_v$. Since $\hat{\psi} = \hat{\varphi}^*$, we have $\operatorname{rk}\hat{\varphi} = \operatorname{rk}\hat{\psi}$.

(ii) This follows from (i).

(iii) Let us apply functor $(-)^*$ to the exact sequences (4). □

(1.8) Now take into consideration a connected algebraic group G, such that $\mathfrak{g} = \operatorname{Lie}G$, i.e. henceforth we assume $\mathfrak{g}$ is an algebraic Lie algebra. Suppose $G \to \operatorname{GL}(V)$ and let ρ be the differential of this representation of G. First, we present a simple result on connections between geometry of the natural action of G on V and homological properties of E and E'. Recall that an action (or representation) $(G:V)$ is said to be (i) *locally free*, if $\max_{v \in V} \dim Gv = \dim G$ and (ii) *locally transitive*, if $\max_{v \in V} \dim Gv = \dim V$.

THEOREM. (i) *If action* $(G:V)$ *is locally free, then* $\operatorname{Ker}\hat{\varphi} = 0$, $\operatorname{pd}_R E = 1$, *and* E' *is a torsion module*; (ii) *If action* $(G:V)$ *is locally transitive, then* $\operatorname{Ker}\hat{\psi} = 0$, $\operatorname{pd}_R E' = 1$, *and* E *is a torsion module.*

Proof. This follows from (5) and 1.7. □

The statements of this theorem are dual to each other. But such a symmetry fails in the sequel. Apparently, this means that E, $\mathfrak{E}$, and $\operatorname{Ker}\hat{\psi}$ are more important, than E', $\mathfrak{E}'$, and $\operatorname{Ker}\hat{\varphi}$. For example, if G is semisimple, then almost all representations of G are locally free, but locally transitive ones appears finitely many times (may be 0). Therefore it is rather seldom that $\operatorname{Ker}\hat{\psi} = 0$ and it may be useful to find conditions when $\operatorname{Ker}\hat{\psi}$ is a free R-module.

Let $J = R^G$ be the subalgebra of G-invariant functions. Put $V/\!/G = \operatorname{Spec}J$ and let $\pi: V \to V/\!/G$ be the morphism, induced by the inclusion $J \hookrightarrow R$. (Here we assume J is finitely generated. This is always the case, if G is reductive.) Define

$$U = \{v \in V \mid \pi(v) \text{ is a smooth point and } d\pi_v \text{ is surjective}\}$$

(1.9) THEOREM. *Let* $G \subset \operatorname{GL}(V)$ *be a connected algebraic group such that*

(i) *J is a polynomial algebra;*

(ii) $\operatorname{codim}_V(V \backslash U) \geqslant 2$;

(iii) $\max_{v \in V} \dim Gv = \dim V - \dim V/\!/G$.

Then $\operatorname{Ker}\hat{\psi}$ *is a free R-module of rank* $\dim V/\!/G$, *generated by the differentials of free generators of J. In particular,* $\operatorname{pd}_R E' \leqslant 2$.

Proof. If $p \in R$, then the differential dp lies in $\operatorname{Mor}(V, V^*)$. Moreover, if $p \in J$, then $dp \in \operatorname{Ker}\hat{\psi}$. Indeed, in this case p is constant on G-orbits in V and $\mathfrak{g}v$ is the tangent space to Gv at v. Therefore $\langle dp(v), \mathfrak{g}v\rangle = 0$.

Let $p_1, \dots, p_l$ be free generators of J, where $l = \dim V/\!/G$. We shall show that R-module $\operatorname{Ker}\hat{\psi}$ is freely generated by $\mathrm{d}p_i$, $i = 1,\ldots,l$. It follows from (iii) and (1.7), that $\operatorname{rank}\operatorname{Ker}\hat{\psi} = l$. Condition (i) of the theorem means U coincides with a set of $v \in V$ such that $\mathrm{d}p_i(v)$, $i = 1,\ldots,l$ are linearly independent. Since U is on open non-empty subset of V, we have $\mathrm{d}p_i$, $i = 1,\ldots,l$ are linearly independent over R. Therefore they generate a free R-submodule, say F, of $\operatorname{Ker}\hat{\psi}$ of rank l and $\operatorname{Ker}\hat{\psi}/F$ is a torsion module. That is, for every $\sigma \in \operatorname{Ker}\hat{\psi}$ there exist $f_1,\ldots, f_l, f \in R$ without common factors, such that $f\sigma = \Sigma_{i=1}^{l} f_i \mathrm{d}p_i$. Assume $f \notin k^*$ and by D denote the support of the divisor (f). Then $\Sigma_{i=1}^{l} f_i(v)\, \mathrm{d}p_i(v) = 0$ for all $v \in D$. By (ii) we have $D \cap U$ is dense in D, whence $f_i(v) = 0$ for every $v \in D$, $i = 1,\ldots,l$. Thus $f_1,\ldots, f_l, f$ must have a common factor. A contradiction! Therefore $f \in k^*$, $\sigma \in F$, and $\operatorname{Ker}\hat{\psi} = F$. □

The following assertion shows that the previous theorem has a sufficiently large field of applications.

(1.10) COROLLARY. *If G is a connected algebraic group without rational characters and J is polynomial algebra, then* $\operatorname{Ker}\hat{\psi}$ *is a free R-module.*

Proof. The reason is that for connected groups without rational characters conditions (ii), (iii) of Theorem 1.9 are automatically fulfilled. For (iii) this follows from Rosenlicht theorem (see e.g. [K, Ch. 2]) and the equality $Q(J) = (Q(R))^G$. For (ii) this is proved in [Kn, Satz 2]. (In fact, the only semisimple groups are considered in [Kn], but those arguments are also valid in our case.) □

Another case when $\operatorname{Ker}\hat{\psi}$ is free is described in 3.4.

