COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 91, n° 2 (1994), p. 175-199
<http://www.numdam.org/item?id=CM_1994__91_2_175_0>

Compositio Mathematica **91**: 175–199, 1994.

Arithmetical investigations of a certain infinite product

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Received 6 October 1992; accepted in final form 5 April 1993

Introduction and main results

Let K denote an algebraic number field of degree d over $\mathbb{Q}$. For every place v of K we define $d_v := [K_v : \mathbb{Q}_v]$. If a finite place v of K lies over the prime p, we write $v \mid p$, and for an infinite place v of K we write $v \mid \infty$. We normalize the absolute value $|\,.\,|_v$ by

$$|p|_v = p^{-1} \quad \text{if } v \mid p \qquad \text{and} \qquad |\alpha|_v = |\alpha| \quad \text{if } v \mid \infty,$$

where $\alpha \in \mathbb{Q}$ and $|\,.\,|$ denotes the ordinary absolute value in $\mathbb{R}$ or in $\mathbb{C}$. Then, for any $\alpha \in K^{\times}$, we have the product formula

$$\prod_v |\alpha|_v^{d_v} = 1.$$

The absolute height $h(\alpha)$ of $\alpha \in K$ is defined by the formula

$$h(\alpha) := \prod_v \max(1, |\alpha|_v)^{d_v/d},$$

and the absolute height $h(\mathbf{a})$ of the vector $\mathbf{a} = {}^t(a_0, a_1) \in K^2$ by

$$h(\mathbf{a}) := \prod_v \max(1, |\mathbf{a}|_v)^{d_v/d} \quad \text{where } |\mathbf{a}|_v := \max(|a_0|_v, |a_1|_v).$$

For the whole paper we suppose that q is some fixed element from K satisfying $|q|_v > 1$ for some fixed valuation v of K, and furthermore $|q|_w \neq 1$ for all $w \mid \infty$. It is easily seen that the infinite product

$$\prod_{j=1}^{\infty} (1 + zq^{-j})$$

*This research was done while P. Bundschuh was visiting the University of Oulu where he enjoyed the kind hospitality of the Department of Mathematics.

converges in $\mathbb{C}_P$ where P is either ∞ (and then $\mathbb{C}_\infty := \mathbb{C}$) or a prime number p. We denote this infinite product by $E_q(z)$, the q-analogue of the exponential function (see [8]), and it is well known that its Taylor expansion about the origin is

$$\sum_{n=0}^{\infty} z^n \prod_{v=1}^{n} (q^v - 1)^{-1}.$$

The first arithmetical investigations of the function E_q, in the classical case $K = \mathbb{Q}$, $v \mid \infty$, date back at least to Lototsky [10] for qualitative questions, and to one of the present authors [5] for quantitative refinements.

The aim of this paper is to prove two further theorems concerning arithmetical properties of the function E_q, one being of qualitative nature, the other of a quantitative one. We will also give some interesting corollaries.

THEOREM 1. *Suppose v is a place of K and q satisfies the above conditions, and let λ denote the positive real number $(d \log h(q))/(d_v \log |q|_v)$. Suppose further $\alpha \in K^\times$ such that $\alpha \neq -q^j$ for all $j \in \mathbb{N} := \{1, 2, \ldots\}$. Then for each $k \in \mathbb{N}$, $k \geqslant 3$, the dimension of the vector space $KE_q(\alpha) + \cdots + KE_q^{(k-1)}(\alpha)$ over K is at least*

$$k(k-1)/\lambda(k + 6\pi^{-2}(k-1)). \tag{$*$}$$

In the special case $K = \mathbb{Q}$, $v \mid \infty$, $q \in \mathbb{Z} \backslash \{0, \pm 1\}$ (where λ is 1), and $\alpha = -1$ the lower estimate $(*)$ can be replaced by the slightly better bound

$$k(k+1)/(k + 1 + 6\pi^{-2}(k-1)) \tag{$**$}$$

for each $k \in \mathbb{N}$. Further, in this special case with $\alpha \in \mathbb{Q}^\times$, $\alpha \neq -q^j$ for all $j \in \mathbb{N}$, $(*)$ can be replaced by

$$k \cdot \max\{(k-1)/(k + 6\pi^{-2}(k-1)),\ (k+1)/(2k + 6\pi^{-2}(k-1))\}$$

for each $k \in \mathbb{N}$.

REMARK. If k is 1 or 2, $(*)$ gives no non-trivial information, since our hypothesis $\alpha \neq -q^j$ is equivalent with $E_q(\alpha) \neq 0$.

In the second part of Theorem 1 we have, for small values of k, a slightly better bound than $(*)$, e.g. the numbers $E_q(\alpha)$ and $E_q'(\alpha)$ are linearly independent over $\mathbb{Q}$.

This result can be stated equivalently in the following way, going back, even in the general setting adopted earlier, to the original infinite product definition of E_q. By logarithmic differentiation we find

$$L_q(z) := E_q'(z)/E_q(z) = \sum_{j=1}^{\infty} (q^j + z)^{-1}$$

such that L_q is a meromorphic function in $\mathbb{C}$.

Now, the linear independence of $E_q(\alpha)$ and $E_q'(\alpha)$ over $\mathbb{Q}$ is equivalent with the irrationality of $L_q(\alpha)$, and this is exactly Borwein's [4] nice result giving a positive answer to a question of Erdös [7].

From (*) we see that for each $k \geqslant 5$ more than 60% of the numbers $E_q(\alpha), \ldots, E_q^{(k-1)}(\alpha)$ are linearly independent, and in the special case $\alpha = -1$ this amount increases over 62%. In this last case we are even sure that $E_q(-1)$, $E_q'(-1)$, $E_q''(-1)$ are linearly independent over $\mathbb{Q}$.

All these results quoted so far suggest that $E_q(\alpha), \ldots, E_q^{(k-1)}(\alpha)$ should be linearly independent over $\mathbb{Q}$ for each $k \in \mathbb{N}$, and indeed this has been proved very recently by Bézivin [3], at least in the classical case, by a method which is completely different from ours, and which does not seem to allow quantitative refinements. We state now our second main result which is quantitative in nature.

THEOREM 2. *Let v, q, λ and α be as in Theorem 1, and suppose further $\lambda < 3/(2 + 3\pi^{-2})$. Then there exists an effectively computable $\gamma \in \mathbb{R}_+$, independent of* **a**, *such that for each $\mathbf{a} = {}^t(a_0, a_1) \in K^2$ with $h(\mathbf{a})$ sufficiently large we have the inequality*

$$|a_0 E_q(\alpha) + a_1 E_q'(\alpha)|_v \geqslant |\mathbf{a}|_v h(\mathbf{a})^{-3d/(3-\lambda(2+3\pi^{-2}))d_v - \gamma(\log h(\mathbf{a}))^{-1/2}\log\log h(\mathbf{a})}.$$

In the special case $\alpha = -1$ we may even allow $\lambda < (1/2 + 1/\pi^2)^{-1}$, and then we can say

$$|a_0 E_q(-1) + a_1 E_q'(-1)|_v \geqslant |\mathbf{a}|_v h(\mathbf{a})^{-d/(1-\lambda(2^{-1}+\pi^{-2}))d_v - \gamma^*(\log h(\mathbf{a}))^{-1/2}\log\log h(\mathbf{a})}$$

with some γ^* having the same properties as γ above.

Again we note some consequences of Theorem 2 in the special case $K = \mathbb{Q}$, $v \mid \infty$.

