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The Fano surface of the Gushel threefold

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Abstract. The Abel-Jacobi mapping Φ of a Fano threefold X relates the Albanese torus $\mathrm{Alb}(F)$ of its Fano surface $F = F(X)$ to the intermediate Jacobian $J(X)$ of X. For a number of Fano threefolds (see e.g. [CG], [W], [L], [CV], [PB]) the mapping Φ is known to be an isomorphism. In all known cases, the Fano surface F is non-singular. The proofs in [L], [CV], [PB] are based on the Clemens-Letizia criterion (see [L, Prop. 2]) which depends on the degeneration of the non-singular surface F (see also [Co], and [PB] for $\dim X \geqslant 4$). The purpose of this work is to describe a situation in which the generic Fano surface F is singular, and yet we are able to show that there is an Abel-Jacobi isomorphism $\Phi\colon \mathrm{Alb}({}^{n}F) \to J(X)$, where ${}^{n}F$ is a partial desingularization of F. What makes our proof possible is that the extra singularities of the (generic Lefschetz) degeneration of F are disjoint from those of the generic F. The variety X with which we work is the Gushel threefold (the Fano threefold of index 1, degree 10, and of 2-nd kind); the Fano surface $F(X)$ is the birationally non-trivial component of the family of conics on X.

0. Introduction

0.1. Fano surfaces

0.1.1 Fano threefolds

A smooth projective threefold X is called a Fano threefold if the anticanonical divisor $-K_X$ is ample; everywhere in this paper the ground field is supposed to be $\mathbf{C}$.

Let X be a Fano threefold for which $\rho(X) = \operatorname{rank} \operatorname{Pic}(X) = 1$ (see [I]), and let H be the (ample) generator of $\operatorname{Pic}(X)$. In particular, $-K_X = r.H$, for some positive integer $r = r(X) =:$ the index of X. The number $d = d(X) = H^3$ is called degree of X.

0.1.2 Abel-Jacobi mappings (see, e.g, [CG], [L]).

If X is a Fano threefold, then the Hodge number $h^{3,0} = h^{0,3}$ vanishes, and the complex torus (the Griffiths intermediate jacobian of X)

$$J(X) = H^{2,1}(X)^*/(H_3(X, \mathbf{Z})\text{: modulo torsion})$$

is a (principally polarized) abelian variety.

Let $A_1(X)$ be the group (of rational equivalence classes) of algebraic 1-cycles on X which are homologous to 0, and let $\Phi: A_1(X) \to J(X)$ be the Abel-Jacobi map of X. Let, in addition, F be a family of homologous curves on X, let C_0 be a fixed curve of the family, and let $\mathrm{Cl}: F \to A_1(X)$ be the class-map $C \to$ (the rational equivalence class of $C - C_0$). The map $\Phi'_F = F.\mathrm{Cl}$: $F \to J(X)$ is called Abel-Jacobi map for F.

Suppose that F is smooth. If $a: F \to \mathrm{Alb}(F)$ is the Albanese map then (by the universal property of Alb) the map Φ'_F splits into a composition $\Phi'_F = \Phi_F . a$. The map $\Phi_F: \mathrm{Alb}\, F \to J(X)$ is called also Abel-Jacobi map for F.

0.1.3 Abel-Jacobi mappings for Fano surfaces

Let X be a Fano threefold for which $\rho(X) = 1$. According to the classification (see [I]), the index $r(X)$ does not exceed 4. In fact, only the quadric $X_2 \subseteq \mathbf{P}^4$, and the projective space $\mathbf{P}^3$ have indices greater than 2.

In cases $r(X) = 1, 2$, the empiric rule is to study the Abel-Jacobi map for the rationally non-trivial components F of the family $F(X)$ of connected curves on X of degree $2/r$. Any such component F is, in the general case, a surface—a Fano surface of the Fano threefold X. For example, if X is a cubic hypersurface in $\mathbf{P}^4$, or if X is a quartic double solid, then $r(X) = 2$, $F = F(X) = \{$the lines on $X\}$, and $\Phi_F: \mathrm{Alb}(F) \to J(X)$ is an isomorphism ([CG], [W]).

Let $r(X) = 1$, let F be a component of the family of conics on X, and let $\Phi_F: \mathrm{Alb}(F) \to J(X)$ be the Abel-Jacobi map. In [L], Letizia proves that Φ_F is an isomorphism for the family $F = F(X_4)$ of conics on the generic quartic hypersurface $X_4 \subseteq \mathbf{P}^4$. In fact, Letizia proves a general criterion, based on results of Clemens. The criterion gives sufficient conditions for Φ_F to be an isomorphism (see [L, Prop. 2]).

The Clemens-Letizia criterion has been used also by Ceresa and Verra for proving the Abel-Jacobi isomorphism for the sextic double solid $X \to \mathbf{P}^3$, and for $F =$ (the family of half-preimages of the conics in $\mathbf{P}^3$, which are totally tangent to the branch locus) (see [CV]). In higher dimensions, the same criterion has been used by Collino (the Abel-Jacobi isomorphism for the surface of planes on the general cubic fivefold, see [Co]), and by Picco Botta (the A. J. isomorphism for the surface of k-dimensional quadrics on the general complete intersection of three quadrics in $\mathbf{P}^{2k+4}$, see [PB]).

0.2. The Abel-Jacobi mapping for the Fano surface $F(X''_{10})$

The Fano surface F of any $X = X''_{10}$ (see 0.4) is singular along a smooth rational curve ρ (see §4, Lemma A). Let $n: {}^nF \to F$ be the normalization of

F. (It can be seen that ^{n}F is smooth, -see, for example (2.5.4), and 3.2. Anyway, we may suppose that $^{n}F \to F$ is a desingularization of F). Then, by the universal property of Alb, the map $n.\Phi'_F: {}^{n}F \to J(X)$ splits into a composition $\Phi_F.a$, where a is the Albanese map for ^{n}F, and Φ_F: $A(F) := \mathrm{Alb}(^{n}F) \to J(X)$.

We call the map Φ_F the Abel-Jacobi map for $F(X)$, $X = X''_{10}$.

0.3. The results in the paper

Let $X = X''_{10}$ be the Fano threefold of degree 10, of 2nd kind (a Gushel threefold). By definition, X is a double covering $\pi: X \to Y$ of the del Pezzo threefold $Y = Y_5 \subseteq \mathbf{P}^6$ (see 0.4).

In this paper we study the non-trivial component $F(X)$ of the family of conics on $X = X''_{10}$ (see (2.1.1), (2.1.2)). More precisely, we study the degenerations of $F(X)$, when X moves in a general Lefschetz pencil $\{X_t: t \in \mathbf{P}^1\}$ of X''_{10},s (see 4.2). We prove that any $F_t = F(X_t)$ in the pencil is a surface with a prescribed singularity along a smooth rational curve ρ, which parametrizes the curve p of double lines on the del Pezzo threefold Y. Moreover, the generic $F(X_t)$ is smooth outside the curve ρ (see (1.2.2), Prop. 2.4.2, 2.5, and Lemma A in §4).

The codim. 1-degenerations $L, Q, \partial R$, and N of the branch locus S (see (2.1.3), (2.4.1)) give rise to additional singularities of $F(X_t)$ (see §3, and Lemmas B, C, in §4).

Let $X = X''_{10}$ be generic. The double covering $\pi: X \to Y$ defines in a natural way an involution $i: F(X) \to F(X)$ (see 2.2.1). If the branch locus S does not contain conics (and S is otherwise generic), the quotient surface $F(X)/i$ can be embedded in a natural way in the $\mathbf{P}^1$-bundle $Z = \mathbf{P}_{\mathbf{P}(V^*)}(O \oplus O(1))$, as a degeneration locus $D_1(\phi)$ of a well-defined bundle map ϕ (see Prop. 2.3.2(2.b)). The isomorphism $F(X)/i \cong D_1(\phi)$ reduces the local study of $F(X)$ to the local study of the degeneration locus $D_1(\phi)$ (see (2.5.2), (2.5.4)). For instance, the approach (2.5.2), which studies the local behavior of $D_1(\phi)$ at the generic point, is based on the possibility to find appropriate coordinates in one of the points of tangency of the pair of bitangent conics (pf, q) which define the point of $D_1(\phi)$.

In §3 we study the singularities of $D_1(\phi) = F(X)/i$ in the case when $X = X''_{10}$ acquires a node. The main results in this section (Lemma 3.2.3, and Lemma 3.2.4) are used in §4 to prove Lemma C, which describes the degeneration of $F(X)$ for X with a simple node.

The description of the generic Lefschetz degeneration of the Fano surface $F(X''_{10})$ (§4, Lemmas A, B, C, D) is analogous to the sufficient conditions of the Clemens-Letizia criterion (see [L, Prop. 2 (a, b, c, d), and Lemma 1.1]). The only difference is that, when $X = X''_{10}$, the Fano surfaces have prescribed singularity—the curve ρ (see above). The original proof of the criterion is based on a local study of the family of $F(X''_{10})$'s in a neighbourhood of the acquired singularity, the double curve Γ (see §4, Lemma C, and [L, 1.2]). The key point is that, if the Lefschetz pencil is sufficiently general, the curve Γ lies "far from" the prescribed singular curve ρ, and we can repeat the proof of the Clemens-Letizia criterion (see 4.2, Lemma 4.2(***)). As a result, we obtain the following (see §4, Theorem 4.3):

THEOREM. *Let $X = X''_{10}$ be a general Gushel threefold (= a Fano threefold of degree* 10, *of 2nd kind), and let $F(X)$ be the Fano surface of X. Then the Abel-Jacobi mapping $\Phi_F\colon A(F) \to J(X''_{10})$ (see* 0.2) *is an isomorphism.*

0.4. Basic definitions and notation

The following notation will be used everywhere, unless the opposite is stated:

$$\mathbf{P}^4 = \mathbf{P}(V),\ V \cong \mathbf{C}^5;\ V_{k+1} \subseteq V\ (\dim V_{k+1} = k+1,\ k = 1, 3);$$

$P(V^*) = G(4, V) = \{\dim 4\text{-subspaces } V_4 \subseteq V\} = \{\mathbf{P}^3\colon \mathbf{P}^3 = P(V_4) \subseteq \mathbf{P}(V) = \mathbf{P}^4\}$; $G = G(2, V) = \{\dim 2\text{-subspaces } V_2 \subseteq V\} \cong \{\text{the lines } 1 = P(V_2) \subseteq \mathbf{P}^4 = \mathbf{P}(V)\} = G(1\colon \mathbf{P}(V))$; $\mathrm{Pl}\colon G \to \mathbf{P}^9 = \mathbf{P}(\Lambda^2 V)$; $V_{k+1} \subseteq \Lambda^2 V$ ($\dim V_{k+1} = k+1$; $k = 6, 7$), a subspace in a general position; $\mathbf{P}^k = \mathbf{P}(V_{k+1}) \subseteq \mathbf{P}^9 = \mathbf{P}(\Lambda^2 V)$, $k = 6, 7$.

For $Z \subseteq \mathbf{P}^n$, $\langle Z \rangle = \text{span } Z$ means the linear span of Z.

Let $\mathbf{P}^{n-1-a} \subseteq \mathbf{P}^{n-b}$, $n - 1 \geqslant a \geqslant b \geqslant 0$, be a flag of subspaces of $\mathbf{P}^n$; by definition, the basic Schubert cycles, of codimension $a + b$ in $G(1\colon \mathbf{P}^n)$, are:

$$\sigma(\mathbf{P}^{n-1-a}, \mathbf{P}^{n-b}) = \{1\text{-line in } \mathbf{P}^n\colon 1 \cap \mathbf{P}^{n-1-a} \neq \varnothing,\ 1 \subseteq \mathbf{P}^{n-b}\};$$

the classes σ_{ab} of these cycles form a basis of the Chow ring $A.(G(1\colon \mathbf{P}^n))$. The formulae of Pieri and Giambelli (see [GH, Ch. I, §5], or [F, Ch. XIV]) describe the intersection of classes in the ring.

Let $V = \mathbf{C}^5$, $G = G(2, V)$, be as above.

