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The K -groups of λ -rings. Part II. Invertibility of the logarithmic map

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1. Introduction

In [2] the author constructed a map $L: K_2(R, I) \rightarrow K_{2,L}(R, I)$. Here $K_2(R, I)$ denotes the relative algebraic K_2 of a ring R and a radical ideal I , and $K_{2,L}(R, I)$ is a linearized version of this.

This map is constructed as an infinite series. To define each term of this series it is assumed that R has a structure of λ -ring leaving I invariant. To make the series converge it is assumed that I satisfies a certain weak nilpotency condition.

One result of this paper is the theorem that the logarithmic map is an isomorphism if I satisfies a certain strong nilpotency condition. Another result of this paper is the theorem that I^2 satisfies the strong nilpotency condition if I satisfies a similar weak nilpotency condition.

In order to prove these theorems we first have to generalize the main theorem of [2] somewhat in order to cover the case of the universal λ -ring. We take the opportunity to present a proof of this theorem which avoids the trick used in [2] of proving the theorem first for the split case and applying that to the ‘double’ of the ring R along I .

A remark about notation. For the definition of $K_{2,L}(R, I)$ see Section 5 of this paper. As noted in [2] one has $K_{2,L}(R, I) \cong \Omega_{R,I}/\delta I \cong HC_1(R, I)$, if the projection $R \rightarrow R/I$ splits. However $K_{2,L}(R, I)$ and $HC_1(R, I)$ do not agree in general. To see this consider the example where $R = \mathbb{Z}[t]/(t^2)$ and $I = 2tR$. Then $K_{2,L}(R, I)$ is cyclic of order 4, generated by $[2t, t]$. On the other hand $HC_1(R, I)$ is of order 2. For this reason we use in this paper the $K_{2,L}$ notation instead of the HC_1 notation.

We end this introduction with some remarks about the relation between L and invariants defined by other authors. In [2], p. 317 the relation between L and the Chern class map $c_2: K_2(R) \rightarrow \Omega_R^2$ is described.

In [4], pp. 368–373 there is defined a map $\alpha: K_2(R) \rightarrow HC_2^-(R)$. In [6], pp. 382–383, and in [10], p. 541, it is explained that the composition of α with the map $HC_2^-(R) \rightarrow HH_2(R)$ is the Dennis trace map, and that the composition of

α with the maps $HC_2^-(R) \rightarrow HC_2^{\text{per}}(R) \rightarrow HC_{2+2k}(R)$ yields the Karoubi Chern classes.

The following is explained in [4], pp. 350–351. If $\mathbb{Q} \subseteq R$ then the periodic cyclic homology $HC_n^{\text{per}}(R, I)$ vanishes, which implies that the map $\beta: HC_1(R, I) \rightarrow HC_2^-(R, I)$ is an isomorphism. In this situation a map $\beta^{-1}\alpha: K_2(R, I) \rightarrow HC_1(R, I)$ is defined. In [2] and the present paper a map to $HC_1(R, I)$ is defined under a much weaker condition.

The following example shows that L is a sharper invariant than α . Consider the case that $R = \mathbb{Z}[t]/t^2$ and $I = tR$. Then $L: K_2(R, I) \rightarrow K_{2,L}(R, I) \cong \mathbb{Z}/2$ is an isomorphism. On the other hand it is explained in [10], pp. 550–551, that $HC_2^-(R, I) = 0$ in this case.

2. Generalities about λ -rings

A λ -ring is a commutative ring with 1, together with maps $\lambda^n: R \rightarrow R$ for $n = 0, 1, 2, \dots$ such that

$$\begin{aligned} \lambda^0(a) &= 1, \\ \lambda^1(a) &= a, \\ \lambda^n(a + b) &= \sum_{i+j=n} \lambda^i(a)\lambda^j(b), \\ \lambda^n(1) &= 0 \quad \text{for } n > 0, \\ \lambda^n(ab) &= F_n(\lambda^1(a), \dots, \lambda^n(a), \lambda^1(b), \dots, \lambda^n(b)), \\ \lambda^m(\lambda^n(a)) &= F_{m,n}(\lambda^1(a), \dots, \lambda^{mn}(a)), \end{aligned}$$

where the F_n and $F_{m,n}$ are certain universal polynomials. A ring-homomorphism $f: R \rightarrow S$ between λ -rings such that $f \circ \lambda^n = \lambda^n \circ f$ for all n is called a λ -map.

There exists a λ -ring U and an element $u \in U$ such that for any λ -ring R and element $a \in R$ there is a unique λ -map $f: U \rightarrow R$ such that $f(u) = a$. The ring U is the polynomial ring over \mathbb{Z} freely generated by the $\lambda^n(u)$ with $n > 0$. Here \mathbb{Z} denotes the ring of rational integers. We make U a graded ring by declaring $\lambda^n(u)$ to be of degree n . We write E for the ideal of U generated by the elements of positive degree.

An element $\xi \in U$ defines a natural map $\xi_R: R \rightarrow R$ on λ -rings. It is defined by $\xi_R(a) = f(\xi)$, where $f: U \rightarrow R$ is the λ -map mapping u to a . Every natural map $\xi_R: R \rightarrow R$ on λ -rings arises in this way.

The most important examples of such natural maps are the Adams oper-

ations ψ^m defined for $m > 0$ by the Newton formula:

$$\sum_{k+m=n} (-1)^k \lambda^k(a) \cdot \psi^m(a) = (-1)^{n-1} n \lambda^n(a).$$

In [2] the author introduced natural maps $\theta^m: R \rightarrow R$ for $m > 0$ such that

$$\sum_{km=n} m \psi^k(\theta^m(a)) = a^n.$$

There are two extensions of the relation between elements of U and natural maps on λ -rings which have to be considered. The first one concerns maps in several variables. These can be viewed in several ways:

PROPOSITION 1. *There are bijective relations between*

- (1) *Elements of $U \otimes U$,*
- (2) *Natural maps $R \times R \rightarrow R \otimes R$,*
- (3) *Natural maps $R \times R \rightarrow R$,*
- (4) *Natural maps $R \rightarrow R \otimes R$.*

We describe a number of natural maps which were introduced by the author in [2] and which can be viewed in any of the above ways.

In the first place there are natural maps $\eta^m: R \times R \rightarrow R$ for $m > 0$ such that

$$\sum_{km=n} m a^{k-1} \psi^k(\eta^m(a, b)) = a^{n-1} b^n.$$

Given a commutative ring R the module of differentials Ω_R is defined as $(\ker \mu)/(\ker \mu)^2$, where $\mu: R \otimes R \rightarrow R$ is the multiplication map. The universal derivation $\delta: R \rightarrow \Omega_R$ is defined by

$$\delta(a) = 1 \otimes a - a \otimes 1 + (\ker \mu)^2.$$

There are natural maps $\phi^n: \Omega_R \rightarrow \Omega_R$ such that

$$n \phi^n(\delta(a)) = \delta(\psi^n(a)).$$

From η and ϕ were constructed natural maps $v^n: R \times R \rightarrow \Omega_R$ by the formula

$$v^n(a, b) = \sum_{km=n} \theta^k(a) \phi^k(\delta \eta^m(a, b)).$$

Finally the main technical point of [2] was the construction of natural maps

$\beta_d^n: R \times \cdots \times R \rightarrow R$ in d variables such that e.g.

$$\delta\beta_2^n(a, b) = v^n(a, b) + v^n(b, a),$$

$$\delta\beta_3^n(a, b, c) = v^n(a, bc) + v^n(b, ac) + v^n(c, ab),$$

for $a, b, c \in R$.

The second extension concerns ideals of λ -rings. An ideal $I \subseteq R$ such that $\lambda^n(I) \subseteq I$ for all $n > 0$ is called a λ -ideal.

PROPOSITION 2. *There is a bijective relation between elements of E and natural maps $I \rightarrow I$.*

A simple example of a combination of these two themes is

PROPOSITION 3. *If $\xi \in E$ and if $\xi_+ : R \times R \rightarrow R$ is defined by*

$$\xi_+(a, b) = \xi(a + b) - \xi(a) - \xi(b),$$

then $\xi_+ \in U \otimes U$ is in fact in $E \otimes E$. So ξ_+ gives rise to a natural map $I \times I \rightarrow I \otimes I$.