COROLLARY 1. *Suppose $q \in \mathbb{Z}\backslash\{0, \pm 1\}$, and $\alpha \in \mathbb{Q}^\times$ such that $\alpha \neq -q^j$ for all $j \in \mathbb{N}$. Then there exists an effectively computable $\gamma \in \mathbb{R}_+$, depending at most on q and on α, such that for each $\mathbf{a} \in \mathbb{Z}^2$ with $|\mathbf{a}| = \max(|a_0|, |a_1|)$ sufficiently large we have*

$$|a_0 E_q(\alpha) + a_1 E_q'(\alpha)| \geqslant |\mathbf{a}|^{-(2\pi^2+3)/(\pi^2-3) - \gamma(\log\log|\mathbf{a}|)(\log|\mathbf{a}|)^{-1/2}}.$$

In the case $\alpha = -1$ *we can replace* $(2\pi^2 + 3)/(\pi^2 - 3) = 3.3101\ldots$ *in the exponent of* $|\mathbf{a}|$ *by* $(\pi^2 + 2)/(\pi^2 - 2) = 1.5082\ldots.$

This means that, under the same hypotheses on q and α as in Corollary 1, the number $L_q(\alpha)$ has an irrationality measure less than 4.311, and for $L_q(-1)$ this is even less than 2.509, which is quite near to its best possible value 2. This should be compared with the upper bound $26/3 = 8.666\ldots$ for the irrationality measure of $L_q(\alpha)$ announced by Borwein [4].

As a further application of Theorem 2 we consider now the case $K = \mathbb{Q}(\sqrt{5})$, $v \mid \infty$, $q = -(3+\sqrt{5})/2$, $|q|_v = (3+\sqrt{5})/2$. Then

$$q^n - 1 = \sqrt{5}F_n((1-\sqrt{5})/2)^{-n},$$

where F_n denotes the nth Fibonacci number. Since, for all complex z with $|z| < |q|$,

$$zL_q(-z) = \sum_{n=1}^{\infty} \frac{z}{q^n - z} = \sum_{n=1}^{\infty} \frac{z^n}{q^n - 1},$$

it follows that

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = \sqrt{5} \sum_{n=1}^{\infty} \frac{(-(1+\sqrt{5})/2)^n}{q^n - 1} = -\frac{5+\sqrt{5}}{2} L_q\left(\frac{1+\sqrt{5}}{2}\right).$$

Further q satisfies $q^2 + 3q + 1 = 0$, and thus $|q|_w \leqslant 1$ for all places $w \neq v$ of K and $|q|_w < 1$ for the other archimedean place w of K. Therefore $\lambda = 1$ in this case and Theorem 2 implies the following

COROLLARY 2. *There exists an effectively computable absolute constant* $\gamma \in \mathbb{R}_+$ *such that for any* $\vartheta \in \mathbb{Q}(\sqrt{5})$ *with* $h(\vartheta)$ *sufficiently large we have*

$$\left|\sum_{n=1}^{\infty} \frac{1}{F_n} - \vartheta\right| > h(\vartheta)^{-6/(1-3\pi^{-2}) - \gamma(\log h(\vartheta))^{-1/2} \log\log h(\vartheta} > h(\vartheta)^{-8.621}.$$

REMARK. It should be mentioned that André-Jeannin [2] proved quite recently the irrationality of $\Sigma\, 1/F_n$. Moreover we note that our theorems are applicable to the function

$$E_q(\sqrt{5}z) = \sum_{n=0}^{\infty} \left(\frac{1-\sqrt{5}}{2}\right)^{n(n+1)/2} \frac{z^n}{F_1 \cdots F_n}$$

at any non-zero point $\alpha \in \mathbb{Q}(\sqrt{5})$ satisfying $\sqrt{5}\alpha \neq -\left(-\frac{3+\sqrt{5}}{2}\right)^m$ for all $m \in \mathbb{N}$.

As in Theorem 2 we can estimate $|a_0 E_q(-1) + a_1 E_q'(-1) + a_2 E_q''(-1)|_v$ from below in terms of $\mathbf{a} \in K^3$. We give the result in the special case $K = \mathbb{Q}$, $v \mid \infty$, $q \in \mathbb{Z}\backslash\{0, \pm 1\}$ where we have

$$|a_0 E_q(-1) + a_1 E_q'(-1) + a_2 E_q''(-1)| \geqslant |\mathbf{a}|^{-6.652}$$

for all $\mathbf{a} \in \mathbb{Z}^3$ with $|\mathbf{a}| := \max(|a_0|, |a_1|, |a_2|)$ sufficiently large.

Finally we should say something why our results become substantially better in the special case $\alpha = -1$ than in the general one. This is intimately connected with the fact, appearing in Section 2 below, that for $\alpha = -1$ we can find much smaller "denominators" $\Omega^*(k, n)$ than the general Ω's are, compare formulae (12) and (12*). It should be pointed out, that also in the case $\alpha = 1$ we could do it better than in the general one, but worse than for $\alpha = -1$. Therefore we omit here the details.

In the following sections we shall always write f instead of E_q, for the sake of shortness.

1. Analytical construction of small linear forms

If the place v of K lies over P, we shall in the following consider all elements of K as elements of $\mathbb{C}_P$ given by a corresponding embedding of K into $\mathbb{C}_P$.

Let $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ be fixed. If $v \mid \infty$, we define the complex integral $I_v(k, n)$ by

$$I_v(k, n) := \frac{1}{2\pi i} \int_{|z|_v = R} f(z) \prod_{\nu=0}^{n} ((z - \alpha q^\nu)^k (z + q^{\nu+1}))^{-1}\, dz, \tag{1_∞}$$

where R has to be larger than $|q|_v^n \max(|\alpha|_v, |q|_v)$. Of course, this integral depends on α and q too, but we do not express this for the moment.

For $v \mid p$, instead of (1_∞), we use the Schnirelman integral

$$I_v(k, n) := \int_{0,R} z f(z) \prod_{\nu=0}^{n} ((z - \alpha q^\nu)^k (z + q^{\nu+1}))^{-1}\, dz, \tag{1_p}$$

where, again, R has to satisfy the above condition. Then we can state the following:

LEMMA 1. *Let $k \in \mathbb{N}$ be fixed. Then for both types of valuations v of K we have the asymptotic relation*

$$\log |I_v(k, n)|_v = -\tfrac{1}{2}(k+1)^2 n^2 \log |q|_v + O(n)$$

as $n \to \infty$.

REMARKS. (1) Here and in all that follows we keep the convention that constants implied in Landau's O as well as constants $c_1, c_2, \dots$ are independent of n. (2) In the next section it will come out that the $I_v(k, n)$ from (1) are indeed linear forms in $f(\alpha), \dots, f^{(k-1)}(\alpha)$ with coefficients from $\mathbb{Q}(\alpha, q)$ which we have to investigate very carefully.

Proof. With $N := (k+1)(n+1)$ we have

$$\prod_{\nu=0}^{n} (z - \alpha q^\nu)^k (z + q^{\nu+1}) = z^N \prod_{\nu=0}^{n} \left(1 - \frac{\alpha q^\nu}{z}\right)^k \left(1 + \frac{q^{\nu+1}}{z}\right),$$

and we write the product of the right-hand side as $1 + w_n(z)$. Then, choosing for both types of valuations $R := |q|_v^N$, we find

$$|w_n(z)|_v = O(|q|_v^{-kn}) \quad \text{on } |z|_v = R \tag{2}$$

for every $n > c_1$. Now, using Popov's trick from [12] or [13], we get *in the case* $v \mid \infty$

$$\begin{aligned} I_v(k, n) &= \frac{1}{2\pi i} \int_{|z|_v = R} z^{-N} f(z)\, dz - \frac{1}{2\pi i} \int_{|z|_v = R} \frac{f(z) w_n(z)\, dz}{z^N (1 + w_n(z))} \\ &=: I_v(k, n, 1) - I_v(k, n, 2), \end{aligned} \tag{3}$$

say.