By definition, a del Pezzo threefold is a smooth intersection $Y = G \cap \mathbf{P}^6$. In fact, any two del Pezzo threefolds are projectively equivalent (see [I]); in this sense, we call Y the del Pezzo threefold. The threefold Y is the unique Fano threefold with invariants $p(Y) = 1$, $r(Y) = 2$, $d(Y) = 5$ (see [I]).

The variety X''_{10}: Let Q be a quadric in $\mathbf{P}^6$, which intersects Y along a surface $S = Q \cap Y$. The pair (Y, S) defines a two-sheeted covering $\pi\colon X \to Y$

with a branch locus S. Any (smooth) threefold $X = X''_{10}$ is a Fano threefold, of index 1, and of degree 10 (of 2nd kind) (see [G]).

The variety X'_{10}: Let $X = X'_{10}$ be a (complete) intersection of $G \subseteq \mathbf{P}^9$ with a subspace $\mathbf{P}^7$, and with a quadric Q. Any (smooth) $X'_{10} = G \cap \mathbf{P}^7 \cap Q$ is a Fano threefold, of index 1, and of degree 10 (of 1st kind) (see [I], [G]).

1. Lines and conics on the del Pezzo threefold $Y \subseteq \mathbf{P}^6$

1.1. Grassmann conics

The description of the conics on a Grassmannian is well-known:

Let q be a connected conic on $G(l:\mathbf{P}^n)$, $n \geqslant 3$, and let $\mathrm{Gr}(q) = \cup\{l : l \in q\}$ be the union of lines $l \subseteq \mathbf{P}^n$, s.t. $l \in q$. Then q belongs to one of the types:

(τ-conics). $\langle \mathrm{Gr}(q) \rangle = \mathbf{P}^3$, and: ($\tau$1): q is smooth, $\mathrm{Gr}(q) \cong \mathbf{P}^1 \times \mathbf{P}^1$, and the set $q = \{l : l \in q\}$ describe the lines of one of the rulings of the quadric $\mathrm{Gr}(q)$; (τ2): $q = l' \cup l''$, $\mathrm{Gr}(q) = P' \cup P''$ is a union of two planes; $P' \cap P'' = \mathbf{P}^1$, and $\exists x', x'' \in \mathbf{P}^1$, $x' \neq x$, such that $l' = \sigma(x', P')$, $l'' = \sigma(x'', P'')$, see 0.4.

(σ-conics). $\langle \mathrm{Gr}(q) \rangle = \mathbf{P}^3$, and: ($\sigma$1): q is smooth, $\mathrm{Gr}(q)$ is a quadratic cone with a vertex $x \in \mathbf{P}^4$, etc.; (σ2): $q = l' \cup l''$, $l' \neq l$, but $x' = x''$, see (τ2).

(ρ-conics). $\langle \mathrm{Gr}(q) \rangle = \mathbf{P}^2$, and ($\rho$1): q is smooth, and the set $\{l : l \in q\}$ describes the family of tangent lines to a smooth conic $q^* \subseteq \mathbf{P}^2$; ($\rho$2). $q = l' \cup l''$, and $P' = P'' = \mathbf{P}^2$, $x' \neq x''$, see (τ2), and (σ2).

(double lines). $q = 2.l$. $l = \sigma(x, \mathbf{P}^2)$.

1.2. Lines and conics on the del Pezzo threefold Y

The family of lines $G(Y)$ on the del Pezzo threefold Y is well-studied. We shall sketch briefly some of the properties of $G(Y)$, which will be useful in the study of the families of conics on Y and X''_{10}.

1.2.1 DESCRIPTION (Lines on the del Pezzo threefold (see [I], [FN], [II]). Let Y be the del Pezzo threefold (see 0.4), and let $G(Y)$ be (the base of) the family of lines on $Y \subseteq \mathbf{P}^6$. Then:

(1) $G(Y) \cong \mathbf{P}^2$. There exists a natural embedding ker: $G(Y) \to \mathbf{P}^4 = \mathbf{P}(V)$, such that: (a) $B := \mathrm{Im}(\ker) \subseteq \mathbf{P}^4$ is a smooth projection of the Veronese surface; moreover $Y = \{1 \in G$: the line $1 \subseteq \mathbf{P}^4$ is a trisecant of $B\}$; (b) If $L = \sigma(x_L, \mathbf{P}^2_L) \in G(Y)$ is a line on Y, then $\ker(L) = x_L =$ the "center" of the line L, and $\mathbf{P}^2_L \cap B = x_L \cup$ (a conic $q(x_L) = q(L)$); the conic $q(L)$ parametrizes the centers of all the lines on Y, which intersect the line L; i.e., $q(L) =$ (the closure of) $\{\ker(L') : L' \in G(Y), L' \neq L$, and $L' \cap L \neq \varnothing\}$. Moreover: (c) Let $G(Y) = \mathbf{P}^2$ be as above, and let L be a line on Y. Then L can

be either a (0,0)-line, or a $(-1,1)$-line, on Y (see [FN]). The set of $(-1,1)$-lines on Y is isomorphic to a smooth conic $q(-1,1) \subseteq \mathbf{P}^2$, and the condition "$x_L \in q(L)$" describes the set of centers $x_L = \ker(L)$ of $(-1,1)$-lines L on Y.

(2) Let $B_{(-1,1)} := \ker(q_{(-1,1)}) \subseteq B$ be the image of $q_{(-1,1)} \subseteq G(Y)$. In particular $B_{(-1,1)}$ is a rational normal quartic curve in $\mathbf{P}^4$, and the points of $B_{(-1,1)}$ are characterized by the property $x_L \in q(L) := q(x_L)$, see (1).

(2) Let $C_{(-1,1)} \subseteq G$ be the curve of tangent lines to $B_{(-1,1)}$. Obviously, the curve $C_{(-1,1)}$ is a rational normal sextic; moreover, $C_{(-1,1)} \subseteq Y$, and the set of $(-1,1)$-lines on Y coincides with the set of tangent lines to $C_{(-1,1)}$.

(3) Let $S_{(-1,1)}$ be the union of tangent lines to $C_{(-1,1)}$ (see also (a)). Then:

(a) $Y - S_{(-1,1)} = \{1 \in Y$: there are exactly three lines on Y through $1\}$;
(b) $S_{(-1,1)} - C_{(-1,1)} = \{1 \in Y$: there are exactly two lines on Y through $1\}$;
(c) $C_{(-1,1)} = \{1 \in Y$: there exists exactly one line on Y through $1\}$.

(1.2.2) PROPOSITION (Description of the family of conics on Y). *Let $F(Y)$ be the family of conics on the del Pezzo threefold Y. Then (see* (0.4)):

(1) *There are no σ-conics, or ρ-conics on Y (see* 1.1); *in particular, all the conics on Y are either τ-conics, or double lines.*
(2) *There exists a natural isomorphism* $\mathrm{sp}: F(Y) \to \mathbf{P}(V^*)$, such that;
(a) $p := \mathrm{sp}\{$*double lines on Y*$\}$ *is a rational normal quartic curve in* $\mathbf{P}(V^*)$;
(b) $\mathrm{Sec} :=$ (*the closure of*) $\mathrm{sp}\{\tau 2$-*conics on* $Y\}$ *is a (singular) cubic hypersurface in* $\mathbf{P}(V^*)$; *moreover* $\mathrm{Sec} =$ (*the union of bisecant lines of p*).

Proof. (1) Let x be a point of $\mathbf{P}^4 = \mathbf{P}(V)$, and let $\sigma(x) = \{1 \in G: x \in L\}$ be the Schubert cycle of lines, which pass through x, see (0.4). The cycle $\sigma(x)$ is isomorphic to $\mathbf{P}^3$, in the Plücker embedding $\mathrm{Pl}: G \to \mathbf{P}^9$. As $Y = G \cap \mathbf{P}^6$ (i.e., $Y = (\sigma_{10})^3$, in $A.(G)$), then $\sigma(x) \cap Y$ can be only a subspace in $\sigma(x) = \mathbf{P}^3$. It follows from (1.2.1)(1) that

$$\dim(\sigma(x) \cap Y) > 0 \Leftrightarrow x \in B = \ker(G(Y));$$

moreover, if $x \in B$, then $\dim(\sigma(x) \cap Y) = 1$, and $\sigma(x) \cap Y =$ the line L, which corresponds to the point $\ker^{-1}(x)$. Remember that $B \subseteq \mathbf{P}^4$ is a smooth projection of the Veronese surface. In particular, Y does not contain σ-conics. In fact, let $x \in \mathbf{P}^4$ be the center of a σ-conic q(= the center x of the cone $\mathrm{Gr}(q)$, or $x = x' = x''$, in case ($\tau 2$)). Then $q \subseteq \sigma(x) \cap Y$, i.e., $\dim(\sigma(x) \cap Y) > 0 \Leftrightarrow x \in B$, and $q \subseteq$ the line $\sigma(x) \cap Y$ – a contradiction. The proof that Y does not contain ρ-conics is similar.

(2) Let $q \subseteq Y$ be a conic, and let $\mathrm{Gr}(q)$ be as in 1.1. It follows from (1)

that $\langle \mathrm{Gr}(q) \rangle = \mathrm{span}\, \mathrm{Gr}(q) \neq \mathbf{P}^3 \Leftrightarrow q$ is a double line. We shall see that the map $\mathrm{sp}: F(Y) \to \mathbf{P}(V^*)$, $q \to \mathrm{span}\langle \mathrm{Gr}(q) \rangle$ is uniquely defined on the subset of double lines on Y. Let $L = \sigma(x_L, \mathbf{P}^2_L) \subseteq Y$ be a line, and let $\mathbf{P}^1(L) := \{\mathbf{P}^3 \subseteq \mathbf{P}^4 : \mathbf{P}^2_L \subseteq \mathbf{P}^3\} \subseteq \mathbf{P}(V^*)$ be the line of hyperplanes through $\mathbf{P}^2_L$. The line $\mathbf{P}^1(L)$ parametrizes, via the map sp, the set of all conics on Y of type $L + L'$, $L' \cap L \neq \varnothing$. Let $q(L) = q(x_L) \subseteq B$ be the conic (of centers of lines L', which intersect the line L)—see (1.2.1)(1). So, we define an isomorphism $q(L) \to \mathbf{P}^1(L)$, $x_{L'} \to \langle \mathrm{Gr}(L + L') \rangle$, which coincides with sp^{-1}, on $q(L)$; here we use the identification $\ker: L \to x_L$. Let L be a $(0,0)$-line on Y. It follows from (1.2.1) that $L' \neq L$, for any line L' which intersects L. In case L is a $(-1,1)$-line, $x_L \in q(L)$, see (1.2.1)(b); obviously, the map sp^{-1} is well-defined also in the point x_L which represents the $(-1,1)$-line L. Therefore, $F(Y) \cong \mathbf{P}(V^*)$, via sp, and the sp-image of the set $\{2L: L \text{ is a } (-1.\ 1)\text{-line on } Y\}$ describes a smooth rational curve $p \subseteq \mathbf{P}(V^*)$.

(2a–2b). Let L, $q(L)$, etc., be as above. Suppose that L is a $(0,0)$-line on Y, and let $F(Y) = \mathbf{P}(V^*)$ be the natural identification. It follows from (1.2.1) that the line $\mathbf{P}^1(L) \subseteq \mathbf{P}(V^*)$ intersects the curve p (see above) in exactly two points. In case L is a $(-1,1)$-line on Y, the line $\mathbf{P}^1(L)$ is tangent to p. It follows that the sp-image of the set of splitting conics on Y coincide with $\mathrm{Sec}(p) =$ (the union of bisecant lines of p). It is not hard to see that $\deg \mathrm{Sec}(p) = 3$. In fact, let $m \subseteq P(V^*)$ be a general line. The line m describes all the hyperplanes in $\mathbf{P}(V)$ through a fixed plane $\mathbf{P}^2$. The plane $\mathbf{P}^2$ intersects B in four points. The, $\deg \mathrm{Sec}(p) = \#(\mathrm{Sec}(p) \cap \mathbf{P}^1) = \#$ (splitting conics through these four points, after the identification $B \cong \mathbf{P}^2$, see (1.2.1)) $= 3$.

