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Crystal bases of Verma modules for quantum affine Lie algebras

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1. Introduction

The *quantized universal enveloping algebra* (or *quantum group*) $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ has been introduced independently by Drinfeld [D] and Jimbo [J] in their study of two dimensional solvable lattice models. The parameter q corresponds to the temperature in the lattice model. In particular, $q = 0$ corresponds to the absolute temperature zero and hence one can expect some simplifications in the zero limit case. Motivated by this observation, one of the authors has introduced the notion of *crystal base* [K1]. The existence of such bases for any integrable representation of $U_q(\mathfrak{g})$ is already known ([K1], [K2], [MM]). In [K2], the *global base* has been introduced. These bases coincide with Lusztig's canonical bases for the finite dimensional Lie algebras of type ADE ([Lu1], [Lu2]). More recently, Lusztig and Grojnowski [LG] have proved that the canonical base and the global base coincide for all Kac-Moody Lie algebras with symmetric generalized Cartan matrices.

The crystal base which can be thought of roughly as a base at $q = 0$ provides a powerful combinatorial tool to study the quantum group $U_q(\mathfrak{g})$ and its integrable representations. For example, the crystal base theory is very useful in decomposing the tensor products. Of course, for this one needs explicit description of the crystal bases. In [KN], the description of the crystal base for any irreducible highest weight modules for $U_q(\mathfrak{g})$ for $\mathfrak{g} = A_n$, B_n, C_n, or D_n is given. In [N], Nakashima has used these descriptions coupled with the properties of crystal bases to obtain Littlewood-Richardson type rules for decomposing the tensor products. The descriptions of crystal bases for irreducible highest weight $U_q(G_2)$-modules and the tensor product decomposition rules are given in [KM1]. Another description of crystal bases for $U_q(\mathfrak{g})$-modules, $\mathfrak{g} = A_n$, B_n, C_n, D_n, E_6, or G_2, using Lakshmibai-Seshadri monomial theory is given in [Li].

Using the Fock space representation, the crystal base for any integrable

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highest weight $U_q(\hat{sl}(n))$-module has been given in [MM] and [JMMO]. It has been observed that the crystal base for these $U_q(\hat{sl}(n))$-modules can be parametrized by certain paths which arise naturally in the context of lattice models. In order to generalize this result to other quantum affine Lie algebras the theory of affine crystals has been introduced in [KMN1] and [KMN2]. Crucial to this theory is the existence of certain crystals known as *perfect crystals*. The existence of at least one such perfect crystal for each integrable highest weight $U_q(\mathfrak{g})$-module for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$, or $D_{n+1}^{(2)}$ has been given in [KMN3]. Each such perfect crystal gives a path realization of the corresponding crystal base (see [KMN2], [KMN3]).

In this paper, we develop the theory of crystals further and give a path realization for the crystal base of $U_q^-(\mathfrak{g})$, $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$, or $D_{n+1}^{(2)}$. This, in particular, gives a path realization for the crystal bases of Verma modules for the corresponding quantum affine Lie algebras. Also note that the crystal base of any integrable highest weight $U_q(\mathfrak{g})$-module is a surjective image of that of $U_q^-(\mathfrak{g})$. One of the key ingredients in this realization is the *energy function* which is necessary to determine the weight of a path. One important result in this paper is the explicit description of the energy function as the maximum of certain linear functions in each case. This result was only known for $\mathfrak{g} = A_n^{(1)}$ (e.g., see [JMMO]) from the corresponding result in the lattice models. It is hoped that these explicit descriptions of the energy functions may provide further information about the corresponding lattice models.

The paper is arranged as follows. In Section 2, we recall basic definitions and develop the theory of crystals further for any symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. It is worth pointing out that in this paper we have axiomatized the crystals as purely combinatorial objects. This might be of independent interest to some researchers. In Section 3, we recall certain results from [KMN2] and [KMN3] which are used in the sequel. The core of the paper is Section 4 where we prove the main results like Theorems 4.7, 4.8, and 4.9. These theorems give the desired path realizations provided we have a suitable classical crystal B_∞ (satisfying (4.1)–(4.5)) and the energy function $H: B_\infty \otimes B_\infty \to \mathbf{Z}$ (see definitions in Section 4). In Section 5, we give the descriptions of these crystals B_∞ which satisfy the required conditions. We also determine the corresponding H-functions for each quantum affine Lie algebra $U_q(\mathfrak{g})$, $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$, $A_{2n-1}^{(2)}, A_{2n}^{(2)}$, or $D_{n+1}^{(2)}$. Thus we have an explicit path realization of the crystal base for $U_q^-(\mathfrak{g})$ for these quantum affine Lie algebras.

2. The algebra $U_q(\mathfrak{g})$ and crystals

Let $\mathfrak{g}$ be any symmetrizable Kac-Moody Lie algebra generated by e_i, f_i ($i \in I = \{0, 1, \ldots, n\}$) and the Cartan subalgebra $\mathfrak{h}$ over $\mathbf{Q}$. Let $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$

and $\{h_i \mid i \in I\} \subset \mathfrak{h}$ denote the simple roots and simple coroots, respectively. We normalize the nondegenerate symmetric invariant bilinear form $(\,,)$ on $\mathfrak{h}^*$ so that $(\alpha_i, \alpha_i) \in \mathbf{Z}_{>0}$. Let P denote the weight lattice and P^* denote the dual lattice. Then $\alpha_i \in P$ and $h_i \in P^*$ for all $i \in I$.

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is then the $\mathbf{Q}(q)$-algebra generated by the symbols e_i, f_i $(i \in I)$ and q^h $(h \in P^*)$ with the following defining relations:

$$q^0 = 1,\ q^h q^{h'} = q^{h+h'} \quad \text{for all } h, h' \in P^*, \tag{2.1}$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i,\ q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i \quad \text{for all } i \in I,\ h \in P^*, \tag{2.2}$$

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where } q_i = q^{(\alpha_i, \alpha_i)} \text{ and } t_i = q^{(\alpha_i, \alpha_i) h_i} = q_i^{h_i}, \tag{2.3}$$

$$\sum_{k=0}^{b} (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = \sum_{k=0}^{b} (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0 \quad \text{for } i \neq j \text{ and } b = 1 - \langle h_i, \alpha_j \rangle. \tag{2.4}$$

Here we use the following notations:

$$[m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}, \quad [k]_i! = \prod_{m=1}^{k} [m]_i \quad \text{and} \quad e_i^{(k)} = e_i^k/[k]_i!,\ f_i^{(k)} = f_i^k/[k]_i!.$$

We understand $e_i^{(k)} = f_i^{(k)} = 0$ for $k < 0$.

It is well known that $U_q(\mathfrak{g})$ has a Hopf algebra structure with a comultiplication Δ defined by

$$\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i,$$

$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i,$$

$$\Delta(q^h) = q^h \otimes q^h$$

for all $i \in I$, $h \in P^*$. The tensor product of two $U_q(\mathfrak{g})$-modules has a structure of $U_q(\mathfrak{g})$-module via this comultiplication. For $i \in I$, let $U_q(\mathfrak{g}_i)$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, t_i and t_i^{-1}.

For a $U_q(\mathfrak{g})$-module M and $\lambda \in P$, the *λ-weight space* of M is defined by $M_\lambda = \{u \in M \mid q^h u = q^{\langle h, \lambda \rangle} u$ for all $h \in P^*\}$. We say that M is *integrable* if $M = \bigoplus_{\lambda \in P} M_\lambda$ and M is a union of finite-dimensional $U_q(\mathfrak{g}_i)$-modules for any $i \in I$. By the representation theory of $U_q(sl(2))$, any element $u \in M_\lambda$ can be uniquely written as

$$u = \sum_{k \geq 0} f_i^{(k)} u_k, \tag{2.5}$$

where $u_k \in \operatorname{Ker} e_i \cap M_{\lambda+k\alpha_i}$. We define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ on M by

$$\tilde{e}_i u = \sum f_i^{(k-1)} u_k, \tag{2.6}$$

$$\tilde{f}_i u = \sum f_i^{(k+1)} u_k. \tag{2.7}$$

Let A be the subring of $\mathbf{Q}(q)$ consisting of the rational functions regular at $q = 0$. A *crystal lattice* of an integrable $U_q(\mathfrak{g})$-module M is a free A-submodule of M such that $M \cong \mathbf{Q}(q) \otimes_A L$, $L = \bigoplus_{\lambda \in P} L_\lambda$ where $L_\lambda = L \cap M_\lambda$, and $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$. A *crystal base* of the integrable $U_q(\mathfrak{g})$-module M is a pair (L, B) such that (i) L is a crystal lattice of M, (ii) B is a $\mathbf{Q}$-base of L/qL, (iii) $B = \sqcup_{\lambda \in P} B_\lambda$ where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$, (iv) $\tilde{e}_i B \subset B \sqcup \{0\}$, $\tilde{f}_i B \sqcup \{0\}$, and (v) for $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$ for $i \in I$. We sometimes replace (ii) by: $B_{ps} = B' \sqcup (-B')$ where B' is a $\mathbf{Q}$-base of L/qL. We call (L, B_{ps}) a *crystal pseudo-base* and $B_{ps}/\{\pm 1\}$ the *associated crystal* of (L, B_{ps}).

For $\lambda \in P_+ = \{\lambda \in P \mid \langle h_i, \lambda \rangle \geqslant 0$ for all $i\}$, let $V(\lambda)$ denote the irreducible integrable $U_q(\mathfrak{g})$-module with highest weight λ. Let u_λ be the highest weight vector of $V(\lambda)$, and let $L(\lambda)$ be the smallest A-submodule of $V(\lambda)$ containing u_λ stable under $\tilde{f}_i$'s. Set $B(\lambda) = \{b \in L(\lambda)/qL(\lambda) \mid b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} u_\lambda \bmod qL(\lambda)\} \setminus \{0\}$. Then $(L(\lambda), B(\lambda))$ is the *crystal base* of $V(\lambda)$ (see [K1]). The associated *crystal graph* is the oriented colored (by I) graph with $B(\lambda)$ as the set of vertices and $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i b$ (hence $\tilde{e}_i b' = b$). This graph completely describes the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $B(\lambda)$. For $b \in B(\lambda)$, we set

$$\varepsilon_i(b) = \max\{k \geqslant 0 \mid \tilde{e}_i^k b \neq 0\}, \tag{2.8}$$

$$\varphi_i(b) = \max\{k \geqslant 0 \mid \tilde{f}_i^k b \neq 0\}. \tag{2.9}$$

Note that $B(\lambda) = \sqcup_{\mu \in P} B(\lambda)_\mu$ and for $b \in B(\lambda)_\mu$, we have $\langle h_i, wt(b) \rangle = \varphi_i(b) - \varepsilon_i(b)$, where $wt(b) = \mu$ denotes the weight of b.