Defining

$$c_2 := \prod_{\nu=1}^{\infty} (1 - q^{-\nu})^{-1}$$

we clearly find for the first integral

$$\begin{aligned} I_v(k, n, 1) &= \prod_{\nu=1}^{N-1} (q^\nu - 1)^{-1} = c_2 q^{-N(N-1)/2} \prod_{\nu=N}^{\infty} (1 - q^{-\nu}) \\ &= c_2 q^{-N(N-1)/2} (1 + O(|q|_v^{-N})). \end{aligned} \tag{4}$$

Furthermore we have on $|z|_v = R$

$$|f(z)|_v \leqslant \prod_{j=1}^{\infty}\left(1+\frac{R}{|q|_v^j}\right) \leqslant c_3 \prod_{j=1}^{N-1}\left(1+\frac{R}{|q|_v^j}\right) \leqslant c_4 \frac{R^{N-1}}{|q|_v^{N(N-1)/2}} = c_4 |q|_v^{N(N-1)/2}$$

and therefore, by (2),

$$|I_v(k, n, 2)|_v = O(|q|_v^{-N(N-1)/2-kn}). \tag{5}$$

This, combined with (4), shows $I_v(k, n, 2) = o(I_v(k, n, 1))$ for $v \mid \infty$, and thus the assertion of Lemma 1 follows in the archimedean case.

Suppose now $v \mid p$. Then we have again a representation (3) for $I_v(k, n)$ with

$$I_v(k, n, 1) := \int_{0,R} z^{1-N} f(z)\, dz = \prod_{\nu=1}^{N-1} (q^\nu - 1)^{-1},$$

and thus $|I_v(k, n, 1)|_v = |q|_v^{-N(N-1)/2}$, see e.g. [1]. Since we have the following estimate on $|z|_v = R$

$$|f(z)|_v = \prod_{j=1}^{\infty} \left|1+\frac{z}{q^j}\right|_v \leqslant \prod_{j=1}^{N-1} \frac{R}{|q|_v^j} = \frac{R^{N-1}}{|q|_v^{N(N-1)/2}} = |q|_v^{N(N-1)/2}, \tag{6}$$

we get immediately from

$$I_v(k, n, 2) := \int_{0,R} \frac{f(z) w_n(z)\, dz}{z^{N-1}(1+w_n(z))}$$

the inequality

$$|I_v(k, n, 2)|_v \leqslant \sup_{|z|_v = R} \left|\frac{f(z) w_n(z)}{z^{N-1}(1+w_n(z))}\right|_v$$

which leads again to (5) taking (2) and (6) into consideration. Therefore we find $|I_v(k, n)|_v = |q|_v^{-N(N-1)/2}$ for all large n, such that Lemma 1 is proved in the non-archimedean case too.

2. Arithmetical properties of the constructed linear forms

First of all, let us mention two well known facts which will be useful later in this section. For each $n \in \mathbb{N}_0$ we have

$$B_{n,v}(q) := \prod_{\mu=1}^{n} (q^{\mu} - 1) \Big/ \left(\prod_{\mu=1}^{v} (q^{\mu} - 1) \prod_{\mu=1}^{n-v} (q^{\mu} - 1) \right) \in \mathbb{Z}[q] \quad (v = 0, \ldots, n), \tag{7}$$

where empty products, as usual, are defined to be 1. Furthermore

$$R_n(q) := \prod_{v=1}^{n} \prod_{d|v} (q^{v/d} - 1)^{\mu(d)},$$

where $\mu(d)$ denotes the Möbius function, satisfies

$$R_n(q)/(q^v - 1) \in \mathbb{Z}[q] \quad (v = 1, \ldots, n), \tag{8}$$

and therefore a fortiori $R_n(q) \in \mathbb{Z}[q]$. The degree of R_n with respect to q, shortly $\deg_q R_n$, is asymptotically

$$\deg_q R_n = \sum_{v=1}^{n} v \sum_{d|v} \frac{\mu(d)}{d} = \frac{3}{\pi^2} n^2 + O(n \log n) \tag{9}$$

as $n \to \infty$. For all this we refer to [9], where only the part $\deg_q R_n \leqslant \cdots$ of (9) is shown, but it is not hard to prove the opposite inequality too.

Secondly, in the proof of the subsequent Lemma 2 we need two facts on our function f. From the definition of f as an infinite product we deduce

$$f(z) q^{k(k+1)/2} = f\left(\frac{z}{q^k}\right) \prod_{v=0}^{k-1} (z + q^{v+1}) \tag{10}$$

for each $k \in \mathbb{N}_0$. Differentiating an appropriate version of this formula σ times we get

$$\frac{1}{\sigma!} f^{(\sigma)}\left(\frac{z}{q^k}\right) = q^{k(k+1)/2 + k\sigma} \cdot \sum_{\tau_0 + \cdots + \tau_k = \sigma} \frac{(-1)^{\sigma - \tau_0}}{\tau_0!} f^{(\tau_0)}(z) \prod_{v=1}^{k} (z + q^v)^{-1-\tau_v}. \tag{11}$$

Now, defining

$$\Omega(k, n) := (k-1)!\alpha^{nk+k-1}q^{(k+1)n(n+1)/2}R_n(q)^{k-1}\cdot\prod_{\nu=1}^{n}(q^\nu-1)^k\cdot\prod_{\nu=1}^{n+1}(q^\nu+\alpha)^k, \tag{12}$$

we can state:

LEMMA 2. *For* $I_v(k, n)$ *defined by* (1) *we have*

$$\Omega(k, n)I_v(k, n) = \sum_{\tau=0}^{k-1} P_\tau(\alpha, q)(-\alpha)^\tau f^{(\tau)}(\alpha) \tag{13}$$

where

$$P_\tau \in \mathbb{Z}[\alpha, q],\ \deg_\alpha P_\tau \leqslant kn+k-1$$

and

$$\deg_q P_\tau \leqslant \left(k+\frac{3}{\pi^2}(k-1)\right)n^2 + O(n\log n),$$

all three properties being true for $\tau = 0, \ldots, k-1$.

Proof. Using (10), and applying then the residue theorem or its analogue in $\mathbb{C}_p$ to the integral (1) we are immediately led to

$$\begin{aligned}
&q^{(n+1)(n+2)/2}I_v(k, n)\\
&= \sum_{\nu=0}^{n}\frac{1}{(k-1)!}\left(\frac{d}{dz}\right)^{k-1}\left\{f\left(\frac{z}{q^{n+1}}\right)\prod_{\substack{\mu=0\\ \mu\neq\nu}}^{n}(z-\alpha q^\mu)^{-k}\right\}\Bigg|_{z=\alpha q^\nu}\\
&= \frac{(-1)^{k-1}}{\alpha^{kn+k-1}}\sum_{\nu=0}^{n}\sum_{\sigma_0+\cdots+\sigma_n=k-1}\frac{(-\alpha)^{\sigma_\nu}}{q^{(n+1)\sigma_\nu}\sigma_\nu!}f^{(\sigma_\nu)}(\alpha q^{\nu-1-n})\\
&\quad\cdot\prod_{\substack{\mu=0\\ \mu\neq\nu}}^{n}\binom{k+\sigma_\mu-1}{k-1}(q^\nu-q^\mu)^{-k-\sigma_\mu}.
\end{aligned}$$

From this we get in virtue of (11)

$$I_v(k,n) = \frac{(-1)^{k-1}}{\alpha^{kn+k-1}} \sum_{v=0}^{n} \sum_{\sigma_0+\cdots+\sigma_n=k-1} \frac{\alpha^{\sigma_v}}{q^{v(n+3/2-v/2+\sigma_v)}}$$
$$\cdot \prod_{\substack{\mu=0\\ \mu\neq v}}^{n} \binom{k+\sigma_\mu-1}{k-1} (q^v - q^\mu)^{-k-\sigma_\mu}$$
$$\cdot \sum_{\tau_0+\cdots+\tau_{n+1-v}=\sigma_v} \frac{(-1)^{\tau_0}}{\tau_0!} f^{(\tau_0)}(\alpha) \prod_{\lambda=1}^{n+1-v} (\alpha+q^\lambda)^{-1-\tau_\lambda}.$$