2. The surface $F(X''_{10})$

2.1. *The families $C^2(X''_{10})$ and $F(X''_{10})$*

(2.1.1) Let $Y = G \cap \mathbf{P}^6$ be the del Pezzo threefold, and let $S = Y \cap Q$, $X = X''_{10}$, $\pi: X \to Y$, etc., be as in (0.4). We shall suppose that the branch locus S has at most finitely many isolated double points. Let H be the hyperplane section of Y, and let $C \subseteq X$ be a curve. By definition, $\deg C := \#(C, \pi^* H)$. Define:

$C^2(X) := \{q \subseteq X: C \text{ is a conic, i.e. } \deg q = 2, \text{ and } q \text{ is connected}\}$.

$F_0 = F(Y)_S :=$ (the closure of) $\{q \in F(Y)$: the conic q is bitangent to $S\}$.

Let $q \in F_0$. In case q does not lie on S, the preimage $p^{-1}(q)$ splits into a union of two conics on $X: \pi^{-1}(q) = q' + q''$. Define:

$F(X) :=$ (the closure of) $\{q' \subseteq X: \exists q \in F_0(\pi^{-1}(q) = q' + q'', \text{ for some } q''\}$.

Now, the following is obvious:

(2.1.2) PROPOSITION. *Let $\pi: X = X''_{10} \to Y$, $C^2(X) =$ the family of conics* *on X, etc., be as above, and let $G(Y) \cong \mathbf{P}^2$ be the family of lines on the del Pezzo threefold Y (see §1). Then*

$$C^2(X) = F(X) \cup \pi^* G(Y),$$

where $\pi^* G(Y) = \{\pi^* L: L\text{-a line on } Y\} \cong G(Y) \cong \mathbf{P}^2$. Let $p \subseteq G(Y) = \mathbf{P}(V^*)$ be the (smooth rational) curve of $(-1, 1)$-lines on Y (see (1.2.2)). The intersection $F(X) \cap \pi^* G(Y) = \pi^* p \cup \{\pi^* L: L = \text{a line on } S\}$.

(2.1.3) REMARK. It is not hard to see that the generic branch locus $S \subseteq Y$ does not contain lines, or conics. Let $I(Y)$ be the graded homogeneous ideal of $Y \subseteq \mathbf{P}^6$. The ideal $I(Y)$ is generated by the component $I_2(Y)$. The component $I_2(Y)$ is isomorphic to $I_2(G)$, for $G = G(2, V) \subseteq \mathbf{P}^9$. The component $I_2(G)$ is canonically isomorphic to $V = \mathbf{C}^5$; the 1-space $V_1 \subseteq V$ corresponds to the Plücker quadric of the 4-space V/V_1. In particular, the parameter space S of all $S = Y \cap Q$, Q = a quadric in $\mathbf{P}^6$, is loc. isomorphic to $\mathbf{P}(S^2 V_7^*)/\mathbf{P}(I_2(Y)) = \mathbf{P}^{22}$. It can be verified directly (see, for example, [CV]) that the subsets

$$L = \{S = Y \cap Q: \exists \text{ a line } L \subseteq S\},$$

$$Q = \{S = Y \cap Q: \exists \text{ a conic } q \subseteq S\}, \text{ and}$$

$$N = \{S = Y \cap Q: S \text{ is singular}\} = (\text{the closure of}) \{S: S \text{ has a node}\},$$

are of codim 1 in S. In particular, we obtain:

(2.1.4) COROLLARY (See the conditions of Prop. 2.1.3). *For the generic $X = X''_{10}$, $F(X) \cap \pi^* G(Y) = (\pi^* p)_{\text{red}}$: the smooth rational curve, which parametrizes the preimages of $(-1, 1)$-lines on Y.*

2.2. The covering $F(X''_{10}) \to F_0$

(2.2.1) It follows from the definitions in (2.1.1) that the covering $\pi: X = X''_{10} \to Y$ defines in a natural way an involution $i: F(X) \to F(X)$; if q is a generic point of F_0, the involution i interchanges the half-preimages q' and q'' of $\pi^{-1}(q)$. One might expect that the induced natural map $\pi(i)$: $F/i \to F_0$ is an isomorphism, at least, for the generic S:

(2.2.2) PROPOSITION. *Let $\pi: X = X''_{10} \to Y$, $F(X)$, F_0, etc., be as in* 2.1.1. *Suppose, moreover that S is generic. Let $\pi(i)$: $F(X)/i \to F_0$ be as above. Then $\pi(i)$ is an isomorphism, outside the finite set $R_0 = \{q \in F_0: q = 1 + m$, and the points of tangency of the lines* 1 *and m, to S, coincide$\}$.*

Proof. It remains to consider the case $q \in R_0$ (obviously, the set R_0 is nonempty for the generic S). In this case,

$$\pi^{-1}(q) = \pi^{-1}(l + m) = \pi^{-1}(l) + \pi^{-1}(m) = l' + l'' + m' + m'',$$

where l', $l''m'$, and m'' are defined in an obvious way. The point is that these four lines have a common point. Consider $l + m$ as a point of F_0. Then, the four points of $F(X)$, over $l + m$, are (the conics): $l' + m'$, $l' + m''$, $l'' + m'$, and $l'' + m''$. The involution i interchanges the components of the pairs $(l' + m', \ l'' + m'')$, and $(l' + m'', \ l'' + m')$. In fact, the natural map $F(Y)/i \to F_0$ desingularizes F_0 in the point $l + m$, which is a point of a simple selfintersection of F_0, etc.

(*) The set R_0 is finite, since R_0 is defined by the closed condition: "the points of tangency of L_1 and L_2 coincide", on the 1-dimensional incidence $\Sigma = \{(L_1, L_2)$: L_1, L_2 – lines on Y, which are tangent to the branch locus $S\}$.

2.3. The surface $F(X''_{10})$ as a double covering of the locus $D_1(\phi)$

(2.3.1) *The bundle M*

In the notation of 2.3 (see also 0.4), let $V_4 \in V^*$, and let $M(V_4) = \Lambda^2 V_4 \cap V_7$. It is not hard to see that $\dim M(V_4) = 3$, for any $V_4 \in V^*$. The rule $V_4 \to M(V_4)$ define a rank 3 vector bundle $M \to \mathbf{P}(V^*)$. We shall define M in a more convenient way. Let

$$i: \tau_{4,V} = \Omega_{\mathbf{P}(V^*)}(1) \to \mathbf{P}(V^*) \times V$$

be the natural embedding of the standard tauthological subbundle on $\mathbf{P}(V^*) = G(4, V)$, and let

$$\Lambda^2 i: \Lambda^2 \tau_{4,V} \to \mathbf{P}(V^*) \times \Lambda^2 V$$

be it 2nd exterior power. The embedding $V_7 \subseteq \Lambda^2 V$ defines a surjective morphism $p: \Lambda^2 V \to E := \Lambda^2 V/V_7$. Evidently, $M = \ker(j . \Lambda^2 i)$.

Let $q \in F(Y)$ be a conic on Y, and let $V_4 = \mathrm{sp}(q)$, (see Prop. 1.2.2). We shall translate the condition "q is bitangent to S" in terms of the bundle M. The Grassmannian $G(2, V_4)$ is embedded:

(a) in G, as the Schubert cycle $\sigma_{11}(\mathbf{P}(V_4)) = \{$the lines in $\mathbf{P}(V_4) \subseteq \mathbf{P}(V)\}$;
(b) in $\mathbf{P}(\Lambda^2 V_4) =$ the fiber $\mathbf{P}(\Lambda^2 \tau_{4,V})(V_4)$, as the Plücker quadric $\mathrm{Pf}(V_4)$.

Therefore (see 0.4): $\mathrm{pf}(V_4) := \mathrm{Pf}(V_4) \cap \mathbf{P}(V_7) \subseteq \mathbf{P}(\Lambda^2 V_4) \cap \mathbf{P}(V_7) = \mathbf{P}(M(V_4))$, is a conic in $\mathbf{P}(M(V_4))$, which lies on Y. It follows from the definition of the map sp that $\mathrm{pf} = \mathrm{sp}^{-1}$, i.e., $q = \mathrm{pf}(V_4(q))$ coincides with the restriction to $\mathbf{P}(M(V_4))$ of the Plücker quadric $\mathrm{Pf}(V_4)$. On the one hand, the equations of the Plücker quadrics in G form a line bundle $\mathrm{Pf} \cong O_{\mathbf{P}(V^*)}(1)$. Therefore, the map $\mathrm{pf}: \mathbf{P}(V^*) \to F(Y)$ can be lifted to a bundle map $\mathrm{pf}: O_{\mathbf{P}(V^*)}(1) \to S^2 M^*$.

On the other hand, let $S = Y \cap Q$ be the branch locus of $\pi: X \to Y$. The set of restrictions $Q(V_4) = Q \cap \mathbf{P}(M(V_4))$, $V_4 \in V^*$, is parametrized by the structure sheaf $O_{\mathbf{P}(V^*)}$, i.e., there exists a map $q: O_{\mathbf{P}(V^*)} \to S^2M^*$, s.t. the conic $(q(V_4) = 0) \subseteq \mathbf{P}(M(V_4))$ coincides with the restriction $Q \cap \mathbf{P}(M(V_4))$. The map

$$\mathrm{pf} \oplus q: O_{\mathbf{P}(V^*)}(1) \oplus O_{\mathbf{P}(V^*)} \to S^2M^*$$

parametrizes the set of all the pencils of conics $\langle \mathrm{pf}(V_4), q(V_4)\rangle$, $V_4 \in \mathbf{P}(V^*)$. In particular, the conic $\mathrm{pf}(V_4)$ is (at least) bitangent to the branch locus $S = Y \cap Q \Leftrightarrow \exists (t_0 : t_1) \in \mathbf{P}^1$, s.t., rank $t_0 . \mathrm{pf}(V_4) + t_1 . q(V_4) \leqslant 1$. As a corollary we obtain:

(2.3.2) PROPOSITION. *Let $\pi: X \to Y$, $S = Y \cap Q$, $i: F(X) \to F(X)$, etc., be as in 2.2. Suppose moreover that the branch locus S does not contain conics, i.e., $S \notin Q$ (see 2.13). Let $Z = \mathbf{P}_{\mathbf{P}(V^*)}(O \oplus O(1))$, and let $\zeta: Z \to \mathbf{P}(V^*)$ be the projection. Then:*

(1) *The points $z = (V_4, s) \in Z$, $s = (s_0, s_1)$, parametrize the conics $s_0 . \mathrm{pf}(V_4) + s_1 . q(V_4)$, as points of the pencils $\langle \mathrm{pf}(V_4), q(V_4)\rangle$, $V_4 \in \mathbf{P}(V^*)$.*

(2) *Let $L_\zeta = \ker(\zeta^*(O \oplus O(1)) \to O_{Z/\mathbf{P}(V^*)}(1))$ be the tautological subsheaf for the projection ζ: $Z \to \mathbf{P}(V^*)$, let $i: L_\zeta \to {}^*(O \oplus O(1))$ be the embedding, and let $\phi = \zeta^*(\mathrm{pf} \oplus q)$: $L_\zeta \to \zeta^* S^2M^*$. Let*

$$D_k(\phi) := \{z \in Z: \text{rank } \phi(z) \leqslant 1\}$$

be the k*th determinantal (the* k*th degeneracy locus) of the symmetric bundle map ϕ, $k = 0, 1, 2$ (see [F, Ch. XIV], [HT], [JLP]). Then: (a) the projection $\zeta: Z \to \mathbf{P}(V^*)$ maps $D_1(\phi)$ onto the family of bitangent conics F_0; moreover, ζ is an isomorphism outside the finite set R_0 (see Prop. 2.2.2); ($\Rightarrow$)(b) Let $i: F(X) \to F(X)$ be the involution on $F(X)$ (see 2.2.1), and let $F(X)/i$ be the factor family. Then $F(X)/i \cong D_1(\phi)$.*

2.4. The prescribed singularities of $D_1(\phi)$

2.4.1 The condition of tangency

Let $E = \mathbf{C}^3$, and let $\mathbf{P}^5 = \mathbf{P}(S^2E^*)$ be the space of conics on $\mathbf{P}^2 = \mathbf{P}(E)$. Let $\mathrm{Ver} \subseteq \mathbf{P}^5$ be the Veronese surface, which represents the conics of rank 1—the double lines, etc. The condition "the conic pf is at least bitangent to the conic q, and $\mathrm{pf} \neq q$" means that the line $\langle \mathrm{pf}, q\rangle$ and the surface Ver have a common point. The case R_0 (see (2.2.2)) corresponds to the case when $\langle \mathrm{pf}, q\rangle$ is a bisecant of Ver, since pf and q are pairs of lines through a common point. In case $\langle \mathrm{pf}, q\rangle$ is a simple secant of Ver, the fibre parameter $s = (s_0, s_1)$, see (2.3.2)(1), separates the two intersection points of $\langle \mathrm{pf}(V_4), q(V_4)\rangle$ and Ver. The set:

$$\partial R = \{S = Y \cap Q : \exists V_4 \in \mathbf{P}(V^*) : \langle \mathrm{pf}(V_4), q(V_4)\rangle \text{ is tangent to Ver}\}$$

is of codim 1 in the parameter space S -see Rem. 2.1.3 and 2.2.2(*).