Motivated by the nice properties of these crystal graphs, we define the notion of a *crystal* as follows.

DEFINITION 2.1. A *crystal* B is a set $B = \sqcup_{\lambda \in P} B_\lambda$ ($wt(b) = \lambda$ if $b \in B_\lambda$) equipped with maps $\tilde{e}_i : B_\lambda \to B_{\lambda+\alpha_i} \sqcup \{0\}$, $\tilde{f}_i : B_\lambda \to B_{\lambda-\alpha_i} \sqcup \{0\}$, $\varepsilon_i : B \to \mathbf{Z} \sqcup \{-\infty\}$, $\varphi_i : B \to \mathbf{Z} \sqcup \{-\infty\}$ for all $i \in I$ such that

for $b \in B_\lambda$, $\varphi_i(b) = \langle h_i, \lambda \rangle + \varepsilon_i(b)$, (2.10)

for $b \in B$, we have (2.11)

$$\begin{aligned} \varepsilon_i(b) &= \varepsilon_i(\tilde{e}_i b) + 1 \quad \text{if } \tilde{e}_i b \neq 0, \\ &= \varepsilon_i(\tilde{f}_i b) - 1 \quad \text{if } \tilde{f}_i b \neq 0, \end{aligned}$$

and

$$\varphi_i(b) = \varphi_i(\tilde{e}_i b) - 1 \quad \text{if } \tilde{e}_i b \neq 0,$$
$$= \varphi_i(\tilde{f}_i b) + 1 \quad \text{if } \tilde{f}_i b \neq 0,$$

$$\text{for } b, b' \in B,\ \tilde{e}_i b' = b \text{ if and only if } b' = \tilde{f}_i b, \tag{2.12}$$

$$\text{for } b \in B,\ \varepsilon_i(b) = \varphi_i(b) = -\infty \text{ implies } \tilde{e}_i b = \tilde{f}_i b = 0. \tag{2.13}$$

Sometimes, for emphasis we will say that B is a *P-weighted crystal.*

DEFINITION 2.2. For two crystals B_1 and B_2, a *morphism* of crystals from B_1 to B_2 is a map $\psi: B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ such that

$$\psi(0) = 0, \tag{2.14}$$

$$\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b) \quad \text{for } b, \tilde{e}_i b \in B_1, \text{ and } \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \quad \text{for } b, \tilde{f}_i b \in B_1, \tag{2.15}$$

$$\text{for } b \in B_1,\ \varepsilon_i(b) = \varepsilon_i(\psi(b)),\ \varphi_i(b) = \varphi_i(\psi(b)) \quad \text{if } \psi(b) \in B_2, \tag{2.16}$$

$$\text{for } b \in B_1,\ wt(b) = wt(\psi(b)) \quad \text{if } \psi(b) \in B_2. \tag{2.17}$$

A morphism of crystals $\psi: B_1 \to B_2$ is called an *embedding* if ψ is injective.

The crystals and their morphisms form a category. For two crystals B_1 and B_2 we define their tensor product as follows. The underlying set is $B_1 \times B_2$. For $b_1 \in B_1$, $b_2 \in B_2$, we write $b_1 \otimes b_2$ for (b_1, b_2). We understand $b_1 \otimes 0 = 0 \otimes b_2 = 0$. We define the maps ε_i, φ_i, $\tilde{e}_i$, $\tilde{f}_i$ for $i \in I$ as follows:

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1)\rangle), \tag{2.18}$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2)\rangle), \tag{2.19}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \tilde{e}_i b_1 \otimes b_2 \quad \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2),$$
$$= b_1 \otimes \tilde{e}_i b_2 \quad \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \tag{2.20}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \tilde{f}_i b_1 \otimes b_2 \quad \text{if } \varphi_i(b_1) > \varepsilon_i(b_2),$$
$$= b_1 \otimes \tilde{f}_i b_2 \quad \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2), \tag{2.21}$$

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2). \tag{2.22}$$

Then the following proposition is immediate.

PROPOSITION 2.3. *If B_1, B_2, and B_3 are crystals, then*

(a) *$B_1 \otimes B_2$ as defined above is a crystal.*

(b) *The map $(B_1 \otimes B_2) \otimes B_3 \to B_1 \otimes (B_2 \otimes B_3)$ given by $(b_1 \otimes b_2) \otimes b_3 \mapsto$*

$b_1 \otimes (b_2 \otimes b_3)$ *is an isomorphism in the category of crystals.* □

Now we give several examples of crystals which will be of interest to us in this paper.

EXAMPLE 2.4. For $\lambda \in P_+$, let $V(\lambda)$ denote the irreducible integrable $U_q(\mathfrak{g})$-module with highest weight λ. Let $(L(\lambda), B(\lambda))$ be the crystal base for $V(\lambda)$. Then the set $B(\lambda)$ is a crystal with the maps $\tilde{e}_i$, $\tilde{f}_i$, ε_i, φ_i defined by (2.6)–(2.9).

EXAMPLE 2.5. Let $U_q^-(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i's. Then $U_q^-(\mathfrak{g})$ has the unique endomorphisms e_i' and e_i'' such that

$$[e_i, u] = \frac{t_i e_i''(u) - t_i^{-1} e_i'(u)}{q_i - q_i^{-1}} \tag{2.23}$$

for any $u \in U_q^-(\mathfrak{g})$. Then e_i' and f_i satisfy the following commutation relations:

$$e_i' f_j = q_i^{-\langle h_i, \alpha_j \rangle} f_j e_i' + \delta_{ij}. \tag{2.24}$$

Here f_j is considered as the left multiplication operator. Then any element in $u \in U_q^-(\mathfrak{g})$ can be uniquely written as

$$u = \sum_{k \geqslant 0} f_i^{(k)} u_k,$$

where $e_i' u_k = 0$. Now we define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i$ on $U_q^-(\mathfrak{g})$ by

$$\tilde{e}_i u = \sum f_i^{(k-1)} u_k, \tag{2.25}$$

$$\tilde{f}_i u = \sum f_i^{(k+1)} u_k. \tag{2.26}$$

Then we have $\tilde{e}_i \tilde{f}_i = 1$. Let $L(\infty)$ be the smallest A-submodule of $U_q^-(\mathfrak{g})$ containing 1 that is stable by $\tilde{f}_i$'s. Let $B(\infty)$ be the subset of $L(\infty)/qL(\infty)$ consisting of the nonzero vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} \cdot 1 \bmod qL(\infty)$. Then $(L(\infty), B(\infty))$ is the crystal base of $U_q^-(\mathfrak{g})$ (see [K2]). For $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_k} \cdot 1 \in B(\infty)$, define

$$\varepsilon_i(b) = \max\{k \geqslant 0 \mid \tilde{e}_i^k b \neq 0\}, \tag{2.27}$$

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle, \tag{2.28}$$

where $wt(b) = -\alpha_{i_1} - \cdots - \alpha_{i_k}$. Then the set $B(\infty)$ with the maps $\tilde{e}_i$, $\tilde{f}_i$, ε_i, φ_i defined in (2.25)–(2.28) is a crystal. We denote by u_∞ the element in $B(\infty)$ that corresponds to $1 \in U_q^-(\mathfrak{g})$. The goal of this paper is to give path realizations of the crystals $B(\infty)$ for the quantum affine Lie algebra of type $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, and $D_{n+1}^{(2)}$.

EXAMPLE 2.6. For $\lambda \in P$, consider the set $T_\lambda = \{t_\lambda\}$ with one element. Define $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$ for $i \in I$ and $wt(t_\lambda) = \lambda$. Then T_λ is a crystal. For any crystals B, $B \otimes T_0 \cong T_0 \otimes B \cong B$.

EXAMPLE 2.7. Given any crystal B, we can form the following crystals by taking the tensor product with the crystal T_λ, $\lambda \in P$:

(a) $B \otimes T_\lambda = \{b \otimes t_\lambda \mid b \in B\}$,

where

$$wt(b \otimes t_\lambda) = \lambda + wt(b),$$
$$\tilde{e}_i(b \otimes t_\lambda) = \tilde{e}_i b \otimes t_\lambda,\ \tilde{f}_i(b \otimes t_\lambda) = \tilde{f}_i b \otimes t_\lambda,$$
$$\varepsilon_i(b \otimes t_\lambda) = \varepsilon_i(b),\ \varphi_i(b \otimes t_\lambda) = \varphi_i(b) + \langle h_i, \lambda\rangle.$$

(b) $T_\lambda \otimes B = \{t_\lambda \otimes b \mid b \in B\}$,

where

$$wt(t_\lambda \otimes b) = \lambda + wt(b),$$
$$\tilde{e}_i(t_\lambda \otimes b) = t_\lambda \otimes \tilde{e}_i b,\ \tilde{f}_i(t_\lambda \otimes b) = t_\lambda \otimes \tilde{f}_i b,$$
$$\varepsilon_i(t_\lambda \otimes b) = \varepsilon_i(b) - \langle h_i, \lambda\rangle,\ \varphi_i(t_\lambda \otimes b) = \varphi_i(b).$$

(c) For $\lambda, \mu \in P$,

$$T_\lambda \otimes B \otimes T_\mu = \{t_\lambda \otimes b \otimes t_\mu \mid b \in B\},$$

where

$$wt(t_\lambda \otimes b \otimes t_\mu) = \lambda + \mu + wt(b),$$
$$\tilde{e}_i(t_\lambda \otimes b \otimes t_\mu) = t_\lambda \otimes \tilde{e}_i b \otimes t_\mu,\ \tilde{f}_i(t_\lambda \otimes b \otimes t_\mu) = t_\lambda \otimes \tilde{f}_i b \otimes t_\mu,$$
$$\varepsilon_i(t_\lambda \otimes b \otimes t_\mu) = \varepsilon_i(b) - \langle h_i, \lambda\rangle,\ \varphi_i(t_\lambda \otimes b \otimes t_\mu) = \varphi_i(b) + \langle h_i, \mu\rangle.$$

