RAYMOND ROSS

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RAYMOND ROSS*

Louisiana State University, Baton Rouge, LA 70803-4918

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Introduction

The Beilinson conjectures ([2], [3]) interpret the special values of the L-functions associated to a projective variety X defined over a number field to the K-theory of X, via a regulator which generalizes that of Dirichlet. In this paper, we examine part of these conjectures in the case where X is the Fermat curve $F_N: x^N + y^N = z^N$ of exponent N, where $N \geqslant 3$. Our investigation is inspired partly by Beilinson's theorem on modular curves ([3], also see [17]), and by a result of Rohrlich on the divisor class group of F_N [12].

Let us recall what Beilinson's theorem says, following the exposition in [17]. Let $X/\mathbf{Q}$ be a modular curve, and let g be the genus of X. For each modulator curve $Y/\mathbf{Q}$ which covers X via a morphism $\theta_Y: Y \to X$ defined over $\mathbf{Q}$, there is a homomorphism $\theta_{Y*}: K_2 Y \to K_2 X$. Let $\mathcal{Q}_Y$ be the subspace $K_2 Y \otimes \mathbf{Q}$ with divisorial support at the cusps of Y (this is made precise in Section 1 below), and let $\mathcal{P}_X$ denote the subgroup of $K_2 X \otimes \mathbf{Q}$ spanned by $\theta_{Y*}(\mathcal{Q}_Y)$ for all such Y. Then $\mathcal{P}_X \subset H^2_{\mathcal{M}}(X, \mathbf{Q}(2))_{\mathbf{Z}}$ and the image of $\mathcal{P}_X$ in $H^2_{\mathcal{D}}(X, \mathbf{R}(2))$ under the regulator homomorphism reg_X is a $\mathbf{Q}$-structure of $H^2_{\mathcal{D}}(X, \mathbf{R}(2))$, and

$$\textstyle\bigwedge^g \mathrm{reg}_X \mathcal{P}_X = c \bigwedge^g H^2_{\mathcal{D}}(X, \mathbf{Q}(2)),$$

with $c \equiv L^{(g)}(0, X) \bmod \mathbf{Q}^*$.

In general, one must look at $\mathcal{P}_X$ and not just $\mathcal{Q}_X$. For example, if p is prime, then $X_0(p)$ has exactly two cusps, and therefore only one (up to scalar multiple) modular unit f. It then follows that $\mathcal{Q}_{X_0(p)}$ is trivial.

For F_N, an analogue of the group of modular units is provided by those functions whose divisorial support is contained in the *points at infinity*, which are those points P on F_N such that $xyz(P) = 0$. In partial analogy with Beilinson's construction, we investigate the subgroup of $K_2 F_N$ generated by the images under the transfer maps $K_2 F_{dN} \to K_2 F_N$ of those elements whose divisorial support is contained in the points at infinity. We show that this subgroup is of positive rank, and that, over $\mathbf{Q}(\mu_{2N})$, it is a cyclic $\mathrm{Aut}\, F_N$-module. Using this, we descend to $\mathbf{Q}$ and obtain a bound for the rank of the corresponding group over $\mathbf{Q}$. In contrast to the modular situation, this

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subgroup is usually of rank smaller than the rank of K_2F_N as predicted by the conjecture, namely, $(N-1)(N-2)/2$, the genus of F_N. The rank of this subgroup, and a set of generators for the vector space obtained by tensoring with **Q**, will be determined in [14].

We now briefly indicate the organization of this paper. We begin by summarizing what we need from K-theory. In Section 2, we exhibit an explicit element in K_2F_N which we show is of infinite order. In Section 3, we describe the subgroup of K_2F_N which is generated by the K-theoretic transfers of those elements in K_2F_{dN} arising from functions with divisorial support on the points at infinity. We will see that this is simply the group of such elements in K_2F_N. Section 4 is an interlude, where we look at the example of F_4. In the final section, we show that the subgroup of K_2F_N under investigation is a principal $\operatorname{Aut} F_N$-module, and use this to determine an upper bound on its rank as an abelian group in case N is an odd prime.

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1. The regulator, Bloch's trick, and symbols with support

We begin by recalling what the regulator of Beilinson and Bloch is in the special case that X is a smooth projective curve defined over **Q**. We fix an algebraic closure $\bar{\mathbf{Q}} \subset \mathbf{C}$ of **Q**. All algebraic extensions of **Q** which arise are tacitly understood to lie in this fixed choice of $\bar{\mathbf{Q}}$.

Let F be a field. By Matsumoto's theorem [10], $K_2F \cong (F^* \otimes F^*)/R$, where R is the subgroup of $F^* \otimes F^*$ generated by the tensors of the form $f \otimes (1-f)$, with $f \neq 0, 1$. The image of $f \otimes h$ in K_2F is denoted $\{f, h\}$, and is referred to as a *symbol.*

Let X be a smooth projective curve of genus g defined over **Q**. The localization sequence provides us with the following exact sequence:

$$\coprod_{P\in X(\bar{\mathbf{Q}})} K_2\mathbf{Q}(P) \to K_2X \to K_2\mathbf{Q}(X) \xrightarrow{\tau} \coprod_{P\in X(\mathbf{Q})} \mathbf{Q}(P)^*$$

where

$$\tau = \coprod_{P\in X(\bar{\mathbf{Q}})} \tau_P,$$

with τ_P being the *tame symbol* at P:

$$\tau_P(\{f, h\}) = (-1)^{(\mathrm{ord}_P f)(\mathrm{ord}_P h)} \frac{f^{\mathrm{ord}_P h}}{h^{\mathrm{ord}_P f}}(P).$$

By Garland's theorem [9], $K_2\mathbf{Q}(P)$ is torsion, whence K_2X and ker τ agree, up to torsion. If k is any finite extension of $\mathbf{Q}$, then by viewing X as a curve over k, these remarks, with $\mathbf{Q}$ replaced with k, remain valid.

In [2], Beilinson defines a regulator for K_2X, which agrees with that of Bloch [6]. This may be viewed as a homomorphism

$$\mathrm{reg}_X: K_2X \to H^1(X(\mathbf{C}), \mathbf{R}(1))^-,$$

which is functorial in X. The superscript denotes the -1-eigenspace of $H^1(X(\mathbf{C}), \mathbf{R}(1))$ under the action of complex conjugation on both $X(\mathbf{C})$ and $\mathbf{R}(1) = 2\pi i\mathbf{R}$. We remark that $H^1(X(\mathbf{C}), \mathbf{R}(1))^-$ is isomorphic to the Deligne cohomology group $H^2_{\mathscr{D}}(X, \mathbf{R}(2))$ mentioned above in the Introduction [18].