Let $p \subseteq F(Y) = \mathbf{P}(V^*)$ be the curve of double lines on Y (see 1.2.2). Denote also by p the (isomorphic) preimage of p, in $D_1(\phi)$. In particular, $p \subseteq D_1(\phi)$ is a smooth rational curve.

(2.4.2) PROPOSITION. *Let L, Q, ∂R, and N be the subsets of S defined in* 2.1.3, *and* (1), *let $S \in S - (L \cup Q \cup \partial R \cup N)$, and let $X = X''_{10}$ be the double covering of Y, ramified over S. Then the locus $D_1(\phi) = F(X''_{10})/i$ (see* 2.2.1, 2.3.2(2)) *is smooth outside the* (*smooth rational*) *curve p.*

Proof. Let $Z = \mathbf{P}_{\mathbf{P}(V^*)}(O \oplus O(1))$, and let $z = (V_4, s) \in D_1(\phi) \subseteq Z$, see 2.3.2. By definition, the point z corresponds to the double line

$$s_0 . \mathrm{pf}(V_4) + s_1 . q(V_4) \subseteq \mathbf{P}(M(V_4)).$$

As the surface S is smooth, and S does not contain lines and conics, we may suppose that $\mathrm{pf}(V_4)$ and $q(V_4)$ have no common components, and that S is smooth in the points of $\mathrm{pf}(V_4) \subseteq S$. There are three possible cases (see Prop. 1.2.2):

(1) $\mathrm{pf}(V_4)$ is a double line on Y, i.e., $V_4 \in p$;
(2) $\mathrm{pf}(V_4) = 1 + m$, i.e. $V_4 \in \mathrm{Sec}$, and either (a) $\mathrm{pf}(V_4) \in \mathrm{Sec} - R_0$, or (b) $\mathrm{pf}(V_4) \in R_0$ (see Prop. 2.2.2);
(3) $\mathrm{pf}(V_4)$ is a smooth conic.

In cases (2)(a) and (3), the line $\langle \mathrm{pf}(V_4), q(V_4)\rangle$ is a simple secant of Ver (see 2.4.1). Let us introduce local coordinates in the point $z = (V_4, s) \in D_1(\phi)$. We may suppose that $q(V_4)$ coincides with the intersection point of $\langle \mathrm{pf}, q\rangle$ and Ver, i.e., we may suppose $q(V_4) \in \mathrm{Ver}$; i.e. $z = (V_4, (0:1))$. Since the conic $\mathrm{pf}(V_4)$ is not a double line, the point $\mathrm{pf}(V_4)$ does not lie on Ver. Then, in the local coordinates, the equations for the 1st order local deformation of $z \in D_1(z)$, inside Z, define an isomorphism between the tangent spaces $T_z D_1(\phi)$, and $T_{q(V_4)}\mathrm{Ver} \cong \mathbf{C}^2$, inside $T_z Z \cong \mathbf{C}^5$; in particular, $D_1(z)$ is smooth in z (see also the local study (2.5.2)).

In case (2)(b), the line $\langle \mathrm{pf}(V_4), q(V_4)\rangle$ is at least a bisecant of Ver. By condition, the branch locus S lies outside the set ∂R; in particular, the line $\langle \mathrm{pf}, q\rangle$ cannot be a tangent line to Ver. The fibre parameter (s), of $z = (V_4, s)$, separates the common points q_1 and q_2 of $\langle \mathrm{pf}, q\rangle$, and Ver. At each $q_i, i = 1, 2$, we can repeat the argument for the cases (2)(a), and (3). Therefore, $D_1(\phi)$ is smooth also in case (2)(b). In fact, in this case the conic

$\mathrm{pf}(V_4)$, respectively V_4, is an isolated point of a selfintersection on the surface F_0, and the projection $\zeta: D_1(\phi) \to F_0$ separates the local sheets of F_0 at the point z (see 2.2.2, and 2.3.2(2)).

In case (1): $V_4 \in p \subseteq \mathbf{P}(V^*)$, we can assume that the line $\langle \mathrm{pf}, q \rangle$ is not tangent of Ver at $\mathrm{pf} = \mathrm{pf}(V_4)$, since S lies outside ∂R. In this case the arguments from above are not valid. In fact, in cases (2) and (3), the conic $q(0)$ had been chosen in a unique way, at least, in some neighbourhood (see case (2)(b)). In case (1) we can replace the conic $q(0)$ with any conic on the line $\langle \mathrm{pf}(0), q(0) \rangle$. This phenomenon can be explained in the terms of (2.3.2). More precisely, $D_1(\phi)$ had been embedded in $Z = \mathbf{P}_{\mathbf{P}(V^*)}(O \oplus O(1))$ as a degeneration locus of the bundle map ϕ. The tautological sheaf L_ζ (see 2.3.2(2)) is invertible, and L_ζ corresponds to the divisor $E =$ (the zero set of the section $\zeta^*\mathrm{pf}) \subseteq Z$. Therefore, the curve $p \subseteq D_1(\phi)$ coincides with the intersection of E and $D_1(\phi)$. On the other hand, the fourfold Z admits a canonical regular projection $\sigma: Z \to \mathbf{P}^5$. The projection σ coincides with a blow-up of $\mathbf{P}^5$ at a point. Obviously, E coincides with the exceptional divisor of σ; in particular, $p = D_1(\phi) \cap E$ is an exceptional rational curve on the normalisation of $D_1(\phi)$. (See also the local study (2.5.4)). The zero-set of the section $\zeta^* q$ (see 2.3.1) corresponds to any (fixed) effective divisor, which represents the invertible sheaf $O_{Z/\mathbf{P}(V^*)}(1)$. In this sense, the section $\zeta^* q$ is not unique, and we can "remove" the conic $q(0)$ along the line $\langle \mathrm{pf}, q \rangle$. Note that pf is the only "forbidden" value of the sections q, since q are sections of the "relative $O(1)$ for the projection ζ", and pf lies on the exceptional section of Z. Anyway, assume that some q takes the value pf. This means that the conic $q = \mathrm{pf}$ lies on the branch locus S of the covering $\pi: X \to Y$, i.e., $S \in Q$ (see Rem. 2.1.3). Therefore, the assumption $S \in Q$ is indispensable in the conditions of Proposition 2.3.2.

2.5. *Local study of* $D_1(\phi)$

We shall introduce appropriate coordinates in the point $z = (V_4, s) \in Z$.

(2.5.1) TECHNICAL LEMMA (see Description 1.2.1(3)). *Let* $Y = G \cap \mathbf{P}(V_7)$ *be the del Pezzo threefold, and let* $P_0 \in Y - S_{(-1,1)}$. *Then:*

(1) *We can choose a coordinate system* (x_i, e_i) *on* V (*see* 0.4) *in such a way that:* (a) $P_0 = e_{34} (= e_3 \wedge e_4)$; (b) $V_7 = (x_{12} - x_{03} = x_{02} - x_{14} = x_{01} - x_{24} + x_{23} = 0)$, in $\Lambda^2 V$.

(2) *Let* L_0, L_1, *and* L_2 *be the three lines, which pass through the point* P_0 (*see* 1.2.1(3)). *Then, up to a permutation of indices* 0, 1, 2: (a) $L_0 = \langle e_{04}, e_{34} \rangle$, $L_1 = \langle e_{13}, e_{34} \rangle$, $L_2 = \langle e_{23} + e_{24}, e_{34} \rangle$; (b) *Let* $V_4(e_k) = (x_k = 0)$, $k = 0, 1, 2$. *Let* $M(V_4(e_k))$ *be the fiber of the bundle* $M \to \mathbf{P}(V^*)$ *over the point* $V_4(e_k) \in \mathbf{P}(V^*) = G(4, V)$. *Let* $\mathrm{Pf}(V_4(e_k))$ *be the Pfaff quadric of the* 4-*space*

$V_4(e_k)$, and let $\mathrm{pf}(V_4(e_k))$ *be the restriction of* $Pf(V_4(e_k))$ *to* $\mathbf{P}(M(V_4(e_k))$ (*see* 2.3.1). *Then* $\mathrm{pf}(V_4(e_0)) = L_1 + L_2$, $\mathrm{pf}(V_4(e_1)) = L_0 + L_2$, $\mathrm{pf}(V_4(e_2)) = L_0 + L_1$.

(3) *Let* $\Sigma a_i . x_i = 0$ *be the equation of* $V_4 \subseteq V$. *Then* (a) *the conic* $\mathrm{pf}(V_4)$ *passes through the point* $P_0 = e_{34} \Leftrightarrow a_3 = a_4 = 0$; (*In fact, the conics in* Y *through* P_0 *are parametrized by the plane* $\mathbf{P}(U_3)$, *where* $U_3 = (V/\langle e_3, e_4\rangle)^*$ $= \{V_4 \subseteq V: \langle e_3, e_4\rangle \subseteq V_4\}$; *the condition a means that* (a_0, a_1, a_2) *can be regarded also as coordinates in* U_3); (b) *Let* $\mathbf{P}(U_3)(a_0, a_1, a_2)$ *be as above, and let* $l_{01} = (a_2 = 0)$, $l_{02} = (a_1 = 0)$, *and* $l_{12} = (a_0 = 0)$. *Then* (i): *The line* (*of conics*) l_{01} *describes the set of conics on* Y *of type* $L_2 + L$, *etc.* (ii): *Let* $(i, j, k) = (0, 1, 2)$, *and let* $v_i = l_{ij} \cap l_{ik}$, $i = 0, 1, 2$. *Then the hyperplane* $V_4(i) \in \mathbf{P}(U_3)$, *with coordinates* $a_j = a_k = 0$, *coincides with the hyperplane* $V_4(e_i)$) (*see* (2), *esp.* (2)(*b*)). *In particular*: (iii): *The conic* $\mathrm{pf}(V_4)$ *through* P_0 *is singular* $\Leftrightarrow V_4 = (a_0x_0 + a_1x_1 + a_2x_2 = 0)$ *and* $a_0 . a_1 . a_2 = 0$.

Proof. In fact, we can assume that e_3, e_4, and $e_3 + e_4$ are the centers of the plane pencils of lines L_0, L_1, and L_2. Then, elementary considerations from the Schubert calculus imply that we can choose e_0, e_1, and e_2 so that $L_0 = \langle e_{04}, e_{34}\rangle$, $L_1 = \langle e_{13}, e_{34}\rangle$, $L_2 = \langle e_{23} + e_{24}, e_{34}\rangle$, see 2(a). Then, up to a diagonal coordinate change, the equations of $V_7 \subseteq \Lambda^2 V$ will be as in 1(b). Part 3 is a direct consequence of 1, 2.

2.5.2 The equations of the tangent space $T_z D_1(\phi)$, $z \notin p$

We shall find the 1st order local deformation in a generic point $z = (V_4, s) \in D_1(\phi)$, inside $Z = \mathbf{P}_{\mathbf{P}(V^*)}(O \oplus O(1))$, (see (3) in the proof of Prop. 2.4.2, and Prop. 2.3.2(2)); i.e., we shall suppose that the conic $\mathrm{pf}(V_4)$ is smooth. Since $\mathrm{pf}(V_4)$ is smooth, the equation of $V_4 \subseteq V$, in the coordinates (2.5.1), is: $a_0x_0 + a_1x_1 + a_2x_2 = 0$, where $a_0 . a_1 . a_2 \neq 0$, see (2.5.1)(3.b). For simplicity, let $a_0 = a_1 = a_2 = 1$; the computations in case $a_0 . a_1 . a_2 \neq 0$ are just the same.