This quite long formula shows that $I_v(k, n)$ is a linear form in $f(\alpha), \ldots, f^{(k-1)}(\alpha)$ as announced in Remark (2) after Lemma 1. It may be written in the following more convenient way

$$I_v(k,n) = \frac{(-1)^{k-1}}{\alpha^{kn+k-1}} \sum_{\tau=0}^{k-1} \frac{(-\alpha)^\tau}{\tau!} f^{(\tau)}(\alpha) \sum_{v=0}^{n} \sum_{\sigma=\tau}^{k-1} \sum_{\Sigma(v)=k-1-\sigma} \frac{\alpha^{\sigma-\tau}}{q^{v(n+3/2-v/2+\sigma)}}$$
$$\cdot \prod_{\substack{\mu=0\\ \mu\neq v}}^{n} \binom{k+\sigma_\mu-1}{k-1} (q^v - q^\mu)^{-k-\sigma_\mu} \sum_{\tau_1+\cdots+\tau_{n+1-v}=\sigma-\tau}$$
$$\cdot \prod_{\lambda=1}^{n+1-v} (\alpha+q^\lambda)^{-1-\tau_\lambda} \tag{14}$$

where $\Sigma(v)$ denotes for a moment the sum $\sigma_0 + \cdots + \hat{\sigma}_v + \cdots + \sigma_n$. In the first product occurring on the right-hand side of (14) we can easily check

$$\prod_{\substack{\mu=0\\ \mu\neq v}}^{n} (q^v - q^\mu)^{k+\sigma_\mu} = q^{kv(v-1)/2+kv(n-v)+\Sigma_{\mu<v}\mu\sigma_\mu+v\Sigma_{\mu>v}\sigma_\mu}$$
$$\cdot \prod_{\mu<v} (q^{v-\mu}-1)^{k+\sigma_\mu} \prod_{\mu>v} (1-q^{\mu-v})^{k+\sigma_\mu}. \tag{15}$$

Since we have $\Sigma(v) \leqslant k-1$ for $v = 0, \ldots, n$, this formula, combined with (7) and (8), makes clear that the factor $R_n(q)^{k-1} \Pi_{v=1}^{n} (q^v - 1)^k$ in definition (12) of $\Omega(k, n)$ is needed to take care for the two products on the right-hand side of (15). In virtue of $\Sigma_{\mu<v} \mu\sigma_\mu + v\Sigma_{\mu>v}\sigma_\mu \leqslant v\Sigma(v) = v(k-1-\sigma)$ for fixed v, σ, we see that the factor q occurs in the denominator of the right-hand side of (14) to a power not larger than

$$v\left(n+\frac{3}{2}-\frac{v}{2}+\sigma\right)+\frac{k}{2}v(v-1)+kv(n-v)+v(k-1-\sigma)$$
$$\leqslant \frac{1}{2}(k+1)n(n+1),$$

and this explains the "pure" power of q in (12).

Our last considerations make evident all assertions of Lemma 2 except for the estimate of the degree of the P_τ's with respect to q which we shall now perform. To do this we point out that $\Omega(k, n)$ is a polynomial in q of exact degree

$$\frac{1}{2}(3k+1)n^2 + \frac{1}{2}(5k+1)n + k + (k-1)\deg_q R_n \tag{16}$$

whereas, for fixed τ, ν, σ, every term on the right-hand side of (14) contains the polynomial

$$q^{\nu(n+3/2-\nu/2+\sigma)} \prod_{\substack{\mu=0\\ \mu\neq\nu}}^{n} (q^\nu - q^\mu)^{k+\sigma_\mu} \cdot \prod_{\lambda=1}^{n+1-\nu} (\alpha + q^\lambda)^{1+\tau_\lambda}$$

in q in the denominator. It is easily checked that the degree of every such polynomial is at least $\frac{1}{2}(k+1)n(n+1) + n$. This, combined with (16) and (9), finally yields the upper bound for $\deg_q P_\tau$ in Lemma 2.

REMARK. In the case $\alpha = -1$, instead of (12), we put

$$\Omega^*(k, n) := (k-1)!\, q^{(k+1)n(n+1)/2} R_{n+1}(q)^{k-1} \prod_{\nu=1}^{n} (q^\nu - 1)^k \cdot \prod_{\nu=1}^{n+1} (q^\nu - 1), \tag{12*}$$

and we can show, similarly to the preceding general case,

$$\Omega^*(k, n) I_v(k, n) = \sum_{\tau=0}^{k-1} P_\tau^*(q) f^{(\tau)}(-1) \tag{13*}$$

with $P_\tau^* \in \mathbb{Z}[q]$ and

$$\deg_q P_\tau^* \leqslant \left(\frac{1}{2}(k+1) + \frac{3}{\pi^2}(k-1)\right) n^2 + O(n \log n) \quad \text{for } \tau = 0, \ldots, k-1.$$

From the definition of the polynomial R_n we see directly

$$|R_n(q)|_v = |q|_v^{\deg_q R_n} \quad \text{for } v \mid p.$$

Since we have

$$|q|_v^{-\nu\sum_{d|\nu}\mu(d)/d} \prod_{d|\nu} |q^{\nu/d} - 1|_v^{\mu(d)} = \prod_{d|\nu} |1 - q^{-d}|_v^{\mu(\nu/d)} = \prod_{\substack{d|\nu\\ \mu(\nu/d)=1}} |\cdots| \Big/ \prod_{\substack{d|\nu\\ \mu(\nu/d)=-1}} |\cdots|$$

$$\leqslant \prod_{d=1}^{\infty} (1 + |q|_v^{-d}) \Big/ \prod_{d=1}^{\infty} (1 - |q|_v^{-d}) =: c_5$$

and analogously the lower bound c_5^{-1} for the expression under consideration, the definition of $R_n(q)$ leads to

$$|R_n(q)|_v = |q|_v^{\deg_q R_n + O(n)} \quad \text{for } v \mid \infty.$$

Thus, using (9), we find for both types of valuations of K

$$\log |R_n(q)|_v = \frac{3}{\pi^2} n^2 \log |q|_v + O(n \log n)$$

as $n \to \infty$. Taking (12) into account we come to

$$\log |\Omega(k, n)|_v = \left(\frac{3k+1}{2} + \frac{3}{\pi^2}(k-1)\right) n^2 \log |q|_v + O(n \log n). \tag{17}$$

REMARK. In the special case $\alpha = -1$ we get from (12*)

$$\log |\Omega^*(k, n)|_v = \left(k + 1 + \frac{3}{\pi^2}(k-1)\right) n^2 \log |q|_v + O(n \log n). \tag{17*}$$

We now estimate $|P_\tau(\alpha, q)|_w$ in the case $w \mid \infty$. From (7), (12), (13), (14) and (15) we obtain

$$\begin{aligned} P_\tau(\alpha, q) = {} & \frac{(-1)^{k-1}(k-1)!}{\tau!} \sum_{\nu=0}^{n} \sum_{\sigma=\tau}^{k-1} \sum_{\Sigma(\nu)=k-1-\sigma} \alpha^{\sigma-\tau} q^{A(k,n,\nu,\sigma_i)} \\ & \cdot \prod_{\substack{\mu=0 \\ \mu \neq \nu}}^{n} \binom{k+\sigma_\mu-1}{k-1} \cdot B_{n,\nu}(q)^k R_n(q)^{k-1} \left\{ \prod_{\mu<\nu} (q^{\nu-\mu}-1)^{\sigma_\mu} \right. \\ & \cdot \left. \prod_{\mu>\nu} (1-q^{\mu-\nu})^{\sigma_\mu} \right\}^{-1} \cdot \sum_{\tau_1+\cdots+\tau_{n+1-\nu}=\sigma-\tau} \prod_{\mu=1}^{n+1} (q^\mu+\alpha)^k \\ & \cdot \prod_{\lambda=1}^{n+1-\nu} (q^\lambda+\alpha)^{-1-\tau_\lambda} \end{aligned}$$

where

$$\begin{aligned} A(k, n, \nu, \sigma_i) := {} & (k+1)n(n+1)/2 - \nu\left(n + \frac{3}{2} - \frac{\nu}{2} + \sigma\right) - k\nu(\nu-1)/2 \\ & - k\nu(n-\nu) - \sum_{\mu<\nu} \mu\sigma_\mu - \nu \sum_{\mu>\nu} \sigma_\mu \geqslant 0. \end{aligned}$$