2. Determinantal conditions and sheets

(2.1) Let an R-module E have a presentation

$$R^m \xrightarrow{\beta} R^n \to E \to 0.$$

Then β may be regarded as a $n \times m$-matrix with entries in R. Let $I_t(\beta)$ be the ideal generated by the t-sized minors of β. Following [HSV] consider the condition on the $I_t(\beta)$'s ($d \geqslant 0$):

$$(\mathscr{F}_d)\quad \operatorname{ht} I_t(\beta) \geqslant \operatorname{rk}\beta - t + 1 + d,\ 1 \leqslant t \leqslant \operatorname{rk}\beta.$$

Clearly, $(\mathscr{F}_d)$ implies $(\mathscr{F}_{d-1})$. A series of sufficient conditions for $S_R(E)$ may be formulated in terms of $(\mathscr{F}_d)$. For instance, if $S_R(E)$ is a domain, then E satisifies $(\mathscr{F}_1)$, while $(\mathscr{F}_0)$ allows us to give a simple expression for $\dim S_R(E)$, etc.

Our aim here is to present a geometric interpretation of these conditions for the Jacobian modules of representations of algebraic groups. We shall show the conditions $(\mathscr{F}_d)$ are rather naturally transferred into the ones on the *sheets* of the group action. Afterwards, a standard invariant theory technique produces a number of representations of reductive groups with $(\mathscr{F}_1)$, as well as sufficient conditions of flatness and of irreducibility of fibers of the moment mapping.

(2.2) Let us come back in our situation: $G \subset \mathrm{GL}(V)$ is a connected algebraic group, E and E' are the Jacobian modules with the presentations (4). Since $\hat{\varphi} = \hat{\psi}^*$, the conditions $(\mathscr{F}_d)$ for E and E' are equivalent. Therefore without loss of generality we shall consider only E in the sequel. By $k(V)^G$ we denote the field of G-invariant rational functions on V.

Recall the terminology on sheets. By definition put

$$V^{(s)} = \{v \in V \mid \dim Gv = s\}.$$

This is a locally closed G-invariant subvariety of V. The irreducible components of $V^{(s)}$ are said to be *sheets*. The number of sheets is finite and since V is irreducible, if follows that there is a unique open sheet. The dimension of the stabilizers of points is upper semi-continuous on V, therefore the closure $\overline{V^{(s)}}$ is contained in $\bigcup_{l \leqslant s} V^{(l)}$. The integer $\mathrm{mod}(G, V) = \max_s(\dim V^{(s)} - s)$ is said to be the *modality* of the action $(G : V)$ (see [Vi2]). If $m_0 = \max_{v \in V} \dim Gv$, then $V^{(m_0)}$ is open sheet and by Rosenlicht theorem $\dim V^{(m_0)} - m_0 = \mathrm{trdeg} k(V)^G$. In particular, $\mathrm{mod}(G, V) \geqslant \mathrm{trdeg}\, k(V)^G$.

(2.3) THEOREM. (i) $\dim \mathfrak{E} = \dim V + \mathrm{mod}(G, V)$;

(ii) $\dim \mathfrak{E}' = \dim \mathfrak{g} + \mathrm{mod}(G, V)$.

Proof. (i) Consider the projection $pr_1 : \mathfrak{E} \to V$. It follows from (1), that $pr_1^{-1}(v) = \{v\} \times \mathrm{Ann}(\mathfrak{g}v)$ for each $v \in V$. That is, the fibres of the projection are affine spaces. Whence, $\mathrm{pr}_1^{-1}(Gv)$ is a variety of dimension $\dim V$. Therefore, if $V_j^{(s)}$ is a sheet such that $\dim V_j^{(s)} - s = \mathrm{mod}(G, V)$, then $pr_1^{-1}(V_j^{(s)})$ has an irreducible component of dimension $\dim V + \mathrm{mod}(G, V)$. Conversely, if $\mathfrak{E}_j$ is an irreducible component of $\mathfrak{E}$, then $pr_1(\mathfrak{E}_j)$ is irreducible and there exist a sheet $V_i^{(l)}$ such that $V_i^{(l)} \cap pr_1(\mathfrak{E}_j)$ is dense in $pr_1(\mathfrak{E}_j)$. Whence, $\dim \mathfrak{E}_j \leqslant \dim V + \dim V_i^{(l)} - l \leqslant \dim V + \mathrm{mod}(G, V)$.

(ii) This goes as well as in (i). □

Since the construction of $\mathfrak{E}$ is symmetric with respect of V and V^*, we get

COROLLARY. $\mathrm{mod}(G, V) = \mathrm{mod}(G, V^*)$. □

(2.4) THEOREM. *Let E be the (first) Jacobian module of a representation $G \to \mathrm{GL}(V)$ and $d \in \{0, 1, 2, \ldots\}$. The following conditions are equivalent:*

(i) *E satisfies $(\mathscr{F}_d)$;*

(ii) *Let Y be a arbitrary closed G-invariant subset of V, such that $Y \subset V \backslash V^{(m_0)}$. Then* $\operatorname{mod}(G, Y) \leqslant \operatorname{mod}(G, V) - d$.

Proof. It follows from (1.7), that the zero set of the ideal $I_t(\hat{\varphi})$ looks as follows.

$$\mathscr{V}(I_t(\hat{\varphi})) = \{v \in V \mid \operatorname{rk} \hat{\varphi}_v < t\} = \{v \in V \mid \dim Gv < t\} = \bigcup_{l<t} V^{(l)}.$$

Since $\operatorname{rk} \hat{\varphi} = m_0$, the condition $(\mathscr{F}_d)$ inverts into

$$\dim V - m_0 \geqslant \dim(\bigcup_{l<t} V^{(l)}) - (t-1) + d,\ 1 \leqslant t \leqslant m_0.$$

If this unequality really holds for every $t \in [1, m_0]$, then this is equivalent to $\operatorname{mod}(G, V) = \operatorname{trdeg} k(V)^G$ and $\operatorname{trdeg} k(V)^G \geqslant \dim V^{(t-1)} - (t-1) + d$, $t \in [1, m_0]$. But the latter is equivalent to the statement (ii) of the theorem for the subvarieties $Y = \overline{V^{(s)}}$, $s < m_0$. Hence this is the case for an arbitrary $Y \subset V \backslash V^{(m_0)}$, because $Y^{(s)} \subset V^{(s)}$ for any s. □

(2.5) COROLLARY. *The following conditions are equivalent*:

