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Convolution *L*-series

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1. Introduction

In the series of papers [4]–[7] we have developed several techniques for estimating the coefficients of *L*-functions which satisfy standard functional equations. Inspired by these works we now begin to examine convolution series formed by multiplying the coefficients.

Suppose we have two Dirichlet series

$$\mathscr{A}(s) = \sum_1^\infty a_n n^{-s}$$

$$\mathscr{B}(s) = \sum_1^\infty b_n n^{-s}$$

which converge absolutely in the half-plane $Re\, s > 1$, which have analytic continuation to entire functions and which satisfy functional equations of the type

$$\mathscr{A}(1 - s) = \Phi(s)\mathscr{A}(s)$$

$$\mathscr{B}(1 - s) = \Psi(s)\mathscr{B}(a).$$

Here $\Phi(s)$ and $\Psi(s)$ are certain holomorphic functions in $Re\, s > 0$ which have at most a polynomial growth on vertical lines. Furthermore we assume that $\mathscr{A}(s)$, $\mathscr{B}(s)$ have Euler products of degree k, l, i.e.,

$$\mathscr{A}(s) = \prod_p \mathscr{A}_p(p^{-s})^{-1}$$

$$\mathscr{B}(s) = \prod_p \mathscr{B}_p(p^{-s})^{-1}$$

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If $Re\, s > 1$, where $\mathscr{A}_p(T), \mathscr{B}_p(T)$ are polynomials in T of degree k, l respectively with the constant term $\mathscr{A}_p(0) = \mathscr{B}_p(0) = 1$. We shall study the convolution series

$$C(s) = \sum_1^\infty a_n b_n n^{-s}.$$

Clearly, this convolution series converges absolutely in $Re\, s > 2$. Our objective will be to prove the absolute convergence in $Re\ s > 1$ and to establish the analytic continuation up to $Re\ s > 1/2$ subject to some further natural conditions.

Motivating examples are the symmetric power L-functions attached to an automorphic form. These have been intensively studied along the lines of Langlands program (see the survey articles by F. Shahidi [8] and D. Bump [2]). In this context our approach to analytic continuation is more elementary but the results are not quite complete since we cannot reach the critical line $Re\ s = 1/2$ and prove a functional equation for the convolution series $C(s)$. Nevertheless our applications illustrate what can be concluded directly from the existence of functional equations for Dirichlet series without appealing to automorphic theory.

As in [4]–[7] our approach requires suitable functional equations for the twisted series

$$\mathscr{A}(s, \chi) = \sum_1^\infty a_n \chi(n) n^{-s}$$

$$\mathscr{B}(s, \chi) = \sum_1^\infty b_n \chi(n) n^{-s}.$$

We assume that for any primitive character χ (mod q) the twisted series are entire functions of finite order and that they satisfy the compatible functional equations

$$\mathscr{A}(1 - s, \chi) = \alpha_\chi q^{k(s-1/2)} \Phi_\nu(s) \mathscr{A}(s, \bar{\chi}) \tag{1}$$

$$\mathscr{B}(1 - s, \chi) = \beta_\chi q^{l(s-1/2)} \Psi_\nu(s) \mathscr{B}(s, \bar{\chi}) \tag{2}$$

with $|\alpha_\chi| = |\beta_\chi| = 1$. Here we allow the factors $\Phi_\nu(s)$, $\Psi_\nu(s)$ to depend on the parity index $\nu = \chi(-1) = \pm 1$ but not on χ itself. It is assumed that $\Phi_\nu(s)$, $\Psi_\nu(s)$ are holomorphic in $Re\, s > 0$ where they have a polynomial growth on vertical lines, i.e.,

$$\Phi_\nu(s),\ \Psi_\nu(s) \ll |s|^B \quad \text{if } Re\ s = \sigma > 0, \tag{3}$$

where $B > 0$ and the implied constant depend on σ. Usually the factors $\Phi_\nu(s)$, $\Psi_\nu(s)$ of functional equations are products of suitable gamma functions but we do not need to assume this property because the s-aspect plays no role in the method.

However the signs α_χ, β_χ of functional equations play the key part. Usually they are expressible in terms of the Gauss sum

$$\tau(\chi) = \sum_{x (\text{mod } q)} \chi(x) e_q(x)$$

where

$$e_q(x) = e\left(\frac{x}{q}\right) = e^{2\pi i x/q}$$

denotes the additive character. Let $\varepsilon_\chi = \tau(\chi) q^{-1/2}$ be the sign of Gauss sum. It satisfies $\bar{\varepsilon}_\chi = \nu\varepsilon_\chi = \varepsilon_\chi^{-1}$ for any primitive character.