3. Perfect crystals and paths

Let $\mathfrak{g}$ be an indecomposable affine Kac-Moody Lie algebra defined over $\mathbf{Q}$. Let $\{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ and $\{h_i \mid i \in I\} \subset \mathfrak{h}$ denote the simple roots and simple coroots, respectively. Thus $\{\alpha_i \mid i \in I\}$ and $\{h_i \mid i \in I\}$ are linearly independent and $\dim \mathfrak{h} = \#I + 1$. Let $Q = \Sigma_i \mathbf{Z}\alpha_i$, $Q_+ = \Sigma_i \mathbf{Z}_{\geq 0}\alpha_i$, and $Q_- = -Q_+$. Let $\delta \in Q_+$ be the generator of null roots and let $c \in \Sigma_i \mathbf{Z}_{\geq 0} h_i$ be the generator of the center. Set

$\mathfrak{h}_{\rm cl} = \bigoplus_i \mathbf{Q}h_i \subset \mathfrak{h}$ and $\mathfrak{h}_{cl}^* = (\bigoplus_i \mathbf{Q}h_i)^*$, and let $\mathrm{cl}: \mathfrak{h}^* \to \mathfrak{h}_{cl}^*$ be the canonical morphism. Then we have an exact sequence

$$0 \longrightarrow \mathbf{Q}\delta \longrightarrow \mathfrak{h}^* \stackrel{cl}{\longrightarrow} \mathfrak{h}_{cl}^* \longrightarrow 0.$$

Fix $i_0 \in I$ and take an integer d such that $\delta - d\alpha_{i_0} \in \Sigma_{i \neq i_0} \mathbf{Z}\alpha_i$. For simplicity, we write 0 for i_0. Let $af: \mathfrak{h}_{cl}^* \to \mathfrak{h}^*$ be a map satisfying $cl \circ af = id$ and $af \circ cl(\alpha_i) = \alpha_i$ for $i \neq 0$. Let Λ_i be the element of $\mathfrak{h}_{cl}^* \subset \mathfrak{h}^*$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$. Hence we have $\alpha_i = \Sigma_j \langle h_j, \alpha_i \rangle af(\Lambda_j) + \delta_{i,0} d^{-1}\delta$. We take $P = \Sigma_i \mathbf{Z} af(\Lambda_i) + \mathbf{Z}d^{-1}\delta \subset \mathfrak{h}^*$ and $P_{\rm cl} = \mathrm{cl}(P) \subset \mathfrak{h}_{cl}^*$. An element of P is called an *afine weight* and an element of $P_{\rm cl}$ is called a *classical weight*. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra associated with P, and let $U_q'(\mathfrak{g})$ be the quantized universal enveloping algebra associated with $P_{\rm cl}$. A P-weighted crystal is called an *affine crystal* and a $P_{\rm cl}$-weighted crystal is called a *classical crystal.*

Let B be a classical crystal. For $b \in B$, we set $\varepsilon(b) = \Sigma_i \varepsilon_i(b)\Lambda_i$ and $\varphi(b) = \Sigma_i \varphi_i(b)\Lambda_i$. Note that $wt(b) = \varphi(b) - \varepsilon(b)$. Set $P_{cl}^+ = \{\lambda \in P_{cl} \mid \langle h_i, \lambda \rangle \geqslant 0$ for all $i \in I\}$ and for $l \in \mathbf{Z}_{\geqslant 0}$ let $(P_{cl}^+)_l = \{\lambda \in P_{cl}^+ \mid \langle c, \lambda \rangle = l\}$.

DEFINITION 3.1. For $l \in \mathbf{Z}_{\geqslant 0}$, we say that B is a *perfect crystal of level l* if

(3.1) $B \otimes B$ is connected.

(3.2) There exists $\lambda_0 \in P_{cl}$ such that $wt(B) \subset \lambda_0 + \Sigma_{i \neq 0} \mathbf{Z}_{\geqslant 0} cl(\alpha_i)$ and that $\#(B_{\lambda_0}) = 1$.

(3.3) There is a finite dimensional $U_q'(\mathfrak{g})$-module with a crystal pseudo-base (L, B_{ps}) such that $B \cong B_{ps}/\pm 1$.

(3.4) For any $b \in B$, we have $\langle c, \varepsilon(b) \rangle \geqslant l$.

(3.5) The maps ε, $\varphi: B^{\min} = \{b \in B \mid \langle c, \varepsilon(b) \rangle = l\} \to (P_{cl}^+)_l$ are bijective.

The elements in $B^{\min}$ are called *minimal elements.*

A $\mathbf{Z}$-valued function H on $B \otimes B$ is called an *energy function* on B if for any $i \in I$ and $b \otimes b' \in B \otimes B$ such that $\tilde{e}_i(b \otimes b') \neq 0$, we have

$$\begin{aligned} H(\tilde{e}_i(b \otimes b')) &= H(b \otimes b') \quad \text{if } i \neq 0, \\ &= H(b \otimes b') + 1 \quad \text{if } i = 0 \text{ and } \varphi_0(b) \geqslant \varepsilon_0(b'), \\ &= H(b \otimes b') - 1 \quad \text{if } i = 0 \text{ and } \varphi_0(b) < \varepsilon_0(b'). \end{aligned} \tag{3.6}$$

The existence of energy functions is proved for the perfect crystals in [KMN2]. From now on, we assume that $\mathfrak{g}$ is of rank $\geqslant 3$. For $\lambda \in P^+$, let $B(\lambda)$

be the affine crystal with highest weight λ, and denote by u_λ the highest weight element of $B(\lambda)$.

THEOREM 3.2 ([KMN2]). *Let B be a perfect crystal of level l, and let b be an element of B. Then we have an isomorphism of classical crystals*

$$B(af(\varepsilon(b))) \otimes B \cong B(af(b))) \tag{3.7}$$

sending $u_{af(\varepsilon(b))} \otimes b$ to $u_{af(\varphi(b))}$. □

For $\mu \in (P_{cl}^+)_l$, let b_μ be the unique element of B such that $\varphi(b_\mu) = \mu$. We define the isomorphism σ of $(P_{cl}^+)_l$ by $\varepsilon(b_\mu) = \sigma\mu$. We lift σ to an isomorphism of $af(P_{cl}^+)_l$. Then for $\lambda \in af(P_{cl}^+)_l$, by Theorem 3.2, we have

$$B(\lambda) \cong B(\sigma\lambda) \otimes B$$

given by $u_\lambda \mapsto u_{\sigma\lambda} \otimes b_{cl(\lambda)}$. We set $\lambda_k = \sigma^k \lambda$ and $b_k = b_{cl(\lambda_{k-1})}$ for $k \geqslant 1$. Thus by applying Theorem 3.2 repeatedly, we obtain an isomorphism of classical crystals

$$\psi_k : B(\lambda) \cong B(\lambda_k) \otimes B^{\otimes k} \tag{3.8}$$

given by

$$u_\lambda \mapsto u_{\lambda_k} \otimes b_k \otimes \cdots \otimes b_2 \otimes b_1.$$

The sequence $(b_1, b_2, b_3, \ldots)$ is called the *ground-state path* of weight λ. A *λ-path a*in B is a sequence $p = (p(k))_{k \geqslant 1}$ in B such that $p(k) = b_k$ for $k \gg 0$. We denote by $\mathfrak{P}(\lambda, B)$ the set of all λ-paths. The crystal graph structure on $\mathfrak{P}(\lambda, B)$ is given by (2.20) and (2.21).

THEOREM 3.3 ([KMN2]). *The crystal $B(\lambda)$ is isomorphic to $\mathfrak{P}(\lambda, B)$ given by $B(\lambda) \ni b \mapsto p \in \mathfrak{P}(\lambda, B)$ where $\psi_k(b) = u_{\lambda_k} \otimes p(k) \otimes \cdots \otimes p(1)$ for $k \gg 0$.* □
The affine weight of elements in $\mathfrak{P}(\lambda, B)$ can be computed by the energy function H as in the following theorem.

THEOREM 3.4 ([KMN2]). *If $b \in B(\lambda)$ corresponds to the λ-path $p = (p(k))_{k \geqslant 1} \in \mathfrak{P}(\lambda, B)$, then we have*

$$wt b = \lambda + \sum_{k=1}^{\infty} (af(wt p(k)) - af(wt b_k))$$
$$-\left(\sum_{k=1}^{\infty} k(H(p(k+1) \otimes p(k)) - H(b_{k+1} \otimes b_k))\right) d^{-1}\delta. \quad □$$

4. Crystal base of $U_q^-(\mathfrak{g})$ and paths

Let $\{B_l\}_{l\geqslant 1}$ be a family of perfect crystals B_l of level l and $B_l^{\min} = \{b \in B_l \mid \langle c, \varepsilon(b)\rangle = l\}$. We take the index set $J = \{(l, b) \mid l \in \mathbf{Z}_{>0}, b \in B_l^{\min}\}$.

DEFINITION 4.1. A classical crystal B_∞ with an element b_∞ is called a *limit* of $\{B_l\}_{l\geqslant 1}$ if it satisfies the following conditions:

$$wt(b_\infty) = 0,\ \varepsilon(b_\infty) = \varphi(b_\infty) = 0, \tag{4.1}$$

for any $(l, b) \in J$, there exists an embedding of crystals (4.2)

$$f_{(l,b)}: T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \to B_\infty$$

sending $t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)}$ to b_∞,

$$B_\infty = \bigcup_{(l,b)\in J} \operatorname{Im} f_{(l,b)}. \tag{4.3}$$

If a limit exists for the family $\{B_l\}$, we say that $\{B_l\}$ is a *coherent family* of perfect crystals. Note that for any $N > 0$, there exists $(l, b) \in J$ such that $\varepsilon_i(b) > N$ and $\varphi_i(b) > N$ for all $i \in I$.