(i) *E satisfies* $(\mathscr{F}_0)$;

(ii) $\operatorname{mod}(G, V) = \operatorname{trdeg} k(V)^G$. □

(2.6) COROLLARY. *Suppose $G \subset \mathrm{GL}(V)$ is reductive, B is a Borel subgroup of G and E_B is the Jacobian module of the representation $B \subset \mathrm{GL}(V)$. Then E_B satisfies $(\mathscr{F}_0)$.*

Proof. The result of E. Vinberg [Vi2] asserts that under these conditions $\operatorname{mod}(B, V) = \operatorname{trdeg} k(V)^B$. □

(2.7) REMARKS. (1) As far as I know, the explicit construction of $\mathfrak{E}$ first appears in [P]. In this paper Pyasetskii has shown that if G acts on V with finitely many orbits, then $\mathfrak{E}$ is a variety of pure dimension $\dim V$ and the number of the irreducible components of $\mathfrak{E}$ is equal to the number of G-orbits in V. Since the construction of $\mathfrak{E}$ is symmetric with respect of V and V^*, he has derived that the number of G-orbits in V is equal to the number of G-orbits in V^*.

(2) An opposite result has been achieved in [Ri]. Richardson proved that for the adjoint representation of a semisimple group the commuting variety $\mathfrak{E}(\cong \mathfrak{E}')$ is irreducible. This result shows the condition that an action is locally free is not necessary for irreducibility (cf. 3.2).

(2.8) EXAMPLE. We shall show that for any $d > 0$ there exist representations such that $(\mathscr{F}_d)$ holds for the Jacobian module E.

Consider $G = SL(W)$ and its representation in $V = nW = W \oplus \cdots \oplus W$, $n \geqslant m = \dim W$. If we fix a base in W, then elements of V are naturally treated as $m \times n$ matrices. The sheets of $(G:V)$ have a rather nice description:

$$V = \coprod_{i=0}^{m} V^{(s_i)},$$

where $V^{(s_i)} = \{$the set of matrices of rank $m - i\}$ and $s_i = \{$dimension of the G-orbit of a matrix of the rank $m - i\}$. It is well-known that $\dim V^{(s_i)} = (m - i)(n + i)$ and it is easy to compute that $s_i = m^2 - im$, if $i \geqslant 1$ and $s_0 = m^2 - 1$. Here $V^{(s_0)}$ is the open sheet and $\dim V^{(s_0)} - s_0 = mn - m^2 + 1$. Put $Y = V \setminus V^{(s_0)}$. By Theorem 2.4 one has to compare $\operatorname{mod}(G, Y)$ and $mn - m^2 + 1$; namely, $(\mathcal{F}_d)$ holds iff $\max_{1 \leqslant i \leqslant m}(\dim V^{(s_i)} - s_i) \leqslant mn - m^2 + 1 - d$. But $\max_{1 \leqslant i \leqslant m}(\dim V^{(s_i)} - s_i) = \max_{1 \leqslant i \leqslant m}[mn - m^2 + i(2m - n + i)] = nm - m^2 + 1 + \max_{1 \leqslant i \leqslant m}[i(2m - n + i) - 1]$. Whence

$$(\mathcal{F}_d) \text{ holds} \Leftrightarrow \max_{1 \leqslant i \leqslant m}[i(2m - n + i) - 1] \leqslant -d$$

and trivial calculation give the answer:

(a) if $m \leqslant n < 2m - 2$, then already $(\mathcal{F}_0)$ does not hold;
(b) if $n \geqslant 2m - 2$, then $(\mathcal{F}_{n-2m+2})$ holds.

3. Jacobian modules for reductive group actions

Hereafter we assume G is reductive. In this case the quotient map $\pi: V \to V/\!/G$ (see 1.8) possesses a number of nice properties (see [VP] or [K]).

The action $(G:V)$ is said to be (a) *stable* whenever almost all fibers of π are G-orbits and (b) *visible*, if $\pi^{-1}(\pi(0))$ contains finitely many G-orbits; then this is the case for all the fibers of π. A subgroup $H \subset G$ is said to be the *stabiliser of general position* (s.g.p.), if there exists an open subset $\Omega \subset V$ such that G_x is conjugated to H for any $x \in \Omega$. The s.g.p. is always exists for linear actions of reductive groups. Moreover, if the action is stable, then H is reductive.

(3.1) THEOREM. (i) *Suppose the action $(G:V)$ is visible. Then the Jacobian module E satisfies $(\mathcal{F}_0)$.*

(ii) *Suppose the action $(G:V)$ is stable and visible. Then the Jacobian module E satisfies $(\mathcal{F}_1)$.*

Proof. (i) Let Y be an irreducible closed G-invariant subvariety of V. Then $\pi(Y) \cong Y/\!/G$ is closed in $V/\!/G$. Since the induced action $(G:Y)$ is visible, we have $\dim Y/\!/G = \operatorname{trdeg} k(Y)^G$. Whence $\operatorname{trdeg} k(Y)^G \leqslant \operatorname{trdeg} k(V)^G$. Considering that this is the case for any Y we get the condition (ii) of theorem 2.4 for $d = 0$.

(ii) If in addition to (i) $(G:V)$ is stable and Y is a proper subset of V, then it follows from the stability that $Y//G \neq V//G$. Whence $\operatorname{trdeg} k(Y)^G \leqslant \operatorname{trdeg} k(V)^G - 1$, i.e. we get the condition (ii) of Theorem 2.4 for $d = 1$. □

REMARK. There exist tables of visible irreducible representations of connected reductive groups ([Kac]) and stability criteria for actions of semisimple groups (see e.g. [VP]). This provides numerous examples of representations with property $(\mathscr{F}_1)$ (see also 3.4). Clearly, a first example of this kind is the adjoint representation of a semisimple group.