Let us give two examples. If $\mathscr{A}(s)$ is the $k - 1$th symmetric power L-function attached to a cusp form for the modular group then one expects (in accordance with the Langlands program) the functional equation for $\mathscr{A}(s, \chi)$ to have the sign $\alpha_\chi = \varepsilon_\chi^k$. Another interesting example is the shifted Riemann zeta-function $\mathscr{A}(s) = \zeta(ks - (k-1)/2)$ which is useful for generating kth powers. In this case we have the functional equation for $\mathscr{A}(s, \chi) = L(ks - (k-1)/2, \chi^k)$ with the sign $\alpha_\chi = \varepsilon_{\chi^k}$ provided χ^k is primitive.

What truly matters in our argument is the Fourier transform

$$K_q(c) = \sum_{\chi (\text{mod } q)}^{*} \chi(c) \alpha_\chi \bar{\beta}_\chi \tag{4}$$

where the star restricts the summation to primitive characters. We can compute $K_q(c)$ in two cases:

Case 1. Suppose $\alpha_\chi = \beta_\chi$. Then if $(c, q) = 1$ we have

$$K_q(c) = \sum_{\chi (\text{mod } q)}^{*} \chi(c) = \sum_{w | (q, c-1)} \varphi(w) \mu(q/w). \tag{5}$$

For $c = 1$ this yields the number of primitive characters to modulus q,

$$K_q(1) = \sum_{sw = q} \mu(s) \varphi(w) = q \prod_{p \| q} \left(1 - \frac{2}{p}\right) \prod_{p^2 | q} \left(1 - \frac{1}{p}\right)^2.$$

If $c \neq 1$ then $K_q(c)$ is bounded on average in q. Moreover one senses a "reciprocity law" for $K_q(c)$ as w is switched into the complementary divisor of $|c-1|$ in (5).

Case 2. Suppose $\alpha_\chi = \nu^{-h}\varepsilon_\chi^k$ and $\beta_\chi = \varepsilon_\chi^l$ with $h = l - k > 0$. Then $2k$ Gauss sums out of $l + k$ annihilate themselves leaving $\alpha_\chi \bar{\beta}_\chi = \varepsilon_{\bar{\chi}}^h$. Using (5) we infer that

$$\begin{aligned} q^{h/2}K_q(c) &= \sum_{\chi(\text{mod } q)}^{*} \bar{\chi}(c)\left(\sum_{x(\text{mod } q)} \chi(x)e_q(x)\right)^h \\ &= \sum_{sw=q} \mu(s)\varphi(w) \sum_{\substack{x_1,\dots,x_h(\text{mod } q)\\ x_1\cdots x_h \equiv c(\text{mod } w)}}^{*} e_q(x_1 + \cdots + x_h) \\ &= \sum_{\substack{sw=q\\(s,w)=1}} \mu(s)^{h+1}\varphi(w) \sum_{\substack{x_1,\dots,x_h(\text{mod } w)\\ x_1\cdots x_h \equiv c(\text{mod } w)}}^{*} e_w((x_1 + \cdots + x_h)\bar{s}) \end{aligned}$$

where $\bar{s}$ denotes the multiplicative inverse of s modulo w. Observe that the innermost sum is the generalized Kloosterman sum for which P. Deligne [3] has established the bound $\tau_h(w)w^{(h-1)/2}$ (for prime modulus only but the extension to all moduli is straightforward). Employing Deligne's bound we get

$$|K_q(c)| \leqslant \tau_h(q)q^{1/2}.$$

Of particular interest is the case of $h = 1$ because the sum

$$q^{1/2}K_q(c) = \sum_{\substack{sw=q\\(s,w)=1}} \mu^2(s)\varphi(w)e_w(c\bar{s}) \quad \text{if } (c, q) = 1 \tag{6}$$