Here we have, by the above considerations on $R_n(q)$ and our assumption

$|q|_w \neq 1$ for all $w \mid \infty$,

$$\left| R_n(q)^{k-1} \left\{ \prod_{\mu<\nu} (q^{\nu-\mu} - 1)^{\sigma_\mu} \prod_{\mu>\nu} (1 - q^{\mu-\nu})^{\sigma_\mu} \right\}^{-1} \right|_w \leqslant c_6 |R_n(q)^{k-1}|_w$$
$$\leqslant c_7^n \max\{1, |q|_w\}^{(k-1)\deg_q R_n}.$$

Further, $|B_{n,\nu}(q)|_w \leqslant 2^n \max\{1, |q|_w\}^{\deg_q B_{n,\nu}}$, see [13]. Since the number of $(x_1, \ldots, x_n) \in \mathbb{N}_0^n$ satisfying $x_1 + \cdots + x_n = k$ is

$$\binom{n+k-1}{k}$$

and

$$A(k, n, \nu, \sigma_i) + k \deg_q B_{n,\nu} + k(n+1)(n+2)/2 - (n+1-\nu)(n+2-\nu)/2$$
$$\leqslant kn^2 + O(n),$$

it follows that

$$|P_\tau(\alpha, q)|_w \leqslant c_8^n \max\{1, |\alpha|_w\}^{O(n)} \max\{1, |q|_w\}^{kn^2 + (k-1)\deg_q R_n + O(n)}.$$

Combining now this estimate, Lemma 1 and Lemma 2 we may state the following:

LEMMA 3. *With the polynomials $P_0, \ldots, P_{k-1}$ from Lemma 2 we have the asymptotic relation*

$$\log \left| \sum_{\tau=0}^{k-1} P_\tau(\alpha, q)(-\alpha)^\tau f^{(\tau)}(\alpha) \right|_v$$
$$= -(k-1)\left(\frac{k}{2} - \frac{3}{\pi^2}\right) n^2 \log |q|_v + O(n \log n) \tag{18}$$

and the inequalities

$$\log |P_\tau(\alpha, q)|_v \leqslant \left(k + \frac{3}{\pi^2}(k-1)\right) n^2 \log |q|_v + O(n \log n) \tag{19}$$

for $\tau = 0, \ldots, k-1$. Moreover, for any place w of K, we have

$$\log |P_\tau(\alpha, q)|_w \leqslant \left(\left(k + \frac{3}{\pi^2}(k-1)\right) n^2 + O(n \log n)\right) \log_+ |q|_w$$
$$+ O(n) \log_+ |\alpha|_w + \delta(w) O(n \log n) \tag{19'}$$

for $\tau = 0, \ldots, k-1$, *where* $\delta(w)$ *is* 1 *for* $w \mid \infty$, *and* 0 *for* $w \mid p$, *and* $\log_+ x := \max(0, \log x)$ *for* $x \in \mathbb{R}_+$.

REMARK. In the special case $\alpha = -1$ we find

$$\log \left| \sum_{\tau=0}^{k-1} P_\tau^*(q) f^{(\tau)}(-1) \right|_v = -(k-1)\left(\frac{k+1}{2} - \frac{3}{\pi^2}\right) n^2 \log |q|_v + O(n \log n) \tag{18*}$$

and the inequalities

$$\log |P_\tau^*(q)|_v \leqslant \left(\frac{k+1}{2} + \frac{3}{\pi^2}(k-1)\right) n^2 \log |q|_v + O(n \log n) \tag{19*}$$

for the P_τ^* from (13*). Again, for any place w of K, we have

$$\log |P_\tau^*(q)|_w \leqslant \left(\left(\frac{k+1}{2} + \frac{3}{\pi^2}(k-1)\right) n^2 + O(n \log n)\right) \log_+ |q|_w + O(n) \log_+ |\alpha|_w + \delta(w) O(n \log n) \tag{19*$'$}$$

for $\tau = 0, \ldots, k-1$.

In the case $K = \mathbb{Q}$, $v \mid \infty$, $q \in \mathbb{Z} \backslash \{0, \pm 1\}$ we can easily get our Theorem 1 using the following Lemma 4 which is essentially due to Nesterenko [11], see also [6].

LEMMA 4. *Suppose* $\mathbf{w} \in \mathbb{R}^k \backslash \{\mathbf{0}\}$. *If there exist* $n_0 \in \mathbb{N}$, $\tau \in \mathbb{R}_+$, *an unbounded, monotonically increasing function* $F: \mathbb{N} \to \mathbb{R}_+$ *with* $\limsup_{n \to \infty} F(n+1)/F(n) \leq 1$, *and a sequence* $(L_n)_{n \geqslant n_0}$ *of integral linear forms satisfying*

$$\log |L_n(\mathbf{w})| + \tau F(n) = o(F(n)) \quad \textit{and} \quad \log \|L_n\| \leqslant F(n), \tag{20}$$

then $\dim_{\mathbb{Q}}(\mathbb{Q} w_1 + \cdots + \mathbb{Q} w_k) \geqslant 1 + \tau$, *where* $\mathbf{w} = {}^t(w_1, \ldots, w_k)$.

If $L_n(\mathbf{x}) = a_{n1} x_1 + \cdots + a_{nk} x_k$, then $\|L_n\|$ denotes the Euclid-norm of the vector ${}^t(a_{n1}, \ldots, a_{nk})$.

We put

$$\alpha = \frac{r}{s}, \; w_\tau := (-\alpha)^{\tau-1} f^{(\tau-1)}(\alpha) \quad \text{for } \tau = 1, \ldots, k,$$

and take

$$L_n(\mathbf{x}) := \sum_{\tau=0}^{k-1} s^{kn+k-1} P_\tau(\alpha, q) x_{\tau+1},$$

compare Lemma 2 (or $L_n^*(\mathbf{x}) := \Sigma_{\tau=0}^{k-1} P_\tau^*(q) x_{r+1}$ in the special case $\alpha = -1$). Then the hypotheses of Lemma 4 are satisfied, especially (20) with

$$\tau := (k-1)\left(\frac{k}{2} - \frac{3}{\pi^2}\right) \Big/ \left(k + \frac{3}{\pi^2}(k-1)\right)$$

in the general case of α, or with

$$\tau := (k-1)\left(\frac{k+1}{2} - \frac{3}{\pi^2}\right) \Big/ \left(\frac{k+1}{2} + \frac{3}{\pi^2}(k-1)\right) \quad \text{for } \alpha = -1.$$

Thus we get Theorem 1 in the special case indicated before Lemma 4.