(3.2) THEOREM. *Suppose $G \subset \mathrm{GL}(V)$ and the action $(G:V)$ is visible, stable, and locally free. Then*

(i) *the moment mapping $\varphi: V \times V^* \to \mathfrak{g}^*$ is surjective and equidimensional;*

(ii) *all the fibers of φ are irreducible reduced complete intersections in $V \times V^*$. In particular, this is the case for $\mathfrak{E}$.*

Proof. Take a point $v \in V$, such that $\dim \mathfrak{g}v = \dim \mathfrak{g}$. Then $\dim \varphi(\{v\} \times V^*) = \dim V - \dim \mathrm{Ann}(\mathfrak{g}v) = \dim \mathfrak{g}$, i.e. φ is surjective. Therefore all irreducible components of all fibers of φ has the dimension greater or equal $2\dim V - \dim \mathfrak{g} = \dim V + \dim V//G$.

By 3.1 the Jacobian module E satisfies $(\mathscr{F}_1)$. Therefore by 2.3, 2.5 $\dim \mathfrak{E} = \dim V + \operatorname{trdeg} k(V)^G$. Since the action $(G:V)$ is stable, we have $\operatorname{trdeg} k(V)^G = \dim V//G$, whence $\mathfrak{E}$ is a variety of *pure* dimension $\dim V + \dim V//G$. Suppose $\mathfrak{E}_i$ is an irreducible component of $\mathfrak{E}$ and let $V_j^{(s)}$ be the sheet, such that $pr_1(\mathfrak{E}_i) \cap V_j^{(s)}$ is dense in $pr_1(\mathfrak{E}_i)$ (see 2.3). Then $\dim V + \mathit{dim}\, V//G = \dim \mathfrak{E}_i \leqslant \dim V + \dim V_j^{(s)} - s$. Since $(\mathscr{F}_1)$ holds, the latter is possible iff $V_j^{(s)} = V^{(m_0)}$ and $pr_1(\mathfrak{E}_i)$ is dense in V. But there exists at most one irreducible component of $\mathfrak{E}$ with this property, because all the fibers $pr_1^{-1}(v)$, $v \in V$ are irreducible. Thus $\mathfrak{E} = \mathfrak{E}_i$.

In order to prove that $\varphi^{-1}(0)$ is reduced, it suffices to find a point $p \in \mathfrak{E}$, such that $d\varphi_p$ is surjective, since $\mathfrak{E}$ is irreducible and has the right dimension. Obviously, we can take $p = (v, 0)$, where $v \in V^{(m_0)}$ (see the first paragraph of the proof). Thus everything is proved for $\varphi^{-1}(0)$. So far as φ is equidimensional and is determined by homogeneous polynomials (of degree 2), the method of associated cones [K, Ch. 2, 4.2] allow us to transfer the desired properties on all fibers of φ. In particular all the fibers are complete intersections. □

(3.3) Suppose Y is an affine G-variety and L is a G-module. Then G act on $\mathrm{Mor}(Y, L)$ by the formula:

$$(g \cdot \gamma)(y) = g(\gamma(g^{-1}y)), \quad \gamma \in \mathrm{Mor}(Y, L),\ g \in G,\ y \in Y.$$

By $\mathrm{Mor}_G(Y, L)$ we denote the set of G-equivariant morphisms $\gamma: Y \to L$. This is the subspace of G-invariant elements in $\mathrm{Mor}(Y, L)$, i.e.

$$\mathrm{Mor}_G(Y, L) = \{\gamma \in \mathrm{Mor}(Y, L) \mid g \cdot \gamma = \gamma\}.$$

It is well-known that $\mathrm{Mor}_G(Y, L)$ is a finitely generated $k[Y]^G$-module, called the *module of covariants* (of type L).

The following assertion may be treated as an application of 1.9. Recall that $J = k[V]^G$.

(3.4) THEOREM. *Suppose $(G : V)$ is a stable action and let H be the s.g.p. If* $\dim V^H = \dim V//G$, *then* $\mathrm{Mor}_G(V, V^*)$ *is a free J-module, generated by the differentials of the generators of J.*

Proof. (a) First we show that conditions of Theorem 1.9 are satisfied here. Put $W = N_G(H)/H$. This group effectively acts on V^H and by [LR] $V^H// W \cong V//G$. Therefore W is finite and by [Pa] J is a polynomial algebra and the quotient morphism π is equidimensional.

Assume $V \backslash U$ contains a divisor D. Clearly D is G-invariant and therefore $\pi(D)$ is closed in $V//G$. Since $(G : V)$ is stable, we have $\pi(D) \neq V$ and then it follows from equidimensionality of π that $\pi(D)$ is a divisor in $V//G$. Since $V//G$ is factorial, there exists $f \in J$ with $D = \mathscr{V}(f)$. Thus D is determined by a G-invariant polynomial. Now the arguments of [Kn, Satz 2] give us a point $v \in D$ such that $d\pi_v$ is surjective. This contradicts the definition of D. Hence $\mathrm{codim}_V(V \backslash U) \geqslant 2$.

(b) The homorphisms $\hat{\varphi}$ and $\hat{\psi}$, defined in Section 1, are evidently G-equivariant. Since G is reductive, the functor $(-)^G$ is exact. Therefore applying $(-)^G$ to (4) we get the exact sequence of J-modules:

$$0 \to (\mathrm{Ker}\,\hat{\psi})^G \to \mathrm{Mor}_G(V, V^*) \xrightarrow{\hat{\psi}} \mathrm{Mor}_G(V, \mathfrak{g}^*) \to (E')^G \to 0. \qquad (6)$$

According to Theorem 1.9 R-module $\mathrm{Ker}\,\hat{\psi}$ is generated by G-invariant elements. Therefore $(\mathrm{Ker}\,\hat{\psi})^G$ is a free J-module of rank $\dim V//G$, generated by the differentials of the free generators of J.