can be transformed by means of the following 'reciprocity' formula

$$e_w(c\bar{s})e_s(c\bar{w}) = e_q(c). \tag{7}$$

2. Statement of results

In this paper we explore Case 2 for $k = 2$ and $l = 3$. It has been shown in [5] that both series formed by squaring the coefficients of $\mathscr{A}(s)$ and $\mathscr{B}(s)$ converge absolutely in $Re\, s > 1$. Hence by Cauchy's inequality the convolution series $\mathscr{C}(s)$ also converges absolutely in $Re\, s > 1$. Now we look deeper into the critical strip to prove analytic continuation in $Re\, s > 1/2$. For simplicity we shall assume more than the absolute convergence in $Re\, s > 1$, namely that the local polynomials $\mathscr{A}_p(T)$ and $\mathscr{B}_p(T)$ have bounded coefficients. This is indeed the case for L-functions attached to holomorphic cusp forms (the Ramanujan conjecture proved by P. Deligne [3]). In general, one can probably avoid this condition by using the bounds on average.

THEOREM 1. *Suppose* $\mathcal{A}(s)$, $\mathcal{B}(s)$ *are Euler products of degree two and three with bounded coefficients such that* $\mathcal{A}(s,\chi)$, $\mathcal{B}(s,\chi)$ *are entire functions of finite order which satisfy the functional equations with signs* $\alpha_\chi = \varepsilon_\chi^2$ *and* $\beta_\chi = \varepsilon_\chi^3$ *respectively for all primitive characters. Then the convolution series* $\mathcal{C}(s)$ *has analytic continuation without poles to the region* $Re\, s > 1/2$.

If we take for $\mathcal{A}(s)$ the Hecke L-function attached to a cusp form for the modular group

$$L_1(s) = \prod_p (1 - \alpha_p p^{-s})^{-1}(1 - \beta_p p^{-s})^{-1}$$

and for $\mathcal{B}(s)$ we take the Shimura symmetric square L-function

$$L_2(s) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1}(1 - \alpha_p \beta_p p^{-s})^{-1}(1 - \beta_p^2 p^{-s})^{-1}$$

then the convolution series $\mathcal{C}(s)$ becomes $L_1(s)L_3(s))P(s)$ where

$$L_3(s) = \prod_p (1 - \alpha_p^3 p^{-s})^{-1}(1 - \alpha_p^2 \beta_p p^{-s})^{-1}(1 - \alpha_p \beta_p^2 p^{-s})^{-1}(1 - \beta_p^3 p^{-s})^{-1}$$

is the symmetric cube L-function and $P(s)$ is given by the product

$$P(s) = \prod_p (1 - \alpha_p p^{-s})(1 - \beta_p p^{-s})(1 - (\alpha_p + \beta_p) p^{-s})^{-1}$$

which converges absolutely in $Re\, s > 1/2$. By Theorem 1 we infer

COROLLARY. *The symmetric cube L-function* $L_3(s)$ *attached to a Hecke eigencusp-form for the modular group has meromorphic continuation to the region* $Re\, s > 1/2$ *whereas* $L_1(s)L_3(s)$ *is holomorphic.*

REMARKS. The corollary is not new, it was proved in greater generality by F. Shahidi and others (see [8]) using quite different methods.

3. Applying the δ-symbol

Our approach to analytic continuation of $\mathcal{C}(s)$ features estimates for finite sums of the type

$$\mathcal{D}(X) = \sum_n a_n b_n \eta^2 \left(\frac{n}{X}\right)$$

where η is any smooth function supported in the interval $[1/2, 1]$. We shall prove that

$$\mathscr{D}(X) \ll X^{1/2+\varepsilon} \tag{8}$$

and this shows through the Mellin transform that $\mathscr{C}(s)$ is holomorphic in $Re\, s > 1/2$.

We begin by writing

$$\mathscr{D}(X) = \sum_m \sum_n a_m b_n \eta\left(\frac{m}{X}\right) \eta\left(\frac{n}{X}\right) \delta_{mn}$$

where δ_{mn} is the diagonal symbol of Kronecker. Then, as in [4], we use the formula

$$Y\delta_{mn} = \sum_{q|(m-n)} \left(\omega(q) - \omega\left(\frac{|m-n|}{q}\right)\right)$$

where ω is any smooth function, compactly supported in $\mathbb{R}^+$ and $Y = \Sigma\, \omega(q)$. We choose $\omega(z) = \eta(z/\sqrt{X})$, so $Y \asymp \sqrt{X}$ and

$$Y\mathscr{D}(X) = \sum_q \sum_{m \equiv n} \sum_{(\bmod q)} a_m b_n f\left(\frac{m}{X}, \frac{n}{X}, \frac{q}{\sqrt{X}}\right)$$

where

$$f(x, y, z) = \eta(x)\eta(y)\left(\eta(z) - \eta\left(\frac{|x-y|}{z}\right)\right).$$

Notice that $f(x, y, z)$ is smooth and supported in the box $[1/2, 1] \times [1/2, 1] \times [0, 1]$.