To describe reg_X, we assume for simplicity that $\{f, h\} \in \ker \tau$. For details, we refer the reader to [8] or [11]. We view $H^1(X(\mathbf{C}), \mathbf{C})$ as the space of $\mathbf{C}$-valued functionals on $H_1(X(\mathbf{C}), \mathbf{Z})$. Let γ be a closed path on $X(\mathbf{C})$, and fix a basepoint $P_0 \in \gamma$; assume that f and h are both regular and nonzero on γ. Then

$$\mathrm{reg}_X(\{f, h\})(\gamma) = i \,\mathrm{Im}\left(\int_\gamma \log f d \log h - \log|h(P_0)| \int_\gamma d \log f\right). \tag{1}$$

Here, we choose fixed branches of $\log f$ and $\log h$ on some neighborhood of γ, and the integrals are taken starting at P_0. One may show that this definition depends only upon the class of $\gamma \in H_1(X(\mathbf{C}), \mathbf{Z})$.

Let Ω_X^{1+} denote the subspace of $H^0(X(\mathbf{C}), \Omega^1_{X/\mathbf{C}})$ which is invariant under the action of complex conjugation on both $X(\mathbf{C})$ and $\Omega^1_{X/\mathbf{C}}$. Under the identification $H^1(X(\mathbf{C}), \mathbf{R}(1))^-$ with $\mathrm{Hom}_{\mathbf{R}}(\Omega_X^{1+} \mathbf{R})$, we view $\mathrm{reg}_X(\{f, h\})$ as a functional on real 1-forms. One then obtains the following formula for the regulator:

$$\mathrm{reg}_X(\{f, h\})(\omega) = \frac{1}{2\pi i} \int_{X(\mathbf{C})} \log|f| \overline{d \log h} \wedge \omega. \tag{2}$$

One can check that this integral converges for any pair of functions on X, and we thereby obtain a homomorphism

$$\mathrm{reg}_X: K_2\mathbf{C}(X) \to H^1(X(\mathbf{C}), \mathbf{C})$$

which is functorial in X and extends the regulator as defined above. As an immediate consequence of this functoriality, we have the following:

LEMMA 1. *Let* $\phi: X \to \mathbf{P}^1$ *be any morphism. Then* $\mathrm{reg}_X(\phi^*(\alpha)) = 0$ *for all* $\alpha \in K_2\mathbf{C}(\mathbf{P}^1)$.

It follows from this, or directly from (2), that the regulator vanishes on those symbols of which one entry is constant.

We now turn to the transfer map in K-theory, and describe those facts which we will need in the sequel.

Let F/L be a finite Galois extension of fields with Galois group G, and denote by ϕ the inclusion $L \hookrightarrow F$. The two maps

$$\phi_*: K_nF \to K_nL$$

and

$$\phi^*: K_nL \to K_nF$$

are then related to the action of Galois on K_nF in the following manner:

$$(\phi^* \circ \phi_*)(\alpha) = \prod_{\sigma \in G} \alpha^\sigma, \quad \alpha \in K_nF \tag{3}$$

$$(\phi_* \circ \phi^*)(\beta) = \beta^{[F:L]}, \quad \beta \in K_nL. \tag{4}$$

This follows from the fact that ϕ^* is induced from the extension of scalars functor $\underline{P}(L) \to \underline{P}(F)$, and ϕ_* is induced from the restriction of scalars functor $\underline{P}(F) \to \underline{P}(L)$, where $\underline{P}(A)$ is the category of finitely generated projective A-modules for any commutative ring A.

We will also need to know how the transfer behaves with respect to the regulator map, as follows. Let $k/\mathbf{Q}$ be a finite Galois extension, and let $\psi: \mathbf{Q}(X) \hookrightarrow k(X)$ be the field inclusion.

LEMMA 2. *Let* $\alpha \in K_2k(X)$. *Then*

$$\mathrm{reg}_X(\psi_*\alpha) = \mathrm{reg}_X(\psi^*\psi_*\alpha) = \mathrm{reg}_X(\alpha^\Sigma)$$

where

$$\Sigma = \sum_{\sigma \in \mathrm{Gal}(k/\mathbf{Q})} \sigma.$$

Proof. Since the regulator is defined over $\mathbf{C}$ and ψ^* is induced by the field inclusion, we have $\mathrm{reg}_X(\beta) = \mathrm{reg}_X(\psi^*\beta)$ for all $\beta \in K_2\mathbf{Q}(X)$. Letting $\beta = \psi_*\alpha$, the lemma then follows from (3). □

In particular, if $\text{reg}_X(\alpha) = 0$, then $\text{reg}_X(\psi_*\alpha) = 0$.

We now recall Bloch's trick [4]. Let $S = \{P_0, \ldots, P_n\} \subset X(\bar{\mathbf{Q}})$. Choose an embedding $X \hookrightarrow \text{Jac}\, X$ such that P_0 corresponds to the identity element of $\text{Jac}\, X$. Assume that the images of the P_i under this embedding are torsion points.

Let f and g be functions in $\bar{\mathbf{Q}}(X)$ whose divisors consist entirely of points in S. One version of Bloch's trick asserts that there is an integer N, functions $\phi_i \in \bar{\mathbf{Q}}(X)^*$, and constants $c_i \in \bar{\mathbf{Q}}^*$ such that $\alpha = \{f, g\}^N \Pi_i \{\phi_i, c_i\}$ is in the kernel of the tame symbol. We will refer to such an α as a *normalization* of $\{f, g\}$. Note that $\alpha \in K_2L(X)$ for some finite extension L of $\mathbf{Q}$; by taking the K-theoretic transfer from $L(X)$ to $\mathbf{Q}(X)$ we obtain an element in $K_2\mathbf{Q}(X) \cap \ker \tau$. We remark that if k is a finite extension of $\mathbf{Q}$, $S \subset X(k)$, and $f, g \in k(X)^*$, then ϕ_i and c_i can be chosen to be defined over k.

Utilizing this version of Bloch's trick, we now construct a subgroup of $K_2X \otimes \mathbf{Q}$ which is associated to S.