3. Proof of Theorem 2

To prepare this we note that the case $k = 2$ of (18), (19), or (18*), (19*) means that for each $n \in \mathbb{N}$ we are given a linear form

$$J(n) := P_0(\alpha, q) f(\alpha) - \alpha P_1(\alpha, q) f'(\alpha) \tag{21}$$

satisfying

$$\log |J(n)|_v = -Bn^2 \log |q|_v + O(n \log n) \tag{22}$$

and, for $\tau = 0, 1$

$$\log |P_\tau(\alpha, q)|_v \leqslant An^2 \log |q|_v + O(n \log n), \tag{23}$$

where the definitions of A and B are obvious from (18), (19) or from (18*), (19*). Further, for any place w of K, we have

$$\log |P_\tau(\alpha, q)|_w \leqslant (An^2 + O(n \log n)) \log_+ |q|_w + O(n) \log_+ |\alpha|_w + \delta(w) O(n \log n). \qquad (23)'$$

Suppose $\mathbf{a} \in K^2$ with $h(\mathbf{a})$ large enough, and define

$$L := a_0 f(\alpha) + a_1 f'(\alpha). \tag{24}$$

We have to show the lower bound for $|L|_v$ which we asserted in Theorem 2. To do this we define

$$\Delta := a_1 P_0 + a_0 \alpha P_1, \tag{25}$$

and find the following equation, using (21) and (24),

$$\alpha P_1 L = \Delta f(\alpha) - a_1 J(n). \tag{26}$$

We note that $f(\alpha) \neq 0$ by our hypotheses on α, and discuss now the two cases $\Delta \neq 0$ and $\Delta = 0$ separately.

Suppose first $\Delta \neq 0$. Then we assert

$$|a_1 J(n)|_v < 2^{-\delta(v)} |\Delta f(\alpha)|_v. \tag{27}$$

Assume, to the contrary of (27), that we have

$$|a_1 J(n)|_v \geqslant 2^{-\delta(v)} |\Delta f(\alpha)|_v. \tag{28}$$

Since $\Delta \in K^\times$, by (25), we may apply to it the product formula, and taking logarithms in (28) leads us via (22) to (observe $a_1 \neq 0$, by (28))

$$\begin{aligned} &\frac{d_v}{d}(\log |a_1|_v - Bn^2 \log |q|_v) + O(n \log n) \\ &\qquad \geqslant \frac{d_v}{d} \log |\Delta|_v = -\sum_{w \neq v} \frac{d_w}{d} \log |\Delta|_w. \end{aligned} \tag{29}$$

By the definition (25) of Δ we have

$$|\Delta|_w \leqslant 2^{\delta(w)} \max(1, |\alpha|_w) \cdot \max(|a_0|_w, |a_1|_w) \cdot \max(|P_0|_w, |P_1|_w)$$

for any valuation w of K, and therefore, using now (23)′ we find

$$\begin{aligned} \log |\Delta|_w \leqslant \log |\mathbf{a}|_w &+ (An^2 + O(n \log n)) \log_+ |q|_w \\ &+ O(n) \log_+ |\alpha|_w + \delta(w) O(n \log n). \end{aligned} \tag{30}$$

Applying this estimate on the right-hand side of (29), and using the definition of $h(\mathbf{a})$, we find

$$\log h(\mathbf{a}) \geqslant (A + B) \frac{d_v}{d} n^2 \log |q|_v - An^2 \log h(q) + O(n \log n)$$

$$\geq \left(A + B - A \frac{d \log h(q)}{d_v \log |q|_v}\right) \frac{d_v}{d} n^2 \log |q|_v + O(n \log n) \tag{31}$$

because of $\log h(q) = \Sigma_w (d_w/d) \log_+ |q|_w$, compare the definition of the absolute height. Here the factor of A in the parentheses is our λ, and the upper bound for λ from Theorem 2 is easily seen to be equivalent with the condition $A + B > \lambda A$. With an appropriate $\gamma_1 \in \mathbb{R}_+$, independent of $\mathbf{a}$ and n, we may write (31) under the form

$$\log h(\mathbf{a}) \geqslant (A + B - \lambda A) \frac{d_v}{d} n^2 \log |q|_v - \gamma_1 n \log n.$$

We suppose from now on, that $\mathbf{a}$ satisfies the inequality

$$\log h(\mathbf{a}) \geqslant (A + B - \lambda A) \frac{d_v}{d} \log |q|_v.$$

Looking for given such vectors $\mathbf{a}$ at the inequality

$$\log h(\mathbf{a}) < (A + B - \lambda A) \frac{d_v}{d} n^2 \log |q|_v - \gamma_1 n \log n, \tag{32}$$

it is clear that it will be satisfied for all $n \in \mathbb{N}\backslash\{1\}$ from some point on. Now we *define* $n := n(\mathbf{a})$ as the smallest positive integer such that for this and for all larger integers inequality (32) is satisfied. For this n inequality (28) cannot hold, and therefore (27) must be true. Of course, we have to keep in mind, that the n in (27) is our $n(\mathbf{a})$ we defined right now.

Combining (26) and (27) we find

$$|\alpha P_1 L|_v \geqslant 2^{-\delta(v)} |\Delta f(\alpha)|_v,$$

and taking logarithms we find via (23) and (30) after a short calculation

$$\log |L|_v \geqslant \log |\Delta|_v - \log |P_1|_v - \gamma_2 = -\frac{d}{d_v} \sum_{w \neq v} \frac{d_w}{d} \log |\Delta|_w$$

$$-\log |P_1|_v - \gamma_2 \geqslant \log_+ |\mathbf{a}|_v - \frac{d}{d_v} (\log h(\mathbf{a}) + An^2 \log h(q)) + O(n \log n). \tag{33}$$

By the definition of our $n \geqslant 2$ in (32) we have

$$(A + B - \lambda A)\,\frac{d_v}{d}\,(n-1)^2 \log |q|_v \leqslant \log h(\mathbf{a}) + \gamma_1(n-1)\log(n-1),$$

and therefore

$$(A + B - \lambda A)\,\frac{d_v}{d}\,n^2 \log |q|_v \leqslant \log h(\mathbf{a}) + \gamma_3 n \log n \tag{34}$$

with a new $\gamma_3 > \gamma_1$. If $h(\mathbf{a})$ is large enough, then this holds for n too, by (32), and our last inequality implies $n \leqslant \gamma_4(\log h(\mathbf{a}))^{1/2}$ such that we find from (33)

$$\log |L|_v \geqslant \log|\mathbf{a}|_v - \frac{d}{d_v}\left(1 + \frac{\lambda A}{A+B-\lambda A}\right)\log h(\mathbf{a}) - \gamma_5(\log h(\mathbf{a}))^{1/2} \log\log h(\mathbf{a}) \tag{35}$$

which implies both lower bounds in Theorem 2, in the general case of α and for $\alpha = -1$ as well, taking the definitions of A and B into account.

We come now to the case $\Delta = 0$, in which (26) reduces to $\alpha P_1 L = a_1 J(n)$. If also $a_1 = 0$, then $P_1 = 0$ by (25). Therefore $J(n) = P_0 f(\alpha) - \alpha P_1 f'(\alpha) = P_0 f(\alpha)$, which implies $|J(n)|_v = |P_0 f(\alpha)|_v$. Using the inequalities (22), (23) and (23)$'$ we obtain $P_0 \neq 0$,

$$\begin{aligned}
-B\,\frac{d_v}{d}\,n^2 \log |q|_v + O(n \log n) &\geq \frac{d_v}{d}\log |P_0|_v = -\sum_{w \neq v} \frac{d_w}{d}\log |P_0|_w \\
&\geqslant -\sum_{w \neq v} \frac{d_w}{d} A n^2 \log_+ |q|_w + O(n \log n) \\
&= -An^2 \log h(q) + \frac{d_v}{d} A n^2 \log |q|_v + O(n \log n).
\end{aligned}$$

This implies the inequality

$$O(n \log n) \geqslant (A + B - \lambda A)\,\frac{d_v}{d}\,n^2 \log |q|_v,$$

which is impossible if n is large enough (we suppose $h(\mathbf{a})$ sufficiently large). This means that $a_1 \neq 0$ if $\Delta = 0$.