The free deformation of $z = (V_4, s)$, inside Z, is described by the parameters $(t; s)$, where: (i) $t = (t_1, t_2, t_3, t_4)$ is the parameter of the deformation of $V_4(t)$, inside $\mathbf{P}(V^*)$ (here $V_4(0) = V_4$); (ii) s is the parameter on the fiber of $\zeta: Z \to \mathbf{P}(V^*)$, see (2.3.2). More precisely, $z(t; s) = (V_4(t), s . \mathrm{pf}(t) + q(t))$, where: (1)

$$V_4(t)\colon\ x_0 + (1 + t_1)x_1 + (1 + t_2)x_2 + t_3x_3 + t_4x_4 = 0,$$

$$\mathrm{pf}(t) = \mathrm{pf}(V_4(t)) \subseteq \mathbf{P}(M(V_4(t))) = \mathbf{P}(M(t)),$$

and

$$q(t) = \text{the restriction of } Q \text{ on } \mathbf{P}(M(t));$$

we may suppose, as in the proof of Proposition 2.4.2, that rank $q(0) = 1$, i.e., $q(0)$ is a double line, $0 = (0, 0, 0, 0)$. The condition: (2)

$$\operatorname{rank}[s.\operatorname{pf}(t) + q(t)] \leqslant 1$$

describes the points $s.\operatorname{pf}(t) + q(t)$, in the fiber $\zeta^{-1}(V_4(t))$, which lie on the 1st determinantal $D_1(\phi) \subseteq Z$; in particular, $z = z(0; 0) = (V_4(0); q(0))$. By definition (see (2.3.1), (2.3.2)), the conics $\operatorname{pf}(t) = \operatorname{pf}(V_4(t))$, and $q(t)$ lie on the plane $\mathbf{P}(M(t)) = \mathbf{P}(M(V_4(t)) = \Lambda^2 V_4(t) \cap V_7$. In the coordinates of Lemma 2.5.1, $M(t)$ is defined by: (3)

$$\begin{aligned} M(t)\colon & x_{23} - x_{24} + (1 + t_2)x_{03} + t_3x_{13} + t_4x_{14} = 0 \\ & x_{14} + (1 + t_1)x_{03} - t_3x_{23} - t_4x_{24} = 0 \\ & x_{03} + (1 + t_1)x_{13} + (1 + t_2)x_{23} - t_4x_{34} = 0 \\ & x_{04} + (1 + t_1)x_{14} + (1 + t_2)x_{24} + t_3x_{34} = 0, \end{aligned}$$

in $V_7(x_{03}, x_{13}, x_{23}, x_{04}, x_{14}, x_{24}, x_{34})$

(see 2.5.1(1)(b)). Since $z = z(0; 0) = (V_4(0); q(0))$, the condition (2), up to $O(t^2)$, becomes: (4)

$$\operatorname{rank}[s.\operatorname{pf}(0) + q(t)] \leqslant 1.$$

(5) *Linear coordinates on* $M(t)$

Let (x, y, z) be linear coordinates on $M(t) = M(V_4(t))$, i.e., x, y, z are linear forms of $x_{ij} = x_i \Lambda x_j$. We may suppose that $z = x_{34}$. In particular, if $(t) = 0$, then the point $P_0 = e_{34}$ (see 2.5.1) has coordinates $(0, 0, 1)$ in $M(0)$. Let $Q = q_{03,03}x_{03}^2 + q_{03,13}x_{03}x_{13} + \dots$, be an equation of (any) quadric Q, s.t. $Q \cap Y = S$. Then $q_{34,34} = 0$, since $P_0 \in S$.

Let $q(t) = q(0) + d(t)$. Then the double line $q(0) \subseteq \mathbf{P}(M(0))$ passes through $P_0(0:0:1)$, i.e. $q(0) = (ax + by)^2$; i.e. $q(t) \subseteq \mathbf{P}(M(t))$ has the form

$$q(t) = (ax + by)^2 + d_{xx}(t).x^2 + d_{xy}(t).xy + \cdots + d_{zz}(t).z^2.$$

The conic $\operatorname{pf}(0)$ can be represented as a restriction, to $\mathbf{P}(M(0))$, of the Plücker quadric

$$\operatorname{Pf}(e_0) = x_{12}.x_{34} - x_{13}.x_{24} + x_{14}.x_{23} = x_{03}.z - x_{13}.x_{24} + x_{14}.x_{23};$$

in particular, the coefficient at z^2, of $\operatorname{pf}(0) = p_{xx}.x^2 + p_{xy}.xy + \dots$, is 0, in any (linear) coordinates (x, y, z), $z = x_{34}$.

Suppose that $z(t) = (V_4(t),\ s.\mathrm{pf}(0) + q(t)) \in D_1(\phi)$. Then, in coordinates (x, y, z), the double line $s.\mathrm{pf}(t) + q(t)$ has the form

$$(a + \alpha(t))x + (b + \beta(t))y + \gamma(t)z)^2 = 0,$$

for some infinitesimal parameters $\alpha(t)$, $\beta(t)$, $\gamma(t)$. The condition $z(t) \in D_1(t)$, equivalently, the condition (5) can be rewritten in the form: (6)

$$(s.\mathrm{pf}(0) + q(0))(x,\ y, z) = ((a + \alpha(t))x + (b + \beta(t))y + \gamma(t)z)^2 = 0, \mathrm{mod}\, O(t^2).$$

In coordinates x, y, $z = x_{34}$, (6) takes the form:

$$\left.\begin{aligned} s.p_{xx} + d_{xx}(t) &= 2a.\alpha(t)\\ s.p_{yy} + d_{xy}(t) &= 2a.\beta(t) + 2b.\alpha(t)\\ s.p_{yy} + d_{yy}(t) &= 2b.\beta(t)\\ s.p_{xz} + d_{xz}(t) &= 2a.\gamma(t)\\ s.p_{yz} + d_{yz}(t) &= 2b.\gamma(t)\\ d_{zz}(t) &= 0. \end{aligned}\right\} \quad \mathrm{mod}\, O(t^2) \qquad (7)$$

The Eq. (7) define the tangent space $T_zD_1(\phi)$ as a subspace of the tangent space $T_zZ = \mathbf{C}^5$. Suppose $a.b \neq 0$ (in fact, the study of case $a.b = 0$ do not differ substantially from that in the "generic" case).

(8) *The coefficients* $d_{zz}(t)$
Evidently, the locus $D_1(\phi)$ is a surface. Suppose that $\dim T_zD_1(\phi) \geqslant 3$, i.e., suppose that $D_1(\phi)$ is singular in z. Then (7) implies that the deformation coefficient $d_{zz}(t) \equiv 0$, $\mathrm{mod}\, O(t^2)$. The last must be true in any linear coordinates (x, y, z), $z = x_{34}$. We shall compute $d_{zz}(t)$ in three systems of linear coordinates:

(a) $x = x_{03}$, $y = x_{24}$ $(z = x_{34})$.

In this case $\mathrm{pf}(0) = xz + x^2 - xy + y^2$, and

$$d_{zz}(t) = d_{zz}(Q(t)) = -t_3.q_{04,34} + t_4.q_{14,34},\ \mathrm{mod}\, O(t^2);$$

(b) $x = x_{03}$, $y = x_{04}$ $(z = x_{34})$

here, $d_{zz}(t) = t_3.(q_{13,34} - q_{23,34} - q_{24,34}) + t_4.q_{13,34},\ \mathrm{mod}\, O(t^2)$;

(c) $x = x_{03}$, $y = x_{13}$ $(z = x_{34})$

then, $d_{zz}(t) = t_4.(q_{23,34} + q_{24,34} - q_{04,34}) - t_3.q_{04,34}$, mod $O(t^2)$.

(9) *The singular quadric* $Q^{\sim}$
On the one hand, the condition $\dim D_1(\phi) \geqslant 3$ implies $d_{zz}(t) = 0$ in coordinates (a), (b), (c). Therefore,

$$q_{13,34} = q_{04,34} = q_{24,34} + q_{23,34} = 0.$$

It follows that the only nonzero coefficients $q_{ij,34}$ can be $q_{03,34}$, $q_{14,34}$, and $q_{24,34} = -q_{23,34}$.

On the other hand, the component $I_2(Y)$ of the graded ideal $I(Y)$ of $Y \subseteq \mathbf{P}^6 = \mathbf{P}(V_7))$ is spanned on the (restriction of) the Plücker quadrics

$$\mathrm{Pf}(e_0) = x_{12}.x_{34} - x_{13}.x_{24} + x_{14}.x_{23} = x_{12}.z - \cdots$$

$$\mathrm{Pf}(e_1) = \cdots = x_{14}.z - \cdots$$

$$\mathrm{Pf}(e_2) = \cdots = (x_{24} - x_{23}).z - \cdots, \ \mathrm{Pf}(e_3), \text{ and } \mathrm{Pf}(e_4);$$

(see also Remark 2.1.3). Since the quadric Q is defined up to $I_2(Y)$, we may substitute Q by $Q^{\sim} := Q - q_{03,34}.\mathrm{Pf}(e_0) - q_{14,34}.\mathrm{Pf}(e_1) - q_{24,34}.\mathrm{Pf}(e_2)$. Then $Q^{\sim} \cap Y = Q \cap Y = S$, and all the coefficients $q^{\sim}_{ij,34}$ vanish.

In particular (it follows from dim $T_z D_1(\phi) \geqslant 3$ and pf(0) smooth that), the branch locus S is singular in the point P_0.

(2.5.3) REMARK. The local study of cases (2)(a) and (1), in the proof of Proposition 2.4.2, can be performed in a similar way. Especially, in case the points of tangency of pf(0) and S lie on $S_{(-1,1)}$, (see 1.2), the technical Lemma 2.5.1 cannot be applied. Anyway, in this case we also may find appropriate coordinate system (x_i, e_i), etc. (see, e.g. 2.5.4). This way, we repeat the proof of Proposition 2.4.2. Moreover, the final conclusion in 2.5.2(9) shows that:

(*) COROLLARY. *If the branch locus S of the covering $\pi: X = X''_{10} \to Y$ is singular at the point $P_0 \in Y - S_{(-1,1)}$, then the locus $D_1(\phi) = F(X)/i$ is singular at any point $z = (V_4, s_0.\mathrm{pf} + s_1.q)$, for which the conic* pf *passes through P_0.*

In other words, the surface F_0 of conics on Y, which are (at least) bitangent to the branch locus S, is singular in any point pf, which represents a conic pf through the singular point P_0 of S.

2.5.4 The curve of double lines
Let pf(0) = pf(V_4(0)) be a double line. Just as in Lemma 2.5.1, we can find a (canonical) coordinate system (x_i, e_i), such that:

$V_4 = V_4(0) = (x_0 = 0)$,

$V_7 = (x_{12} - x_{03} = x_{13} - x_{24} = x_{14} - x_{02} = 0)$.

Therefore, see 2.5.1(2)(3), $\mathrm{Pf}(0) = \mathrm{Pf}(e_0) = -(x_{24})^2$. Introduce linear coordinates $x = x_{23}$, $y = x_{24}$, $z = x_{34}$ (see 2.5.2(5)–(8)). In particular, $\mathrm{pf}(0) = y^2$. The free deformation $V_4(t)$ of V_4, inside $\mathbf{P}(V^*)$, is:

$$V_4(t) = (x_0 = t_1x_1 + t_2x_2 + t_3x_3 + t_4x_4).$$

Let, for example, the line $L_0 = (y = 0)$ be a simple secant of the branch locus S; we can suppose that $L_0 \cap S = \{(0:0:1), (1:0:0)\}$. We can suppose that $q(0)$ is any of the conics $q(\kappa) = \kappa y^2 + (xy + xz + yz)$, where κ is a parameter (due to the nonuniqueness of the choice of $q(0)$ (*see the proof of* 2.4.2)). Then, just as in 2.5.2

$$\mathrm{pf}(t) = -(t_3x^2 + t_4xy + y^2 - t_2xz - t_1yz + t_4z^2).$$

The condition $s.q(\kappa) - \mathrm{pf}(t) \equiv 0 : \mathrm{mod}\, O(t^2)$, $\kappa \in \mathbf{C}$, describes the linear deformation of the point $z = (\mathrm{pf}(0), (1:0)) \in D_1(\phi)$. Now, the analogue of 2.5.2(7) implies $(s; \alpha, \beta, \gamma; t_1, t_2, t_3, t_4) = (s; s/2,\ \kappa.s/2,\ (s - t_1)/2; t_1, s, 0, 0)$. Therefore, $\dim T_zD_1(\phi) \cong \mathbf{C}^3$, at $z = (\mathrm{pf}(0),\ (1:0)) \in p \cong$ (the curve of double lines on Y); see also the proof of Proposition 2.4.2, case (1).