Choose k so that $S \subset X(k)$, let $\mathcal{U}_S = \{f \in \bar{\mathbf{Q}}(X)^* : \text{ord}_Q(f) = 0 \text{ for all } Q \notin S\}$, and let $\{f_1, \ldots, f_m\}$ be generators for $\mathcal{U}_S/\bar{\mathbf{Q}}^*$ defined over k. Let $\{\mathcal{U}_S, \mathcal{U}_S\}$ denote the subgroup of $K_2\bar{\mathbf{Q}}(X)$ generated by symbols $\{f, g\}$ such that $f, g \in \mathcal{U}_S$. Then $\{\mathcal{U}_S, \mathcal{U}_S\}$ is generated by symbols of the following three types:

$$\{c, d\} \quad c, d \in \bar{\mathbf{Q}}^* \tag{5}$$

$$\{f_i, c\} \quad c \in \bar{\mathbf{Q}}^* \tag{6}$$

$$\{f_i, f_j\} \tag{7}$$

The symbols in (5) are torsion, being in $K_2\bar{\mathbf{Q}} = \varinjlim K_2L$, with the limit being taken over all finite extensions L of $\mathbf{Q}$ in $\bar{\mathbf{Q}}$. The symbols in (6) are pullbacks from $K_2\bar{\mathbf{Q}}(\mathbf{P}^1)$; we may therefore apply Bloch's trick on $\mathbf{P}^1$ and thereby obtain a symbol in $K_2\bar{\mathbf{Q}}(X)$ which is a pullback from $K_2\mathbf{P}^1/\bar{\mathbf{Q}}$. This all occurs over some finite extension of $\mathbf{Q}$, whence this symbol is torsion.

Let $\mathcal{N}_{XS}(k)$ be the group generated by a normalization of each of those symbols in (7). This group depends on the normalizations chosen; see the remarks below for more about this dependence. Observe, however, that if $\mathcal{N}_{X,S}$ denotes the group generated by all the symbols in (5)–(7), then $\mathcal{N}_{X,S}(k) \otimes \mathbf{Q}$ and $\mathcal{N}_{X,S} \otimes \mathbf{Q}$ have the same image under the regulator map.

We now put $\mathcal{T}_{X,S}(\mathbf{Q}) = \text{tr}\, \mathcal{N}_{X,S}(k)$, where tr is the transfer map from $K_2k(X)$ to $K_2\mathbf{Q}(X)$. By (3), $\mathcal{T}_{X,S}(\mathbf{Q}) \subset \ker \tau$, and we may therefore identify $\mathcal{T}_{X,S}(\mathbf{Q}) \otimes \mathbf{Q}$ with a subspace of $K_2X \otimes \mathbf{Q}$, which we shall refer to as the *subspace of* $K_2X \otimes \mathbf{Q}$ *with divisorial support on* S.

REMARKS. Bloch's trick, as stated in [4], does not specify how to choose N and c_i, or even that the modifications to $\{f, g\}$ to yield an element of $\ker \tau$ are of the type that we have selected. Any two normalizations of $\{f, g\}$ would differ by a rational multiple and an element of $\Gamma = \ker \tau \cap \ker \mathrm{reg}_X$. Very little, if anything, seems to be known about Γ; Beilinson conjectures that $(\Gamma \otimes \mathbf{Q}) \cap H^2_{\mathscr{M}}(X, \mathbf{Q}(2))_{\mathbf{Z}} = 0$. If X is such that $H^2_{\mathscr{M}}(X, \mathbf{Q}(2))_{\mathbf{Z}} = H^2_{\mathscr{M}}(X, \mathbf{Q}(2))$, then *conjecturally* our construction yields the same vector space as any version of Bloch's trick. In particular, the Fermat curves satisfy this condition, since their Jacobians have CM: Letting $\mathscr{F}_N$ denote a regular proper model for F_N over $\mathbf{Z}$, we have the localization exact sequence in K-theory:

$$\coprod_p K_2'\mathscr{F}_{N,p} \otimes \mathbf{Q} \to H^2_{\mathscr{M}}(F_N, \mathbf{Q}(2))_{\mathbf{Z}} \to H^2_{\mathscr{M}}(F_N, \mathbf{Q}(2)) \to \coprod_p K_1'\mathscr{F}_{N,p} \otimes \mathbf{Q}.$$

By ([19], Theorem 3) the Euler factor $L_p(F_{N,s})$ has no pole at $s = 0$; therefore, the right-most vector space is zero ([11], Proposition 4.7.9). Finally, the left-most vector space is zero ([1]).

2. An element of infinite order

Let $\zeta = e^{2\pi i/N}$, and let $A_{i,j}$ denote the automorphism

$$(x, y) \mapsto (\zeta^i x, \zeta^j y) \tag{8}$$

of F_N. Let $t^{1/N}$ denote the principal branch of the Nth root function, and let γ denote the following path from $(1, 0)$ to $(0, 1)$ on $F_N(\mathbf{C})$:

$$\gamma\colon [0, 1] \to F_N(\mathbf{C})$$
$$\gamma\colon t \mapsto ((1 - t)^{1/N}, t^{1/N}).$$

For integers m and n, let $\gamma_{m,n}$ denote the following closed path on $F_N(\mathbf{C})$:

$$\gamma_{m,n} = \gamma - A_{m,0}\gamma + A_{m,n}\gamma - A_{0,n}\gamma.$$

By a slight abuse of notation, we will also denote by $\gamma_{m,n}$ the corresponding element of $H_1(F_N(\mathbf{C}), \mathbf{Z})$.

We will need the beta function $B(u, v)$, which is defined for positive real numbers u and v by

$$B(u, v) = \int_0^1 t^{u-1}(1 - t)^{v-1}\, dt.$$

THEOREM 1. *The symbol* $\{1-x, 1-y\}^{2N}$ *belongs* $\ker \tau$ *and has non-zero image under* reg_{F_N}. *In particular,* $\{1-x, 1-y\}^{2N}$ *represents an element in* K_2F_N *of infinite order.*

Proof. An easy calculation shows that $\{1-x, 1-y\}^{2N} \in \ker \tau$. We show that $\{1-x, 1-y\}^{2N}$ has nontrivial image under the regulator homomorphism by computing its value on the 1-cycle $\gamma_{1,1} + \gamma_{1,-1}$. Let $\varepsilon > 0$ be small, and choose a representative path $\gamma_{m,n,\varepsilon}$ for $\gamma_{m,n}$ such that the initial point of $\gamma_{m,n,\varepsilon}$ is $((1-\varepsilon)^{1/N}, \varepsilon^{1/N})$, and such that both $1-x$ and $1-y$ are nonzero and regular on $\gamma_{m,n,\varepsilon}$. Let $\sigma = \{1-x, 1-y\}^{2N}$. Choosing the branches of $\log(1-x)$ and $\log(1-y)$ which are real-valued on γ, we find that $\mathrm{reg}_{F_N}(\sigma)(\gamma_{m,n})$ is equal to:

$$2iN \,\mathrm{Im}\left(\int_{\gamma_{m,n,\varepsilon}} \log(1-x)d\log(1-y) - \log(1-\varepsilon^{1/N})\int_{\gamma_{m,n,\varepsilon}} d\log(1-x)\right).$$