Let B be a classical crystal with an element $b_0 \in B$ satisfying the following properties:

for any $N > 0$, there exists $(l_N, b_N) \in J$ such that $\varepsilon_i(b_N) > N$, $\varphi_i(b_N) > N$

for all $i \in I$ and that there exists an embedding (4.4)

$$f_N: T_{\varepsilon(b_N)} \otimes B_{l_N} \otimes T_{-\varphi(b_N)} \to B$$

sending $t_{\varepsilon(b_N)} \otimes b_N \otimes t_{-\varphi(b_N)}$ to b_0,

$$B = \bigcup_{N\geqslant 1} \operatorname{Im} f_N. \tag{4.5}$$

Since each B_{l_N} is connected, B is also connected.

LEMMA 4.2. *If* (B, b_0) *satisfies* (4.4) *and* (4.5), *then*

(a) *for any finite subset* F *of* B, *there exists* $N > 0$ *such that* $F \subset \operatorname{Im} f_N$,

(b) $\tilde{e}_i B \subset B$, $\tilde{f}_i B \subset B$ *for all* $i \in I$.

Proof. We may assume that F is a connected subcrystal of B containing b_0. Take N such that $\varepsilon_i(b) \geqslant -N$ and $\varphi_i(b) \geqslant -N$ for all $i \in I$ and $b \in F$. In this case, we will show that $F \subset \operatorname{Im} f_N$, $\tilde{e}_i F \subset B$, and $\tilde{f}_i F \subset B$. For this, it suffices to show that

(i) for $b \in F \cap \operatorname{Im} f_N$, we have $\tilde{e}_i b \in \operatorname{Im} f_N$,
(ii) for $b \in F \cap \operatorname{Im} f_N$, we have $\tilde{f}_i b \in \operatorname{Im} f_N$.
Let $b = f_N(t_{\varepsilon(b_N)} \otimes b' \otimes t_{-\varphi(b_N)})$ for some $b' \in B_{l_N}$. Then $\varepsilon_i(b') = \varepsilon_i(b) + \varepsilon_i(b_N) > 0$, which implies $\tilde{e}_i b' \in B_{l_N}$. Hence $\tilde{e}_i b \in \operatorname{Im} f_N$. The proof for (ii) is similar. □

LEMMA 4.3. *Let (B_∞, b_∞) be a limit of a coherent family $\{B_l\}_{l \geqslant 1}$ and suppose that (B, b_0) satisfies the conditions* (4.4) *and* (4.5). *Then there exists a unique isomorphism $B \to B_\infty$ of crystals sending b_0 to b_∞.*

Proof. By Lemma 4.2, any finite connected subcrystal F of B containing b_0 is contained in $\operatorname{Im} f_N$ for some $N \gg 0$. Hence there exists an embedding of crystals

$$F \xrightarrow{f_N^{-1}} T_{\varepsilon(b_N)} \otimes B_{l_N} \otimes T_{-\varphi(b_N)} \xrightarrow{f_{(l_N, b_N)}} B_\infty,$$

which yields an embedding $B \to B_\infty$. This is evidently an isomorphism. □

COROLLARY 4.4. *A limit (B_∞, b_∞) of a coherent family $\{B_l\}_{l \geqslant 1}$ is unique up to isomorphism.* □

The following proposition is an immediate consequence of the properties of the perfect crystal B_l (see Definition 3.1).

PROPOSITION 4.5.
(a) *$B_\infty \otimes B_\infty$ is connected.*
(b) *For any $b \in B_\infty$, $\langle c, \varepsilon(b) \rangle \geqslant 0$.*
(c) *The maps ε and φ give bijections from $B_\infty^{\min} = \{b \in B_\infty \mid \langle c, \varepsilon(b) \rangle = 0\}$ to $(P_{cl})_0$.* □

Let $b_0 \in B_\infty^{\min}$. Then $b_0 \in \operatorname{Im} f_{(l,b)}$ for some $(l, b) \in J$ with $\varepsilon_i(b) \gg 0$, $\varphi_i(b) \gg 0$. Define $b' \in B_l$ by

$$f_{(l,b)}(t_{\varepsilon(b)} \otimes b' \otimes t_{-\varphi(b)}) = b_0. \tag{4.6}$$

Then $\varepsilon(b') = \varepsilon(b) + \varepsilon(b_0)$ and $\varphi(b') = \varphi(b) + \varphi(b_0)$. Hence $b' \in B_l^{\min}$ and $\varepsilon(b')$, $\varphi(b') \gg 0$. Moreover, the map $T_{-\varepsilon(b_0)} \otimes f_{(l,b')} \otimes T_{\varphi(b_0)}$ gives an embedding of crystals

$$T_{\varepsilon(b')} \otimes B_l \otimes T_{-\varphi(b')} \to T_{\varepsilon(b_0)} \otimes B_\infty \otimes T_{-\varphi(b_0)}$$

such that

$$t_{\varepsilon(b')} \otimes b' \otimes t_{-\varphi(b')} \mapsto t_{\varepsilon(b_0)} \otimes b_0 \otimes t_{-\varphi(b_0)}. \tag{4.7}$$

Therefore,

$$(T_{\varepsilon(b_0)} \otimes B_\infty \otimes T_{-\varphi(b_0)}, t_{\varepsilon(b_0)} \otimes b_0 \otimes t_{-\varphi(b_0)})$$

satisfies the conditions (4.4) and (4.5). Hence we obtain:

LEMMA 4.6. *For any $b \in B_\infty^{\min}$, there exists a unique isomorphism*

$$\theta_b : B_\infty \to T_{\varepsilon(b)} \otimes B_\infty \otimes T_{-\varphi(b)}$$

such that

$$\theta_b(b_\infty) = t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)}. \tag{4.8}$$

□

For $b', b'' \in B_\infty^{\min}$, let b be an element of B_∞ such that

$$\theta_{b''}(b') = t_{\varepsilon(b'')} \otimes b \otimes t_{-\varphi(b'')}.$$

Then $\varepsilon(b) = \varepsilon(b') + \varepsilon(b'')$ and $\varphi(b) = \varphi(b') + \varphi(b'')$. Hence we have a well-defined linear automorphism σ on $(P_{cl})_0$ such that $\sigma\varphi(b) = \varepsilon(b)$ for $b \in B_\infty^{\min}$.

As in Section 3, a $\mathbf{Z}$-valued function H on $B_\infty \otimes B_\infty$ is called an *energy function* if $H(b_\infty \otimes b_\infty) = 0$ and if for any $i \in I$ and $b \otimes b' \in B_\infty \otimes B_\infty$ such that $\tilde{e}_i(b \otimes b') \neq 0$, we have

$$\begin{aligned} H(\tilde{e}_i(b \otimes b')) &= H(b \otimes b') \quad \text{if } i \neq 0, \\ &= H(b \otimes b') + 1 \quad \text{if } i = 0 \quad \text{and} \quad \varphi_0(b) \geqslant \varepsilon_0(b'), \\ &= H(b \otimes b') - 1 \quad \text{if } i = 0 \quad \text{and} \quad \varphi_0(b) < \varepsilon_0(b'). \end{aligned} \tag{4.9}$$

Then the existence and the uniqueness of the energy function on B_∞ follows immediately from the corresponding result on B_l.

Take $\lambda \in af(P_{cl}^+)$ such that $\lambda(h_i) \gg 0$ for all i, and let $B(\lambda)$ be the affine crystal with highest weight λ. By Theorem 3.2, we have

$$B(\lambda) \cong B(\sigma\lambda) \otimes B_l$$

given by $u_\lambda \mapsto u_{\sigma\lambda} \otimes b_0$, where b_0 is the unique element in $B_l^{\min}$ such that $\varphi(b_0) = \lambda$, $\varepsilon(b_0) = \sigma\lambda$. Thus we have an isomorphism of crystals

$$B(\lambda) \otimes T_{-\lambda} \cong B(\sigma\lambda) \otimes T_{-\sigma\lambda} \otimes T_{\sigma\lambda} \otimes B_l \otimes T_{-\lambda} \tag{4.10}$$

given by

$$u_\lambda \otimes t_{-\lambda} \mapsto u_{\sigma\lambda} \otimes t_{-\sigma\lambda} \otimes t_{\sigma\lambda} \otimes b_0 \otimes t_{-\lambda}.$$

Note that $B(\infty) = \varinjlim_\lambda B(\lambda) \otimes T_{-\lambda}$ and $B_\infty = \varinjlim_\lambda T_{\sigma\lambda} \otimes B_l \otimes T_{-\lambda}$. Hence by taking the limit of (4.10) we obtain:

THEOREM 4.7. *There is an isomorphism of crystals*

$$B(\infty) \cong B(\infty) \otimes B_\infty$$

sending u_∞ *to* $u_\infty \otimes b_\infty$. □

By applying Theorem 4.7 repeatedly, we obtain an isomorphism of crystals

$$\psi_k : B(\infty) \cong B(\infty) \otimes B_\infty^{\otimes k}$$

given by $u_\infty \mapsto u_\infty \otimes b_\infty \otimes \cdots \otimes b_\infty \otimes b_\infty$.

The sequence $(b_\infty, b_\infty, b_\infty, \ldots)$ is called the *ground-state path*. A *path* in B_∞ is a sequence $p = (p(k))_{k \geq 1}$ in B_∞ such that $p(k) = b_\infty$ for $k \gg 0$. We denote by $\mathfrak{P}(B_\infty)$ the set of all paths in B_∞. The actions of $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in I)$ are given by (2.20) and (2.21). Then we have

THEOREM 4.8. *The crystal* $B(\infty)$ *is isomorphic to* $\mathfrak{P}(B_\infty)$ *given by* $B(\infty) \ni b \mapsto p \in \mathfrak{P}(B_\infty)$ where $\psi_k(b) = u_\infty \otimes p(k) \otimes \cdots \otimes p(1)$ *for* $k \gg 0$. □

Now we will compute the weight of a path in $\mathfrak{P}(B_\infty)$. Let $p = (p(k))_{k \geq 1} \in \mathfrak{P}(B_\infty)$ be a path. By Theorem 4.8, we can find an $l \gg 0$ and $b_0 \in B_l^{\min}$ such that for all $k \geq 1$, $p(k) = f_{(l,b)}(p_k)$ for $p_k \in T_{\varepsilon(b_0)} \otimes T_{-\varphi(b_0)}$. Since $p(k) = b_\infty$ for $k \gg 0$, we also have $p_k = t_{\varepsilon(b_0)} \otimes b_0 \otimes t_{-\varphi(b_0)}$ for $k \gg 0$. Note that if $f : B \to B'$ is an embedding of crystals with energy functions H and H', respectively, then $f \otimes f : B \otimes B \to B' \otimes B'$ is also an embedding of crystals and we have $wt f(b) = wt b$ for $b \in B$ and $H'(f(b_1) \otimes f(b_2)) = H(b_1 \otimes b_2)$ for b_1, $b_2 \in B$. Therefore the following theorem is an immediate consequence of Theorem 3.4.