In Section 3, we study the degeneration of $D_1(\phi)$, respectively the degeneration of the 2-dimensional component $F(X''_{10})$ in case S acquires an ordinary node. We shall see that, in case X''_{10} is generic with a node, the surface $D_1(\phi)$ is irreducible and $D_1(\phi)$ has a transversal selfintersection along the smooth curve Γ_0 of ($\geq$ bitangent) conics through the node.

3. The nodal X''_{10}

3.1. Canonical equations of the projection of X through the node

(1) Suppose that the branch locus $S = Y \cap Q$ of the covering $\pi: X = X''_{10} \to Y$ has a simple node P_0; let S be, otherwise, generic. In particular, $P_0 \in Y - S_{(-1,1)}$, see 1.2.1(3). Therefore, according to (2.5.1), we may choose a (canonical) coordinate system (x_i, e_i), in the point $P_0 = e_{34}$. Moreover, see 2.5.2(9), there exists a quadric $\tilde{Q} \equiv Q$ mod. $I_2(Y)$, such that $\tilde{q}_{ij,34} = 0$, $\forall ij$; we may suppose that $Q = \tilde{Q}$. In particular, Q is a cone with vertex P_0. In "canonical" coordinates

$$(y) = (x_{03}, x_{13}, x_{23}, x_{04}, x_{14}, x_{24}), \quad x_{34} = z$$

(see 2.5.1(1b), 2.5.2(3)), the equation of $Q \subseteq \mathbf{P}(V_7)$ becomes:

$$Q(y) := Q(y;0) = Q(y;z) = 0.$$

Let

$$\mathrm{Pf}(e_3) = (x_{24} - x_{23})x_{24} - x_{14}^2 + x_{03}x_{04} = 0$$

and

$$\mathrm{Pf}(e_4) = (x_{24} - x_{23})x_{23} - x_{13}x_{14} + x_{03}^2 = 0,$$

be the equations of the Pfaffians $\mathrm{Pf}(e_3)$ and $\mathrm{Pf}(e_4)$, in $\mathbf{P}(V_7)$, see Remark 2.1.3 and 2.5.2(9). Let $\underline{Y} \subseteq \mathbf{P}^5(y)$ be the image of $Y \subseteq \mathbf{P}^6(y;z)$, after the projection "through $P_0 = (0;1)$":

$$p : \mathbf{P}^6(y;z) - - \rightarrow \mathbf{P}^5(y).$$

Then $\underline{Y} = \mathrm{Pf}(e_3) \cap \mathrm{Pf}(e_4)$, in $\mathbf{P}^5(y)$; i.e., $\underline{Y}$ is an intersection of two quadrics in $\mathbf{P}^5(y)$. The p-image $\underline{S}$ of $S \subseteq Y$, in $\underline{Y}$ (!), is an intersection of three quadrics in $\mathbf{P}^5(t)$ — the quadrics $\mathrm{Pf}(e_3)$, $\mathrm{Pf}(e_4)$, and the quadric $Q(y)$.

(2) Let $\pi : X \rightarrow Y$ be as usual; denote also by P_0 the node of X, over P_0. We may suppose that $X \subseteq \mathbf{P}^7(y;z;w)$ (w — a formal parameter), and $\mathbf{P}^6(y;z)$ is the polar hyperplane of the point (0; 0; 1), with respect to the quadric $W = w^2 - Q$.

Let $\underline{X}$ be the image of X, after the projection "through $P_0 = (0;1;0)$": $p : \mathbf{P}^7(y;z;w) - - \rightarrow \mathbf{P}^6(y,w)$. Then $\underline{X}$ becomes a (singular) intersection of three quadrics, in $\mathbf{P}^6(y;w)$ — the quadrics $\mathrm{Pf}(e_3)$, $\mathrm{Pf}(e_4)$, and the quadric $w^2 - Q(y)$. We call $\mathrm{Pf}(e_3) = \mathrm{Pf}(e_4) = w^2 - Q(y)$ canonical equations of $\underline{X}$. The coordinates $(y;z)$ are supposed to be "canonical" — see above, and 2.5.1(1b), 2.5.2(3). Obviously, the covering $\underline{\pi} : \underline{X} \rightarrow \underline{Y}$ defined by the quadric $w^2 - Q(y)$ is also the double covering of $\underline{Y}$ induced by π.

3.2. The curve Γ_0 of tangent conics through the node P_0

3.2.1. The plane $\mathbf{P}^2_*$ of conics through P_0

In the notation of 3.1, let $F(Y)_{P_0} \subseteq F(Y) \cong \mathbf{P}(V^*)$ be the set of conics on Y, which pass through the point $P_0 = e_{34}$. The linear isomorphism $F(Y) \rightarrow \mathbf{P}(V^*)$, see Prop. 1.2.2, maps the subset $F(Y)_{P_0}$ isomorphically to the plane $\mathbf{P}^2_* := \mathbf{P}((V/\langle e_3, e_4\rangle)^*) \subseteq \mathbf{P}(V^*)$. We shall use the notation of 2.5.1(3). For example, in canonical coordinates (x_i, e_i), the equation of the element $V_4 \in U_3 = (V/\langle e_3, e_4\rangle)^*$ (as a hyperspace in V) is:

$$a_0x_0 + a_1x_1 + a_2x_2 = 0, \quad \mathbf{P}^2_* = \mathbf{P}(U_3), \text{ etc.}$$

3.2.2. Definition of Γ_0

Let $\pi: X \to Y$, $S = Y \cap Q$, etc., be as in 6.1. Let $M \to \mathbf{P}^2_*$ be the restriction, on $\mathbf{P}^2_* \subseteq \mathbf{P}(V^*)$, of the bundle $M \to \mathbf{P}(V^*)$, see (2.3.1). Just as in (2.3.2), the restrictions of the quadric Q, to the fibers of M, are parametrized by the sheaf $O_{\mathbf{P}^2_*}$, i.e., the quadric Q defines a section $q: O_{\mathbf{P}^2_*} \to S^2M^*$, such that $\forall V_4 \in \mathbf{P}^2_*$: the restriction $Q \cap \mathbf{P}(M(V_4))$ is defined, on $\mathbf{P}(M(V_4))$, by the equation $q(V_4) = 0$.

Let $V_4 \in \mathbf{P}^2_*$. Since the quadric Q is singular in the node P_0, the conic $q(V_4)$ cannot be smooth. In fact, $q(V_4)$ is a pair of (possibly coincident) lines through P_0. Let $\Gamma_0 = \{V_4 \in \mathbf{P}^2_* : \text{rank } q(V_4) \leqslant 1\}$.

The branched locus $S \subseteq Y$ has been chosen generic with a node in P_0. In particular, S does not contain lines, i.e., rank $q(V_4) \geqslant 1$, for any $V_4 \in \mathbf{P}^2_*$.

(3.2.3) LEMMA. *In the notation of* (3.2.2): $\Gamma_0 \subseteq \mathbf{P}^2_*$ *is a smooth plane curve of degree* 6.

Proof. (deg $\Gamma_0 = 6$): Let $\mathbf{P}(M)$ be the projectivisation of the bundle M. Each $\mathbf{P}(M(V_4))$ passes (as a plane in $\mathbf{P}(V_7)$) through the point $P_0 = e_{34}$. Therefore, we can project the bundle M through the subspace $\langle e_{34}\rangle$, which represents the point P_0. This way we obtain a 2-bundle $\underline{M}$, and a projection $p: M \to \underline{M}$. The section $q \in H^0(\mathbf{P}^2_*, S^2M^*)$ (see 3.2.2) defines, via the projection $p: M \to \underline{M}$, a section $\underline{q} \in H^0(\mathbf{P}^2_*, S^2\underline{M}^*)$. Obviously, $\Gamma_0 = D_1(\underline{q}) =$ (the 1st determinantal of $\underline{q}$) $= \{V_4 \in \mathbf{P}^2_* : (\underline{q}(V_4) = 0)$ is a double point on the line $\mathbf{P}(M(V_4))\}$. The bundle M is defined as a restriction of $\Lambda^2\tau_{4,V}$, on $V_7 \subseteq \Lambda^2 V$ (see 2.3.1). In the Chow ring $A.(\mathbf{P}^2_*) = Z[h]/(h^3 = 0)$, the Chern polynomial $c(M) = 1 - 3h + 5h^2$. Therefore $c(\underline{M}) = 1 - 3h$, and $\Gamma_0 \subseteq \mathbf{P}^2_*$ coincides with the 1st determinantal of the symmetric bundle map $\underline{q}$. In particular $\deg(\Gamma_0) = 2.c_1(M^*) = 6$ (see [JLP], [HT]).

(Γ_0 is smooth): The proof repeats the arguments from 2.5. More precisely, we have to find the 1st order deformation of the point $V_4(0) \in \Gamma_0$, inside the tangent space of $\mathbf{P}^2_*$ in $V_4(0)$. In this case $t_3 = t_4 = 0$ (see 6.2.1), and

$$V_4(t): x_0 + (1 + t_1)x_1 + (1 + t_2)x_2 = 0$$

(see 2.5.2(1)).

Note that the projection $p: M \to \underline{M}$ "deletes" the coordinate $x_{34} = z$. Now, just as in 2.5.2(5)–(7), we introduce linear coordinates (x, y)—in $\underline{M}(t)$. The condition rank$[\underline{q}(t)] = 1$ can be rewritten in the form (see 2.5.2(5)–(7)):

(1) $d_{xx}(t) = 2a.\alpha(t)$, $d_{xy}(t) = 2a.\beta(t) + 2b.\alpha(t)$, $d_{yy}(t) = 2b.\beta(t) : \text{mod}.O(t^2)$. Here

$$q(0) = (ax + by)^2,\ V_4(0) = (x_0 + x_1 + x_2 = 0),$$

$$q(t) = q(0) + d_{xx}(t)x^2 + d_{xy}(t)xy + d_{yy}(t)y^2, \text{ etc.}$$

It remains to be seen that the equations (1) involve (in the generic case) a non-trivial linear relation between t_1 and t_2. The rest repeats 2.5.2(7), (9).

(*) REMARK. Let $V_4 \in \mathbf{P}^2_*$. Then $\mathrm{Pf}(V_4)$ is smooth in P_0, except for $V_4 = V_4(e_i)$, $i = 0, 1, 2$ (see 2.5.1(2b), (3b)). Fortunately, if S is generic with a node, the curve Γ_0 does not pass through these points. Otherwise, Γ_0 must acquire a singularity in the point $V_4(e_i) \in R_0$ (see Prop. 2.2.2). By definition, the curve Γ_0 lies in $\mathbf{P}^2_* \subseteq \mathbf{P}(V^*) \cong F(Y)$, see 1.2.2. It follows from (2.2.2) and (2.3.2)(2) that (the generic case $\Rightarrow \Gamma_0$ does not intersect $R_0 \Rightarrow$) Γ_0 is naturally embedded in the 1st determinantal $D_1(\phi) = F(X)/i$.

(3.2.4) LEMMA. *Let $z \in \Gamma_0 \subseteq D_1(\phi)$. Then $z \in \mathrm{Sing}\, D_1(\phi)$. Moreover, if K_z is the tangent cone to $D_1(\phi)$ in z, then* $\mathrm{rank}(K_z) = 2$.

Proof. Let $z = (V_4, q(V_4))$, $V_4 = V_4(0)$, and let $\mathrm{pf}(0) = \mathrm{pf}(V_4(0))$.