3. Some identities

In this section we apply the ideas of the last section to prove some identities which are needed in the proof of the first theorem. They involve the map $\Delta: I \otimes_R I \rightarrow I \otimes_R \Omega_R$ defined by

$$\Delta(v \otimes w) = v \otimes \delta w + w \otimes \delta v.$$

First we need the following variation of proposition 3:

PROPOSITION 4. *If $\xi \in U \otimes E$ and if $\xi_x : R \times R \times R \rightarrow R$ is defined by*

$$\xi_x(a, b, c) = \xi(c, a + b - abc) - \xi(c, a) - \xi(c, b),$$

then ξ_x is in $E \otimes E \otimes U$.

This means that ξ_x gives rise to a natural map $I \times I \times R \rightarrow I \otimes I \otimes R$. Let $\pi: I \otimes I \otimes R \rightarrow I \otimes_R I$ be the map defined by

$$\pi(x \otimes y \otimes z) = x \otimes yz = xz \otimes y.$$

Then $\pi \circ \zeta_x$ is a natural map $I \times I \times R \rightarrow I \otimes_R I$.

In particular $\Delta\pi\beta_{2,\times}^n$ is a natural map $I \times I \times R \rightarrow I \otimes_R \Omega_R$. We shall express this map in terms of a map $v_\pi^n: I \times I \times R \rightarrow I \otimes_R \Omega_R$ defined by the formula

$$v_\pi^n(a, b, c) = \sum_{km=n} \theta^k(c) \cdot (\psi^k \otimes \phi^k) \Delta\pi\eta_\times^m(a, b, c),$$

and a map $v_\otimes^n: I \times R \rightarrow I \otimes_R \Omega_R$ defined by the formula

$$v_\otimes^n(x, y) = \sum_{km=n} \theta^k(x) \otimes \phi^k \delta\eta^m(x, y).$$

LEMMA 1. For $a, b \in I$ and $c \in R$ one has

$$\Delta\pi\beta_{2,\times}^n(a, b, c) = v_\pi^n(a, b, c) + v_\otimes^n(a + b - abc, c) - v_\otimes^n(a, c) - v_\otimes^n(b, c).$$

Proof. This follows by using naturality from the case $R = U \otimes U \otimes U$, $I = R(u \otimes 1 \otimes 1) + R(1 \otimes u \otimes 1)$, $a = u \otimes 1 \otimes 1$, $b = 1 \otimes u \otimes 1$, $c = 1 \otimes 1 \otimes u$. But in this situation the map $\mu_\Omega: I \otimes_R \Omega_R \rightarrow \Omega_R$ defined by $\mu_\Omega(a \otimes \omega) = a\omega$ is injective since Ω_R is a free R -module. Thus it suffices to check that the identity holds after applying μ_Ω . But that is an immediate consequence of the main property of β_2^n since

$$\mu_\Omega \Delta\pi\beta_{2,\times}^n(a, b, c) = \delta\beta_2^n(c, a + b - abc) - \delta\beta_2^n(c, a) - \delta\beta_2^n(c, b),$$

$$\mu_\Omega v_\pi^n(a, b, c) = v^n(c, a + b - abc) - v^n(c, a) - v^n(c, b),$$

$$\mu_\Omega v_\otimes^n(x, y) = v^n(x, y).$$

To prove these three identities it is useful to introduce a few abbreviations. Given a ring R and an ideal $I \subseteq R$ we write

$$\Omega' = \Omega_R \otimes I \otimes R \oplus I \otimes \Omega_R \otimes R \oplus I \otimes I \otimes \Omega_R.$$

Let $\pi_\Omega: \Omega' \rightarrow I \otimes_R \Omega_R$ be the map defined by

$$\pi_\Omega(\omega \otimes b \otimes c) = bc \otimes \omega \quad \text{for } \omega \in \Omega_R, b \in I, c \in R,$$

$$\pi_\Omega(a \otimes \omega \otimes c) = ac \otimes \omega \quad \text{for } \omega \in \Omega_R, a \in I, c \in R,$$

$$\pi_\Omega(a \otimes b \otimes \omega) = ab \otimes \omega \quad \text{for } \omega \in \Omega_R, a \in I, b \in I.$$

Let $\delta': I \otimes I \otimes R \rightarrow \Omega'$ be the map defined by

$$\delta' = \delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta.$$

Let $\mu': I \otimes I \otimes R \rightarrow I$ be the map defined by $\mu'(a \otimes b \otimes c) = abc$. Then it is clear that

$$\Delta \circ \pi = \pi_\Omega \circ \delta',$$

$$\mu_\Omega \circ \pi_\Omega \circ \delta' = \delta \circ \mu',$$

$$\mu_\Omega \circ (\psi^k \otimes \phi^k) = \phi^k \circ \mu_\Omega. \quad \square$$

For the second identity we need the following variation of Proposition 3.

PROPOSITION 5. *If $\xi \in E$ is of degree n and if $\xi_\psi: R \times R \rightarrow R$ is defined by $\xi_\psi(a, b) = \xi(ab) - \xi(a)\psi^n(b)$, then in fact $\xi_\psi \in E^2 \otimes U$.*

Proof. We must show that $\xi(ab) - \xi(a)\psi^n(b) \in I^2$ for $a \in I, b \in R$. For $\xi \in E^2$ this is obvious; but every ξ of degree n is modulo E^2 a multiple of λ^n . For $\xi = \lambda^n$ it is just Lemma 1.7(c) of [2]. \square

LEMMA 2. *There is natural map $\beta_{3,\pi}^n: I \times R \times R \rightarrow I \otimes_R I$ such that*

$$\Delta \beta_{3,\pi}^n(a, b, c) = v_{\otimes}^n(a, bc) - v_{\otimes}^n(ab, c) - v_{\otimes}^n(ac, b).$$

Proof. According to the construction of the β_d in Proposition 9.1 of [2] one has

$$\beta_{3,\pi}^n(a, b, c) = \sum_{km=n} (\beta_2^k(a, b)\psi^k\eta^m(ab, c) + \beta_2^k(ab, c)\psi^k\eta^m(abc, 1)).$$

Now assume $a \in I$. Then all the terms with $m > 1$ are in I^2 . From this and Proposition 5 it follows that

$$\beta_{3,\pi}^n(a, b, c) - \beta_2^n(ab, c) - \beta_2^n(ac, b) \in I^2.$$

In particular for $R = U \otimes U \overset{\circ}{\otimes} U, I = E \otimes U \otimes U, a = u \otimes 1 \otimes 1, b = 1 \otimes u \otimes 1$ and $c = 1 \otimes 1 \otimes u$ one gets an element of $E^2 \otimes U \otimes U$. Choose an element of $E \otimes E \otimes U \otimes U$ mapping to this element under the multiplication map. This element gives rise to a natural map $I \times R \times R \rightarrow I \otimes I \otimes R \otimes R$. We define $\beta_{3,\pi}^n$ as the composition of this map with the map $\pi': I \otimes I \otimes R \otimes R \rightarrow I \otimes_R I$ defined by $\pi'(x \otimes y \otimes z_1 \otimes z_2) = x \otimes yz_1z_2$.

To check the formula for $\Delta \beta_{3,\pi}^n$ it suffices to check it after applying μ_Ω as in the proof of Lemma 1. But in Ω_R it is a direct consequence of the main property of β_3 . \square

4. Continuity

The ‘logarithmic map’ from K_2 to linearized K_2 is an infinite sum of terms involving the aforementioned natural maps so we turn now to convergence questions.

An ideal I of a λ -ring R is called a ψ -ideal if $\psi^n(I) \subseteq I$ for all $n > 0$. Any λ -ideal is a ψ -ideal but the reverse is not true.

A filtered λ -ring is a λ -ring R together with a sequence of ψ -ideals $J_1 \supseteq J_2 \supseteq J_3 \dots$ such that each $\theta^m: R \rightarrow R$ is continuous i.e. for all m and M there exists some N such that $\theta^m(J_N) \subseteq J_M$.

EXAMPLE 1. Let R be a λ -ring and let J be a ψ -ideal. Then the pair $(R, \{J^n\})$ is a filtered λ -ring. This follows from Proposition 5.2 of [2].

EXAMPLE 2. Let U be the universal λ -ring. We write J_n^U for the ideal generated by all elements of degree at least n . In this case one has even $\psi^m(J_n^U) \subseteq J_{mn}^U$ and $\theta^m(J_n^U) \subseteq J_{mn}^U$.

