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JEAN-MARIE MORVAN
LOUIS NIGLIO

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Isotropic characteristic classes

JEAN-MARIE MORVAN[1] AND LOUIS NIGLIO[2]

[1] *Université de Lyon I., 43 Bd. du 11 Novembre 1918, 69622 Villeurbanne, France; e-mail: mowan @ celeste. univ-lyon 1. fr*

[2] *Université d'Avignon, 33 Rue Louis Pasteur, 84000 Avignon, France*

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This article is devoted to the study of cohomological invariants which arise in symplectic geometry in the theory of isotropic submanifolds of a symplectic manifold, or couples of isotropic subbundles of a symplectic vector bundle. The origin of this interest is the study of Lagrangian subbundles, in which the first cohomological invariant appears: The Maslov class, which is a generator of the first cohomology group $H^1(\Lambda, \mathbb{R})$ of the Lagrangian Grassmannian Λ. Other cohomology classes appear in higher dimensions $(4l + 1)$, describing the cohomology of Λ, and can be interpreted as secondary characteristic classes. All these classes have the same geometric property: On a symplectic bundle, endowed with two Lagrangian subbundles, they are obstructions to their transversality. Their topological study is in [Fu]. In [Mo 1], [Mo 2], the first author proved that the (usual) Maslov class of a Lagrangian submanifold of a complex vector space is nothing but the class defined by the dual of the mean curvature vector of the immersion. In [M-N 1] we could prove that all the Maslov classes (of any degree $4l + 1$) can be spanned by closed forms built with the second fundamental form of the immersion. The generalisation to couples of Lagrangian subbundles is obvious. In the present work, we extend these results to the isotropic case. In the first part, we study the De Rham cohomology of the isotropic Grassmannian $\mathscr{T}G_n(\mathbb{C}^{n+k})$ of isotropic n-dimensional oriented real subspaces of $\mathbb{C}^{n+k}$. We use geometric methods, in order to be able to describe explicitely the cohomology classes in terms of closed differential forms. (An algebraic point of view is summarised in a Note [Mo 3], and a topological study is in [La]). This is much more delicate than in the Lagrangian case, essentially because $\mathscr{T}G_n(\mathbb{C}^{n+k})$ is not a symmetric space. Basically, the cohomology is spanned by classes of degree $4l + 1$, like in the Lagrangian case, but the minimal degree $4l_0 + 1$ satisfies $k < 2l_0 + 1$.

As an application, we define in the second part, cohomology forms and classes on isotropic submanifolds of $\mathbb{C}^{n+k}$, by pulling back these forms through the Gauss map of the immersion. These classes can be expressed in terms of the curvature and the second fundamental form of the immersion.

In the third part, we extend the results obtained for isotropic submanifolds to symplectic bundles endowed with two isotropic subbundles, (of the same dimension). Using Chern-Weil theory, this can be interpreted as follows: To each isotropic subbundle corresponds a reduction to $SO(n) \times U(k)$ of the $U(n+k)$-principal frame bundle associated to the symplectic bundle. This leads us to construct secondary characteristic forms and classes, using adapted connections. Unfortunately, for technical reasons, we need to assume the existence of a third isotropic subbundle endowed with a flat connexion.

We examine, in the fourth part, interesting properties of these classes. As classic characteristic classes, if one characteristic class of a couple (I, I_0) is not null, then it is not possible to deform I into I_0 through isotropic subbundles. Another property is a generalisation of the Lagrangian transversality property: Let $I_0^\perp$ be the coisotropic subbundle of E which is orthogonal (for an adapted metric) to I_0. If one of the characteristic classes of (I, I_0) is non zero, then I and $I_0^\perp$ are not transverse everywhere; (more precisely I cannot be deformed through isotropic subbundles into a subbundle I' which is transverse to $I_0^\perp$ everywhere).

If the dimension of I is odd, the last isotropic class has a particular interest. We show, in the fourth part, that it is also an obstruction to the deformation of I in I_0 through any (oriented) subbundle (of constant rank) of E.

Finally, we would like to conclude that this kind of framework can be generalised in the very large context of reduction of the structural group of a principal bundle. This point of view will be adopted by the second author in a forthcoming paper.

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1. The real cohomology of the isotropic Grassmannian

1.1. *The isotropic Grassmannian* $\mathcal{T}G_n(\mathbb{C}^{n+k})$

Let $\mathbb{C}^{n+k}$ be the complex vector space of real dimension $2(n+k)$ endowed with its canonical scalar product $<, >$, its complex structure J and its symplectic structure σ, given by

$$\sigma(X, Y) = \langle JX, Y\rangle, \quad \forall X, Y \in \mathbb{C}^{n+k}.$$

A real vector subspace I of $\mathbb{C}^{n+k}$ is called isotropic if $I \subset I^\circ$, where "$\circ$" denotes the orthogonal for the symplectic structure.

A real vector subspace C of $\mathbb{C}^{n+k}$ is called coisotropic if $C^\circ \subset C$.

A real vector subspace L of $\mathbb{C}^{n+k}$ is called Lagrangian if $L^\circ = L$.