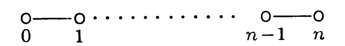
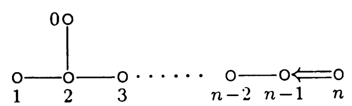
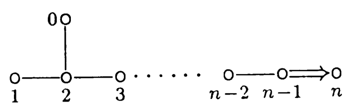
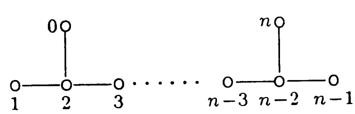
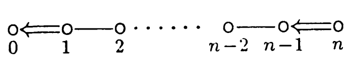
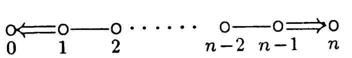
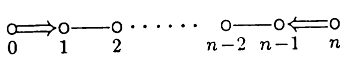
THEOREM 4.9. *If* $b \in B(\infty)$ *corresponds to the path* $p = (p(k))_{k \geq 1} \in \mathfrak{P}(B_\infty)$, *then we have*

$$wt b = \sum_{k=1}^{\infty} af(wt p(k)) - \sum_{k=1}^{\infty} kH(p(k+1) \otimes p(k)) d^{-1}\delta.$$ □

Here d is an integer such that $\delta - d\alpha_0 \in \Sigma_{i\neq 0}\mathbf{Z}\alpha_i$.

5. The crystals B_∞ and energy function

In this section we will give a description of B_∞ and the energy functions by using a coherent family of perfect crystals given in [KMN3] for $\mathfrak{g} = A_n^{(1)}$, $A_{2n-1}^{(2)}$, $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$ and $C_n^{(1)}$. We give the proof only in the $A_n^{(1)}$-case. The other cases can be proved similarly. The details of these proofs can be seen in our RIMS preprint [KKM]. In the table below we list the Dynkin diagrams and the corresponding coherent families of perfect crystals without the 0-arrow.

Lie algebra	Dynkin diagram	Perfect crystals of level l
$A_n^{(1)}$		$B(l\Lambda_1)$
$A_{2n-1}^{(2)}$		$B(l\Lambda_1)$
$B_n^{(1)}$		$B(l\Lambda_1)$
$D_n^{(1)}$		$B(l\Lambda_1)$
$A_{2n}^{(2)}$		$B(l\Lambda_1)\oplus B((l-1)\Lambda_1)\oplus\cdots\oplus B(0)$
$D_{n+1}^{(2)}$		$B(l\Lambda_1)\oplus B((l-1)\Lambda_1)\oplus\cdots\oplus B(0)$
$C_n^{(1)}$		$B(2l\Lambda_1)\oplus B(2(l-1)\Lambda_1)\oplus\cdots\oplus B(0)$

5.1. $A_n^{(1)}$ $(n \geqslant 2)$

Let $\mathscr{I} = \mathbf{Z}/(n+1)\mathbf{Z}$ and

$$B_\infty = \left\{(v_i)_i \in \mathbf{Z}^{\mathscr{I}} \,\middle|\, \sum_i v_i = 0\right\}, \quad b_\infty = (0, 0, \ldots, 0).$$

Define $\delta_j \in \mathbf{Z}^{\mathscr{I}}$ by $(\delta_j)_i = \delta_{ji}$.

For $b = (v_i)_i$ and $b' = (v'_i)_i$ in B_∞,

$$\begin{aligned}\tilde{e}_i b &= b + \delta_i - \delta_{i+1} \\ \tilde{f}_i b &= b - \delta_i + \delta_{i+1}\end{aligned} \tag{5.1}$$

$$\begin{aligned}wt(b &= \sum_i (v_i - v_{i+1})\Lambda_i, \\ \varphi_i(b) &= v_i,\ \varepsilon_i(b) = v_{i+1}.\end{aligned} \tag{5.2}$$

$H(b \otimes b') = \max\{\theta_j(b \otimes b') \mid 0 \leqslant j \leqslant n\}$, where

$$\theta_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k) + v'_{j+1}.$$

5.2. $A^{(2)}_{2n-1}$ $(n \geqslant 3)$

$$B_\infty = \left\{(v_1,\dots,v_n,\bar{v}_n,\dots,\bar{v}_1) \in \mathbf{Z}^{2n} \,\middle|\, \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \bar{v}_i = 0\right\}, \quad b_\infty = (0, 0,\dots,0)$$

For $b = (v_1,\dots,v_n,\bar{v}_n,\dots,\bar{v}_1)$ and $b' = (v'_1,\dots,v'_n,\bar{v}'_n,\dots,\bar{v}_1)$ in B_∞,

$$\begin{aligned}
\tilde{e}_0 b &= (v_1, v_2 - 1,\dots,\bar{v}_2, \bar{v}_1 + 1) \quad \text{if } v_2 > \bar{v}_2, \\
&= (v_1 - 1, v_2,\dots,\bar{v}_2 + 1, \bar{v}_1) \quad \text{if } v_2 \leqslant \bar{v}_2, \\
\tilde{e}_n b &= (v_1,\dots,v_n + 1, \bar{v}_n - 1,\dots,\bar{v}_1), \\
\tilde{e}_i b &= (v_1,\dots,v_i + 1, v_{i+1} - 1,\dots,\bar{v}_1) \quad \text{if } v_{i+1} > \bar{v}_{i+1}, \\
&= (v_1,\dots,\bar{v}_{i+1} + 1, \bar{v}_i - 1,\dots,\bar{v}_1) \quad \text{if } v_{i+1} \leqslant \bar{v}_{i+1}, \\
\tilde{f}_0 b &= (v_1, v_2 + 1,\dots,\bar{v}_2, \bar{v}_1 - 1) \quad \text{if } v_2 \geqslant \bar{v}_2, \\
&= (v_1 + 1, v_2,\dots,\bar{v}_2 - 1, \bar{v}_1) \quad \text{if } v_2 < \bar{v}_2, \\
\tilde{f}_n b &= (v_1,\dots,v_n - 1, \bar{v}_n + 1,\dots,\bar{v}_1), \\
\tilde{f}_i b &= (v_1,\dots,v_i - 1, v_{i+1} + 1,\dots,\bar{v}_1) \quad \text{if } v_{i+1} \geqslant \bar{v}_{i+1}, \\
&= (v_1,\dots,\bar{v}_{i+1} - 1, \bar{v}_i + 1,\dots,\bar{v}_1) \quad \text{if } v_{i+1} < \bar{v}_{i+1},
\end{aligned}$$

$$\begin{aligned}
wt(b) &= (\bar{v}_1 - v_1 + \bar{v}_2 - v_2)\Lambda_0 \\
&\quad + \sum_{i=1}^{n-1} (v_i - \bar{v}_i + \bar{v}_{i+1} - v_{i+1})\Lambda_i + (v_n - \bar{v}_n)\Lambda_n, \\
\varphi_0(b) &= \bar{v}_1 + (\bar{v}_2 - v_2)_+,\ \varepsilon_0(b) = v_1 + (v_2 - \bar{v}_2)_+, \\
\varphi_i(b) &= v_i + (\bar{v}_{i+1} - v_{i+1})_+ \quad \text{for } i = 1,\dots,n-1,
\end{aligned}$$

$\varepsilon_i(b) = \bar{v}_i + (v_{i+1} - \bar{v}_{i+1})_+ \quad \text{for } i = 1,\ldots,n-1,$

$\varphi_n(b) = v_n,\ \varepsilon_n(b) = \bar{v}_n,$

where by definition $x_+ = \max(x, 0)$.

$$H(b \otimes b') = \max(\{\theta_j(b \otimes b'),\ \theta'_j(b \otimes b') \mid 1 \leqslant j \leqslant n-1\} \cup \{\eta_j(b \otimes b'),$$
$$\eta'_j(b \otimes b') \mid 1 \leqslant j \leqslant n\}),$$

where

$$\theta_j(b \otimes b') = \sum_{k=1}^{j} (\bar{v}_k - \bar{v}'_k),$$

$$\theta'_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k),$$

$$\eta_j(b \otimes b') = \sum_{k=1}^{j} (\bar{v}_k - \bar{v}'_k) + (\bar{v}'_j - v_j),$$

$$\eta'_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k) + (v_j - \bar{v}'_j).$$

5.3. $B_n^{(1)}$ $(n \geqslant 3)$

$$B_\infty = \left\{(v_1,\ldots,v_n,\ v_0,\ \bar{v}_n,\ldots,\bar{v}_1) \in \mathbf{Z}^{2n} \times \{0, 1\} \mid v_0 = 0 \text{ or } 1,\right.$$
$$\left.\sum_{i=1}^{n} v_i + v_0 + \sum_{i=1}^{n} \bar{v}_i = 0\right\}, \quad b_\infty = (0, 0,\ldots,0).$$