We shall write $U(d)$ for the subring of U generated by the elements of degree $\leq d$ and thus by the $\lambda^i(u)$ with $i \leq d$. Then for every $\xi \in U$ there is some d such that $\xi \in U(d)$.

PROPOSITION 6. Let R be a filtered λ -ring. Then for every M and d there exists an N such that every $\xi \in U(d)$ induces a well defined map $R/J_N \rightarrow R/J_M$.

Proof. From Lemma 1.7 of [2] one sees that if the maps $\theta^m: R \rightarrow R$ are continuous, then so are the maps $\lambda^k: R \rightarrow R$. But then there is for every M an every m a number N such that $\lambda^k(J_N) \subseteq J_M$ for $0 < k \leq m$. It follows from the formula for $\lambda^k(a + b)$ that λ^k induces a well defined map $R/J_N \rightarrow R/J_M$. So the same is true for sums of products of these. □

Let F be a functor from commutative rings to abelian groups. If $(R, \{J_n\})$ is a filtered ring, then the system of groups $F(R/J_n)$ constitute a pro-object $F^{\text{pro}}(R)$ in the category of abelian groups. This inverse system gives rise to an inverse limit group $F^{\text{top}}(R)$. If F transforms ring surjections to group surjections, then $F^{\text{pro}}(R)$ is determined by the topological group $F^{\text{top}}(R)$ up to isomorphism. For these facts about pro-groups see [1] and [5]. In fact one can view $F^{\text{top}}(R)$ as the completion of $F(R)$ for the filtration given by the subgroups $F(R, J_n) = \ker(F(R) \rightarrow F(R/J_n))$. In this paper we consider only functors with the above property. An example is $F = \Omega$:

LEMMA 3. Each ϕ^n is continuous for this filtration on Ω_R and thus induces maps $\Omega_R^{\text{pro}} \rightarrow \Omega_R^{\text{pro}}$ and $\Omega_R^{\text{top}} \rightarrow \Omega_R^{\text{top}}$.

Proof. Given M there exists an N such that $\lambda^i(J_N) \subseteq J_M$ for $i \leq m$. Substitu-

ting this in the formula 4.8 of [2]

$$\phi^m(a\delta b) = (-1)^{m-1}\psi^m(a) \sum_{i=1}^m \lambda^{m-i}(-b)\delta\lambda^i(b),$$

one sees that $\phi^m(a\delta b)$ vanishes in Ω_{R/J_M} if a or b vanishes in R/J_N . So $\phi^m(\Omega_{R,J_N}) \subseteq \Omega_{R,J_M}$. □

Let $(R, \{J_n\})$ be a filtered λ -ring. A λ -ideal $I \subseteq R$ is called λ -nilpotent if for all m there exists some N such that $J_N^U(I) \subseteq J_M$. The purpose of this condition is that $\sum_{n=0}^\infty \xi^n(a)$ now has a meaning in $\varprojlim R/J_M$ if $a \in I$ and if each ξ^n is a natural map of degree n .

EXAMPLE 3. The augmentation ideal $E \subset U$ is λ -nilpotent in the situation of Example 2.

In the next sections we only need to give a meaning to $\sum_{n=0}^\infty \xi^n(a)$ for $a \in I$ for some special maps ξ^n . Therefore a weaker nilpotency condition is more useful.

The ring V is defined as the subring of U generated by the elements $\psi^i(\theta^k(u))$. We write $J_n^V = J_n^U \cap V$. A λ -ideal $I \subseteq R$ is called θ -nilpotent if for all M there exists some N such that $J_N^V(I) \subseteq J_M$.

5. The K -groups and the logarithmic map

The starting point of this paper as well as [2] is the following theorem of [9] and [7].

Let R be a commutative ring with 1, and let I be an ideal. Consider the abelian group $D(R, I)$ defined by the following presentation. The generators are the symbols $\langle a, b \rangle$ with $a \in I$ and $b \in R$. The relations are

$$\begin{aligned} \langle a, b \rangle + \langle b, a \rangle & \text{ for } a \in I, b \in I, \\ \langle a, bc \rangle - \langle ab, c \rangle - \langle ac, b \rangle & \text{ for } a \in I, b \in R, c \in R, \\ \langle c, a + b - abc \rangle - \langle c, a \rangle - \langle c, b \rangle & \text{ for } a \in R, b \in R, c \in I, \\ \langle a + b - abc, c \rangle - \langle a, c \rangle - \langle b, c \rangle & \text{ for } a \in I, b \in I, c \in R. \end{aligned}$$

Then $K_2(R, I)$ is isomorphic to $D(R, I)$ if I is contained in the Jacobson radical of R .

Now suppose that $(R, \{J_n\})$ is a filtered λ -ring and I is a θ -nilpotent λ -ideal. Then $(I + J_M)/J_M$ is a nilpotent ideal of the ring R/J_M in the usual sense, so the above presentation is applicable to the groups $K_2(R/J_M, (I + J_M)/J_M)$ occurring in the definition of $K_2^{\text{lop}}(R, I)$.

In analogy with the above presentation the author introduced in [2] the abelian group $K_{2,L}(R, I)$ defined by the following presentation. The generators are the symbols $[a, b]$ with $a \in I$ and $b \in R$. The relations are

$$\begin{aligned} & [a, b] + [b, a] \quad \text{for } a \in I, b \in I, \\ & [a, bc] - [ab, c] - [ac, b] \quad \text{for } a \in I, b \in R, c \in R, \\ & [c, a + b] - [c, a] - [c, b] \quad \text{for } a \in R, b \in R, c \in I, \\ & [a + b, c] - [a, c] - [b, c] \quad \text{for } a \in I, b \in I, c \in R. \end{aligned}$$

There is an isomorphism $\iota: K_{2,L}(R, I) \rightarrow \text{cok}(\Delta)$ which is given by the formula $\iota[a, b] = a \otimes \delta b$; see Proposition 7.1 of [2].

The following theorem is a generalization of the main thorem in [2]. We need this generality in order to be able to apply the theorem to the case of the universal λ -ring.

THEOREM 1. *Let $(R, \{J_n\})$ be a filtered λ -ring, and let I be a θ -nilpotent λ -ideal. Then for every M there exists a P such that the formula*

$$\iota L\langle a, b \rangle = \sum_{m,k} \theta^m(a) \otimes \phi^m(\delta \eta^k(a, b))$$

defines a well defined map

$$L: K_2(R/J_P, (I + J_P)/J_P) \rightarrow K_{2,L}(R/J_M, (I + J_M)/J_M).$$

Thus L induces maps $K_2^{\text{pro}}(R, I) \rightarrow K_{2,L}^{\text{pro}}(R, I)$ and $K_2^{\text{top}}(R, I) \rightarrow K_{2,L}^{\text{top}}(R, I)$.

Proof. First we shall describe a map

$$v_{\otimes}: I \times R \rightarrow K_{2,L}(R/J_M, (I + J_M)/J_M).$$

Then we shall check that it induces a map on $D(R, I)$ because it maps the defining relations to zero. Finally the image of $v_{\otimes}(a, b)$ will be shown to depend only on the classes of a and $b \text{ mod } J_P$ for some large P ; therefore we get a map defined on $D(R/J_P, (I + J_P)/J_P) = K_2(R/J_P, (I + J_P)/J_P)$.

In order to define v_{\otimes} consider the expression

$$v_{\otimes}^n(a, b) = \sum_{km=n} \theta^k(a) \otimes \phi^k \delta \eta^m(a, b),$$

for $a \in I$ and $b \in R$. There is an N_1 such that the class of $\theta^k(a)$ in R/J_M vanishes for $k > N_1$. According to Lemma 3 there is an N_2 such that ϕ^k is a well defined map $\Omega_{R/J_{N_2}} \rightarrow \Omega_{R/J_M}$ for each $k \leq N_1$. Finally there is an N_3 such that $\eta^m(a, b)$ vanishes in R/J_{N_2} for $m > N_3$. This means that the expression vanishes in

$R/J_M \otimes_R \Omega_{R/J_M}$ for $n > N_1 N_3$. So $\sum_{n=1}^N v_{\otimes}^n(a, b)$ is independent of N for large N and can be taken as definition for $v_{\otimes}(a, b)$.