Note that as ε tends toward zero, the imaginary part of the second integral tends toward zero. Since the value of this integral depends only on the homology class of $\gamma_{m,n}$ and not on the basepoint chosen, we conclude that

$$\mathrm{reg}_{F_N}(\sigma)(\gamma_{m,n}) = 2iN\,\mathrm{Im}\int_{\gamma_{m,n}} \log(1-x)d\log(1-y).$$

For any $a, b \in \mathbf{Z}$, let

$$I_{a,b} = \int_0^1 \log(1-\zeta^a(1-t)^{1/N})d\log(1-\zeta^b t^{1/N}).$$

Then

$$\begin{aligned} I_{a,b} &= \int_0^1 \frac{1}{N}\sum_{k=1}^{\infty}\frac{1}{k}\zeta^{ak}(1-t)^{k/N}\sum_{j=1}^{\infty}\zeta^{jb}t^{j/N-1}\,\mathrm{d}t \\ &= \frac{1}{N}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{k}\zeta^{ka+jb}\int_0^1 (1-t)^{k/N}t^{j/N-1}\,\mathrm{d}t. \end{aligned}$$

The interchange of the two sums and the integral may be justified by a double application of the Lebesgue dominated convergence theorem. Moreover, the double sum converges absolutely, which may be seen by considering the case $a = b = 0$. Noting that the last integral above is $B(j/N, k/N + 1)$ and using the identity

$$B(u, v+1) = \frac{v}{u+v}B(u, v),$$

we obtain

$$I_{a,b} = \frac{1}{N} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{k+j} \zeta^{ka+jb} B\left(\frac{j}{N}, \frac{k}{N}\right).$$

Returning to $\gamma_{m,n}$, we find that

$$\begin{aligned} \mathrm{reg}_{F_N}(\sigma)(\gamma_{m,n}) &= 2iN \operatorname{Im}(I_{0,0} - I_{m,0} + I_{m,n} - I_{0,n}) \\ &= 2i \operatorname{Im} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{k+j} (1 - \zeta^{km})(1 - \zeta^{jn}) B\left(\frac{j}{N}, \frac{k}{N}\right). \end{aligned}$$

Therefore

$$\mathrm{reg}_{F_N}(\sigma)(\gamma_{1,1} + \gamma_{1,-1}) = -4i \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{k+j} \sin \frac{2\pi k}{N} \left(1 - \cos \frac{2\pi j}{N}\right) B\left(\frac{j}{N}, \frac{k}{N}\right).$$

Define b_k by

$$b_k = \sum_{j=1}^{\infty} \frac{1}{k+j} \left(1 - \cos \frac{2\pi j}{N}\right) B\left(\frac{j}{N}, \frac{k}{N}\right),$$

and for $1 \leqslant K \leqslant N$, let

$$\beta_K = \sum_{k \equiv K \bmod N} b_k.$$

Note that $b_k > 0$ for all k and that the sequence $\{b_k\}$ is monotonically decreasing, so $\beta_1 > \beta_2 > \cdots > \beta_N > 0$. Let N' be defined by

$$N' = \begin{cases} \dfrac{N-1}{2} & \text{if } N \text{ is odd} \\ \dfrac{N}{2} - 1 & \text{if } N \text{ is even.} \end{cases}$$

Then

$$\begin{aligned} \mathrm{reg}_{F_N}(\sigma)(\gamma_{1,1} + \gamma_{1,-1}) &= -4i \sum_{K=1}^{N} \beta_K \sin \frac{2\pi K}{N} \\ &= -4i \sum_{K=1}^{N'} (\beta_K - \beta_{N-K}) \sin \frac{2\pi K}{N} \end{aligned}$$

which is nonzero, since each summand is positive. □

3. Symbols with support at infinity and the Fermat tower

We now turn our attention to the elements in K_2F_N which have divisorial support on the points at infinity. By the *points at infinity* we mean the set $S_N = \{P \in F_N(\bar{\mathbf{Q}}): xyz(P) = 0\}$. There are $3N$ such points, given in projective coordinates by $[\zeta^j, 0, 1]$, $[0, \zeta^j, 1]$, and $[\xi\zeta^j, 1, 0]$, for $1 \leqslant j \leqslant N$, where ζ is a primitive Nth root of unity and ξ is a primitive $2N$th root of unity, which we shall always take to satisfy $\xi^2 = \zeta$.

Let $k = \mathbf{Q}(\boldsymbol{\mu}_{2N})$, and, to fix matters, choose an embedding $\iota: F_N \hookrightarrow \operatorname{Jac} F_N$ defined over $\mathbf{Q}$ and sending the point $[0, 1, 1]$ to the origin. A theorem of Rohrlich [12] asserts that $\iota(S_N)$ is torsion and, in the notation of Section 1, the group $\mathscr{U}_S/\bar{\mathbf{Q}}^*$ is generated by the following functions, all of which are defined over k:

$$x \qquad y \qquad x - \zeta^j \qquad y - \zeta^j \qquad x - \xi\zeta^j y \tag{9}$$

$$\left(\prod_{j=0}^{N-1} ((x - \zeta^j)(y - \zeta^j))^j\right)^{1/N} \tag{10}$$

$$\left(\prod_{j=0}^{N-1} ((x - \zeta^j)(x - \xi\zeta^j y))^j\right)^{1/N} \tag{11}$$

$$\left(\prod_{j=0}^{N-1} ((x - \zeta^j)(y - \zeta^j)(x - \xi\zeta^j y))^{j(j+1)/2}\right)^{1/E(N)} \tag{12}$$

with $E(N) = N$ if N is odd and $E(N) = N/2$ if N is even.

We are interested in the space $\mathscr{T}_{F_N, S_N}(\mathbf{Q}) \otimes \mathbf{Q}$, the space of symbols in $K_2F_N \otimes \mathbf{Q}$ with divisorial support at infinity. Recall that this space is $\psi_*(\mathscr{N}_{F_N, S_N}(k)) \otimes \mathbf{Q}$, where $\mathscr{N}_{F_N, S_N}(k)$ is the group generated by normalizations of the symbols obtained from the functions in (9)–(12), and ψ_* is the transfer map induced by the field inclusion $\psi^{\to}: \mathbf{Q}(F_N) \hookrightarrow k(F_N)$.