In the case $\Delta = 0$ we thus have, again by (22) and (23),

$$\begin{aligned}\log |L|_v &= \log |a_1|_v + \log |J(n)|_v - \log |P_1|_v - \gamma_6 \\ &\geqslant \log |a_1|_v - (A + B)n^2 \log |q|_v - \gamma_7 n \log n \\ &\geqslant \log |a_1|_v - \frac{A+B}{A+B-\lambda A}\frac{d}{d_v} \log h(\mathbf{a}) - \gamma_8(\log h(\mathbf{a}))^{1/2} \log\log h(\mathbf{a}),\end{aligned}$$

if we use (34) and $n \leqslant \gamma_4(\log h(\mathbf{a}))^{1/2}$. Thus we have the lower bounds of Theorem 2, but $|\mathbf{a}|_v$ is replaced by $|a_1|_v$. In particular, this means that $f'(\alpha) \neq 0$. Using this fact we may suppose without loss of generality that $|\mathbf{a}|_v = |a_1|_v$. (If necessary, we change the roles of $f(\alpha)$ and $f'(\alpha)$ in the above proof.) The proof of Theorem 2 is now completed.

4. Construction of more linear forms. Proof of Theorem 1

In this section, for fixed $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we consider k linear forms of type (1) instead of only one. We write things down only in the archimedean case, the necessary modifications for the non-archimedean being obvious. For every $x \in \mathbb{C}$ and $j = 1, \ldots, k$ let us define

$$I_j(x) := \frac{1}{2\pi i} \int_{|z|=R} f(xz)(z - \alpha q^n)^{-j} \prod_{\nu=0}^{n-1} (z - \alpha q^\nu)^{-k} \, dz \tag{36}$$

where we now suppose $R := |q|_v^N$, $N := kn + j$ (compare $I_k(q^{-n-1})$ with (1_∞)). From (10) we get

$$f(zq^\nu) = f(z)S_\nu(z) \quad \text{with } S_\nu(z) := \prod_{j=0}^{\nu-1} (q^j z + 1) \text{ for each } \nu \in \mathbb{N}_0. \tag{37}$$

If we put $k_\nu := k$ for $\nu = 0, \ldots, n-1$, $k_n := j$ and furthermore

$$r(\nu, \kappa) := \frac{1}{(k_\nu - 1 - \kappa)!} \left(\frac{d}{dz}\right)^{k_\nu - 1 - \kappa} \prod_{\substack{\mu=0 \\ \mu \neq \nu}}^{n} (z - \alpha q^\mu)^{-k_\mu} \Big|_{z = \alpha q^\nu} \tag{38}$$

for $\kappa = 0, \ldots, k_\nu - 1$; $\nu = 0, \ldots, n$, we find from (36) via the residue theorem

$$\begin{aligned}I_j(x) = &\sum_{\lambda=0}^{k-1} \frac{f^{(\lambda)}(\alpha x)}{\lambda!} \sum_{\kappa=\lambda}^{k-1} \frac{x^\kappa}{(\kappa-\lambda)!} \sum_{\nu=0}^{n-1} \frac{r(\nu, \kappa)}{q^{\nu\kappa}} S_\nu^{(\kappa-\lambda)}(\alpha x) \\ &+ \sum_{\lambda=0}^{j-1} \frac{f^{(\lambda)}(\alpha x)}{\lambda!} \sum_{\kappa=\lambda}^{j-1} \frac{x^\kappa}{(\kappa-\lambda)!} \frac{r(n, \kappa)}{q^{n\kappa}} S_n^{(\kappa-\lambda)}(\alpha x)\end{aligned} \tag{39}$$

where we used (37) to replace $f^{(\kappa)}(\alpha x q^{\nu})$ by the $f^{(\lambda)}(\alpha x)$ with $\lambda = 0, \ldots, \kappa$. By the definition of the S_ν we have

$$\frac{1}{(\kappa-\lambda)!} S_\nu^{(\kappa-\lambda)}(z) = \binom{\nu}{\kappa-\lambda} q^{\nu(\nu-1)/2} z^{\nu-\kappa+\lambda} + \cdots$$

and therefore

$$\sum_{\kappa=\lambda}^{k-1} \frac{x^\kappa}{(\kappa-\lambda)!} \sum_{\nu=0}^{n-1} \cdots$$
$$= x^\lambda \sum_{\kappa=\lambda}^{k-1} \sum_{\nu=0}^{n-1} \frac{r(\nu,\kappa)}{q^{\nu\kappa}} \left(\binom{\nu}{\kappa-\lambda} q^{\nu(\nu-1)/2} \alpha^{\nu-\kappa+\lambda} x^\nu + \cdots \right)$$

where the double sum on the right-hand side is a polynomial in x of degree less than n whereas from

$$\sum_{\kappa=\lambda}^{j-1} \frac{x^\kappa}{(\kappa-\lambda)!} \frac{r(n,\kappa)}{q^{n\kappa}} S_n^{(\kappa-\lambda)}(\alpha x)$$
$$= x^\lambda \sum_{\kappa=\lambda}^{j-1} \frac{r(n,\kappa)}{q^{n\kappa}} \left(\binom{n}{\kappa-\lambda} q^{n(n-1)/2} \alpha^{n-\kappa+\lambda} x^n + \cdots \right)$$

we see that the sum over κ on the right-hand side is a polynomial in x of degree not exceeding n. For $\lambda = j-1$ the leading coefficient of this polynomial is

$$q^{n(n+1-2j)/2} \alpha^{n(1-k)} \prod_{\mu=0}^{n-1} (q^n - q^\mu)^{-k} \neq 0. \tag{40}$$

These considerations make evident that (39) can be written as

$$I_j(x) = \sum_{\lambda=0}^{k-1} \frac{x^\lambda}{\lambda!} f^{(\lambda)}(\alpha x) \cdot Q_{j,\lambda+1}(x) \quad (j = 1, \ldots, k)$$

with an obvious definition of the polynomials $Q_{jm}(j, m = 1, \ldots, k)$. It is clear that

$$D(x) := \det(Q_{jm}(x))_{j,m=1,\ldots,k}$$

is a polynomial of exact degree kn. On the other hand we find

$$f(\alpha x) D(x) = \det \begin{pmatrix} I_1 & Q_{12} & \cdots & Q_{1k} \\ \vdots & \vdots & & \vdots \\ I_k & Q_{k2} & \cdots & Q_{kk} \end{pmatrix} (x).$$

Since we know $f(0) \neq 0$ and since we will show in a moment that all functions $I_1, \ldots, I_k$ have a zero of order at least kn at the origin, it is clear that we have

$$D(x) = cx^{kn}$$

with some $c \neq 0$ which could be given explicitly using (40). To show the assertion on the order of the I's we start from (36) and make there the transformation $w = z^{-1}$ of the integration variable leading us to

$$I_j(x) = \frac{1}{2\pi i} \int_{0+} f\left(\frac{x}{w}\right) g(w) w^{kn+j-2} \,\mathrm{d}w, \tag{41}$$

where 0^+ indicates a small circle around the origin in the positive sense, and where $g(w)$ denotes the function

$$(1 - \alpha q^n w)^{-j} \prod_{\nu=0}^{n-1} (1 - \alpha q^\nu w)^{-k}$$

which is holomorphic inside and on the boundary of 0^+. If $f(z) = \Sigma_{s=0}^{\infty} f_s z^s$ and $g(w) = \Sigma_{t=0}^{\infty} g_t w^t$, it follows from (41) that

$$I_j(x) = \sum_{\substack{s,t \geq 0 \\ kn+j+t-1=s}} f_s g_t x^s = x^{kn+j-1} \sum_{t=0}^{\infty} f_{kn+j+t-1} g_t x^t$$

is the Taylor series of the entire function I_j around the origin. Since we have $g_0 = 1$ and all $f_s \neq 0$ we may even say $\mathrm{ord}_0 I_j = kn + j - 1$.