The natural map $\beta_2^n: R \times R \rightarrow R$ is in $E \otimes E$ and thus gives rise to a natural map $\beta_{2,\otimes}^n: I \times I \rightarrow I \otimes_R I$. From the main property of β_2^n one deduces easily in the manner of Lemma 1 that in $I \otimes_R \Omega_R$ one has

$$\Delta \beta_{2,\otimes}^n(a, b) = v_{\otimes}^n(a, b) + v_{\otimes}^n(b, a),$$

for $a, b \in I$. So if we define $\beta_{2,\otimes}(a, b) = \sum_{n=1}^N \beta_{2,\otimes}^n(a, b)$ for the same N as before, then

$$\Delta \beta_{2,\otimes}(a, b) = v_{\otimes}(a, b) + v_{\otimes}(b, a),$$

if $a, b \in I$. This means that the first relation is satisfied.

In a similar way we define $\beta_{3,\pi}(a, b, c) = \sum_{n=1}^N \beta_{3,\pi}^n(a, b, c)$. It follows from Lemma 2 that in $(I + J_M)/J_M \otimes_R \Omega_{R/J_M}$ one has

$$\Delta \beta_{3,\pi}(a, b, c) = v_{\otimes}(a, bc) - v_{\otimes}(ab, c) - v_{\otimes}(ac, b),$$

if $a \in I$ and $b, c \in R$. This means that the second relation is satisfied.

In order to prove the other relations it is useful to work in the ring of formal power series $R[[t]]$ equipped with the obvious λ -ring structure and derivation $\delta: R[[t]] \rightarrow \Omega_R[[t]]$. In this ring the expression $\eta(x, ty) = \sum_{m=1}^{\infty} \eta^m(x, ty) = \sum_{m=1}^{\infty} \eta^m(x, y)t^m$ makes sense for every $x, y \in R[[t]]$. The main property of the η^m is Proposition 3.3 of [2] saying that

$$\eta(c, ta + tb - t^2 abc) - \eta(c, ta) - \eta(c, tb) = 0,$$

for $a, b, c \in R$. Now let $c \in I$; then terms with $m > N_3$ vanish when viewed in $(R/J_{N_2})[[t]]$. So this formula can be rewritten as

$$\sum_{m=1}^{N_3} (\eta^m(c, ta + tb - t^2 abc) - \eta^m(c, ta) - \eta^m(c, tb)) = 0.$$

Therefore in $((I + J_M)/J_M \otimes_R \Omega_{R/J_M})[[t]]$ one has

$$\begin{aligned} \sum_{k=1}^{N_1} \sum_{m=1}^{N_3} (\theta^k(c) \otimes \phi^k \delta \eta^m(c, ta + tb - t^2 abc) \\ - \theta^k(c) \otimes \phi^k \delta \eta^m(c, ta) - \theta^k(c) \otimes \phi^k \delta \eta^m(c, tb)) = 0. \end{aligned}$$

Putting $t = 1$ in this formula yields

$$v_{\otimes}(c, a + b - abc) - v_{\otimes}(c, a) - v_{\otimes}(c, b) = 0.$$

This means that the third relation is satisfied.

The main property of the η^m implies that for all $a, b \in I$ and $c \in R$ one has $\sum_{m=1}^{\infty} \pi \eta^m_x(ta, tb, c) = 0$ in $(I \otimes_R I)[[t]]$. When viewed modulo J_{N_2} the terms with $m > N_3$ vanish. From this it follows that in $((I + J_M)/J_M \otimes_R \Omega_{R/J_M})[[t]]$ one has

$$\sum_{k=1}^{N_1} \sum_{m=1}^{N_3} \theta^k(c) \cdot (\psi^k \otimes \phi^k) \Delta \pi \eta^m_x(ta, tb, c) = 0.$$

Now consider the sum $\sum_{n=1}^N v^n_{\otimes}(ta, tb, c)$ for $N \geq N_1 N_3$. It contains the above sum and the remaining terms have either $k > N_1$ or $m > N_3$. Both kinds of terms vanish e.g. in the first case because $\psi^k(I) \subseteq J_M$. Therefore the above expression vanishes for large N . Combining this fact with Lemma 1 we see that for such N one has

$$\sum_{n=1}^N (v^n_{\otimes}(ta + tb - t^2 abc, c) - v^n_{\otimes}(ta, c) - v^n_{\otimes}(tb, c)) = \Delta \sum_{n=1}^N \pi \beta_{2, \times}^n(ta, tb, c).$$

Putting $t = 1$ in this formula yields

$$v_{\otimes}(a + b - abc, c) - v_{\otimes}(a, c) - v_{\otimes}(b, c) \in \text{im}(\Delta).$$

Therefore the last relation is satisfied.

The formula for $v^n_{\otimes}(a, b)$ involves $\theta^k(a)$ and $\phi^k \eta^m(a, b)$ only for $k \leq N_1$ and $m \leq N_3$. According to Proposition 6 there exists a P such that the classes of $\theta^k(a)$ in R/J_M and of $\eta^m(a, b)$ in R/J_{N_2} depend only on the classes of a, b in R/J_P . So $v_{\otimes}(a, b)$ depends only on these classes. □

The expressions $\beta_{2, \otimes}^n(a, b)$ and $\beta_{3, \pi}^n(a, b, c)$ can be shown to vanish in $(I + J_M)/J_M \otimes_R (I + J_M)/J_M$ for large n and thus $\beta_{2, \otimes}^n(a, b)$ and $\beta_{3, \pi}^n(a, b, c)$ do not depend on N for large N . However this fact does not seem to be needed in the proof of the above theorem.

In the formulation and proof of this theorem we used the identification ι of $K_{2, L}$ with $\text{cok}(\Delta)$. For later reference we now describe L itself.

PROPOSITION 7. *The map L is given by the formula*

$$L\langle a, b \rangle = \sum_{n=1}^{\infty} \sum_{km=n} \sum_{i=1}^m (-)^{m-1} [\theta^m(a) \lambda^{m-i} (-\eta^k(a, b)), \lambda^i(\eta^k(a, b))].$$

Proof. We shall show that for $x \in I$ and $y \in R$ one has

$$\iota^{-1}(x \otimes \phi^m(\delta y) + \text{im}(\Delta)) = (-1)^{m-1} \sum_{i=0}^m [x\lambda^{m-i}(-y), \lambda^i(y)].$$

Here we may omit the term with $i = 0$ since it vanishes. In combination with the definition of v_{\otimes} this yields the stated formula.

The map ϕ^m is defined in Section 4 of [2] as the map induced by $(-1)^{m-1}\lambda^m$ on $(\ker(\mu))/(\ker(\mu))^2$. Thus one has

$$\begin{aligned} \phi^m(\delta y) &= (-1)^{m-1}\lambda^m(1 \otimes y - y \otimes 1) + (\ker(\mu))^2 \\ &= (-1)^{m-1} \sum_{i=0}^m \lambda^{m-i}(-y) \otimes \lambda^i(y) + (\ker(\mu))^2. \end{aligned}$$

We refer to Proposition 7.1 of [2]. We identified $(\ker(\mu))/(\ker(\mu))^2$ with the cokernel of $D:R \otimes R \otimes R \rightarrow R \otimes R$ by the inclusion map. Furthermore we identified $x \otimes (y_1 \otimes y_2 + \text{im}(D)) \in I \otimes_R \text{cok}(D)$ with $xy_1 \otimes y_2 + \text{im}(D_I)$ in the cokernel of $D_I: I \otimes R \otimes R \rightarrow I \otimes R$. Finally we identified $z \otimes y_2 + \text{im}(T_I) + \text{im}(D_I)$ in the common cokernel of D_I and $T_I: I \otimes I \rightarrow I \otimes R$ with $[z, y_2] \in K_{2,L}(R, I)$. □

6. Truncated polynomial rings

In this section we shall show that the map L in Theorem 1 is an isomorphism if R is a polynomial ring and I its augmentation ideal. In the next section we shall apply this result to universal examples and deduce that the map L is always an isomorphism if R is a λ -ring and I a λ -nilpotent λ -ideal.

In this section we consider a polynomial ring $R = A[X]$ generated by a set X over a ring A . We equip R with a grading with the aid of a map $d: X \rightarrow \mathbb{N}$, assuming that X has only finitely many elements in each degree. Furthermore we assume that R a λ -ring in such a way that $\lambda^i(a) \in A$ for $a \in A$, and $d(\lambda^k(x)) = kd(x)$ for $k \in \mathbb{N}$ and $x \in X$. We defined I as the ideal of R generated by X , and J_M as the ideal generated by all homogeneous elements of degree $\geq M$.