Next we write by means of multiplicative characters

$$Y\mathscr{D}(X) = \sum_{qrt} \varphi(qt)^{-1} \sideset{}{^*}\sum_{\chi(\bmod q)} \sum\sum_{(mn,t)=1} \chi(m)\bar{\chi}(n) a_{rm} b_{rn} f\left(\frac{rm}{X}, \frac{rn}{X}, \frac{qrt}{\sqrt{X}}\right).$$

Further transformation of $\mathscr{D}(X)$ requires factoring the coefficients a_{rm}, b_{rn} as well as relaxing the condition $(mn, t) = 1$. To this end we exploit the Euler products for $\mathscr{A}(s)$ and $\mathscr{B}(s)$. There are numbers $a_r(a) \ll r^\varepsilon$ defined for $a \mid r$ such that for all m it holds that

$$a_{rm} = \sum_{am'=m} a_r(a) a_{m'}.$$

Also there are numbers $b_r(b) \ll r^\varepsilon$ defined for $b \mid r^2$ such that for all n it holds that

$$b_{rn} = \sum_{bn' = n} b_r(b) b_{n'}.$$

The above factorizations yield

$$Y\mathscr{D}(X) = \sum_{qrt} \varphi(qt)^{-1} \sum_{(ab,qt)=1} a_r(a) b_r(b) \times \sideset{}{^*}\sum_{\chi (\mathrm{mod}\, q)} \sum\sum_{(mn,t)=1} \chi(am)\bar{\chi}(bn) a_m b_n f\left(\frac{arm}{X}, \frac{brn}{X}, \frac{qrt}{\sqrt{X}}\right).$$

To relax the co-primality condition we again appeal to the Euler products. One can define numbers $c_t(c) \ll t^\varepsilon$ for $c \mid t^2$ with the property that

$$a_m = \sum_{cm' = m} c_t(c) a_{m'}$$

if $(m, t) = 1$, or else the sum vanishes. Also one can define numbers $d_t(d) \ll t^\varepsilon$ for $d \mid t^3$ with the property that

$$b_n = \sum_{dn' = n} d_t(d) b_{n'}$$

if $(n, t) = 1$, or else the sum vanishes. The above relations yield

$$Y\mathscr{D}(X) = \sum_{qrt} \varphi(qt)^{-1} \sum_{(ab,qt)=1} a_r(a) b_r(b) \sum_{(cd,q)=1} c_t(c) d_t(d) \times \sideset{}{^*}\sum_{\chi (\mathrm{mod}\, q)} \sum_m \sum_n \chi(acm)\bar{\chi}(bdn) a_m b_n f\left(\frac{acrm}{X}, \frac{bdrn}{X}, \frac{qrt}{\sqrt{X}}\right). \tag{9}$$

4. Applying the functional equations

Now we are ready to execute the summation in m and n. Let us first consider a general character sum of the type

$$\Delta(\chi) = \sum_m \sum_n \chi(m)\bar{\chi}(n) a_m b_n g(m, n)$$

where g is a smooth function, compactly supported in $\mathbb{R}^+ \times \mathbb{R}^+$. Employing the functional equations for $\mathscr{A}(s, \chi)$ and $\mathscr{B}(s, \chi)$ by way of Mellin's transform we infer

$$\Delta(\chi) = \alpha_\chi \bar{\beta}_\chi \sum_m \sum_n \bar{\chi}(m)\chi(n) a_m b_n g_\nu(mq^{-2}, nq^{-3}) q^{-5/2}$$

where g_ν is an integral transform of g given by

$$g_\nu(x, y) = \frac{-1}{4\pi^2} \iint_{(\sigma_1,\sigma_2)} \breve{g}(s_1, s_2)\Phi_\nu(s_1)\Psi_\nu(s_2) x^{-s_1} y^{-s_2}\, ds_1\, ds_2$$