(c) We have already proved that J is a polynomial algebra and π is equidimensional. This implies that all modules of covariants, in particular $\mathrm{Mor}_G(V, V^*)$ and $\mathrm{Mor}_G(V, \mathfrak{g}^*)$, are free. For stable actions there is a simple formula for the rank of modules of covariants:

$$\mathrm{rank}\,\mathrm{Mor}_G(V, L) = \dim L^H.$$

Hence it follows from our assumptions that $\mathrm{rank}(\mathrm{Ker}\,\hat{\psi})^G = \mathrm{rank}\ \mathrm{Mor}_G(V, V^*)$, because H is reductive and $\dim V^H = \dim(V^*)^H$. Now, since $\mathrm{Mor}_G(V, V^*)/ (\mathrm{Ker}\,\hat{\psi})^G$ is a torsion module and $\mathrm{Mor}_G(V, \mathfrak{g}^*)$ is free, it follows from (6) that $\hat{\psi}|_{\mathrm{Mor}_G(V, V^*)} = 0$. □

COROLLARY (of the proof). *If $(G : V)$ is stable, J is a polynomial algebra, and*

π is equidimensional, then $\operatorname{Ker}\hat{\psi}$ *is a free R-module.* □

(3.5) EXAMPLES. (1) Suppose $\mathfrak{g}$ is a reductive Lie algebra and θ is an automorphism of order 2. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the corresponding $\mathbf{Z}_2$-gradation. If G_0 is the connected subgroup of G with $\operatorname{Lie} G_0 = \mathfrak{g}_0$, then the restriction on G_0 of the adjoint representation of G induces the representation $G_0 \to GL(\mathfrak{g}_1)$. This is the *isotropy representation of the symmetric space* G/G_0. It is known that this one is visible and stable [KR]. Therefore the condition $(\mathscr{F}_1)$ always holds here. The moment mapping coincides with the commutator:

$$\varphi = [\,,\,]\colon \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0 \cong \mathfrak{g}_0^*. \tag{7}$$

This representation is locally free iff θ has the maximal rank, i.e. there exists a Cartan subalgebra, lying in $\mathfrak{g}_1$. A well-known fact is that for every simple Lie algebra there is a unique (up to conjugation) involution θ of the maximal rank. In particular, if $\mathfrak{g} = \mathfrak{sl}_n$, then $\mathfrak{g}_1 = S_n$ and we come to the Example 3 in 0.1. Therefore Theorem 3.2 may be regarded as a generalization of 3.1 and 3.3 in [BPV].

(2) Let us present the example of involution θ such that the commuting variety $\mathfrak{C} \subset \mathfrak{g}_1 \times \mathfrak{g}_1$ appears to be reducible. This means the condition that the action is locally free (or some substitution of it) is essential in 3.2. Suppose $\mathfrak{g} = \mathfrak{sl}_n (n > 2)$ and θ is determined by conjugation on the matrix $\operatorname{diag}(-1, \ldots, -1, 1)$. Then $\mathfrak{g}_0 = \mathfrak{gl}_{n-1}$ and $\dim \mathfrak{g}_1 = 2n - 2$. It is not difficult to calculate that $\mathfrak{C}$ has 3 irreducible components and their dimensions are $2n - 1$, $2n - 2$, $2n - 2$.

(3) The first example admits a generalization, which has been investigated in [Vi1]. Let θ be an automorphism of order m and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{m-1}$ be the corresponding $\mathbf{Z}_m$-gradation (the indices is being considered modulo m). The image of the natural representation $G_0 \to GL(\mathfrak{g}_1)$ is called the θ-group. Here the moment mapping also coincides with commutator

$$\varphi = [\,,\,]\colon \mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0.$$

The action $(G_0\colon \mathfrak{g}_1)$ is visible, but is not always stable, if $m > 2$. Nevertheless, there exists a large number of examples of θ-groups that satisfy all the conditions of Theorem 3.2.

The important property of θ-groups is that $k[\mathfrak{g}_1]^{G_0}$ is always a polynomial algebra [Vi1]. Therefore $\operatorname{pd}_R E' \leqslant 2$ for semisimple or stable θ-groups. It is worthwhile mentioning that the adjoint group is also a θ-group, appearing when $\theta = \mathrm{id}$.

(3.6) For stable locally free θ-groups we are able to produce (1) the description of $\mathfrak{C} \subset \mathfrak{g}_1 \times \mathfrak{g}_{-1}$, similar to the one of the obvious commuting variety in $\mathfrak{g} \times \mathfrak{g}$ [Ri] and (2) the Chevalley-type theorem for the quotient variety $\mathfrak{E}//G_0$.

First recall basic result on θ-groups [Vi1]. Bearing in mind our purposes, we always assume $(G_0 : \mathfrak{g}_1)$ is stable. Suppose $\mathfrak{c}_1 \subset \mathfrak{g}_1$ is the maximal subspace that consists of commuting semisimple elements. Then $\mathfrak{c}_1$ is called a *Cartan subspace* of the θ-group. Put

$$N(\mathfrak{c}_1)_0 = \{s \in G_0 \mid s\mathfrak{c}_1 \subset \mathfrak{c}_1\},$$

$$Z(\mathfrak{c}_1)_0 = \{s \in G_0 \mid sx = x \text{ for any } x \in \mathfrak{c}_1\}.$$

Then $W_1 = N(\mathfrak{c}_1)_0/Z(\mathfrak{c}_1)_0$ is a finite group, called the *Weyl group of the graded Lie algebra* $\mathfrak{g}$. The main assertions on θ-groups are:

(*) Let $x \in \mathfrak{g}_1$ be an arbitrary element. Then G_0x is closed iff $G_0x \cap \mathfrak{c}_1 \neq \varnothing$ iff x is semisimple.

(**) Linear group $W_1 \subset \mathrm{GL}(\mathfrak{c}_1)$ is generated by (pseudo)reflections.

(***) The inclusion $\mathfrak{c}_1 \hookrightarrow \mathfrak{g}_1$ induces the isomorphism $\mathfrak{c}_1/W_1 \cong \mathfrak{g}_1//G_0$. In particular, $\dim \mathfrak{c}_1 = \dim \mathfrak{g}_1//G_0$.