Because J_M is a λ -ideal one gets in fact maps

$$L: K_2(R/J_M, I/J_M) \rightarrow K_{2,L}(R/J_M, I/J_M),$$

and these should already be isomorphisms. If one computes these latter groups one discovers that one gets certain artifacts from the truncation. For this reason we consider different groups where these artifacts have been killed.

We define $G^{(M)}(R, I)$ as the quotient of $K_2(R/J_M, I/J_M)$ by the subgroup generated by all $\langle a, b \rangle$ with $a \in J_p$ and $b \in J_q$ with $p + q \geq M$. If N satisfies

$2N \leq M + 1$, then one must have $p \geq N$ or $q \geq N$, so $G^{(M)}(R, I)$ surjects onto $K_2(R/J_N, I/J_N)$. Therefore the $G^{(M)}(R, I)$ have $K_2^{\text{top}}(R, I)$ as inverse limit.

Similarly we define $G_L^{(M)}(R, I)$ as the quotient of $K_{2,L}(R/J_M, I/J_M)$ by the subgroup generated by all $[a, b]$ with $a \in J_p$ and $b \in J_q$, where $p + q \geq M$. Again the $G_L^{(M)}(R, I)$ have $K_{2,L}^{\text{top}}(R, I)$ as inverse limit. If $a \in J_p$ and $b \in J_q$, then $L\langle a, b \rangle$ is a sum of terms $[a', b']$ with $a' \in J_p$ and $b' \in J_q$ according to the formula at the end of Section 5. This means that L induces a homomorphism $L^{(M)}: G^{(M)}(R, I) \rightarrow G_L^{(M)}(R, I)$. We shall show that $L^{(M)}$ is an isomorphism for every M .

First we list some useful consequences of the relations in the presentation of $K_2(R/J_M, I/J_M)$.

LEMMA 4. If $a_i \in I$ for all i and $b \in R$, then

$$\left\langle \prod_j a_j, b \right\rangle = \sum_i \left\langle a_i, b \prod_{j \neq i} a_j \right\rangle.$$

LEMMA 5. The map $h: J_p \times J_q \rightarrow G^{(M)}(R, I)$ defined by $h(a, b) = \langle a, b \rangle$ is additive in both entries, if $p + q \geq M - 1 \geq 1$ and $p \geq 1$.

Proof. The third relation in the presentation of $K_2(R/J_M, I/J_M)$ implies that for $a \in J_p$ and $b_1, b_2 \in J_q$ one has

$$\langle a, b_1 \rangle + \langle a, b_2 \rangle = \langle a, b_1 + b_2 \rangle + \langle a, c \rangle,$$

if c is such that $b_1 + b_2 - ab_1b_2 = (b_1 + b_2) + c - a(b_1 + b_2)c$. But then $c = -(1 - ab_1 - ab_2)^{-1} ab_1b_2 \in J_{p+2q}$ and so $\langle a, c \rangle$ vanishes in $G^{(M)}(R, I)$ since $p + (p + 2q) \geq 2M - 2 \geq M$. A similar reasoning applies to the other entry. \square

Let $e: X \rightarrow \mathbb{Z}$ be a map such that $e(x) \geq 0$ for every $x \in X$, and such that $\text{supp}(e) = \{x \in X; e(x) > 0\}$ is finite and nonempty. Then we associate to it the monomial $x^e = \prod_{y \in X} y^{e(y)}$; this is a homogeneous element of $A[X]$ of degree $d(e) = \sum_{x \in X} d(x)e(x)$. Furthermore we associate to it the quotient group

$$H^{(e)}(A) = \frac{\Omega_A \oplus A^{\text{supp}(e)}}{\{(\delta a, y \mapsto e(y)a); a \in A\}}.$$

PROPOSITION 8. If $d(e) = M - 1 \geq 1$, then there is a nontrivial natural homomorphism $\Phi^{(e)}: H^{(e)}(A) \rightarrow G^{(M)}(R, I)$.

Proof. We define a map $h_x: A \times A \rightarrow G^{(M)}(R, I)$ by the formula $h_x(a, b) = h(ax^e, b)$. Lemma 5 applied to $p = M - 1$ and $q = 0$ implies that h_x extends to a homomorphism $h_\otimes: A \otimes A \rightarrow G^{(M)}(R, I)$. It is clear from the second relation in the presentation of $K_2(R/J_M, I/J_M)$ that h_\otimes vanishes on the

image of $D: A \otimes A \otimes A \rightarrow A \otimes A$. Therefore h_\otimes induces a homomorphism $h_\Omega: \Omega_A \rightarrow G^{(M)}(R, I)$.

For every $y \in \text{supp}(e)$ we define a map $h_y: A \rightarrow G^{(M)}(R, I)$ by the formula $h_y(a) = -h(y, y^{-1}x^e a)$. Here $y^{-1}x^e$ is an abbreviation for $y^{e(y)-1} \prod_{x \neq y} x^{e(x)}$. Lemma 5 applied to $p = 1$ and $q = M - 2$ says that h_y is a homomorphism.

By Lemma 4 one has

$$h_\Omega(\delta a) = \langle x^e, a \rangle = \sum_y e(y) \langle y, y^{-1}x^e a \rangle = -\sum_y e(y) h_y(a).$$

Therefore the homomorphism $\Omega_A \oplus A^{\text{supp}(e)} \rightarrow G^{(M)}(R, I)$ which one gets by combining h_Ω and all h_y induces one on the quotient group $H^{(e)}(A)$. \square

We write $H^{(M)}(A)$ for the direct sum of all $H^{(e)}(A)$ for which $d(e) = M - 1$. Thus all $\Phi(e)$ together define a homomorphism

$$\Phi^{(M)}: H^{(M)}(A) \rightarrow G^{(M)}(R, I).$$

PROPOSITION 9. *If $M \geq 2$, then the image of $\Phi^{(M)}$ is the kernel of the canonical surjection $G^{(M)}(R, I) \rightarrow G^{(M-1)}(R, I)$.*

Proof. Consider the canonical map

$$\pi: K_2(R/J_M, I/J_M) \rightarrow K_2(R/J_{M-1}, I/J_{M-1}),$$

which is clearly a surjection. If $z \in K_2(R/J_M, I/J_M)$ is mapped to zero in $G^{(M-1)}(R, I)$, then one must have

$$\pi(z) = \sum_i \langle a_i + J_{M-1}, b_i + J_{M-1} \rangle,$$

for certain $a_i \in J_p$ and $b_i \in J_q$ with $p + q \geq M - 1$ and $p \geq 1$. Now consider the element $z' = \sum_i \langle a_i + J_M, b_i + J_M \rangle \in K_2(R/J_M, I/J_M)$. Then $z - z'$ is in the image of $K_2(R/J_M, J_{M-1}/J_M)$ and therefore of the form $z - z' = \sum_j \langle a_j + J_M, b_j + J_M \rangle$ with $a_j \in J_{M-1}/J_M$ and $b_j \in R/J_M$. Taken together one has $z = \sum_k h(a_k, b_k)$ in $G^{(M)}(R, I)$ for certain $a_k \in J_p$ and $b_k \in J_q$ with $p + q \geq M - 1$ and $p \geq 1$.

By Lemma 5 we may rewrite each term in this sum as a similar sum in which every a_k and b_k is of the form cx^e with $c \in A$. By the second relation we may rewrite each term in this sum as a similar sum in which every b_k is in A or in X . In the first case the term is in the image of h_Ω , in the second it is by the first relation in the image of some h_y . \square

In a similar way as above one has maps $\Phi_L^{(e)}: H^{(e)}(A) \rightarrow G_L^{(e)}(R, I)$ and $\Phi_L^{(M)}: H^{(M)}(A) \rightarrow G_L^{(M)}(R, I)$, by using square brackets $[x, y]$ instead of angle brackets $\langle x, y \rangle$. But in this case more is true.