with $\sigma_1, \sigma_2 > 0$ and

$$\breve{g}(s_1, s_2) = \iint g(u, v) u^{-s_1} v^{-s_2}\, du\, dv.$$

Note that g_ν depends only on the parity index $\nu = \chi(-1) = \pm 1$ but not on the character itself. We put $g^+ = g_1 + g_{-1}$ and $g^- = g_1 - g_{-1}$ so $2g_\nu = g^+ + \nu g^-$. Summing over the primitive characters we evaluate the Fourier transform of $\Delta(\chi)$ as follows

$$\sum_{\chi (\mathrm{mod}\, q)}^* \chi(e)\Delta(\chi) = \frac{1}{2} \sum\sum_{(mn,q)=1} a_m b_n K_q(\pm e\bar{m}n) g^\pm(mq^{-2}, nq^{-3}) q^{-5/2}$$

for any $(e, q) = 1$. In particular this together with (9) gives

$$2Y\mathscr{D}(X) = X^2 \sum_{rt<\sqrt{X}} \sum_{\substack{a|r\\(ab,t)=1}} \sum_{b|r^2} \sum_{\substack{c|t^2\\d|t^3}} a_r(a) b_r(b) c_t(c) d_t(d) (abcd)^{-1} \mathscr{E} \tag{10}$$

where we have put

$$\mathscr{E} = \sum_{\substack{m\\(abcdmn,q)=1}} \sum_n \sum_q q^{-5/2} \varphi(qt)^{-1} a_m b_n K_q(\pm acn\overline{bdm}) F^\pm \left(\frac{mX}{acrq^2}, \frac{nX}{bdrq^3}, \frac{qrt}{\sqrt{X}}\right)$$

and $F^\pm = F_1 \pm F_{-1}$, where $F(u, v, z)$ are the integral transforms of $f(x, y, z)$ given by

$$F_\nu(u, v, z) = \frac{-1}{4\pi^2} \iint_{(\sigma_1,\sigma_2)} \breve{f}(s_1, s_2, z)\Phi_\nu(s_1)\Psi_\nu(s_2) u^{-s_1} v^{-s_2}\, ds_1\, ds_2 \tag{11}$$

with

$$\breve{f}(s_1, s_2, z) = \iint f(x, y, z) x^{-s_1} y^{-s_2}\, dx\, dy$$

on the lines $Re\, s_1 = \sigma_1 > 0$ and $Re\, s_2 = \sigma_2 > 0$.

5. Applying the reciprocity transformation

Let $(uv, q) = 1$. By (6) and (7) we get

$$q^{1/2}K_q(u\bar{v}) = e\left(\frac{u}{vq}\right)\sum_{\substack{sw=q\\(s,w)=1}} \mu^2(s)\varphi(w)e\left(\frac{-u\bar{w}}{vs}\right).$$

Since $\varphi(qt)^{-1} = \varphi(st)^{-1}\varphi(w)^{-1}\sigma((t, w))$ with $\sigma(h) = \varphi(h)h^{-1}$ we obtain

$$\mathscr{E} = \sum_m \sum_n a_m b_n \sum_{\substack{(s,w)=1\\(sw,abcdmn)=1}}\sum \mu^2(s)\frac{\sigma((t, w))}{\varphi(st)} e\left(\pm\frac{acn\bar{w}}{bdms}\right) G(m, n, sw)$$

where for notational simplicity we have put

$$G(m, n, q) = q^{-3}e\left(\frac{\pm acn}{bdmq}\right)F^{\pm}\left(\frac{mX}{acrq^2}, \frac{nX}{bdrq^3}, \frac{qrt}{\sqrt{X}}\right). \tag{12}$$

In the sequel we shall denote $u = \pm acn\delta^{-1}$, $v = bdm\delta^{-1}$ where $\delta = (acn, bdm)$. Furthermore we split $\mathscr{E} = \mathscr{E}_0 + \mathscr{E}_1$, where

$$\mathscr{E}_0 = \sum_m \sum_n a_m b_n \sum_{\substack{(s,w)=1\\(sw,\delta uv)=1}}\sum \frac{\mu(vs)\sigma((t, w))}{\varphi(st)\varphi(sv)} G(m, n, sw),$$

$$\mathscr{E}_1 = \sum_m \sum_n a_m b_n \sum_{\substack{(s,w)=1\\(sw,\delta uv)=1}}\sum \mu^2(s)\frac{\sigma((t, w))}{\varphi(st)}\left(e\left(\frac{u\bar{w}}{vs}\right) - \frac{\mu(vs)}{\varphi(vs)}\right) G(m, n, sw).$$