For $b = (v_1,\ldots,v_n,\ v_0,\ \bar{v}_n,\ldots,\bar{v}_1)$ and $b' = (v'_1,\ldots,v'_n,\ v'_0,\ v'_n,\ldots,\bar{v}_1)$ in B_∞,

$$\tilde{e}_0 b = (v_1,\ v_2 - 1,\ldots,\bar{v}_2,\ \bar{v}_1 + 1) \quad \text{if } v_2 > \bar{v}_2,$$
$$= (v_1 - 1,\ v_2,\ldots,\bar{v}_2 + 1,\ \bar{v}_1) \quad \text{if } v_2 \leqslant \bar{v}_2,$$
$$\tilde{e}_n b = (v_1,\ldots,v_n,\ v_0 + 1,\ \bar{v}_n - 1,\ldots,\bar{v}_1) \quad \text{if } v_0 = 0,$$
$$= (v_1,\ldots,v_n + 1,\ v_0 - 1,\ \bar{v}_n,\ldots,\bar{v}_1) \quad \text{if } v_0 = 1,$$
$$\tilde{e}_i b = (v_1,\ldots,v_i + 1,\ v_{i+1} - 1,\ldots,\bar{v}_1) \quad \text{if } v_{i+1} > \bar{v}_{i+1},$$
$$= (v_1,\ldots,\bar{v}_{i+1} + 1,\ \bar{v}_i - 1,\ldots,\bar{v}_1) \quad \text{if } v_{i+1} \leqslant \bar{v}_{i+1},$$

$$\tilde{f}_0 b = (v_1,\ v_2 + 1,\ldots,\bar{v}_2,\ \bar{v}_1 - 1) \quad \text{if } v_2 \geqslant \bar{v}_2,$$
$$= (v_1 + 1,\ v_2,\ldots,\bar{v}_2 - 1,\ \bar{v}_1) \quad \text{if } v_2 < \bar{v}_2,$$

$$\tilde{f}_n b = (v_1, \ldots, v_n - 1, v_0 + 1, \bar{v}_n, \ldots, \bar{v}_1) \quad \text{if } v_0 = 0,$$
$$= (v_1, \ldots, v_n, v_0 - 1, \bar{v}_n + 1, \ldots, \bar{v}_1) \quad \text{if } v_0 = 1,$$
$$\tilde{f}_i b = (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} \geqslant \bar{v}_{i+1},$$
$$= (v_1, \ldots, \bar{v}_{i+1} - 1, \bar{v}_i + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} < \bar{v}_{i+1},$$

$$wt(b) = (\bar{v}_1 - v_1 + \bar{v}_2 - v_2)\Lambda_0 + \sum_{i=1}^{n-1} (v_i - \bar{v}_i + \bar{v}_{i+1} - v_{i+1})\Lambda_i$$
$$+ 2(v_n - \bar{v}_n)\Lambda_n,$$

$$\varphi_0(b) = \bar{v}_1 + (\bar{v}_2 - v_2)_+, \ \varepsilon_0(b) = v_1 + (v_2 - \bar{v}_2)_+,$$
$$\varphi_i(b) = v_i + (\bar{v}_{i+1} - v_{i+1})_+ \quad \text{for } i = 1, \ldots, n-1,$$
$$\varepsilon_i(b) = \bar{v}_i + (v_{i+1} - \bar{v}_{i+1})_+ \quad \text{for } i = 1, \ldots, n-1,$$
$$\varphi_n(b) = 2v_n + v_0, \ \varepsilon_n(b) = 2\bar{v}_n + v_0,$$

$$H(b \otimes b') = \max(\{\theta_j(b \otimes b'), \theta'_j(b \otimes b') \mid 1 \leqslant j \leqslant n-1\} \cup \{\eta_j(b \otimes b'), \eta'_j(b \otimes b') \mid 1 \leqslant j \leqslant n\}),$$

where

$$\theta_j(b \otimes b') = \sum_{k=1}^{j} (\bar{v}_k - \bar{v}'_k),$$
$$\theta'_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k),$$
$$\eta_j(b \otimes b') = \sum_{k=1}^{j} (\bar{v}_k - \bar{v}'_k) + (\bar{v}'_j - v_j),$$
$$\eta'_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k) + (v_j - \bar{v}'_j).$$

5.4. $D_n^{(1)}$ $(n \geqslant 4)$

$$B_\infty = \left\{(v_1, \ldots, v_n, v_{n-1}, \ldots, \bar{v}_1) \in \mathbf{Z}^{2n-1} \,\middle|\, \sum_{i=1}^{n} v_i + \sum_{i=1}^{n-1} \bar{v}_i = 0\right\},$$
$$b_\infty = (0, 0, .., 0).$$

For $b = (v_1, \ldots, v_n, v_{n-1}, \ldots, \bar{v}_1)$ and $b' = (v'_1, \ldots, v'_n, \bar{v}'_{n-1}, \ldots, \bar{v}_1)$ in B_∞,

$$\begin{aligned}
\tilde{e}_0 b &= (v_1, v_2 - 1, \dots, \bar{v}_2, \bar{v}_1 + 1) \quad \text{if } v_2 > \bar{v}_2,\\
&= (v_1 - 1, v_2, \dots, \bar{v}_2 + 1, \bar{v}_1) \quad \text{if } v_2 \leqslant \bar{v}_2,\\
\tilde{e}_{n-1} b &= (v_1, \dots, v_{n-1} + 1, \bar{v}_n - 1, \bar{v}_{n-1}, \dots, \bar{v}_1),\\
\tilde{e}_n b &= (v_1, \dots, v_n + 1, v_{n-1} - 1, \dots, \bar{v}_1),\\
\tilde{e}_i b &= (v_1, \dots, v_i + 1, v_{i+1} - 1, \dots, \bar{v}_1) \quad \text{if } v_{i+1} > \bar{v}_{i+1},\\
&= (v_1, \dots, \bar{v}_{i+1} + 1, \bar{v}_i - 1, \dots, \bar{v}_1) \quad \text{if } v_{i+1} \leqslant \bar{v}_{i+1},\\
\tilde{f}_0 b &= (v_1, v_2 + 1, \dots, \bar{v}_2, \bar{v}_1 - 1) \quad \text{if } v_2 \geqslant \bar{v}_2,\\
&= (v_1 + 1, v_2, \dots, \bar{v}_2 - 1, \bar{v}_1) \quad \text{if } v_2 < \bar{v}_2,\\
\tilde{f}_{n-1} b &= (v_1, \dots, v_{n-1} - 1, v_n + 1, \bar{v}_{n-1}, \dots, \bar{v}_1),\\
\tilde{f}_n b &= (v_1, \dots, v_n - 1, \bar{v}_{n-1} + 1, \dots, \bar{v}_1),\\
\tilde{f}_i b &= (v_1, \dots, v_i - 1, v_{i+1} + 1, \dots, \bar{v}_1) \quad \text{if } v_{i+1} \geqslant \bar{v}_{i+1},\\
&= (v_1, \dots, \bar{v}_{i+1} - 1, \bar{v}_i + 1, \dots, \bar{v}_1) \quad \text{if } v_{i+1} < \bar{v}_{i+1},\\
wt(b) &= (\bar{v}_1 - v_1 + \bar{v}_2 - v_2)\Lambda_0 + \sum_{i=1}^{n-2} (v_i - \bar{v}_i + \bar{v}_{i+1} - v_{i+1})\Lambda_i\\
&\quad + (v_{n-1} + \bar{v}_{n-1} - v_n)\Lambda_{n-1} + (v_{n-1} - \bar{v}_{n-1} + v_n)\Lambda_n,\\
\varphi_0(b) &= \bar{v}_1 + (\bar{v}_2 - v_2)_+, \ \varepsilon_0(b) = v_1 + (v_2 - \bar{v}_2)_+,\\
\varphi_i(b) &= v_i + (\bar{v}_{i+1} - v_{i+1})_+ \quad \text{for } i = 1, \dots, n-2,\\
\varepsilon_i(b) &= \bar{v}_i + (v_{i+1} - \bar{v}_{i+1})_+ \quad \text{for } i = 1, \dots, n-2,\\
\varphi_{n-1}(b) &= v_{n-1}, \ \varepsilon_{n-1}(b) = v_n + \bar{v}_{n-1},\\
\varphi_n(b) &= v_{n-1} + v_n, \ \varepsilon_n(b) = \bar{v}_{n-1},
\end{aligned}$$

$$\begin{aligned}
H(b \otimes b') = \max(\{0_j(b \otimes b'), 0'_j(b \otimes b') \mid 1 \leqslant j \leqslant n-2\} \cup \{\eta_j(b \otimes b'),\\
\eta'_j(b \otimes b') \mid 1 \leqslant j \leqslant n\}),
\end{aligned}$$

where

$$\begin{aligned}
0_j(b \otimes b') &= \sum_{k=1}^{j} (\bar{v}_k - \bar{v}'_k) \quad \text{for } j = 1, \dots, n-2,\\
0'_j(b \otimes b') &= \sum_{k=1}^{j} (v'_k - v_k) \quad \text{for } j = 1, \dots, n-2,
\end{aligned}$$

$$\eta_j(b \otimes b') = \sum_{k=1}^{j} (\bar{v}_k - \bar{v}'_k) + (\bar{v}'_j - v_j) \quad \text{for } j = 1, \ldots, n-1,$$

$$\eta'_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k) + (v_j - \bar{v}'_j) \quad \text{for } j = 1, \ldots, n-1,$$

$$\eta_n(b \otimes b') = \sum_{k=1}^{n-1} (\bar{v}_k - \bar{v}'_k) + v_n,$$

$$\eta'_n(b \otimes b') = \sum_{k=1}^{n-1} (v'_k - v_k) - v_n.$$

5.5. $A_{2n}^{(2)}$ $(n \geqslant 2)$

$$B_\infty = \{(v_1, \ldots, v_n, \bar{v}_n, \ldots, \bar{v}_1) \mid v_i, \bar{v}_i \in \mathbf{Z}\} = \mathbf{Z}^{2n}, \quad b_\infty = (0, 0, \ldots, 0).$$