PROPOSITION 10. *The map $\Phi_L^{(M)}$ is an isomorphism from $H^{(M)}(A)$ to the kernel of the canonical surjection $\pi_L: G_L^{(M)}(R, I) \rightarrow G_L^{(M-1)}(R, I)$.*

Proof. We associate to $ax^e \in R/J_M$ with $a \in A$ the multidegree $e \in \mathbb{Z}$. In this way R/J_M becomes a multigraded ring. The map $D: R/J_M \otimes R/J_M \otimes R/J_M \rightarrow R/J_M \otimes R/J_M$ preserves total multidegree; therefore $\text{cok}(D) = \Omega_{R/J_M}$ is a direct sum of its homogeneous parts. The map $\Delta: I/J_M \otimes_R I/J_M \rightarrow I/J_M \otimes_R \Omega_{R/J_M}$ also preserves total multidegree; therefore $\text{cok}(\Delta) = K_{2,L}(R/J_M, I/J_M)$ is also a direct sum of its homogeneous parts. Finally $G_L^{(M)}(R, I)$ is the quotient of this group by the summands associated to the multidegrees e with $d(e) \geq M$; therefore it is still a direct sum of its homogeneous parts. It is clear that the direct summand associated to a multidegree e with $d(e) = M - 1$ can be identified with $H^{(e)}(A)$ using $\Phi_L^{(e)}$. \square

The Propositions 8, 9 and 10 are true without reference to a λ -ring structure on R . If R is a λ -ring as at the start of this section, then we can use L to compare the G groups and G_L groups.

PROPOSITION 11. *The maps $L: G^{(M)}(R, I) \rightarrow G_L^{(M)}(R, I)$ and the maps $L: K_2(R/J_N, I/J_N) \rightarrow K_{2,L}(R/J_N, I/J_N)$ are isomorphisms.*

Proof. If $a \in J_p$ and $b \in J_q$, then it follows from Proposition 7 that $L\langle a, b \rangle - [a, b]$ is a sum of terms $[a', b']$ with $a' \in J_r$ and $b' \in J_s$ with $r + s > p + q$. In particular the expression

$$L\Phi^{(e)}(adb) - \Phi_L^{(e)}(adb) = L\langle ax^e, b \rangle - [ax^e, b]$$

is a sum of terms $[a', b']$ with $a' \in J_r$ and $b' \in J_s$ and $r + s > M - 1$. These terms vanish in $G_L^{(M)}(R, I)$. Therefore $L^{(M)} \circ \Phi^{(M)} = \Phi_L^{(M)}$ on the Ω_A part of $H^{(e)}(A)$; and a similar reasoning applies to the $A^{\text{supp}(e)}$ part. Thus one has a commutative diagram with exact rows

$$\begin{array}{ccccccc} H^{(M)}(A) & \xrightarrow{\Phi^{(M)}} & G^{(M)}(R, I) & \xrightarrow{\pi} & G^{(M-1)}(R, I) & \longrightarrow & 0 \\ \downarrow 1 & & \downarrow L^{(M)} & & \downarrow L^{(M-1)} & & \\ 0 & \longrightarrow & H^{(M)}(A) & \xrightarrow{\Phi_L^{(M)}} & G_L^{(M)}(R, I) & \xrightarrow{\pi_L} & G_L^{(M-1)}(R, I) \longrightarrow 0. \end{array}$$

The first statement follows now by induction using the five lemma, starting with the case $M = 1$ where both groups are trivial.

The group $K_2(R/J_N, I/J_N)$ is the quotient of $G^{(2N-1)}(R, I)$ by the subgroup generated by all $\langle a, b \rangle$ with $a \in J_N$ and $b \in R$. A similar statement holds for $K_{2,L}(R/J_N, I/J_N)$. If $a \in J_N$ and $b \in R$, then it follows from Proposition 7 that $L\langle a, b \rangle - [a, b]$ is a sum of terms $[a', b']$ which vanish in $G_L^{(2N-1)}(R, I)$, since for the terms with $m > 1$ one has $a' \in J_{2N}$ and for the terms with $k > 1$ one has $a' \in J_N$ and $b' \in J_N$. This proves the second statement. \square

We end this section with a proposition illuminating the structure of the groups $H^{(e)}(A)$.

PROPOSITION 12. Let l be the cardinality of $\text{supp}(e)$, and let e' be the greatest common divisor of the $e(x)$ with $x \in X$. Then one has

$$H^{(e)}(A) \cong \frac{\Omega_A \oplus A}{\{(\delta a, e'a); a \in A\}} \oplus_{i=1}^{l-1} A.$$

7. The exponential map

Let $U' = U \otimes U$ and $J'_n = J_n^U \otimes U$, and consider the filtered λ -ring $(U', \{J'_n\})$ and the λ -ideal $E' = E \otimes U$. One can apply the theory of the last section by taking $A = \mathbb{Z} \otimes U$ and $X = \{\lambda^n(u) \otimes 1; n > 0\}$. Thus $L: K_2^{\text{top}}(U', E') \rightarrow K_{2,L}^{\text{top}}(U', E')$ is an isomorphism. We write ε for the element of $K_2^{\text{top}}(U', E')$ for which $L\varepsilon = [u \otimes 1, 1 \otimes u] \in K_{2,L}^{\text{top}}(U', E')$. We write ε_n for its image in $K_2(U'/J'_n, E'/J'_n)$.

Consider a filtered λ -ring $(R, \{J_n\})$ and a λ -nilpotent λ -ideal I . Given $a \in I$ and $b \in R$ there is a unique λ -map $f: U' \rightarrow R$ such that $f(u \otimes 1) = a$ and $f(1 \otimes u) = b$. Given any $M \in \mathbb{N}$, there is some $n \in \mathbb{N}$ such that $J_n^U(I) \subseteq J_M$ and so $f(J'_n) \subseteq J_M$. Then f induces a map $f^{n,M}: U'/J'_n \rightarrow R/J_M$. Thus each $\exp_M(a, b) = f_*^{n,M}(\varepsilon_n)$ is an element of $K_2(R/J_M, (I + J_M)/J_M)$. Together they define an element $\exp(a, b) \in K_2^{\text{top}}(R, I)$.

LEMMA 6. The map $\exp_M: I \times R \rightarrow K_2(R/J_M, (I + J_M)/J_M)$ vanishes on the relations defining $K_{2,L}(R, I)$.

Proof. Consider the λ -ring $U'' = U \otimes U \otimes U$ which is filtered by the ideals $J''_n = \sum_{k+m=n} J_k^U \otimes J_m^U \otimes U$, and consider the λ -ideal $E'' = E \otimes U \otimes U + U \otimes E \otimes U$. One can apply the theory of the last section by taking $A = \mathbb{Z} \otimes \mathbb{Z} \otimes U$ and $X = \{\lambda^k(u) \otimes \lambda^m(u) \otimes 1; k + m > 0\}$. Thus

$L: K_2(U''/J''_n, E''/J''_n) \rightarrow K_{2,L}(U''/J''_n, E''/J''_n)$ is an isomorphism for every n .

Let $\alpha, \beta, \gamma: U' \rightarrow U''$ be the unique λ -maps such that

$$\begin{aligned} \alpha(u \otimes 1) &= u \otimes 1 \otimes 1, & \alpha(1 \otimes u) &= 1 \otimes 1 \otimes u, \\ \beta(u \otimes 1) &= 1 \otimes u \otimes 1, & \beta(1 \otimes u) &= 1 \otimes 1 \otimes u, \\ \gamma(u \otimes 1) &= u \otimes 1 \otimes 1 + 1 \otimes u \otimes 1, & \gamma(1 \otimes u) &= 1 \otimes 1 \otimes u. \end{aligned}$$

They induce maps $\alpha^n, \beta^n, \gamma^n: U'/J'_n \rightarrow U''/J''_n$ mapping E'/J'_n to E''/J''_n and thus induce maps of relative K_2 and $K_{2,L}$ groups. Since L is natural with respect to such maps one has

$$\begin{aligned}
 L\gamma_*^n \varepsilon &= \gamma_*^n L\varepsilon = \gamma_*^n [u \otimes 1, 1 \otimes u] \\
 &= [u \otimes 1 \otimes 1 + 1 \otimes u \otimes 1, 1 \otimes 1 \otimes u] \\
 &= [u \otimes 1 \otimes 1, 1 \otimes 1 \otimes u] + [1 \otimes u \otimes 1, 1 \otimes 1 \otimes u] \\
 &= \alpha_*^n [u \otimes 1, 1 \otimes u] + \beta_*^n [u \otimes 1, 1 \otimes u] \\
 &= \alpha_*^n L\varepsilon_n + \beta_*^n L\varepsilon_n = L\alpha_*^n \varepsilon_n + L\beta_*^n \varepsilon_n
 \end{aligned}$$

and so $\gamma_*^n \varepsilon = \alpha_*^n \varepsilon + \beta_*^n \varepsilon$.