6. Estimating $G(m, n, q)$

First let us estimate the transform (11). The partial derivatives of $f(x, y, z)$ are bounded by

$$f^{(ijk)}(x, y, z) \ll z^{-i-j-k}$$

with the implied constant depending on i, j, k only. Hence, by a repeated partial integration the Mellin transform satisfies

$$z^k\frac{\partial^k}{\partial z^k}\check{f}(s_1, s_2, z) \ll (1 + |s_1|z)^{-A}(1 + |s_2|z)^{-A}$$

for s_1, s_2 on the vertical lines $Re\, s_1 = \sigma_1 > 0$, $Re\, s_2 = \sigma_2 > 0$, where A is an arbitrary positive number, the implied constant depending on σ_1, σ_2, A and k only. Since $\Phi_\nu(s_1)$ and $\Psi_\nu(s_2)$ have at most a polynomial growth we obtain

$$u^i v^j z^k F_\nu^{(ijk)}(u, v, z) \ll u^{-\sigma_1} v^{-\sigma_2} z^{-2B}$$

for any σ_1, $\sigma_2 > 0$ and some constant $B > 0$. This yields

$$u^i v^j z^k F_\nu^{(ijk)}(u, v, z) \ll (1+u)^{-A}(1+v)^{-A}(uz^2)^{-B}(uv)^{-\varepsilon}$$

for any ε, $A > 0$ and some constant $B > 0$. Finally, by a change of variables we conclude from the above and (12) that

$$m^i n^j q^k G^{(ijk)}(m, n, q) \ll q^{-3}\left(1+\frac{mX}{acrq^2}\right)^{-A}\left(1+\frac{nX}{bdrq^3}\right)^{-A} X^\varepsilon \tag{13}$$

for $m, n, q \geqslant 1$, where ε, $A > 0$ are arbitrary and the implied constant depends on ε, A, i, j, k only.

Recall that $G(m, n, q)$ vanishes if $qrt > \sqrt{X}$. Therefore, by (13) all terms in $\mathscr{E}_0$ and $\mathscr{E}_1$ are very small except for

$$\begin{aligned} &1 \leqslant m \leqslant M \quad \text{with } M = acr^{-1}t^{-2}X^\varepsilon \\ &1 \leqslant n \leqslant N \quad \text{with } N = bdr^{-2}t^{-3}X^{1/2+\varepsilon} \\ &QX^{-\varepsilon} < q < Q \quad \text{with } Q = (rt)^{-1}X^{1/2}. \end{aligned} \tag{14}$$

Notice that the above conditions imply $M \leqslant X^\varepsilon$, $N \leqslant X^{1/2+\varepsilon}$, and $Q < X^{1/2}$.

7. Estimating $\mathscr{E}_0$

First we execute the summation over n in $\mathscr{E}_0$ by an appeal to the following general result:

LEMMA 1. *Let $G(n)$ be a smooth function on $\mathbb{R}^+$ whose derivatives satisfy*

$$n^j G^{(j)}(n) \ll \left(1+\frac{n}{N}\right)^{-A}$$

for some $N \geqslant 1$ and any $A > 0$. Then we have

$$\sum_{(n,t)=1} b_{rn} G(n) \ll (rtN)^\varepsilon. \tag{15}$$

Proof. Using the functional equation for $\mathscr{B}(s)$ by contour integration we infer that

$$\sum_n b_n G(n) \ll N^\varepsilon.$$

Hence

$$\sum_{(n,t)=1} b_{rn} G(n) = \sum_{\substack{b|r^2 \\ (b,t)=1}} b_r(b) \sum_{d|t^3} d_t(d) \sum_n b_n G(bdn) \ll (rtN)^\varepsilon.$$

The sum over n in $\mathscr{E}_0$ is of the type (15). More precisely we have

$$\mathscr{E}_0 = \sum_m a_m \sum_{\delta|bdm} \sum_{\substack{(s,w)=1 \\ (sw,abcdmn)=1}}\sum \frac{\mu(vs)\sigma((t, w))}{\varphi(st)\varphi(sv)} \sum_{\substack{(n,sw)=1 \\ (acn,bdm)=\delta}} b_n G(m, n, sw).$$