We will see below (Theorem 3) that for a suitable choice of normalizations,

$$\mathscr{N}_{F_N, S_N}(k) \otimes \mathbf{Q} = \mathbf{Q}[\Gamma_N] \cdot \{1 - x, 1 - y\}$$

where Γ_N is the automorphism group of F_N. Let $\mathscr{S}_{F_N, S_N}(k)$ denote the $\mathbf{Q}$-vector space $\mathbf{Q}[\Gamma_N]\{1 - x, 1 - y\}$, and let $\mathscr{Q}_{F_N, S_N}(\mathbf{Q}) = \psi_* \mathscr{S}_{F_N, S_N}(k)$. It follows from Theorem 3 that $\mathscr{T}_{F_N, S_N}(\mathbf{Q}) \otimes \mathbf{Q} = \mathscr{Q}_{F_N, S_N}(\mathbf{Q})$. We will assume these facts in this section, since the results below do not depend on those discussed here.

For each positive integer d, let

$$\phi_d : F_{dN} \to F_N$$

be the morphism given by

$$\phi_d : (x, y) \mapsto (x^d, y^d).$$

Let u and v be the standard coordinate functions on F_N, such that $u = x^d$ and $v = y^d$. Put $k_d = \mathbf{Q}(\boldsymbol{\mu}_{2dN})$, so $k_d(F_{dN}) = k_d(F_N)(x, y)$ is an abelian extension of $k_d(F_N)$ with Galois group $G_d \cong \boldsymbol{\mu}_d \times \boldsymbol{\mu}_d$.

For each integer $n \geqslant 1$, let $A_{i,j}(n)$ (i and j read modulo n), $\sigma(n)$, and $\eta(n)$ denote the following elements of $\Gamma_n = \operatorname{Aut} F_n$:

$$A_{i,j}(n) : (x, y) \mapsto (\zeta_n^i x, \zeta_n^j y)$$
$$\sigma(n) : (x, y) \mapsto (y, x)$$
$$\eta(n) : (x, y) \mapsto \left(\frac{1}{y}, \xi_n^{-1} \frac{x}{y}\right)$$

where ζ_n is a primitive nth root of unity and $\xi_n^2 = \zeta$, $\xi_n^n = -1$. Then Γ_n is generated by $A_{i,j}(n)$, $\sigma(n)$, and $\eta(n)$, and is isomorphic to a semidirect product of $\boldsymbol{\mu}_n \times \boldsymbol{\mu}_n$ and the symmetric group S_3.

Now fix N, let ζ_N and ξ_N be as above, and for each integer $d \geqslant 1$, choose ζ_{dN} and ξ_{dN} such that $\zeta_{dN}^d = \zeta_N$ and $\xi_{dN}^d = \xi_N$. Then the map $\Gamma_{dN} \to \Gamma_N$ given by

$$A_{i,j}(dN) \mapsto A_{i,j}(N)$$
$$\sigma(dN) \mapsto \sigma(N)$$
$$\eta(dN) \mapsto \eta(N)$$

is a homomorphism, the kernel of which is the relative automorphism group $\operatorname{Aut}(F_{dN}/F_N)$. We view $\mathscr{S}_{F_N,S_N}(k)$ as a Γ_{dN}-module via this map. Let $j_d : k_1(F_N) \hookrightarrow k_d(F_N)$ be the field inclusion.

PROPOSITION 1. $\phi_{d*}\mathscr{S}_{F_{dN},S_{dN}}(k_d) = j_d^*\mathscr{S}_{F_N,S_N}(k)$.

Proof. A direct calculation using (3) shows that

$$\phi_d^* \circ \phi_{d*} : \mathscr{S}_{F_{dN},S_{dN}}(k_d) \otimes \mathbf{Q} \to \mathscr{S}_{F_{dN},S_{dN}}(k_d) \otimes \mathbf{Q}$$

is Γ_{dN}-equivariant, and that

$$(\phi_d^* \circ \phi_{d*})\{1 - x, 1 - y\} = (\phi_d^* \circ j_d^*)\{1 - u, 1 - v\}.$$

Therefore

$$\begin{aligned}(\phi_d^* \circ \phi_{d*})\mathscr{S}_{F_{dN},S_{dN}}(k_d) &= (\phi_d^* \circ \phi_{d*})(\mathbf{Q}[\Gamma_{dN}]\{1-x, 1-y\}) \\ &= \mathbf{Q}[\Gamma_{dN}](\phi_d^* \circ \phi_{d*})\{1-x, 1-y\} \\ &= \mathbf{Q}[\Gamma_{dN}](\phi_d^* \circ j_d^*)\{1-u, 1-v\} \\ &= (\phi_d^* \circ j_d^*)\mathbf{Q}[\Gamma_N]\{1-u, 1-v\} \\ &= (\phi_d^* \circ j_d^*)\mathscr{S}_{F_N,S_N}(k).\end{aligned}$$

It follows from (4) that ker ϕ_d^* is trivial (recall that we have tensored with $\mathbf{Q}$). Therefore $\phi_{d*}\mathscr{S}_{F_{dN},S_{dN}}(k_d) = j_d^*\mathscr{S}_{F_N,S_N}(k)$. □

We now descend to $\mathbf{Q}$. Let $\psi_d: \mathbf{Q}(F_{dN}) \hookrightarrow k_d(F_{dN})$ be the field inclusion.

THEOREM 2. $\phi_{d*}\mathscr{Q}_{F_{dN},S_{dN}}(\mathbf{Q}) = \mathscr{Q}_{F_N,S_N}(\mathbf{Q})$.