Given $a, b \in I$ and $c \in R$ there is a unique λ -map $g: U'' \rightarrow R$ such that

$$g(u \otimes 1 \otimes 1) = a, \quad g(1 \otimes u \otimes 1) = b, \quad g(1 \otimes 1 \otimes u) = c.$$

Given M there is some n such that $g(J''_n) \subseteq J_M$, so that g induces a map $g^{n,M}: U''/J''_n \rightarrow R/J_M$. Now $g\gamma$ is the unique λ -map $f: U' \rightarrow R$ such that $f(u \otimes 1) = a + b$ and $f(1 \otimes u) = c$, and thus $f^{n,M} = g^{n,M}\gamma^n$. Thus

$$\exp_M(a + b, c) = f_*^{n,M} \varepsilon_n = g_*^{n,M} \gamma_*^n \varepsilon_n.$$

Similarly $\exp_M(a, c) = g_*^{n,M} \alpha_*^n \varepsilon_n$ and $\exp_M(b, c) = g_*^{n,M} \beta_*^n \varepsilon_n$. Therefore one has

$$\exp_M(a + b, c) = \exp_M(a, c) + \exp_M(b, c).$$

The other three identities can be proven in a very similar way. □

LEMMA 7. *For every M there exist a P such that \exp_M induces a well defined map $K_{2,L}(R/J_P, (I + J_P)/J_P) \rightarrow K_{2,L}(R/J_M, (I + J_M)/J_M)$. Thus \exp is a continuous map $K_{2,L}^{\text{op}}(R, I) \rightarrow K_{2,L}^{\text{op}}(R, I)$.*

Proof. Let n be such that $J_n^U(I) \subseteq J_M$. Then ε_n is a finite sum of terms $\langle \xi'_k + J_n^U \otimes U, \xi''_k + J_n^U \otimes U \rangle$ with $\xi'_k, \xi''_k \in U \otimes U$. There exists some $d \in I$ such that $U(d) \otimes U(d)$ contains all ξ'_k and ξ''_k .

According to Proposition 6 there exists some P such that for all $\xi \in U(d) \otimes U(d)$ the class of $\xi(a, b)$ in R/J_M only depends on the classes of a and b in R/J_P . □

THEOREM 2. *Let $(R, \{J_n\})$ be a filtered λ -ring and let I be a λ -nilpotent λ -ideal. Then the maps $L: K_{2,L}^{\text{pr}^0}(R, I) \rightarrow K_{2,L}^{\text{pr}^0}(R, I)$ and $\exp: K_{2,L}^{\text{pr}^0}(R, I) \rightarrow K_{2,L}^{\text{pr}^0}(R, I)$ are each others inverses.*

Proof. We prove the equivalent statement for $K_{2,L}^{\text{op}}$. Let $a \in I$ and $b \in R$ and let $f: U' \rightarrow R$ be the unique λ -map such that $f(u \otimes 1) = a$ and $f(1 \otimes u) = b$; this map is continuous. Since L is natural for continuous λ -maps one has

$$L \circ \exp[a, b] = L \circ f_* \varepsilon = f_* L \varepsilon = f_* [u \otimes 1, 1 \otimes u] = [a, b].$$

But L and \exp are continuous, and finite sums of terms $[a, b]$ are dense in $K_{2, \mathbb{Z}}^{\text{lop}}(R, I)$. Therefore $L \circ \exp = 1$.

In particular one has $L \circ \exp \circ L \langle u \otimes 1, 1 \otimes u \rangle = L \langle u \otimes 1, 1 \otimes u \rangle$. Since $L: K_2^{\text{lop}}(U', E') \rightarrow K_{2, \mathbb{Z}}^{\text{lop}}(U', E')$ is an isomorphism this implies that

$$\exp \circ L \langle u \otimes 1, 1 \otimes u \rangle = \langle u \otimes 1, 1 \otimes u \rangle.$$

Since L and \exp are natural for continuous λ -maps one has for a, b, f as above

$$\begin{aligned} \exp \circ L \langle a, b \rangle &= \exp \circ L f_* \langle u \otimes 1, 1 \otimes u \rangle = f_* \exp \circ L \langle u \otimes 1, 1 \otimes u \rangle \\ &= f_* \langle u \otimes 1, 1 \otimes u \rangle = \langle a, b \rangle. \end{aligned}$$

But L and \exp are continuous, and finite sums of terms $\langle a, b \rangle$ are dense in $K_2^{\text{lop}}(R, I)$. Therefore $\exp \circ L = 1$. □

8. Partitions

In this section we review some combinatorial concepts which we need in the next section in order to give sufficient conditions for a λ -ideal to be λ -nilpotent.

A map $\pi: \mathbb{N} \rightarrow \mathbb{Z}$ is called a list if $\pi(i) \geq 0$ for all i and if $\text{supp}(\pi) = \{i \in \mathbb{N}; \pi(i) > 0\}$ is finite. We call the cardinality of $\text{supp}(\pi)$ its length $l(\pi)$, $\max\{\pi(i)\}$ its height $h(\pi)$ and $\sum \pi(i)$ its degree $d(\pi)$. If σ is an element of the group $\mathcal{S}(\mathbb{N})$ of permutations of \mathbb{N} , then $\pi \circ \sigma^{-1}$ is again a list. An equivalence class of lists under the action of $\mathcal{S}(\mathbb{N})$ is called a partition. It has well defined length, height and degree. Any partition can be represented by a list π for which $\pi(1) \geq \pi(2) \geq \dots \geq \pi(s) > 0$ and $\pi(i) = 0$ for $i > s$; we call that an ordered list.

The ring U is the polynomial ring freely generated by the $\lambda^i(u)$ for $i > 0$. Therefore the element $\lambda^d(u \otimes u) \in U \otimes U$ can be written as a sum $\sum \xi^\pi \otimes \lambda^{\pi(1)}(u) \lambda^{\pi(2)}(u) \dots \lambda^{\pi(s)}(u)$ for certain $\xi^\pi \in U$ of degree d . Here the sum is over all ordered lists π of degree d , and thus length $s \leq d$. These ξ^π can be viewed as natural maps $R \rightarrow R$ defined for each partition π and each λ -ring R so that

$$\lambda^d(ab) = \sum \xi^\pi(a) \lambda^{\pi(1)}(b) \lambda^{\pi(2)}(b) \dots \lambda^{\pi(s)}(b),$$

for every $a, b \in R$.

In order to discuss the properties of these maps we recall the construction of the universal λ -ring U . Let P_n be the polynomial ring $\mathbb{Z}[t_1, t_2, \dots, t_n]$ with the λ -ring structure for which $\lambda^i(t_j) = 0$ for $i > 1$. Let $f_n: P_n \rightarrow P_{n-1}$ be the homomorphism given by $f_n(t_j) = t_j$ for $j < n$ and $f_n(t_n) = 0$. Let P_∞ be the

inverse limit of this system. Then P_∞ is a λ -ring since the maps f_n are λ -maps. There is unique λ -map $\gamma_\infty: U \rightarrow P_\infty$ such that $\gamma(u) = \sum_{j=1}^\infty t_j$. This map is injective, which means that one can identify U with its image in P_∞ which consists of all symmetric elements. On $U(d)$ the corresponding map to P_d is already injective. This means that one can characterize an element $\xi \in U(d)$ by its value on $\sum_{j=1}^n t_j \in P_n$ provided that $n \geq d$.

Let π be a list such that $\pi(i) = 0$ for $i > n$. Then one can associate to π the monomial $\prod_{j=1}^n t_j^{\pi(j)} \in P_n$. Let $\sigma \in \mathcal{S}(\mathbb{N})$ be a permutation such that $\sigma(i) = i$ for $i > n$. Then $\pi \circ \sigma^{-1}$ is again a list as above. So the subgroup $\mathcal{S}(n)$ of $\mathcal{S}(\mathbb{N})$ consisting of these σ acts on the set of these π .