We shall say $x \in \mathfrak{c}_1$ is generic, if x does not lie in the union of the reflecting hyperplanes of W_1. Take a generic $x \in \mathfrak{c}_1$ and put $\mathfrak{c}_{-1} := \mathrm{Ann}(\mathfrak{g}_0 x) \subset \mathfrak{g}_{-1} \cong \mathfrak{g}_1^*$. Then $\dim \mathfrak{c}_1 = \dim \mathfrak{c}_{-1}$, $\mathfrak{c}_{-1}$ is a Cartan subspace in $\mathfrak{g}_{-1}$, and $\mathfrak{c}_{-1}$ does not depend on the choice of generic x. Therefore $\mathfrak{c}_1 \times \mathfrak{c}_{-1} \subset \mathfrak{E}$. By definition put $\mathfrak{K} := \overline{G_0(\mathfrak{c}_1 \times \mathfrak{c}_{-1})}$. This is a G_0-invariant irreducible subvariety of $\mathfrak{g}_1 \times \mathfrak{g}_{-1}$.

(3.7) PROPOSITION. *If θ-group is stable, then $\mathfrak{K}$ is an irreducible component of $\mathfrak{E}$.*

Proof. Since θ-groups are visible, we have (1) $k(\mathfrak{g}_1)^{G_0} = Q(k[\mathfrak{g}_1]^{G_0})$ and (2) $\dim \mathfrak{E} = \dim \mathfrak{g}_1 + \operatorname{trdeg} k(\mathfrak{g}_1)^{G_0}$. Whence $\dim \mathfrak{E} = \dim \mathfrak{g}_1 + \dim \mathfrak{c}_1$. On the other hand, for a generic $x \in \mathfrak{c}_1$ and $\xi = (x, y) \in \mathfrak{c}_1 \times \mathfrak{c}_{-1}$ we have $\dim G_0\xi = \dim G_0 x$ and $G_0\xi \cap (\mathfrak{c}_1 \times \mathfrak{c}_{-1})$ is finite. Thus $\dim \mathfrak{K} = 2\dim \mathfrak{c}_1 + \dim G_0 x$ and the stability assumption imply that $\dim G_0 x = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_1//G_0$. Hence $\dim \mathfrak{K} = \dim \mathfrak{E}$. □

COROLLARY. *If $G_0 \subset \mathrm{GL}(\mathfrak{g}_1)$ is a stable locally free θ-group, then* $\mathfrak{E} = \overline{G_0(\mathfrak{c}_1 \times \mathfrak{c}_{-1})}$. □

It follows from definitions that W_1 respects also $\mathfrak{c}_{-1}$. Consider the diagonal action $(W_1 : \mathfrak{c}_1 \times \mathfrak{c}_{-1})$.

(3.8) THEOREM. *The injection $\mathfrak{c}_1 \times \mathfrak{c}_{-1} \hookrightarrow \mathfrak{K}$ induces the surjective birational morphism*

$$\tau\colon \mathfrak{c}_1 \times \mathfrak{c}_{-1}/W_1 \to \mathfrak{K}//G_0.$$

Proof. (a) In order to establish surjectivity, it suffices to prove that all closed G-orbits in $\mathfrak{R}$ meet with $\mathfrak{c}_1 \times \mathfrak{c}_{-1}$. Take an arbitrary $\xi = (x, y) \in \mathfrak{R}$. Let $x = x_s + x_n$ and $y = y_s + y_n$ be the Jordan decompositions. Then x_s, $x_n \in \mathfrak{c}_1$ and $y_s, y_n \in \mathfrak{c}_{-1}$. Since $[x, y] = 0$, we have also $[x_s, y_s] = 0$, etc. Therefore the standard arguments of theory of θ-groups imply that (x_s, y_s) lies in the closure of $G_0\xi$. Therefore, if $G_0\xi$ is closed, then both x and y are semisimple. By 3.6(*) we may assume $x \in \mathfrak{c}_1$. Then applying again 3.6(*) to the $\mathbf{Z}_m$-graded reductive subalgebra $\mathfrak{g}_x$, we get $(G_x)_0 y \cap \mathfrak{c}_{-1} \neq \emptyset$.

(b) Birationality is an easy consequence of 3.6(***), because for generic $x \in \mathfrak{c}_1$ we have $G_0\xi \cap (\mathfrak{c}_1 \times \mathfrak{c}_{-1}) = W_1\xi$. □

Since $(\mathfrak{c}_1 \times \mathfrak{c}_{-1})/W_1$ is normal, the Richardson lemma [LR] give us:

COROLLARY. *τ is an isomorphism iff $\mathfrak{R}//G_0$ is normal.* □

Clearly it suffices to have normality of $\mathfrak{R}$.

(3.9) Methods of invariant theory allow us to produce also negative results. Concerning the notion of Luna stratification of the quotient variety $V//G$ see e.g. [VP].

PROPOSITION. *Suppose the action $(G : V)$ is visible and $V//G$ has a Luna stratum of codimension $d > 0$. Then E does not satisfy $(\mathscr{F}_{d+1})$.*

Proof. By 2.4 and 3.1(i) it is sufficient to find a closed G-invariant subvariety $Y \subset V \backslash V^{(m_0)}$, such that $\dim Y//G = \dim V//G - d$. Suppose Z is a stratum of codimension d in $V//G$. Then take $Y = \overline{\pi^{-1}(Z)} \backslash V^{(m_0)}$. It follows from the properties of π that $Y//G = \bar{Z}$. □

If $G_0 \to \mathrm{GL}(\mathfrak{g}_1)$ is a θ-group, then the theory developed in [Vi1] yields that $\mathfrak{g}_1//G_0$ always has strata of codimension 1 (they correspond to the reflecting hyperplanes of the Weyl group of a graded Lie algebra). Therefore we get

COROLLARY. *If E is the Jacobian module of a θ-group, then $(\mathscr{F}_2)$ is never satisfied.* □

4. On normality of commuting varieties for θ-groups

(4.1) Provided that $\mathfrak{C}$ is irreducible, it is rather natural to investigate singularities of it. In this section we shall show that $\mathfrak{C}$ (and also other fibers of φ) is normal for the isotropy representation of a symmetric space of the maximal rank (see 3.5 Example 1).