Let (x_i, e_i) be a canonical coordinates system in $P_0 = e_{34} \in Y - S_{(-1,1)}$ (see 3.1(1)). Suppose that pf(0) is smooth. The rest repeats considerations from (2.5.2). In particular, we may assume $a_0 = a_1 = a_2 = 1$, and:

$$V_4(t) = x_0 + (1 + t_1)x_1 + (1 + t_2)x_2 + t_3x_3 + t_4x_4, \text{ etc.}$$

(The only difference in case pf(0) is non-smooth will be $a_0.a_1a_2 = 0$.)

We shall find the 2nd order deformation of $z \in D_1(\phi)$, inside

$$Z = \mathbf{P}_{\mathbf{P}(V^*)}(O \oplus O(1)).$$

Let, for example, $x = x_{03}$, $y = x_{24}$, $z = x_{34}$, be the coordinates 2.5.2(8a). Let

$$v = (x_{23}, x_{14}, x_{04}, x_{13})^t, \quad u = (x, y, z)^t = (x_{03}, x_{24}, x_{34})^t.$$

Then, the equations 2.5.2(3) can be rewritten in the form:

$$(E + T)v = (B + t)u, \tag{*}$$

where E is the unit, and

$$T = [T_{23}; T_{14}; T_{04}; T_{13}]^t, \quad B = [B_x, B_y, B_z], \quad \mathbf{t} = [\mathbf{t}_x, \mathbf{t}_y, \mathbf{t}_z],$$

where

$$T_{23} = (0, t_4, 0, t_3), \quad T_{14} = (-t_3, 0, 0, 0), \quad T_{04} = (t_3, t_1, 0, 0),$$

$$T_{13} = (t_2, -t_4, 0, t_1 - t_3); \quad B_x = (-1, -1, 1, 0)^t, \quad B_y = (1, 0, -1, -1)^t,$$

$$B_z = (0, 0, 0, 0)^t; \quad \mathbf{t}_x = (-t_2, -t_1, t_1, t_2)^t,$$

$$\mathbf{t}_y = (0, t_4, -t_2 - t_4, 0)^t, \quad \mathbf{t}_z = (0, 0, -t_3, t_4)^t.$$

Remember that the quadric $Q = \tilde{Q}$ is chosen to be singular in the point $P_0 = e_{34}$ (see 3.1(1)). It follows from 2.5.3(*) that the locus $D_1(\phi)$ is singular in the point z. Moreover, see 2.5.2(8), $\dim T_z D_1(\phi) \geqslant 3$. It is not hard to see that (S — generic with a node $\Rightarrow$) $\dim T_z D_1(\phi) = 3$, at any point $z \in \Gamma_0$.

Let $\mathbf{C}^3_z \subseteq T_z Z = \mathbf{C}^5$ be the space of solutions of 2.5.2(7). Let $F_0 \subseteq \mathbf{P}(V^*)$ be as in 2.1.1. It follows from Rem. 3.2.3(*) that $D_1(\phi)$ and $F_0 \subseteq \mathbf{P}(V^*)$ are isomorphic in some neighbourhood of the curve Γ_0 (see also Prop. 2.2.2). In particular, if $z = (V_4, q(0)) \in \Gamma_0$, then the tangent space $\mathbf{C}^3_z = T_z D_1(\phi)$ can be embedded isomorphically in the tangent space $T_{V_4}\mathbf{P}(V^*) \cong \mathbf{C}^4(t)$. We shall find the equation of the tangent cone $K_z \subseteq \mathbf{C}^3_z$.

Let $Q = Q(v; u)$ be the quadratic form Q, in coordinates (v, u). Let $q(t) = q(V_4(t)) \subseteq \mathbf{P}(M(t))$ be as in 2.5.2(5). In particular, $q(t)(x, y, z) = q(t)(u) = q(v(t)(u), u)$, where $v(t)(u)$ is the formal solution of (*).

Let $d_{zz}(t) = \Sigma\, [d_{zz}(t)]_k$ be the power expansion of the coefficient of $q(t) = q(t)(x, y, z)$ at z^2, in the graded algebra $\mathbf{C}[t] = \oplus_{k \geqslant 0} \mathbf{C}[t]_k$. For example, $[d_{zz}(t)]_1 = d_{zz}(t) \equiv 0$, since $z = (V_4, q(0)) \in \operatorname{Sing} D_1(\phi)$.

Let us turn back to the substitutions (a, b, c) in 2.5.2(8). We can check directly that the coefficient of pf(0), at yz, vanishes after each of these substitutions. For example, in case (a): $\mathrm{pf}(0) = xz + x^2 - xy + y^2$, etc.

Let $q(0) = (ax + by)^2$ be the equation of the double line $q(0) \subseteq \mathbf{P}(M(0))$. There are two possible cases: *Case* 1: $(ax + by = 0)$ is a simple secant of pf(0), i.e., $b \neq 0$; *Case* 2: the line $(ax + by = 0)$ is tangent to pf(0), at P_0 ($\Leftrightarrow$ the "infinitesimal" R_0). We shall find the equation of K_z in coordinates 2.5.2(8a):

Case 1 ($b \neq 0$; we may suppose $a = b = 1$). In this case (see (*)), the equation of the tangent cone K_z, in coordinates (*), is:

$$K_z = (4\,.\,[d_{zz}(t)]_2 - d_{yz}(t))^2 = 0) \subseteq \mathbf{C}^3_z$$

where

$$1/2\,.\,d_{yz}(t) = (q_{23,04} - q_{04,04} - q_{13,04}) \cdot (-t_3) + (q_{13,23} - q_{04,13} - q_{13,13}) \cdot t_4,$$

$$[d_{zz}(t)]_2 = q_{04,04} \cdot (-t_3)^2 + q_{04,13} \cdot (-t_3)\,.\,t_4 + q_{13,13} \cdot t_4^2 \text{ (here } q_{ij,kl} = q_{kl,ij}).$$

In case the discriminant is $\neq 0$, we obtain $\operatorname{rank} K_z = 2$. The rest repeats 2.5.2(9). In fact, $\operatorname{rank} K_z \leqslant 2$ since (in Case 1) the quadratic form K_z is a function of the linearly independent parameters t_3 and t_4. The point is that the branch locus $S = Q \cap Y$ is generic with a node in P_0, and the condition $\operatorname{rank} K_z \leqslant 1$ imposes additional requirements on the coefficients of Q (see also 2.5.2(9)).

Case 2 ($b = 0$; we may suppose $a = 1$ ($\Leftarrow S$ does not contain lines). In this case, the free deformation of the double line

$q(0) = (x^2 = 0)$ is $((1+\alpha)x + \beta y + yz)^2$.

In this case, $d_{yz}(t) = 0$, and the equations 2.5.2(7) imply:

$$0 = 1/2 . d_{yz}(t) = (q_{24,04} + q_{23,04} + q_{04,04} - q_{13,04}) . t_3$$
$$+ (q_{13,24} + q_{23,04} - q_{13,04} - q_{13,13}) . t_4.$$

(remember that $p_{yz} = 0$); i.e., t_3 and t_4 are linearly dependent, and the equation of K_z becomes: $4 . [d_{zz}(t)]_2 - (d_{xz}(t) - d_{yy}(t))^2 = 0$, etc.—see Case 1.

4. Lefschetz degenerations of the Fano surface $F = F(X''_{10})$

In this paragraph, we prove that the Abel-Jacobi mapping Φ_F is an isomorphism for the generic X''_{10} (see Th. 4.3). We shall use the Clemens-Letizia criterion (see [L]) in the particular case $X = X''_{10}$. The case $X = X''_{10}$ is somewhat peculiar—the Fano surface $F(X''_{10})$ has a prescribed singularity, and we cannot apply directly the criterion (in its conditions, the generic "Fano" surface should be nonsingular). Fortunately, this singularity can be ignored. This is the reason to reformulate:

4.1. The conditions of the Clements-Letizia criterion (see [L, Prop. 2 (a, b, c, d), Lemma 1.1 and 1.2]) *in the particular case* $X = X''_{10}$.

(A) LEMMA (see [L, Prop. 2a]). *Let* $\pi : X''_{10} \to Y$, $S \subseteq Y$, $D_1(\phi) = F(X)/i$, *etc., be as in §2. Suppose moreover that the branch locus S is generic; more precisely, suppose that* $S \notin L \cup Q \cup \partial R \cup N$ (see Prop. 2.4.2). *Then, the only singularities of the Fano surface* $F(X)$ *are the points of a smooth rational curve* ρ, *which parametrize the* π*-preimages of the* $(-1, 1)$*-lines on* Y.

Proof. Since $S \notin Q$, the branch locus S does not contain conics, and we are in the conditions of Proposition 2.3.2. In particular, let $i : F(X) \to F(X)$ be the natural involution (see 2.2.1). Then, the quotient surface $F(X)/i$ is isomorphic to the degeneration locus $D_1(\phi)$ (see 2.3.2(2). Evidently, the set of the points of the smooth rational curve $\rho := (\pi^* p)_{\mathrm{red}}$ (see (2.1.2), (2.1.4)) and the set of the fixed points of the involution i are coincident. Now, the lemma follows from Proposition 2.4.2; see also the proof of 3.2.4.

The following is standard (see, e.g. [L], [CV]):

(B) LEMMA (see [L, Prop. 2b]). *If the branch locus S is generic with the property* $\{S \in L \cup Q \cup \partial R\}$, *then the Fano surface* $F(X)$ *has, besides* ρ, *only a finite number of isolated singularities.*

(C) LEMMA (see [L, Prop. 2c, d]). *If the branch locus S is generic with* *the property* $\{S \in N\}$, *i.e., if* $X = X''_{10}$ *is generic with a node, then* $F(X)$ *is an irreducible surface which is singular (besides* ρ*) only along a smooth irreducible curve* Γ; *moreover, any suitably small complex neighbourhood of* Γ *is analytically reducible in two smooth component meeting transversally.*

Proof. Let $\pi = \pi(i): F(X) \to F(X)/i = D_1(\phi)$ be, as usual, the natural double covering. Evidently, the curve $\Gamma \subseteq F(X)$ coincides with the preimage of the curve $\Gamma_0 \subseteq D_1(\phi)$, see Lemma 3.2.4. Since $S \in N$ is supposed to be generic, the curves Γ_0 and ρ are disjoint ($\Rightarrow \Gamma \cap \rho = \emptyset$)—see the proof of 4.2(***). It rests to prove that the curve Γ, and the surface $F(X)$ are irreducible.

The curve Γ *is smooth and irreducible*: It follows from $\Gamma \cap \rho = \emptyset$ that $\pi: \Gamma = \pi^*\Gamma_0 \to \Gamma_0$ is a two-sheeted unbranched covering. Moreover, the curve Γ_0 is isomorphic to a smooth plane curve (see Lemma 3.2.3 and Rem. 3.2.3(*)). In particular, Γ is smooth. It remains to see that Γ is connected. The rest is similar to the proof of [CV, Prop. 2.35]: Suppose that Γ is not connected. Then the double covering $\pi: \Gamma \to \Gamma_0$ is trivial, i.e., Γ is a disjoint union of two isomorphic preimages of $\Gamma_0: \Gamma = \Gamma' \sqcup \Gamma''$. Let $S(\Gamma_0) \subseteq Y$ be the surface $S(\Gamma_0) = \cup\{\mathrm{pf}(V_4): V_4 \in \Gamma_0\}$. It is not hard to see that $S(\Gamma_0)$ is a (singular) surface on Y of degree 30. Let $S_X(\Gamma_0) = \pi^{-1}S(\Gamma_0)$ be the preimage of $S(\Gamma_0)$ on X. Since the covering $\Gamma \to \Gamma_0$ is trivial, $S_X(\Gamma_0)$ splits into two isomorphic preimages of $S(\Gamma_0): S_X(\Gamma_0) = S' \cup S''$.

Let $H \subseteq Y$ be a generic hyperplane section of Y, which does not pass through the node $P_0 \in S \subseteq Y$; in particular, H is a smooth del Pezzo surface of degree 5. Let $B = \pi^{-1}(H)$ be the preimage of H in X, and let $\pi: B \to H$ be the induced double covering, branched along the curve $C_H = S \cap H$. Since H is generic, the curve C_H is smooth (canonical curve of degree 10 on $H = H_5 \subseteq \mathbf{P}^5$); therefore, B is a smooth $K3$-surface. Let $C = H \cap S(\Gamma_0)$. It follows from $C \subseteq S(\Gamma_0)$ that the curve $\pi^{-1}(C)$ (also) splits into a disjoint union: $\pi^{-1}(C) = C' \sqcup C''$ of two isomorphic preimages, interchangeable under the action of the involution i. Moreover, C' and C'' lie on the smooth $K3$-surface B.