Proof. What we must show is that

$$(\phi_{d*} \circ \psi_{d*})\mathscr{S}_{F_{dN},S_{dN}}(k_d) = \psi_{1*}\mathscr{S}_{F_N,S_N}(k).$$

From the following diagram of field inclusions:

$$\begin{array}{ccccc} \mathbf{Q}(F_{dN}) & \xrightarrow{\psi_d} & k_d(F_{dN}) & & \\ \uparrow \phi_d^* & & & \nwarrow \phi_d^* & \\ & & & & k_d(F_N) \\ & & & \nearrow j_d & \\ \mathbf{Q}(F_N) & \xrightarrow{\psi_1} & k_1(F_N) & & \end{array}$$

we obtain the following commutative diagram in K-theory:

$$\begin{array}{ccccc} & & K_2k_d(F_{dN}) & \xrightarrow{\psi_d} & K_2\mathbf{Q}(F_{dN}) \\ & \swarrow \phi_{d*} & & & \\ K_2k_d(F_N) & & & & \downarrow \phi_{d*} \\ & \searrow j_{d*} & & & \\ & & K_2k_1(F_N) & \xrightarrow{\psi_1} & K_2\mathbf{Q}(F_N). \end{array}$$

Note that by (4), $(j_{d*} \circ j_d^*)\mathscr{S}_{F_N,S_N}(k) = \mathscr{S}_{F_N,S_N}(k)$. Therefore

$$\begin{aligned}(\phi_{d*} \circ \psi_{d*})\mathscr{S}_{F_{dN},S_{dN}}(k_d) &= (\psi_{1*} \circ j_{d*} \circ \phi_{d*})\mathscr{S}_{F_{dN},S_{dN}}(k_d) \\ &= (\psi_{1*} \circ j_{d*} \circ j_d^*)\mathscr{S}_{F_N,S_N}(k) \\ &= \psi_{1*}\mathscr{S}_{F_N,S_N}(k)\end{aligned}$$

as desired. □

4. The Fermat curve of exponent 4

In this section, we examine in some detail the curve F_4. In particular, we will exhibit a symbol $\{f, g\} \in \mathscr{Q}_{F_4S_4}(\mathbf{Q})$ such that the divisor of f contains points which are of infinite order under the canonical embedding $F_4 \to \operatorname{Jac} F_4$; compare with [15]. In some contrast to [15], we can relate this symbol to the L-value $L^{(3)}(F_4, 0)$ in the manner predicted by the Beilinson conjecture.

We begin by showing that the symbols

$$\alpha_1 = \{x^2 + 1, 1 - y\}^8, \quad \alpha_2 = \{1 - x, y^2 + 1\}^8, \quad \alpha_3 = \{1 - x, 1 - y\}^8$$

generate a rank 3 subgroup of K_2F_4. Note that these symbols lie in $\mathscr{Q}_{F_4,S_4}(\mathbf{Q})$.

Let $\gamma_{m,n}$ be as in Section 2. Calculating as in the proof of Theorem 1, we find that

$$\operatorname{reg}_{F_4}(\alpha_1)(\gamma_{m,n}) = \begin{cases} 8i(\beta_1 - \beta_3) & \text{if } n = 1,\ m \text{ odd} \\ 8i(\beta_3 - \beta_1) & \text{if } n = 3,\ m \text{ odd} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\beta_J = \sum_{j \equiv J \bmod 4} \sum_{k=1}^{\infty} \frac{1}{4k + j - 2} B\left(k - \frac{1}{2}, \frac{j}{4}\right).$$

Note that $\beta_1 > \beta_2 > \beta_3 > \beta_4 > 0$; thus in the first two cases the regulator value is non-zero.

A similar calculation for α_2 yields:

$$\operatorname{reg}_{F_4}(\alpha_2)(\gamma_{m,n}) = \begin{cases} 8i(\beta_1 - \beta_3) & \text{if } m = 1,\ n \text{ odd} \\ 8i(\beta_3 - \beta_1) & \text{if } m = 3,\ n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Define B_{odd} and B_{even} by

$$B_{\text{odd}} = \sum_{\substack{k \text{ odd} \\ k \geq 1}} \sum_{j=1}^{\infty} \frac{1}{j+k-1} B\left(\frac{2j-1}{4}, \frac{2k-1}{4}\right)$$

$$B_{\text{even}} = \sum_{\substack{k \text{ even} \\ k \geq 2}} \sum_{j=1}^{\infty} \frac{1}{j+k-1} B\left(\frac{2j-1}{4}, \frac{2k-1}{4}\right).$$

Note that $B_{\text{odd}} > B_{\text{even}} > 0$. Calculating as in the proof of Theorem 1, one finds that

$$\operatorname{reg}_{F_4}(\alpha_3)(\gamma_{2,1}) = 2i(B_{\text{odd}} - B_{\text{even}}),$$

which is nonzero. Finally, one may verify that these elements are linearly independent by considering their values on the paths $\gamma_{2,1}$, $\gamma_{1,3}$ and $\gamma_{3,3}$.

It is not difficult to write down an explicit isomorphism $X_0(64) \to F_4$ which is defined over $\mathbf{Q}$, and is such that the cusps of $X_0(64)$ correspond precisely with the points at infinity on F_4. Thus α_1, α_2, and α_3 generate the subspace $\mathscr{P}_{X_0(64)}$ of $K_2X_0(64) \otimes \mathbf{Q}$ described in the Introduction. In particular, the images of α_1, α_2, and α_3 under reg_{F_4} define a $\mathbf{Q}$-structure on $H^2_{\mathscr{D}}(F_4, \mathbf{R}(2))$, with volume equal to a rational multiple of $L^{(3)}(F_4, 0)$.

On the other hand, we can approach K_2F_4 "from below". The Jacobian $J_0(64)$ of $X_0(64)$ is isogenous over $\mathbf{Q}$ to $E \times E' \times E'$, where E and E' are the elliptic curves

$$E: Y^2 = X^3 - 4X$$

$$E': Y^2 = X^3 + 4X,$$

and we have morphisms $\rho: F_4 \to E$, $\beta: F_4 \to E'$, and $\beta': F_4 \to E'$ given by:

$$\rho(x, y) = \left(2\,\frac{y^2+1}{x^2}, 4\,\frac{y(y^2+1)}{x^3}\right)$$

$$\beta(x, y) = \left(2\,\frac{1-y^2}{x^2}, 4\,\frac{1-y^2}{x^3}\right)$$

$$\beta'(x, y) = \left(2\,\frac{1-x^2}{y^2}, 4\,\frac{1-x^2}{y^3}\right).$$

One computes that

$$\beta'^* \left\{1 - \frac{2X}{Y}, \frac{8X}{Y^2}\right\}^8 = \alpha_1^{-1}$$

$$\beta^* \left\{1 - \frac{2X}{Y}, \frac{8X}{Y^2}\right\}^8 = \alpha_2.$$

Noting that the divisors of the pullbacks to F_4 of $8X/Y^2$ and $1 - 2X/Y$ by β and β' consist of points at infinity, we conclude that these functions have torsion divisorial support on E'. A calculation using Rohrlich's explicit version [13] of Bloch's theorem [5] shows that

$$\mathrm{reg}_{E'}\left(\left\{1 - \frac{2X}{Y}, \frac{8X}{Y^2}\right\}^8\right)$$

defines a $\mathbf{Q}$-structure on $H^2_{\mathcal{D}}(E', \mathbf{R}(2))$ with volume equal to $c'L'(0, E')$, where $c' \in \mathbf{Q}^*$. The conjectures of Beilinson imply that

$$\left\{1 - \frac{2X}{Y}, \frac{8X}{Y^2}\right\}$$

is a generator for $K_2E' \otimes \mathbf{Q}$.