Therefore, by Lemma 1 we obtain

$$\begin{aligned} \mathscr{E}_0 &\ll \sum_m |a_m| \sum_{q>Q} t^{-1} q^{-3} \left(1 + \frac{mX}{acrq^2}\right)^{-A} X^\varepsilon \\ &\ll t^{-1} M Q^{-2} X^\varepsilon \ll acrt^{-1} X^{\varepsilon-1}. \end{aligned} \tag{16}$$

8. Estimating $\mathscr{E}_1$

We replace $\sigma((t, w))$ by

$$\sigma((t, w)) = \sum_{\tau|(t,w)} \tau^{-1}\mu(\tau)$$

and relax the condition $(w, \delta u) = 1$ using Möbius inversion to get

$$\begin{aligned} \mathscr{E}_1 = \sum_m \sum_n a_m b_n & \sum_{\substack{\tau|t \quad v|\delta u \\ (\tau,\delta uv)=(v,v)=1}} \sum \tau^{-1}\mu(\tau)\mu(v) \sum_{(s,\tau\delta uv)=1} \mu^2(s)\varphi(st)^{-1} \\ & \times \sum_{(w,sv)=1} \left(e\left(\frac{\overline{u\tau vw}}{vs}\right) - \frac{\mu(vs)}{\varphi(vs)}\right) G(m, n, \tau vsw). \end{aligned}$$

First, we shall show that small s contribute very little. To this end we establish the following general result:

LEMMA 2. *Let $G(w)$ be a smooth function supported in $[0, W]$ whose derivatives* *satisfy* $G^{(j)} \ll W^{-j}$. *Let* $g < W^{1-\varepsilon}$. *Then*

$$\sum_{w \equiv a (\bmod g)} G(w) = \frac{1}{g} \sum_{w} G(w) + O(W^{-A}).$$

Proof. The Poisson summation gives

$$\frac{1}{g} \sum_{h} e\left(\frac{-ah}{g}\right) \hat{G}\left(\frac{h}{g}\right)$$

where $\hat{G}$ is the Fourier transform of G. Integrating by parts one proves that

$$\hat{G}(y) \ll (1 + |y|W)^{-A}.$$

Hence our sum is equal to $g^{-1}\hat{G}(0) + O(W^{-A})$. This shows that the sum does not depend on a (mod g) up to the error term $O(W^{-A})$, giving the result.

COROLLARY. *If* $(a, g) = 1$ *and* $g < W^{1-\varepsilon}$ *then*

$$\sum_{(w,g)=1} \left(e\left(\frac{a\bar{w}}{g}\right) - \frac{\mu(g)}{\varphi(g)}\right) G(w) \ll W^{-A}. \tag{17}$$

The hypothesis of the above corollary to Lemma 2 is satisfied for $G(w) = G(m, n, \tau v s w)$ in the range $g = vs \leqslant (\tau v s)^{-1} Q X^{-2\varepsilon}$ by virtue of (13), where $Q = (rt)^{-1} X^{1/2}$. We put

$$S = (\tau v v)^{-1/2} Q^{1/2} X^{-\varepsilon},$$

so (17) can be used for all $s \leqslant S$. Therefore, the contribution of this range to $\mathscr{E}_1$ is

$$\mathscr{E}_{11} \ll X^{-A}. \tag{18}$$

In the remaining range of $s > S$ we estimate the contribution of the part $-\mu(vs)\varphi(vs)^{-1}$ trivially as follows:

$$\begin{aligned} \mathscr{E}_{12} &\ll \sum_{m} \sum_{n} |a_m b_n| \sum_{\tau | t} \tau^{-1} \sum_{v | \delta u} \varphi(t)^{-1} \varphi(v)^{-1} S^{-2} \sum_{q} |G(m, n, \tau v q)| \tau(q) \\ &\ll t^{-1} M N Q^{-3} X^{\varepsilon} \ll abcd t^{-3} X^{\varepsilon - 1}. \end{aligned} \tag{19}$$

Now we are left with

$$\mathscr{E}_{13} = \sum_m \sum_n a_m b_n \sum_{\substack{\tau|t \quad v|\delta u \\ (\tau,\delta uv)=(v,v)=1}} \sum \tau^{-1}\mu(\tau)\mu(v)$$

$$\times \sum_{\substack{(s,\tau\delta uv)=1 \\ s>S}} \mu^2(s)\varphi(st)^{-1} \sum_{(w,sv)=1} e\left(\frac{u\overline{\tau vw}}{vs}\right) G(m, n, \tau vsw).$$