Throughout this section $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the $\mathbf{Z}_2$-gradation of maximal rank of a reductive Lie algebra, θ is the corresponding involutive automorphism of $\mathfrak{g}$, φ is the map (7), and $\mathfrak{C} = \{(x, y) \in \mathfrak{g}_1 \times \mathfrak{g}_1 \mid [x, y] = 0\}$ is the commuting variety. By $\mathfrak{z}_\mathfrak{g}(x)$ we denote the centralizer of $x \in \mathfrak{g}$. If $x \in \mathfrak{g}_1$ is semisimple, then $\mathfrak{z}_{\mathfrak{g}}(x)$ is a reductive θ-invariant subalgebra of $\mathfrak{g}$ and $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(x)_0 \oplus \mathfrak{z}_{\mathfrak{g}}(x)_1$ is $\mathbf{Z}_2$-gradation of the maximal rank. In our old notations (see 1.1) we have $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{g}_x$ and $\mathfrak{g}_{\mathfrak{g}}(x)_0 = (\mathfrak{g}_0)_x$. The following formula is a direct consequence of existence of the Kirillov form on (co)adjoint orbits. For any $x \in \mathfrak{g}_1$ we have

$$\dim[\mathfrak{g}_0, x] + \dim \mathfrak{z}_{\mathfrak{g}}(x)_1 = \dim \mathfrak{g}_1. \tag{8}$$

The assumption of the maximality of rank means that $\dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 = \operatorname{rk} \mathfrak{g}$. Therefore 3.2(i) implies

$$\dim \mathfrak{C} = \dim \mathfrak{g}_1 + \operatorname{rk} \mathfrak{g}. \tag{9}$$

(4.2) By $\mathfrak{C}_{\text{sing}}$ we denote the singular locus of $\mathfrak{C}$. After 3.2 we know $\mathfrak{C}$ is an irreducible complete intersection. Therefore normality of $\mathfrak{C}$ is equivalent to smoothness in codimension 1. First we give a simple description of $\mathfrak{C}_{\text{sing}}$.

PROPOSITION. *Suppose* $\xi = (x, y) \in \mathfrak{C}$. *Then*

$$(x, y) \in \mathfrak{C}_{\text{sing}} \Leftrightarrow \mathfrak{z}_{\mathfrak{g}}(x)_0 \cap \mathfrak{z}_{\mathfrak{g}}(y)_0 \neq \{0\}.$$

Proof. We know φ is equidimensional and surjective, hence

$$(x, y) \in \mathfrak{C}_{\text{sing}} \Leftrightarrow d\varphi_{\xi} \text{ is not surjective.}$$

Since φ is bilinear, $\operatorname{Im} d\varphi_{\xi} = [\mathfrak{g}_1, x] + [\mathfrak{g}_1, y] \subset \mathfrak{g}_0$. By $(-)^{\perp}$ denote the orthogonal complement relative to a scalar product on $\mathfrak{g}_0$, which is a 0-part of an invariant scalar product on $\mathfrak{g}$. Then $[\mathfrak{g}_1, x]^{\perp} = \mathfrak{z}_{\mathfrak{g}}(x)_0$ and we get $(\operatorname{Im} d\varphi_{\xi})^{\perp} = \mathfrak{z}_{\mathfrak{g}}(x)_0 \cap \mathfrak{z}_{\mathfrak{g}}(y)_0$. □

As a matter of fact, this is a particular case of a more general assertion for arbitrary moment mappings.

(4.3) Recall that $\pi: \mathfrak{g}_1 \to \mathfrak{g}_1//G_0$ is the quotient morphism and the action $(G_0 : \mathfrak{g}_1)$ is visible, i.e. $\mathfrak{N} := \pi^{-1}(0))$ contains finitely many G-orbits.

THEOREM. $\mathfrak{C}$ *is smooth in codimension* 1.

Proof. We imitate partially the arguments of [Ri], that have been used to establish irreducibility of the commuting variety in $\mathfrak{g} \times \mathfrak{g}$. The proof is by induction on semisimple rank of $\mathfrak{g}$, $\operatorname{srk} \mathfrak{g} := \operatorname{rk}[\mathfrak{g}, \mathfrak{g}]$. Clearly, adding of the central torus does not change codimension of the set of singularities. Therefore we may assume at the beginning that $\mathfrak{g}$ is semisimple.

(a) Suppose $\operatorname{rk} \mathfrak{g} = 1$. Then, clearly, $\mathfrak{g} = \mathfrak{sl}_2$ and $\operatorname{codim}_{\mathfrak{C}} \mathfrak{C}_{\text{sing}} = 3$.

(b) Suppose $\operatorname{rk} \mathfrak{g} > 1$ and $\xi = (x, y) \in \mathfrak{C}$. Let $x = x_s + x_n$ be the Jordan decomposition of x. It is known, that also $x_s, x_n \in \mathfrak{g}_1$ [KR].

(1) Assume $x_s \neq 0$. Put $\mathfrak{L} = \mathfrak{z}_{\mathfrak{g}}(x_s)$. This is a $\mathbf{Z}_2$-graded Lie algebra and $\operatorname{srk} \mathfrak{L} < \operatorname{rk} \mathfrak{g}$. Let $\mathfrak{E}(\mathfrak{L}) \subset \mathfrak{L}_1 \times \mathfrak{L}_1$ be the commuting variety for $\mathfrak{L}$. Properties of the Jordan decomposition imply $(x, y) \in \mathfrak{E}(\mathfrak{L})$ and $\mathfrak{z}_{\mathfrak{g}}(x)_0 \cap \mathfrak{z}_{\mathfrak{g}}(y)_0 = \mathfrak{z}_{\mathfrak{L}}(x)_0 \cap \mathfrak{z}_{\mathfrak{L}}(y)_0$. Whence,

$$(x, y) \in \mathfrak{E}_{\text{sing}} \Leftrightarrow (x, y) \in \mathfrak{E}(\mathfrak{L})_{\text{sing}}.$$

Considering that $\mathfrak{g}$ contains finitely many conjugacy classes of Levi subalgebras, we see that singular points with $x_s \neq 0$ are contained in a finite union of subsets of the form $G_0 \cdot \mathfrak{E}(\mathfrak{L})_{\text{sing}}$. By the induction hypothesis $\dim \mathfrak{E}(\mathfrak{L})_{\text{sing}} \leqslant \mathfrak{E}(\mathfrak{L}) - 2$. Therefore $\dim G_0 \mathfrak{E}(\mathfrak{L})_{\text{sing}} \leqslant \dim G_0 + \dim \mathfrak{E}(\mathfrak{L})_{\text{sing}} - \dim \mathfrak{L}_0 \leqslant \dim G_0 + \dim \mathfrak{E}(\mathfrak{L}) - \dim \mathfrak{L}_0 - 2 = \dim \mathfrak{E} - 2$. The latter equality follows from (9), because $\operatorname{rk} \mathfrak{g} = \operatorname{rk} \mathfrak{L}$ and the induced gradation on $\mathfrak{L}$ also has maximal rank.