For $b = (v_1, \ldots, v_n, \bar{v}_n, \ldots, \bar{v}_1)$ and $b' = (v'_1, \ldots, v'_n, \bar{v}'_n, \ldots, \bar{v}_1)$ in B_∞,

$$\begin{aligned}
\tilde{e}_0 b &= (v_1 - 1, v_2, \ldots, \bar{v}_1) \quad \text{if } v_1 > \bar{v}_1,\\
&= (v_1, \ldots, \bar{v}_2, \bar{v}_1 + 1) \quad \text{if } v_1 \leqslant \bar{v}_1,\\
\tilde{e}_n b &= (v_1, \ldots, v_n + 1, \bar{v}_n - 1, \ldots, \bar{v}_1),\\
\tilde{e}_i b &= (v_1, \ldots, v_i + 1, v_{i+1} - 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} > \bar{v}_{i+1},\\
&= (v_1, \ldots, \bar{v}_{i+1} + 1, \bar{v}_i - 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} \leqslant \bar{v}_{i+1},
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_0 b &= (v_1 + 1, v_2, \ldots, \bar{v}_1) \quad \text{if } v_1 \geqslant \bar{v}_1,\\
&= (v_1, \ldots, \bar{v}_2, \bar{v}_1 - 1) \quad \text{if } v_1 < \bar{v}_1,\\
\tilde{f}_n b &= (v_1, \ldots, v_n - 1, \bar{v}_n + 1, \ldots, \bar{v}_1),\\
\tilde{f}_i b &= (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} \geqslant \bar{v}_{i+1},\\
&= (v_1, \ldots, \bar{v}_{i+1} - 1, \bar{v}_i + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} < \bar{v}_{i+1},
\end{aligned}$$

$$wt(b) = 2(\bar{v}_1 - v_1)\Lambda_0 + \sum_{i=1}^{n-1} (v_i - \bar{v}_i + \bar{v}_{i+1} - v_{i+1})\Lambda_i + (v_n - \bar{v}_n)\Lambda_n,$$

$$\varphi_0(b) = -s(b) + (\bar{v}_1 - v_1)_+, \; \varepsilon_0(b) = -s(b) + (v_1 - \bar{v}_1)_+,$$

$$\varphi_i(b) = v_i + (\bar{v}_{i+1} - v_{i+1})_+ \quad \text{for } i = 1, \ldots, n-1,$$

$$\varepsilon_i(b) = \bar{v}_i + (v_{i+1} - \bar{v}_{i+1})_+ \quad \text{for } i = 1, \ldots, n-1,$$

$$\varphi_n(b) = v_n, \; \varepsilon_n(b) = \bar{v}_n,$$

where by definition $s(b) = \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \bar{v}_i$.

$$H(b \otimes b') = \max\{0_j(b \otimes b'),\ 0'_j(b \otimes b'),\ \eta_j(b \otimes b'),\ \eta'_j(b \otimes b') | 1 \leqslant j \leqslant n\},$$

where

$$0_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (\bar{v}_k - \bar{v}'_k) + s(b') - s(b),$$

$$0'_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (v'_k - v_k) + s(b) - s(b'),$$

$$\eta_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (\bar{v}_k - \bar{v}'_k) + 2(\bar{v}_j - v_j) + s(b') - s(b),$$

$$\eta'_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (v'_k - v_k) + 2(v_j - \bar{v}'_j) + s(b) - s(b').$$

5.6. $D_{n+1}^{(2)}$ $(n \geqslant 2)$

$$B_\infty = \{(v_1, \ldots, v_n, v_0, \bar{v}_n, \ldots, \bar{v}_1) \mid v_0 = 0 \text{ or } 1,\ v_i,\ \bar{v}_i \in \mathbf{Z}\} = \mathbf{Z}^{2n} \times \{0, 1\},$$
$$b_\infty = (0, 0, \ldots, 0) \in B_\infty.$$

For $b = (v_1, \ldots, v_n, v_0, \bar{v}_n, \ldots, \bar{v}_1)$ and $b' = (v'_1, \ldots, v'_n, v'_0, \bar{v}'_n, \ldots, \bar{v}_1)$ in B_∞,

$$\begin{aligned}
\tilde{e}_0 b &= (v_1 - 1, v_2, \ldots, \bar{v}_1) \quad \text{if } v_1 > \bar{v}_1,\\
&= (v_1, \ldots, \bar{v}_2, \bar{v}_1 + 1) \quad \text{if } v_1 \leqslant \bar{v}_1,\\
\tilde{e}_n b &= (v_1, \ldots, v_n, v_0 + 1, \bar{v}_n - 1, \ldots, \bar{v}_1) \quad \text{if } v_0 = 0,\\
&= (v_1, \ldots, v_n + 1, v_0 - 1, \bar{v}_n, \ldots, \bar{v}_1) \quad \text{if } v_0 = 1,\\
\tilde{e}_i b &= (v_1, \ldots, v_i + 1, v_{i+1} - 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} > \bar{v}_{i+1},\\
&= (v_1, \ldots, \bar{v}_{i+1} + 1, \bar{v}_i - 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} \leqslant \bar{v}_{i+1},
\end{aligned}$$

$$\begin{aligned}
\tilde{f}_0 b &= (v_1 + 1, v_2, \ldots, \bar{v}_1) \quad \text{if } v_1 \geqslant \bar{v}_1,\\
&= (v_1, \ldots, \bar{v}_2, \bar{v}_1 - 1) \quad \text{if } v_1 < \bar{v}_1,\\
\tilde{f}_n b &= (v_1, \ldots, v_n - 1, v_0 + 1, \ldots, \bar{v}_1) \quad \text{if } v_0 = 0,\\
&= (v_1, \ldots, v_n, v_0 - 1, \bar{v}_n + 1, \bar{v}_n, \ldots, \bar{v}_1) \quad \text{if } v_0 = 1,\\
\tilde{f}_i b &= (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} \geqslant \bar{v}_{i+1},\\
&= (v_1, \ldots, \bar{v}_{i+1} - 1, \bar{v}_i + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} < \bar{v}_{i+1},
\end{aligned}$$

$$wt(b) = 2(\bar{v}_1 - v_1)\Lambda_0 + \sum_{i=1}^{n-1} (v_i - \bar{v}_i + \bar{v}_{i+1} - v_{i+1})\Lambda_i + 2(v_n - \bar{v}_n)\Lambda_n,$$

$$\varphi_0(b) = -s(b) + 2(\bar{v}_1 - v_1)_+, \ \varepsilon_0(b) = -s(b) + 2(v_1 - \bar{v}_1)_+,$$

$$\varphi_i(b) = v_i + (\bar{v}_{i+1} - v_{i+1})_+ \quad \text{for } i = 1,\ldots,n-1,$$

$$\varepsilon_i(b) = \bar{v}_i + (v_{i+1} - \bar{v}_{i+1})_+ \quad \text{for } i = 1,\ldots,n-1,$$

$$\varphi_n(b) = 2v_n + v_0, \ \varepsilon_n(b) = 2\bar{v}_n + v_0,$$

where by definition $s(b) = \sum_{i=1}^{n} v_i + v_0 + \sum_{i=1}^{n} \bar{v}_i$.

$$H(b \otimes b') = \max\{\theta_j(b \otimes b'), \theta'_j(b \otimes b'), \eta_j(b \otimes b'), \eta'_j(b \otimes b') | 1 \leqslant j \leqslant n\},$$

where

$$\theta_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (\bar{v}_k - \bar{v}'_k) + s(b') - s(b),$$

$$\theta'_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (v'_k - v_k) + s(b) - s(b'),$$

$$\eta_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (\bar{v}_k - \bar{v}'_k) + 2(\bar{v}_j - v_j) + s(b') - s(b),$$

$$\eta'_j(b \otimes b') = 2 \sum_{k=1}^{j-1} (v'_k - v_k) + 2(v_j - \bar{v}'_j) + s(b) - s(b').$$

5.7. $C_n^{(1)}$ $(n \geqslant 2)$

$$B_\infty = \left\{(v_1,\ldots,v_n, \bar{v}_n,\ldots,\bar{v}_1) \in \mathbf{Z}^{2n} \,\middle|\, \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \bar{v}_i \in 2\mathbf{Z}\right\},$$

$$b_\infty = (0, 0,\ldots,0) \in B_\infty.$$

For $b = (v_1,\ldots,v_n, \bar{v}_n,\ldots,\bar{v}_1)$ and $b' = (v'_1,\ldots,v'_n, \bar{v}'_n,\ldots,\bar{v}_1)$ in B_∞, we have

$$\begin{aligned}
\tilde{e}_0 b &= (v_1 - 2, v_2,\ldots,\bar{v}_2, \bar{v}_1) \quad \text{if } v_1 \geqslant \bar{v}_1 + 1,\\
&= (v_1 - 1, v_2,\ldots,\bar{v}_2, \bar{v}_1 - 1) \quad \text{if } v_1 = \bar{v}_1 + 1,\\
&= (v_1, v_2,\ldots,\bar{v}_2, \bar{v}_1 + 2) \quad \text{if } v_1 \leqslant \bar{v}_1,\\
\tilde{e}_n b &= (v_1,\ldots,v_n + 1, \bar{v}_n - 1,\ldots,\bar{v}_1),\\
\tilde{e}_i b &= (v_1,\ldots,v_i + 1, v_{i+1} - 1,\ldots,\bar{v}_1) \quad \text{if } v_{i+1} > \bar{v}_{i+1},\\
&= (v_1,\ldots,\bar{v}_{i+1} + 1, \bar{v}_i - 1,\ldots,\bar{v}_1) \quad \text{if } v_{i+1} \leqslant \bar{v}_{i+1},
\end{aligned}$$

$$\tilde{f}_0 b = (v_1 + 2, v_2, \ldots, \bar{v}_2, \bar{v}_1) \quad \text{if } v_1 \geqslant \bar{v}_1,$$
$$= (v_1 + 1, v_2, \ldots, \bar{v}_2, \bar{v}_1 - 1) \quad \text{if } v_1 = \bar{v}_1 - 1,$$
$$= (v_1, v_2, \ldots, \bar{v}_2, \bar{v}_1 - 2) \quad \text{if } v_1 \leqslant \bar{v}_1 - 2,$$
$$\tilde{f}_n b = (v_1 \ldots, v_i - 1, v_{i+1} + 1, \ldots, \bar{v}_1),$$
$$\tilde{f}_i b = (v_1, \ldots, v_i - 1, v_{i+1} + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} \geqslant \bar{v}_{i+1},$$
$$= (v_1, \ldots, \bar{v}_{i+1} - 1, \bar{v}_i + 1, \ldots, \bar{v}_1) \quad \text{if } v_{i+1} < \bar{v}_{i+1},$$

$$wt(b) = (\bar{v}_1 - v)\Lambda_0 + \sum_{i=1}^{n-1} (v_i - \bar{v}_i + \bar{v}_{i+1} - v_{i+1})\Lambda_i + (v_n - \bar{v}_n)\Lambda_n,$$

$$\varphi_0(b) = -\tfrac{1}{2}s(b) + (\bar{v}_1 - v_1)_+,$$
$$\varepsilon_0(b) = -\tfrac{1}{2}s(b) + (v_1 - \bar{v}_1)_+,$$
$$\varphi_i(b) = v_i + (\bar{v}_{i+1} - v_{i+1})_+ \quad \text{for } i = 1, \ldots, n-1,$$
$$\varepsilon_i(b) = \bar{v}_i + (v_{i+1} - \bar{v}_{i+1})_+ \quad \text{for } i = 1, \ldots, n-1,$$
$$\varphi_n(b) = v_n, \ \varepsilon_n(b) = \bar{v}_n,$$

where by definition $s(b) = \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} \bar{v}_i$.