By (39) we now obtain

$$I_j(1) = \sum_{\lambda=0}^{k-1} \frac{1}{\lambda!} Q_{j,\lambda+1}(1) f^{(\lambda)}(\alpha) \quad (j = 1, \ldots, k),$$

where

$$Q_{j,\lambda+1}(1) := \sum_{\kappa=\lambda}^{k-1} \sum_{\nu=0}^{n-1} \frac{r(\nu,\ \kappa)}{q^{\nu\kappa}} \frac{S_\nu^{(\kappa-\lambda)}(\alpha)}{(\kappa-\lambda)!} + \sum_{\kappa=\lambda}^{j-1} \frac{r(n,\ \kappa)}{q^{n\kappa}} \frac{S_n^{(\kappa-\lambda)}(\alpha)}{(\kappa-\lambda)!}$$

for $\lambda < j$, and

$$Q_{j,\lambda+1}(1) := \sum_{\kappa=\lambda}^{k-1} \sum_{\nu=0}^{n-1} \frac{r(\nu,\ \kappa)}{q^{\nu\kappa}} \frac{S_\nu^{(\kappa-\lambda)}(\alpha)}{(\kappa-\lambda)!}$$

for $\lambda \geq j$. Using the definition of $S_\nu(z)$ in (37) we see that each $S_\nu^{(\kappa-\lambda)}(\alpha)/(\kappa - \lambda)!$

is a polynomial in $\mathbb{Z}[\alpha, q]$ of $\deg_q \leqslant \nu(\nu - 1)/2$ and $\deg_\alpha \leqslant \nu - \kappa + \lambda$. The term $r(\nu, \kappa)$ defined in (38) is already considered in the proof of Lemma 2, see (14) and (15). By these results we deduce analogously to the proof of Lemma 2 that the definition

$$\Omega(k, n) := (k-1)!\alpha^{kn+k-1} q^{kn(n+1)/2 - n} R_n(q)^{k-1} \prod_{\nu=1}^{n} (q^\nu - 1)^k$$

gives us linear forms

$$J_j(n) := \Omega(k, n) I_j(1) = \sum_{\lambda=0}^{k-1} P_{j,\lambda}(\alpha, q) f^{(\lambda)}(\alpha) \quad (j = 1, \ldots, k), \tag{42}$$

where all $P_{j,\lambda} \in \mathbb{Z}[\alpha, q]$ satisfy

$$\deg_q P_{j,\lambda} \leqslant \left(\frac{k}{2} + 3\pi^{-2}(k-1)\right) n^2 + O(n \log n), \ \deg_\alpha P_{j,\lambda} = O(n).$$

Since $D(1) \neq 0$, our linear forms (42) are linearly independent. Further, as in Lemma 1, we have

$$|I_j(1)| \leqslant |q|_v^{-k^2 n^2/2 + O(n)}.$$

Therefore we obtain the following analogue of Lemma 3, the proof of which follows from the above considerations together with the estimates

$$|S_\nu^{(\kappa-\lambda)}(\alpha)/(\kappa - \lambda)!|_w \leqslant c_9^n \max\{1, |\alpha|_w\}^{O(n)} \max\{1, |q|_w\}^{\nu(\nu-1)/2},$$

$$|\Omega(k, n) r(\nu, \kappa)/q^{\nu\kappa}|_w \leqslant c_{10}^n \max\{1, |\alpha|_w\}^{O(n)} \max\{1, |q|_w\}^{A(k,n,\nu)},$$

$$A(k, n, \nu) := kn(n+1)/2 - k\nu(\nu-1)/2 - k\nu(n-\nu) + k \deg_q B_{n,\nu} + (k-1)\deg_q R_n + O(n),$$

valid for all $w \mid \infty$ (see the proof of Lemma 3).

LEMMA 5. *The linearly independent linear forms* (42) *satisfy the estimates*

$$\log|J_j(n)|_v \leqslant -(\tfrac{1}{2}k(k-2) - 3\pi^{-2}(k-1))n^2 \log|q|_v + O(n \log n), \tag{43}$$

$$\log|P_{j,\lambda}(\alpha, q)|_v \leqslant \left(\frac{k}{2} + 3\pi^{-2}(k-1)\right) n^2 \log|q|_v + O(n \log n) \tag{44}$$

for all $j,\ \lambda+1=1,\ldots,k$. *Furthermore, the polynomials* $P_{j,\lambda}\in\mathbb{Z}[\alpha,q]$ *satisfy the inequality*

$$\log|P_{j,\lambda}(\alpha,q)|_w \leqslant \left(\left(\frac{k}{2}+3\pi^{-2}(k-1)\right)n^2+O(n\log n)\right)\log_+|q|_w + O(n)\log_+|\alpha|_w+\delta(w)O(n\log n) \tag{44'}$$

for any place w *of* K.

REMARK. From now on we suppose that $k\geqslant 3$, since in the case $k=2$ inequality (43) is too weak to give any non-trivial result.

Proof of Theorem 1 (continued from p. 15). We define

$$A:=\frac{k}{2}+3\pi^{-2}(k-1),\quad B:=\frac{1}{2}k(k-2)-3\pi^{-2}(k-1). \tag{45}$$

Suppose that the dimension m of the vector space $Kf(\alpha)+\cdots+Kf^{(k-1)}(\alpha)$ over K satisfies

$$m<(A+B)/(\lambda A). \tag{46}$$

Then there exist $M:=k-m$ linearly independent relations

$$a_{j0}f(\alpha)+a_{j1}f'(\alpha)+\cdots+a_{j,k-1}f^{(k-1)}(\alpha)=0\quad(j=1,\ldots,M)$$

with coefficients $a_{j\lambda}\in O_K$. Further, without loss of generality, we may assume that

$$\Delta:=\det\begin{pmatrix} P_{10}(\alpha,q) & \cdots & P_{1,k-1}(\alpha,q)\\ \vdots & & \vdots\\ P_{m0}(\alpha,q) & \cdots & P_{m,k-1}(\alpha,q)\\ a_{10} & \cdots & a_{1,k-1}\\ \vdots & & \vdots\\ a_{M0} & \cdots & a_{M,k-1}\end{pmatrix}\neq 0.$$

We now have

$$f(\alpha)\Delta=J_1(n)\Delta_1+\cdots+J_m(n)\Delta_m,$$

where, by the estimate (44),

$$\log |\Delta_j|_v \leqslant (m-1)An^2 \log |q|_v + O(n \log n).$$

The product formula together with (43) and (44)′ then implies

$$\frac{d_v}{d}((m-1)A - B)n^2 \log |q|_v + O(n \log n)$$

$$\geqslant \frac{d_v}{d} \log |\Delta|_v = -\sum_{w \neq v} \frac{d_w}{d} \log |\Delta|_w$$

$$\geqslant -\sum_{w \neq v} \frac{d_w}{d}((mAn^2 + O(n \log n)) \log_+ |q|_w$$

$$+ O(n) \log_+ |\alpha|_w + \delta(w) O(n \log n))$$

$$= -\sum_w \frac{d_w}{d} mAn^2 \log_+ |q|_w + \frac{d_v}{d} mAn^2 \log |q|_v + O(n \log n)$$

$$= -mAn^2 \log h(q) + \frac{d_v}{d} mAn^2 \log |q|_v + O(n \log n).$$

Thus we have an inequality

$$-(A + B - \lambda mA)n^2 \frac{d_v}{d} \log |q|_v > O(n \log n),$$

which, by (46), gives a contradiction for all sufficiently large n. We therefore deduce that (46) is not true. This implies an inequality

$$m \geqslant (A + B)/(\lambda A) = (k^2 - k)/\lambda(k + 6\pi^{-2}(k-1))$$

proving our Theorem 1 completely.

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