Let $\{l_1, \dots, l_{10}\}$ be the 10 lines, and let $L_i = \{q_{k,t}: t \in \mathbf{P}^1\}$, $k = 1, \dots, 5$, be the 5 pencils of (mutually disjoint) conics, on the smooth del Pezzo surface $H = H_5 \subseteq \mathbf{P}^5$. Since H is chosen to be generic, we may suppose that the surface $S(\Gamma_0)$ does not pass through each of l_i, and each of $q_{k,t}$. In particular, l_i and $q_{k,t}$ cannot be components of the curve $C = H \cap S(\Gamma_0)$. Therefore, the components C' and C'' of $\pi^{-1}(C)$ cannot be a pair of disjoint curves on the $K3$-surface $B = \pi^{-1}(H)$; note that, by virtue of the general choice of H, the surface $B = \pi^{-1}(H)$ does not contain pairs of mutually disjoint curves, other than the preimages of l_i and $q_{k,t}$; q.e.d.

The proof that *the surface* $F(X)$ *is irreducible* is similar.

(D) LEMMA (see [L, Lemma 1.1]). *If $X = X''_{10}$ is generic, then the Abel-Jacobi map $\Phi : \mathrm{Alb}\, F(X) \to J(X)$ is not a constant map.*

Proof (see [L], [CV]). Since the way is standard, we shall omit the details. It is sufficient to see that the transpose α of the differential of $\Phi' : F(X) \to J(X)$ is not the zero map. Let $W = \mathbf{P}_Y(O \oplus O(1))$, and let $p : W \to Y$ be the projection. Then $X = X''_{10}$ can be embedded in W as a zero-set of a section of $O_{W/Y}(2)$, and the covering $\pi : X \to Y$ is induced by p. The map α can be included in the commutative diagram:

$$\begin{array}{ccc} H^0(X,\ N_{X/W} \otimes K_X) & \stackrel{R}{\to} & H^1(X,\ \Omega^2_X) \cong \Omega_{J(X)}|_0 \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \alpha} \\ H^0(q,\ N_{q/W} \otimes K_X) & \stackrel{b}{\to} & H^0(q,\ N_{q/X})^* \cong \Omega_{F(X)}|_q, \end{array}$$

where r is the restriction map, R is the residue homomorphism, and b is the connecting morphism in the sequence:

$$0 \to H^1(N_{q/X})^* \to H^0(N_{q/W} \otimes K_X) \to H^0(q,\ N_{X/W} \otimes K_X) \stackrel{b}{\to}$$
$$\to H^0(N_{q/X})^* \to H^1(N_{q/W} \otimes K_X) \to H^1(q,\ N_{X/W} \otimes K_X) \to 0.$$

Here

$$K_X = O_{W/Y}(-1) \otimes O_X,\ N_{q/X} = O \oplus O$$

(q — a generic conic on X, see [I]),

$$O_{W/Y}(1) \otimes O_q = O(2)\ (\leftrightarrow q \text{ is a conic}), \quad N_{q/W} = O(1) \oplus O(1) \oplus O(2),$$

i.e., the only nontrivial part of the sequence is (the conic q is smooth generic):

$$0 \to \mathbf{C} \to \mathbf{C}^3 \stackrel{b}{\to} \mathbf{C}^2 \to 0,$$

i.e., b is surjective. Moreover, the restriction map r is, of course, surjective. Therefore, $\alpha . R = b . r$ is surjective $\Rightarrow \alpha \neq 0$.

4.2. The generic Lefschetz pencil of X''_{10}, S

Let $\mathbf{P}^1 \subseteq \mathbf{P}(S^2 V_7^*)$ be a line, which does not intersect the subspace $\mathbf{P}(I_2(Y)) \cong \mathbf{P}(V)$, see Rem. 2.1.3. Let $\{Q_t : t \in \mathbf{P}^1\}$ be the corresponding pencil of quadrics, let $S_t = Q_t \cap Y$, $t \in \mathbf{P}^1$, and let $\pi_t : X_t \to Y$ be the double covering, branched over the surface S_t. Let $W = P_Y(O \oplus O(1))$. We may assume also that the pencil $\mathbf{P}^1$ of X''_{10},s corresponds to a (generic) pencil of sections of the sheaf $O_{W/Y}(2)$ (see the proof of Lemma D).

Let S, L, Q, ∂R, and N be as in 2.1.3 and 2.4.1. The line $L = \mathbf{P}^1(t)$ is embedded in a natural way in S. Moreover, the sets $B = \mathbf{P}^1 \cap (S \cup Q \cup \partial R)$ and $C = \mathbf{P}^1 \cap N$ are finite. Define $A := \mathbf{P}^1 - (B \cup C)$. This way, we obtained a partition of $\mathbf{P}^1$ into three disjoint subsets $\mathbf{P}^1 = A \cup B \cup C$, such that B and C are finite, and for each $t \in A$ (resp. $t \in B$ or $t \in C$), the Fano surface $F_t = F(X_t)$ has the properties stated in Lemma A (resp. Lemma B or Lemma C). The last is equivalent to the Clemens-Letizia criterion, in case the singularity along the curve ρ does not exist. Nevertheless, the singularity along the curve ρ can be ignored, since:

(1) the singularity of $F(X)$ along ρ is prescribed—it does not depend on the deformation $\{F_t : t \in \mathbf{P}^1\}$;
(2) the curve ρ, and the singularities, owed to the degenerations of types B and C, are disjoint.

Therefore, we can repeat the arguments in [L, 1.2] ($\Leftrightarrow$ the proof of the criterion).

Let $X = X_{t_0}$ be a fixed (generic smooth) threefold—of type A ($\Leftrightarrow$ $t_0 \in A$). Let $n: {}^nF \to F$ be a minimal desingularization of F. Then the group $H_1({}^nF, \mathbf{Z})$ is generated by (isomorphic) preimages of integer 1-chains on $F(X)$, which do not intersect the contractible rational cycle $\rho = (\pi^* p)_{\mathrm{red}}$ (see Prop. 2.4.2—the proof of case 1). Let $\Phi_F : \mathrm{Alb}({}^nF) \to J(X)$ be the corresponding Abel-Jacobi mapping (see 0.2). In order to prove that Φ_F is an isomorphism, we have only to prove that

The cylinder map $\gamma_{(F(X))} . n : H_1({}^nF, \mathbf{Z}) \to H_1(F(X), \mathbf{Z}) \to H_3(X, \mathbf{Z})$, is an isomorphism modulo torsion (see [L, Prop. 1]). (*)

Since the vanishing cycles generate the kernel of the inclusion map $H_3(X, \mathbf{Z}) \to H_3(W, \mathbf{Z}) = 0$, these cycles generate $H_3(X, \mathbf{Z})$. The degenerations of type A and B do not generate vanishing cycles—neither on $H_1({}^nF, \mathbf{Z})$, nor on $H_3(X, \mathbf{Z})$. Each degeneration of type C determine exactly one, up to sign, vanishing cycle $\sigma \in H_3(X, \mathbf{Z})$ (see [L, 1.2]). Note that the singular curve ρ does not depend on the parameter $t \in \mathbf{P}^1$. Moreover, the acquired singularities—isolated singular points and double curves—lie "far from" the curve ρ (see e.g., the proof of Lemma (***)). Therefore, we can perform the desingularization over ρ uniformly for all the surfaces $F_t, t \in \mathbf{P}^1$.

Let us assume for a moment that the prescribed singularity of $F(X_t)$, $t \in \mathbf{P}^1$, does not exist. Let X_V, $v \in C \subseteq \mathbf{P}^1$, be any of the nodal threefolds in the pencil, and let Γ be the double curve of the Fano surface

$F_v = F(X_v)$. The proof of $(*)$ in [L] rests on the fact that, in case the hypothesis of Lemma C holds, one can choose, *in a neighbourhood of the double curve* $\Gamma \subseteq \bigcup\{F(X_t): t \in \mathbf{P}^1\}$, a (vanishing) cycle $r \in H_1(F(X), \mathbf{Z})$, such that

$$\gamma_{F(X)}(r) = \pm\sigma. \qquad (**)$$

(Here $X = X_{t_0}, t_0 \in A$, should be chosen sufficiently close to the nodal X_v—see below and [L, 1.2]).

Let X_v, $v \in C = \mathbf{P}^1 \cap N$, be any of the nodal threefolds of the pencil $\{X_t: t \in \mathbf{P}^1\}$. Observe now that:

$(***)$ LEMMA. *For the general degeneration* X_v *of type* N ($\Leftrightarrow$ *of type* C, *on the base of the Lefschetz pencil* $\mathbf{P}^1$), *the curve* Γ, *and the prescribed singularity* ρ, *are disjoint.*

Proof. Let X_v, $v \in C = \mathbf{P}^1 \cap N$ be as above.

Let $\Gamma_0 \subseteq D_1(\phi)$ and $p \subseteq D_1(\phi)$, $\phi = \phi_v$, be as in 3.2 and (2.4.1)–(2.4.2). In order to see that $\Gamma \cap \rho = \varnothing$, it is enough to prove that $\Gamma_0 \cap p = \varnothing$ (see the proofs of Lemma A and Lemma C). On the one hand:

(i) the elements of p are exactly the double lines on Y (see Proposition 1.2.2);
(ii) (the supports of) the double lines on Y are exactly the $(-1, 1)$-lines on Y (see 1.2.1, and the proof of 1.2.2);
(iii) the surface $S_{(-1,1)}$ coincides with the union of $(-1, 1)$-lines on Y (see 1.2.1(2),(3)).

On the other hand, the degeneration of type C is generic; in particular:

(iv) the singular point P_0 does not lie on the surface $S_{(-1,1)} \subseteq Y$.

Since the elements of the curve Γ_0 are conics on Y through the node P_0, (i, ii, iii, iv) imply that $\Gamma_0 \cap p = \varnothing$, hence $\Gamma \cap \rho = \varnothing$. q.e.d.

There is nothing surprising that $\Gamma_0 \cap p = \phi$, since the curve p lies on the exceptional section of the projection $\zeta: Z = \mathbf{P}_{\mathbf{P}(V^*)}(O \oplus O(1) \to \mathbf{P}(V^*)$ (see 2.3.2(2) and the proof of 2.4.2—Case 1).

Since the generic X_t of the pencil is of type A, we can choose the fixed $X = X_{t_0}$, $t_0 \in A$, to be sufficiently close to X_v (i.e., $t_0 \in A$ and $|t_0 - v|$ sufficiently small). This makes it possible, in just the same way as in [L, 1.2], to find (in some neighbourhood of the curve $\Gamma \subseteq \bigcup\{F(X_t), t \in \mathbf{P}^1\}$) the cycle $r \in H_1(F(X), \mathbf{Z})$ with the property $(**)$. Moreover, $F(X)$ is nonsingular outside ρ, and $''F \to F$ is a desingularization of $F(X)$. Since the cycle

$r \in H_1(F(X), \mathbf{Z})$ lies "far from" ρ, we may assume that r belongs also to $H_1({}^nF, \mathbf{Z}) :\Rightarrow (*)$. As a consequence, we obtain:

4.3. THEOREM. *Let $Y \subseteq \mathbf{P}^6$ be the del Pezzo threefold* (*see* 0.4), *and let $S \subseteq Y$ be a generic smooth intersection of Y with a quadric. Let $X = X''_{10}$ be the Gushel threefold, defined by the double covering $\pi: X \to Y$, and by the branch locus S* (*see* 0.4). *Let $F = F(X) =$ (the closure of) $\{C \subseteq X : \pi$ maps C isomorphically onto a conic on $Y\}$ be the Fano surface of X* (*see* 2.1). *Then F is nonsingular outside the smooth rational curve ρ which parametrizes the π-preimages of the* $(-1, 1)$-*lines on Y* (*see Lemma A*).

Let ${}^nF \to F$ be the desingularization of F. Then the Abel-Jacobi mapping $\Phi_F : \mathrm{Alb}({}^nF) \to J(X)$ is an isomorphism.

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