$$H(b \otimes b') = \max\{\theta_j(b \otimes b'), \theta'_j(b \otimes b'), \eta_j(b \otimes b'), \eta'_j(b \otimes b') \mid 1 \leqslant j \leqslant n\},$$

where

$$\theta_j(b \otimes b') = \sum_{k=1}^{j-1} (\bar{v}_k - \bar{v}'_k) + \tfrac{1}{2}(s(b') - s(b)),$$

$$\theta'_j(b \otimes b') = \sum_{k=1}^{j-1} (v'_k - v_k) + \tfrac{1}{2}(s(b') - s(b)),$$

$$\eta_j(b \otimes b') = \sum_{k=1}^{j-1} (\bar{v}_k - \bar{v}'_k) + (\bar{v}_j - v_j) + \tfrac{1}{2}(s(b) - s(b)),$$

$$\eta'_j(b \otimes b') = \sum_{k=1}^{j-1} (v'_k - v_k) + (v'_j - \bar{v}'_j) + \tfrac{1}{2}(s(b) - s(b')).$$

5.8. The proof in the case $A_n^{(1)}$ $(n \geqslant 2)$

We shall give the proof only in the case of $A_n^{(1)}$ $(n \geqslant 2)$. The other cases are similar. For any positive integer l, we first recall from [KMN3] the description of the perfect crystal B_l which is isomorphic to $B(l\Lambda_1)$ as crystals for $U_q(A_n)$:

$$B_l = \left\{(x_i)_i \in \mathbf{Z}^{\mathscr{I}} \mid x_i \geqslant 0, \sum_i x_i = l\right\}.$$

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ are the same as that of Section 5.1 replacing v with x, as well as ε_i, φ_i and wt. In this case, we have $B_l^{\min} = B_l$. Furthermore, if $\lambda = \Sigma_i k_i \Lambda_i \in (P_{cl}^+)_l$, then $\sigma\lambda = \Sigma_i k_i \Lambda_{i-1}$.

Now let B_∞ be the crystal defined in Section 5.1.

For $b_0 = (m_i)_i \in B_l^{\min}$, let $\lambda = \varphi(b_0)$ and $\sigma\lambda = \varepsilon(b_0)$. Then $\lambda = \Sigma_i m_i \Lambda_i$ and $\sigma\lambda = \Sigma_i m_{i+1}\Lambda_i$. For $b = (x_1, x_2, \ldots, x_{n+1}) \in B_l$, we define the map

$$f_{(l,b_0)}: T_{\sigma\lambda} \otimes B_l \otimes T_{-\lambda} \to B_\infty$$

by $f_{(l,b_0)}(t_{\sigma\lambda} \otimes b \otimes t_{-\lambda}) = b' = (v_i)_i$,

where $v_i = x_i - m_i$.

Note that $f_{(l,b_0)}(t_{\sigma\lambda} \otimes b_0 \otimes t_{-\lambda}) = b_\infty$. It is straightforward to check that $f_{(l,b_0)}$ satisfies the condition (2.15). By Example 2.7, we have

$$\begin{aligned}
wt(t_{\sigma\lambda} \otimes b \otimes t_{-\lambda}) &= wt(b) + \sigma\lambda - \lambda \\
&= \sum_i (x_i - x_{i+1})\Lambda_i - \sum_i (m_i - m_{i+1})\Lambda_i \\
&= wt(b'), \\
\varphi_i(t_{\sigma\lambda} \otimes b \otimes t_{-\lambda}) &= \varphi_i(b) + \langle h_i, -\lambda\rangle = x_i - m_i = \varphi_i(b'), \\
\varepsilon_i(t_{\sigma\lambda} \otimes b \otimes t_{-\lambda}) &= \varepsilon_i(b) - \langle h_i, \sigma\lambda\rangle = x_{x+1} - m_{i+1} = \varepsilon_i(b').
\end{aligned}$$

Hence for any $b_0 \in B_l^{\min}$, $f_{(l,b_0)}: T_{\sigma\lambda} \otimes B_l \otimes T_{-\lambda} \to B_\infty$ is a morphism of crystals. By definition, it is clear that $f_{(l,b_0)}: T_{\sigma\lambda} \otimes B_l \otimes T_{-\lambda} \to B_\infty$ is an embedding and that

$$B_\infty = \bigcup_{(l,b_0)\in J} \operatorname{Im} f_{(l,b_0)}.$$

Therefore B_∞ is the limit of the coherent family of perfect crystals $\{B_l\}_{l\geqslant 1}$ for $U_q(A_n^{(1)})$.

THEOREM 5.1. *For* $b = (v_1, \ldots, v_{n+1})$ *and* $b' = (v'_1, \ldots, v'_{n+1})$ *in* B_∞,

$$H(b \otimes b') = \max\{\theta_j(b \otimes b') \mid 0 \leqslant j \leqslant n\},$$

where

$$\theta_j(b \otimes b') = \sum_{k=1}^{j} (v'_k - v_k) + v'_{j+1}.$$

Proof. Recall that $\varphi_0(b) = v_{n+1}$ and $\varepsilon_0(b') = v'_1$. If $v_{n+1} \geqslant v'_1$, by (2.20) and (5.1), we have

$$\tilde{e}_0(b \otimes b') = \tilde{e}_0 b \otimes b' = (v_1 - 1, \ldots, v_{n+1} + 1) \otimes (v'_1, \ldots, v'_{n+1}).$$

Hence

$$\theta_0(\tilde{e}_0 b \otimes b') = \theta_0(b \otimes b')$$

and for $i = 1, \ldots, n$,

$$\theta_i(\tilde{e}_0 b \otimes b') = \theta_i(b \otimes b') + 1.$$

But since

$$\theta_n(b \otimes b') - \theta_0(b \otimes b') = v_{n+1} - v'_1 \geqslant 0,$$

and $\theta_n(\tilde{e}_0 b \otimes b') - \theta_0(\tilde{e}_0 b \otimes b') = v_{n+1} + 1 - v'_1 > 0,$

we have $H(\tilde{e}_0 b \otimes b') = H(b \otimes b') + 1.$

If $v_{n+1} < v'_1$, then by (2.20) and (5.1), we have

$$\tilde{e}_0(b \otimes b') = b \otimes \tilde{e}_0 b' = (v_1, \ldots, v_{n+1}) \otimes (v'_1 - 1, \ldots, v'_{n+1} + 1).$$

Hence

$$\theta_n(b \otimes \tilde{e}_0 b') = \theta_n(b \otimes b')$$

and for $i = 0, 1, \ldots, n-1$,

$$\theta_i(b \otimes \tilde{e}_0 b') = \theta_i(b \otimes b') - 1.$$

But since

$$\theta_0(b \otimes b') - \theta_n(b \otimes b') = v'_1 - v_{n+1} > 0,$$

and

$$\theta_0(b \otimes \tilde{e}_0 b') - \theta_n(b \otimes \tilde{e}_0 b') = v'_1 - 1 - v_{n+1} \geqslant 0,$$

we have $H(b \otimes \tilde{e}_0 b') = H(b \otimes b') - 1$.

For $i = 1, \ldots, n$, recall that $\varphi_i(b) = v_i$ and $\varepsilon_i(b') = v'_{i+1}$. If $v_i \geqslant v'_{i+1}$, by (2.20) and (5.1), we have

$$\tilde{e}_i(b \otimes b') = \tilde{e}_i b \otimes b' = (v_1, \ldots, v_i + 1, v_{i+1} - 1, \ldots, v_{n+1}) \otimes (v'_1, \ldots, v'_{n+1}).$$

Hence $\theta_i(\tilde{e}_i b \otimes b') = \theta_i(b \otimes b') - 1$

and for $j \neq i$

$$\theta_j(\tilde{e}_i b \otimes b') = \theta_j(b \otimes b').$$

But since

$$\theta_{i-1}(b \otimes b') - \theta_i(b \otimes b') = v_i - v'_{i+1} \geqslant 0,$$

and

$$\theta_{i-1}(\tilde{e}_i b \otimes b') - \theta_i(\tilde{e}_i b \otimes b') = v_i - v'_{i+1} + 1 > 0,$$

we have $H(\tilde{e}_i b \otimes b') = H(b \otimes b')$.

If $v_i < v'_{i+1}$, then by (2.20) and (5.1), we have

$$\tilde{e}_i(b \otimes b') = b \otimes \tilde{e}_i b' = (v_1, \ldots, v_{n+1}) \otimes (v'_1, \ldots, v'_i + 1, v'_{i+1} - 1, \ldots, v'_{n+1}).$$

Hence

$$0_{i-1}(b \otimes \tilde{e}_i b') = 0_{i-1}(b \otimes b') + 1$$

and for $j \neq i-1$

$$0_j(b \otimes \tilde{e}_i b') = 0_i(b \otimes b').$$

But since

$$0_i(b \otimes b') - 0_{i-1}(b \otimes b') = v'_{i+1} - v_i > 0,$$

and

$$0_i(b \otimes \tilde{e}_i b') - 0_{i-1}(b \otimes \tilde{e}_0 b') = v'_{i+1} - v_i - 1 \geqslant 0,$$

we have $H(b \otimes \tilde{e}_i b') = H(b \otimes b')$,

which proves the theorem. □

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