Note that

$$f = 1 - x/y \quad \text{and} \quad g = \frac{1 - y^2}{1 - x^2},$$

while functions on F_4, actually define functions on E, which we also denote by f and g. The divisors of f and g on E consist of torsion points, and computing as above we find that $\mathrm{reg}_E(\{f, g\})$ determines a $\mathbf{Q}$-structure on $H^2_{\mathcal{D}}(E, \mathbf{R}(2))$ with volume equal to $cL'(0, E)$ with $c \in \mathbf{Q}^*$. The Beilinson conjectures imply that $\{f, g\}$ is a generator for $K_2E \otimes \mathbf{Q}$. Put $\tilde{\alpha}_3 = \rho^*\{f, g\}^8$.

There is a decomposition of cohomology induced by ρ, β, and β':

$$H^1(F_4(\mathbf{C}), \mathbf{C}) \cong H^1(E(\mathbf{C}), \mathbf{C}) \oplus H^1(E'(\mathbf{C}), \mathbf{C}) \oplus H^1(E'(\mathbf{C}), \mathbf{C})$$

which is orthogonal with respect to the pairing

$$\langle \omega, \eta \rangle = \frac{1}{2\pi i} \int_{F_4(\mathbf{C})} \omega \wedge \bar{\eta},$$

where we are using DeRham cohomology. Therefore α_1, α_2, and $\tilde{\alpha}_3$ span a 3-dimensional subspace V of $K_2F_4 \otimes \mathbf{Q}$, and the image of V under the regulator is a $\mathbf{Q}$-structure of $H^2_{\mathscr{D}}(F_4, \mathbf{R}(2))$ with volume equal to a rational multiple of $L^{(3)}(F_4, 0)$.

If the Beilinson conjectures are true, we should have $V = \mathscr{Q}_{F_4,S_4}(\mathbf{Q})$ (and both should be equal to $K_2F_4 \otimes \mathbf{Q}$). We now verify that it is indeed the case that $V = \mathscr{Q}_{F_4,S_4}(\mathbf{Q})$. Note that it suffices to check that $\tilde{\alpha}_3 \in \mathscr{T}_{F_4,S_4}(\mathbf{Q}) \otimes \mathbf{Q}$.

A computation using (3) shows that, up to torsion,

$$\alpha_3^2\alpha_1\alpha_2 = \rho^*(\rho_*\alpha_3).$$

Using the algorithm in [16], we compute that, up to torsion,

$$\rho^*(\rho_*\alpha_3) = \tilde{\alpha}_3.$$

Therefore, $\tilde{\alpha}_3 \in \mathscr{Q}_{F_4,S_4}(\mathbf{Q})$, as desired.

In closing, we note that the divisor of $1 - x/y$ contains points which are not points at infinity on F_4; by a result of Coleman [7], these points are of infinite order in $\operatorname{Jac} F_4$.

5. The group of symbols with support at infinity

In this section, we fix our attention on F_N, and let $k = k_1 = \mathbf{Q}(\boldsymbol{\mu}_{2N})$, $\mathscr{N} = \mathscr{N}_{F_N,S_N}(k)$, $\mathscr{S} = \mathscr{S}_{F_N,S_N}(k)$, $\mathscr{T} = \mathscr{T}_{F_N,S_N}(\mathbf{Q})$, $\psi = \psi_1: \mathbf{Q}(F_N) \hookrightarrow k(F_N)$, and $G = G_1 = \operatorname{Gal}(k/\mathbf{Q})$.

In general, $\mathscr{T} \otimes \mathbf{Q}$ cannot be equal to $K_2F_N \otimes \mathbf{Q}$, provided that one accepts the Beilinson conjectures. For the predicted rank of K_2F_N is the genus of F_N, which equals $(N-1)(N-2)/2$, and we have

PROPOSITION 2. *Let p be an odd prime. Then* $\dim_{\mathbf{Q}} \mathscr{T}_{F_p,S_p}(\mathbf{Q}) \otimes \mathbf{Q} \leqslant 3(p-1)$. *In particular, for $p > 7$, we have* $\dim_{\mathbf{Q}} \mathscr{T}_{F_p,S_p}(\mathbf{Q}) \otimes \mathbf{Q} < \operatorname{genus}(F_p)$.

Proposition 2 is a consequence of the following:

THEOREM 3. *With an appropriate choice of normalizations,* $\mathscr{N} \otimes \mathbf{Q} = \mathscr{S} = \mathbf{Q}[\Gamma_N]\{1-x, 1-y\}$.

Proof. By (9)–(12), it is clear that $\mathscr{N} \otimes \mathbf{Q}$ is generated over $\mathbf{Q}$ by normalizations of the following symbols:

$$\{x - \zeta^j, y - \zeta^k\}, \{x - \zeta^j, x - \xi\zeta^k y\}, \{y - \zeta^j, x - \xi\zeta^k y\} \tag{13}$$

$$\{x - \xi\zeta^j y, x - \xi\zeta^k y\}, \{x - \xi\zeta^j y, x\}, \{x - \xi\zeta^j y, y\} \tag{14}$$

$$\{x-\zeta^j, y\}, \{y-\zeta^j, x\}, \{x-\zeta^j, x-\zeta^k\}, \{y-\zeta^j, y-\zeta^k\} \tag{15}$$

$$\{x-\zeta^j, x\}, \{y-\zeta^j, y\}, \{x, y\} \tag{16}$$

for $0 \leqslant j, k \leqslant N-1$. Note that we do not need to include symbols for which one of the entries is a function listed in (10)–(12), because we have tensored with $\mathbf{Q}$.

The symbols in (16) are torsion in $K_2k(F_N)$.

Although not torsion, appropriate powers of the symbols in (15) are pullbacks from $K_2k(\mathbf{P}^1)$, and therefore have trivial normalizations.