After changing the order of summation we estimate as follows:

$$\mathscr{E}_{13} \ll \varphi(t)^{-1} \sum_m |a_m| \sum_{\tau|t} \tau^{-1} \sum_{(v,v)=1} \sum_{\delta|bdm} \mathscr{H} \tag{20}$$

where $\mathscr{H}$ is a sum in s, w, n given by

$$\mathscr{H} = \sum_{\substack{(s,\tau vv)=1 \\ s>S}} \varphi(s)^{-1} \sum_{\substack{(w,sv)=1 \\ \tau vsw<Q}} \left| \sum_{\substack{(n,\tau s)=1, v|acn \\ (acn,bdm)=\delta}} b_n e\left(\frac{u\overline{\tau vw}}{vs}\right) G(m, n, \tau vsw)\right|.$$

By virtue of (13) we can restrict the summation to the range (14) up to a small error term $O(X^{-A})$. Moreover we can separate n from the other variables of $G(m,n,q)$ by any standard technique at no cost. In fact the integral representation (11) yields the desired separation without effort. We obtain

$$\mathscr{H} \ll Q^{-3}X^{\varepsilon} \sum_{\substack{(s,\tau vv)=1 \\ s>S}} s^{-1} \sum_{\substack{(w,sv)=1 \\ \tau vsw<Q}} \left| \sum_{\substack{n>N, v|acn \\ (n,s)=1}} \beta_n e\left(\frac{u\overline{\tau vw}}{vs}\right)\right| + X^{-A}$$

for some $\beta_n \ll b_n$. For estimating this sum we shall use the large sieve inequality (see [1]).

LEMMA 3. *For any complex numbers* β_n *it holds that*

$$\sum_{\substack{s<T \\ (s,v)=1}} \sum_{x(\mathrm{mod}\ vs)}^{*} \left| \sum_{\substack{n<N \\ (n,s)=1}} \beta_n e\left(\frac{nx}{vs}\right)\right|^2 \ll (vT^2 + N) \sum_n |\beta_n|^2.$$

From Lemma 3 by Cauchy's inequality we deduce the following:

COROLLARY. *If* $(ab, v) = 1$ *it holds that*

$$\sum_{\substack{sw<Q \\ (vs,abw)=1 \\ s>S}} s^{-1} \left| \sum_{\substack{n<N \\ (n,s)=1}} \beta_n e\left(an\frac{\overline{bw}}{vs}\right)\right| \ll (vQ)^{1/2}\left(1+\frac{Q}{vS}\right)^{1/2}\left(1+\frac{N}{vS^2}\right)^{1/2}\left(\sum|\beta_n|^2\right)^{1/2}.$$

The corollary provides an estimate for a sum of the type we have in $\mathscr{H}$. It gives

$$\begin{aligned}\mathscr{H} &\ll Q^{-3}X^{\varepsilon}\left(\frac{vQ}{\tau v}\right)^{1/2}\left(1+\frac{Q}{\tau v v s^2}\right)^{1/2}\left(1+\frac{(v,ac)N}{v v S^2}\right)^{1/2}\left(\frac{(v,ac)}{v}N\right)^{1/2}\\ &\ll v^{-1}(v,ac)v^{1/2}Q^{-5/2}(1+NQ^{-1})^{1/2}N^{1/2}X^{\varepsilon}\\ &\ll v^{-1}(v,ac)bdr^{3/2}t(1+bdr^{-1}t^{-2})^{1/2}X^{\varepsilon-1} \ll v^{-1}(v,ac)bdr^2t^{3/2}X^{\varepsilon-1}\end{aligned}$$

because $b|r^2$, $d|t^3$, so $bd \leqslant r^2t^3$. Hence by (20) we derive

$$\mathscr{E}_{13} \ll bdr^2t^{1/2}MX^{\varepsilon-1} \ll abcdrt^{-3/2}X^{\varepsilon-1}. \tag{21}$$

Gathering together (18), (19), and (21) we conclude that

$$\mathscr{E}_1 \ll abcdrt^{-3/2}X^{\varepsilon-1}. \tag{22}$$

From (16) and (22) we obtain

$$\mathscr{E} \ll abcdrt^{-1}X^{\varepsilon-1}. \tag{23}$$

Finally by (10) and (23) we conclude (8). This completes the proof of Theorem 1.

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