(2) Assume $x_s = 0$. Then $x = x_n \in \mathfrak{N}$ [KR]. Clearly, points $(x, y) \in \mathfrak{E}$ with $x = x_n$ are contained in $\mathfrak{E} \cap (\mathfrak{N} \times \mathfrak{g}_1)$. If $\mathfrak{N} = \sqcup_i \mathcal{O}_i$ is finite union of G_0-orbits and $x_i \in \mathcal{O}_i$, then $\mathfrak{E} \cap (\mathfrak{N} \times \mathfrak{g}_1) = \sqcup_i G_0(\{x_i\} \times \mathfrak{z}_{\mathfrak{g}}(x_i)_1)$. Hence it follows from (8) that

$$\dim(\mathfrak{E} \cap (\mathfrak{N} \times \mathfrak{g}_1)) = \dim \mathfrak{g}_1.$$

Therefore (9) and the assumption $\operatorname{rk} \mathfrak{g} > 1$ imply this intersection is of codimension $\geqslant 2$ in $\mathfrak{E}$. □

(4.4) COROLLARY. *All fibers of the moment mapping $\varphi: \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ are normal.*

Proof. It follows from 3.2(ii) and 4.3 that $\mathfrak{E}$ is normal. Again, the method of associated cones and the conditions on φ allow us to transfer this property on the other fibers of φ (Cf. 3.2). □

A Cartan subspace and the Weyl group of $\mathbf{Z}_2$-graded Lie algebra of the maximal rank are nothing else but a Cartan subalgebra and obvious Weyl group of $\mathfrak{g}$ respectively. Therefore comparing 3.2(i), 3.7, 3.8, and 4.4 we get a Chevalley-type statement.

(4.5) THEOREM. *Let $\mathfrak{t}$ be a Cartan subalgebra of reductive Lie algebra $\mathfrak{g}$ and W be the Weyl group relative to $\mathfrak{t}$. Suppose $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the $\mathbf{Z}_2$-gradation of the maximal rank and $\mathfrak{E} \subset \mathfrak{g}_1 \times \mathfrak{g}_1$ is the commuting variety. Then*

$$\mathfrak{t} \times \mathfrak{t}/W \cong \mathfrak{E}//G_0. \qquad \square$$

COROLLARY. *If G_0 is semisimple, then $\mathfrak{E}$ is not factorial.*

Proof. If $\mathfrak{E}$ is factorial, then $\mathfrak{E}//G_0$ must also be factorial. On the other hand, since W as a subgroup of $GL(\mathfrak{t} \times \mathfrak{t})$ does not contain reflections, we get the divisor class group $Cl(\mathfrak{t} \times \mathfrak{t}/W)$ is isomorphic to the character group of W, which is not zero. □

(4.6) In conclusion we state some problems on commuting varieties of θ-groups and their singularities.

(1) Find necessary and sufficient conditions of irreducibility of the commuting variety of a θ-group.

(2) Prove normality of the (obvious) commuting variety in $\mathfrak{g} \times \mathfrak{g}$.

(3) Find the equivariant resolution and prove rationality of singularities of a normal irreducible commuting variety.

Acknowledgement

The author thanks the Max-Planck-Institut für Mathematik for the hospitality and support during the preparation of this paper.

References

[Bo] N. Bourbaki, "Algèbre", Paris, Masson, 1970.
[BPV] J. R. Brennan, M. V. Pinto and W. V. Vasconcelos, The Jacobian module of a Lie algebra, *Trans. Amer. Math. Soc.* 321 (1990), 183–196.
[HSV] J. Herzog, A. Simis and W. V. Vasconcelos, On the arithmetic and homology of algebras of linear type, *Trans. Amer. Math. Soc.* 283 (1984), 661–683.
[Kac] V. G. Kac, Some remarks on nilpotent orbits, *J. Algebra* 64 (1980), 190–213.
[KR] B. Kostant and S. Rallis, Orbits and representations associated with symmetric spaces, *Amer. J. Math.* 93 (1971), 753–809.
[Kn] F. Knop, Über die Glattheit von Quotientenabbildungen, *Manuscripta Math.* 56 (1986), 419–427.
[K] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Aspekte der Mathematik D1, Vieweg-Verlag, Braunschweig 1984.
[LR] D. Luna and R. W. Richardson, A generalization of the Chevalley restriction theorem, *Duke Math. J.* 46 (1979), 487–496.
[Pa] D. I. Panyushev, On orbit spaces of finite and connected linear groups, *Math. USSR-Izv.* 20 (1983), 97–101.
[P] V. S. Pyasetskii, Linear Lie groups acting with finitely many orbits, *Functional Anal. Appl.* 9 (1975), 351–353.
[Ri] R. W. Richardson, Commuting varieties of semisimple Lie algebras and algebraic groups, *Compositio Math.* 38 (1979), 311–327.

[SV] A. Simis and W. V. Vasconcelos, Krull dimension and integrality of symmetric algebras, *Manuscripta Math.* 61 (1988), 63–78.

[Vi1] E. B. Vinberg, The Weyl group of a graded Lie algebra, *Math. USSR-Izv.* 10 (1976), 463–495.

[Vi2] E. B. Vinberg, Complexity of actions of reductive groups, *Functional. Anal. Appl.* 20 (1986), 1–11.

[VP] E. B. Vinberg and V. L. Popov, "Invariant theory", in: *Contemporary problems in Math. Fundamental aspects*, v. 55. Moscow, VINITI, 1989 (Russian). (English translation in: Encyclopaedia of Math. Sci., v. 55, Berlin-Springer, 1994.)