LEMMA 8. *Let π be an ordered list of degree d and let $n \geq d$. Then $\xi^\pi(\sum_{j=1}^n t_j)$ is the sum of all monomials associated to lists in the orbit of π .*

In other words $\xi^\pi(\sum_{j=1}^n t_j) = t_1^{\pi(1)} t_2^{\pi(2)} \dots t_s^{\pi(s)} + \text{permutations}$.

Proof. If one applies the formula for λ^d of a sum to $a = \sum_{j=1}^N t_j$ one gets

$$\begin{aligned} \lambda^d(ab) &= \sum \lambda^{i_1}(t_1 b) \lambda^{i_2}(t_2 b) \dots \lambda^{i_N}(t_N b) \\ &= \sum t_1^{i_1} t_2^{i_2} \dots t_N^{i_N} \lambda^{i_1}(b) \lambda^{i_2}(b) \dots \lambda^{i_N}(b), \end{aligned}$$

where the sum is over all nonnegative integers i_1, i_2, \dots, i_N such that $i_1 + i_2 + \dots + i_N = d$. If π is any ordered list, then the coefficient of $\lambda^{\pi(1)}(b) \lambda^{\pi(2)}(b) \dots \lambda^{\pi(d)}(b)$ is exactly the sum of all terms in the orbit of $t_1^{\pi(1)} t_2^{\pi(2)} \dots t_d^{\pi(d)}$. □

EXAMPLE 4. If $\pi(i) = 1$ for $i \leq n$ and $\pi(i) = 0$ otherwise, then $\xi^\pi = \lambda^n$. If $\rho(i) = k\pi(i)$ for all i , then $\xi^\rho = \psi^k \circ \xi^\pi$.

Lemma 8 says that the ξ^π are just the ‘monomial symmetric functions’ discussed in [8], especially pages 32–34, where the next two lemmas are clarified.

LEMMA 9. *The ξ^π constitute a basis of U as an abelian group.*

Let π', π'' be lists such that $\pi'(i) = 0$ for $i > r$, and $\pi''(j) = 0$ for $j > s$. Then the combination $\pi = \pi' \pi''$ is the list defined by

$$\pi(i) = \begin{cases} \pi'(i) & \text{for } i \leq r, \\ \pi''(i) & \text{for } r < i \leq r + s, \\ 0 & \text{for } r + s < i. \end{cases}$$

Let π be any list. Then the contracted list ρ is defined by

$$\rho(i) = \begin{cases} \pi(1) + \pi(2) & \text{for } i = 1, \\ \pi(i - 1) & \text{for } i > 3. \end{cases}$$

Two lists π' and π'' are called disjoint if $\pi'(i) \neq \pi''(j)$ for all i and j . All these notions carry over to partitions; of course a partition can in general be contracted in several ways.

LEMMA 10. *If π', π'' are disjoint partitions, then*

$$\xi^{\pi'}(a)\xi^{\pi''}(a) = \xi^{\pi'\pi''}(a) + \sum c_\rho \xi^\rho(a),$$

where the coefficient $c_\rho \in \mathbb{Z}$ vanishes unless ρ arises from $\pi'\pi''$ by repeated contraction.

Proof. Obvious from Lemmas 8 and 9. □

PROPOSITION 13. *Let R be a λ -ring and I a λ -ideal. If $a \in I$, then $\xi^\pi(a)$ is in the ideal generated by $\{\psi^m(b); m \geq h(\pi), b \in I\}$.*

Proof. We apply induction with respect to $l(\pi)$. If there are s and k such that $\pi(i) = k$ for $i \leq s$ and $\pi(i) = 0$ for $i > s$, then $\xi^\pi = \psi^k \circ \lambda^s$ and the result is obvious. Otherwise there are disjoint partitions π' and π'' such that $\pi = \pi'\pi''$. By Lemma 10 we have

$$\xi^\pi(a) = \xi^{\pi'}(a)\xi^{\pi''}(a) - \sum c_\rho \xi^\rho(a).$$

For every relevant ρ one has $l(\rho) < l(\pi') + l(\pi'') = l(\pi)$ and thus $\xi^\rho(a)$ is in the ideal generated by $\{\psi^m(b); m \geq h(\rho), b \in I\}$ by induction hypothesis. Since $h(\rho) \geq h(\pi'\pi'')$ this is contained in the stated ideal. By the induction hypothesis $\xi^{\pi'}(a)$ is in the ideal generated by $\{\psi^m(b); m \geq h(\pi'), b \in I\}$, and $\xi^{\pi''}(a)$ is in the ideal generated by $\{\psi^m(b); m \geq h(\pi''), b \in I\}$. Since $h(\pi) = \max(h(\pi'), h(\pi''))$ one of these is contained in the stated ideal, and thus the product is. □

9. Nilpotency

A λ -ideal I in a filtered λ -ring $(R, \{J_n\})$ is called ψ -nilpotent if it satisfies the following conditions:

- (1) For every M there exist N_1 such that $\psi^n(I) \subseteq J_M$ for $n \geq N_1$. This is certainly the case if I is θ -nilpotent.
- (2) For every M there exist N_2 such that $I^{N_2} \subseteq J_M$.

We shall show that in this case the ideal I^2 is λ -nilpotent.

LEMMA 11. *Let $(R, \{J_n\})$ be a filtered λ -ring and let I be a ψ -nilpotent λ -ideal. Then for every M there exist a N such that $\lambda^n(ab) \in J_M$ for $n \geq N$.*

Proof. Let M, N_1, N_2 be as above and consider the identity

$$\lambda^n(ab) = \sum \xi^\pi(a)\lambda^{\pi(1)}(b)\lambda^{\pi(2)}(b) \dots \lambda^{\pi(s)}(b),$$

for $n > (N_1 - 1)(N_2 - 1)$. Since $n = d(\pi) \leq l(\pi)h(\pi)$ one has $s = l(\pi) \geq N_2$ or $h(\pi) \geq N_1$. In the first case $\prod_{i=1}^s \lambda^{\pi(i)}(b) \in I^s \subseteq J_M$. In the second case $\xi^n(a)$ is by Proposition 13 in the ideal generated by some $\psi^m(b)$ with $b \in I$ and $m \geq N_1$. But then $\psi^m(b) \in J_M$. \square

LEMMA 12. *Let $(R, \{J_n\})$ be a filtered λ -ring and let I be a ψ -nilpotent λ -ideal. Then for every M there exist a N such that $\lambda^n(I^2) \subseteq J_M$.*

Proof. Let M, N_1, N_2 be as above and let $N_3 - 1 = (N_1 - 1)(N_2 - 1)$. Let $x \in I^2$ and write $x = \sum_{i=1}^m a_i b_i$ with $a_i, b_i \in I$. Consider the identity

$$\lambda^n(x) = \sum \lambda^{i_1}(a_1 b_1) \lambda^{i_2}(a_2 b_2) \lambda^{i_m}(a_m b_m),$$

where the sum is over all nonnegative integers i_1, i_2, \dots, i_m such that $i_1 + i_2 + \dots + i_m = n$. Assume that $n > (N_2 - 1)(N_3 - 1)$. If the number of nonzero i_k is at least N_2 , then the product is in $I^{N_2} \subseteq J_M$. If not, then $i_k \geq N_3$ for some k , and $\lambda^{i_k}(a_k b_k) \in J_M$ by Lemma 11. \square

THEOREM 3. *Let $(R, \{J_n\})$ be a filtered λ -ring and let I be a ψ -nilpotent λ -ideal. Then for every M there exist N such that $\xi(I^2) \subseteq J_M$ for $\xi \in J_N^U$.*

Proof. Let M, N_2, N_3 be as above and let $N_4 - 1 = (N_2 - 1)(N_3 - 1)$. Let $\xi \in U$ be homogeneous of degree n and let $x \in I^2$. Then $\xi(x)$ is an integral combination of terms $\lambda^{i_1}(x) \lambda^{i_2}(x) \dots \lambda^{i_m}(x)$, where the nonnegative integers i_1, i_2, \dots, i_m satisfy $i_1 + i_2 + \dots + i_m = n$. Assume that $n > (N_2 - 1)(N_4 - 1)$. If the number of nonzero i_k is at least N_2 , then the product is in $I^{N_2} \subseteq J_M$. If not, then $i_k \geq N_4$ for some k , and $\lambda^{i_k}(x) \in J_M$ by Lemma 12. \square

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