The symbols in (14) are also pullbacks from $K_2k(\mathbf{P}^1)$. Indeed, letting $\eta = \eta(N)$, one finds that:

$$\eta^*\{x-\xi\zeta^j y, x-\xi\zeta^k y\}^{2N} = \{1-\zeta^j x, 1-\zeta^k x\}^{2N}\left\{1-x^N, \frac{1-\zeta^j x}{1-\zeta^k x}\right\}^2$$

$$\eta^*\{x-\xi\zeta^j y, x\}^{2N} = \{1-\zeta^j x, 1-x^N\}^{-2}$$

$$(\eta^2)^*\{x-\xi\zeta^{jj} y, y\}^{2N} = \{1-y^N, y-\zeta^j\}^{-2}.$$

Thus the symbols in (14)–(16) all have trivial normalizations. We now show that the symbols in (13) have normalizations which lie in $\mathscr{S}$, and explicitly determine these normalizations. Given a symbol $\{f, g\}$, we will denote a normalization of $\{f, g\}$ by $v(f, g)$.

Since $\{x-\zeta^j, y-\zeta^k\}^{2N} = A^*_{-j,-k}\{1-x, 1-y\}^{2N}$, we take

$$v(x-\zeta^j, y-\zeta^k) = \{x-\zeta^j, y-\zeta^k\}^{2N} = A^*_{-j,-k}\{1-x, 1-y\}^{2N}. \tag{17}$$

In examining the other symbols in (13), we will need the following. Since $\{1-x, y\}^{2N} = \{1-x, 1-x^N\}^2 \in K_2k(x)$, we apply Bloch's trick and select $\delta = \Pi_i\{f_i(x), c_i\}$ with $f_i \in k(x)^*$ and $c_i \in k^*$ such that

$$\{1-x, y\}^{2N} \cdot \delta = \{1-x, 1-x^N\}^2 \cdot \delta \in \ker \tau \cap K_2k(x).$$

Therefore $\{1-x, y\}^{2N} \cdot \delta$ is torsion; let M denote its order. We thus have

$$\{1-x, y\}^{2MN} = \{1-x, 1-x^N\}^{2M} = \delta^{-M}. \tag{18}$$

We note that the image of δ under any automorphism of F_N is a product of symbols in which one entry is constant.

A routine calculation shows that we may select normalizations of the remaining symbols in (13) as follows:

$$\begin{aligned} v(x-\zeta^j, x-\xi\zeta^k y) &= \{x-\zeta^j, x-\xi\zeta^k y\}^{2MN} \cdot [A^*_{-j,-k}(\eta^2)^*](\delta^M) \\ &= [A^*_{-j,-k}(\eta^2)^*]\{1-x, 1-y\}^{-2MN}. \end{aligned} \tag{19}$$

and

$$\begin{aligned} v(y-\zeta^j, x-\xi\zeta^k y) &= \{y-\zeta^j, x-\xi\zeta^k y\}^{2MN}\cdot[\eta^* A^*_{j,-k}\sigma^*](\delta^M) \\ &= [\eta^* A^*_{j,-k}\sigma^*]\{1-x, 1-y\}^{-MN}. \end{aligned} \tag{20}$$

From this it is clear that $\mathcal{N}\otimes\mathbf{Q}\subset\mathcal{S}$; a routine verification using (17)–(20) yields the reverse inclusion. □

Proof of Proposition 2. Note that as **Q**-vector spaces, $\mathcal{T}\otimes\mathbf{Q}\cong\psi^*\mathcal{T}\otimes\mathbf{Q}\cong\psi^*\psi_*\mathcal{N}\otimes\mathbf{Q}$. Let $\Sigma=\Sigma_{\sigma\in G}\,\sigma$. Then by (3) we see that $\mathcal{T}\otimes\mathbf{Q}\cong\mathcal{N}^\Sigma\otimes\mathbf{Q}$; in light of the previous result, we consider $\mathcal{S}^\Sigma$.

We first make a convenient choice of spanning set for $\mathcal{S}$. Since p is odd, $x-\xi\zeta^k y = x+\zeta^{k'}y$, where $k'=(2k+1+p)/2$. In what amounts to a re-indexing, let

$$\begin{aligned} T_{j,k} &= \{x-\zeta^j, y-\zeta^k\} \\ U_{j,k} &= \{x-\zeta^j, x+\zeta^k y\} \\ V_{j,k} &= \{y-\zeta^j, x+\zeta^k y\}. \end{aligned}$$

Then $\mathcal{S}$ is spanned by the normalizations $v((T_{j,k})$, $v(U_{j,k})$, and $v(V_{j,k})$ of these symbols.

Therefore $\mathcal{S}^\Sigma$ is generated by $v(T_{j,k})^\Sigma$, $v(U_{j,k})^\Sigma$, and $v(V_{j,k})^\Sigma$. One finds, using (18), (19), and (20) that $v(T_{j,k})^\Sigma=(T^\Sigma_{j,k})^{2N}$, $v(U_{j,k})^\Sigma=(U^\Sigma_{j,k})^{2MN}$, and $v(V_{j,k})^\Sigma=(V^\Sigma_{j,k})^{2MN}$.

Assume that $(j,k)\neq(0,0)$. Let $B_{j,k}$ denote any one of $T_{j,k}$, $U_{j,k}$, and $V_{j,k}$. If a is an integer relatively prime to p, then $B^\Sigma_{j,k}=B^\Sigma_{aj,ak}$, where the subscripts are read modulo p. This follows from the fact that $B^{\sigma_a}_{j,k}=B_{aj,ak}$, where $\sigma_a\colon\zeta\mapsto\zeta^a$ is the Frobenius automorphism at a.

It follows that $\mathcal{S}^\Sigma$ is generated by the following $3(p+2)$ elements:

$$\begin{array}{lll} T^\Sigma_{0,0} & T^\Sigma_{1,0} & T^\Sigma_{j,1} \\ U^\Sigma_{0,0} & U^\Sigma_{1,0} & U^\Sigma_{j,1} \\ V^\Sigma_{0,0} & V^\Sigma_{1,0} & V^\Sigma_{j,1} \end{array}$$

for $0\leqslant j\leqslant p-1$. Note that we have the following relations:

$$\prod_{j=0}^{p-1} T^{2\Sigma}_{j,1} = 1$$

$$T_{0,0} = T^{-\Sigma}_{0,1}$$

$$T_{0,0} = T^\Sigma_{1,0}$$

So we may discard the elements $T^{\Sigma}_{0,0}$, $T^{\Sigma}_{0,1}$, and $T^{\Sigma}_{1,0}$. The same remark holds, *mutatis mutandis*, for the $U^{\Sigma}_{j,k}$ and the $V^{\Sigma}_{j,k}$, since they are obtained from the $T^{\Sigma}_{j,k}$ by automorphisms. So $\mathcal{T}$ is generated over $\mathbf{Q}$ by $3(p-1)$ symbols, as claimed. □

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