In this article, we shall deal with the Grassmannian $\mathcal{T}G_n(\mathbb{C}^{n+k})$ of real oriented n-dimensional isotropic vector subspaces I of $\mathbb{C}^{n+k}$. It is clear that there is a natural identification between $\mathcal{T}G_n(\mathbb{C}^{n+k})$ and the homogeneous space

$$U(n+k)/SO(n) \times U(k),$$

where $SO(n)$, (resp. $U(k)$), denotes the orthogonal, (resp. unitary) group. $SO(n) \times U(k)$ is the subgroup of $U(n+k)$ given by the identification

$$(M_1, M_2) \rightarrow \begin{pmatrix} M_1 & O \\ O & M_2 \end{pmatrix} \quad \forall M_1 \in SO(n), \forall M_2 \in U(k).$$

Its tangent space at the origin,

$$T_e(\mathscr{T}G_n(\mathbb{C}^{n+k})) \simeq T_e(U(n+k)/SO(n) \times U(k)) \simeq \mathfrak{u}(n+k)/\mathfrak{so}(n) \oplus \mathfrak{u}(k),$$

(where $\mathfrak{u}(n)$, (*resp.* $\mathfrak{so}(n)$), is the Lie algebra of $U(n)$, (resp. $SO(n)$)), can be identified with the space $\mathfrak{m}$ of squared matrices of the following type:

$$\begin{pmatrix} iA & -{}^t\bar{B} \\ B & O \end{pmatrix} \tag{1}$$

where B is a complex (n, k) matrix, and A is a real (n, n) symmetric matrix.

In the following, we shall use the following decomposition:

$$\mathfrak{u}(n+k) = (\mathfrak{so}(n) \oplus \mathfrak{u}(k)) \oplus \mathfrak{m},$$

where $\mathfrak{m}$ is the space of matrices of type (1).

1.2. Real cohomology of a compact Lie group

We recall here some basic facts on the cohomology of a connected compact Lie group G. Let $\mathfrak{g}$ be the Lie algebra of G, and $I(G)$ be the algebra of invariant polynomials on $\mathfrak{g}$. Let $\wedge_{BI}(G)$ be the space of bi-invariant differential forms in G. The *Cartan map* is the linear map

$$\mathscr{C} = I(G) \rightarrow \wedge_{BI}(G)$$

defined on the homogeneous elements of $I(G)$ by

$$\mathscr{C}(f)(X_1, \ldots, X_{2l-1}) = \frac{(-1)^{(l-1)}(l-1)!}{2^{l-1}} f(X_1, [X_2, X_3], \ldots, [X_{2l-2}, X_{2l-1}]),$$

for every invariant polynomial f of degree l.

The image of $\mathscr{C}$ is a vector subspace $P(G) \subset \wedge_{BI}(G)$, called the *Samelson space* of G. It is well known that the cohomology algebra of G, $H^*(G, \mathbb{R})$, is isomorphic to the exterior algebra $\wedge_{BI}(G)$, and also that

$$H^*(G, \mathbb{R}) \simeq \wedge P(G).$$

1.3. Real cohomology of a homogeneous space

Let G be a compact connected Lie group, and H be a compact connected subgroup of G. Let

$$\tau\colon P(G) \to I(G),$$

be a transgression ($\mathscr{C} \circ \tau = \mathrm{Id}_{P(G)}$).

It is well known that the real cohomology of G/H, $H^*(G/H, \mathbb{R})$, is isomorphic to the real cohomology of the algebra

$$W = \wedge P(G) \otimes I(H),$$

endowed with the differential d given by

$$\mathrm{d}(1 \otimes c) = 0,$$
$$\mathrm{d}(x \otimes 1) = 1 \otimes \tau(x)_{|\mathfrak{h}},$$

where $\mathfrak{h}$ is the Lie algebra of H.

Let $P(G, H)$ be the Samelson space of the pair (G, H). The elements of $P(G, H)$ are characterised as follows: let $z \in P(G)$. Then, $z \in P(G, H)$ if and only if there exists $t_1, \ldots, t_m \in P(G)$, $r_1, \ldots, r_m \in I(H)$ of strictly positive degree, such that

$$\mathrm{d}z = (\mathrm{d}t_1).r_1 + \cdots + (\mathrm{d}t_m).r_m.$$

Let

$$\pi\colon G \to G/H$$

be the canonical projection, and

$$\pi^*\colon H^*(G/H, \mathbb{R}) \to H^*(G, \mathbb{R})$$

be the corresponding map in cohomology. We know that the image of π^* is an exterior algebra over $P(G, H)$. Moreover,

$$\dim P(G, H) \leqslant \operatorname{rank}(G) - \operatorname{rank}(H). \tag{2}$$

When (2) is an equality, we say that (G, H) is a *Cartan pair*.

If (G, H) is a Cartan pair, we have

$$H^*(G/H, \mathbb{R}) \simeq \wedge P(G, H) \otimes \mathscr{A},$$

where $\mathscr{A}$ denotes the ring of characteristic classes of the bundle $G \to G/H$.

1.4. *Real cohomology of $\mathcal{T}G_n(\mathbb{C}^{n+k})$*

Using the identification between $\mathcal{T}G_n(\mathbb{C}^{n+k})$ and $U(n+k)/SO(n) \times U(k)$, we have to compute the cohomology of G/H, with $G = U(n+k)$, and $H = SO(n) \times U(k)$. We apply the results of 1.2 and 1.3.

It is well known that:

(i) $I(U(k))$ is isomorphic to $\mathbb{R}[\tilde{c}]$, the algebra of polynomials spanned by the Chern generators $(\tilde{c}_1, \ldots, \tilde{c}_k)$.

(ii) $I(SO(n))$ is isomorphic to $\mathbb{R}[p]$, the algebra of polynomials spanned by the Pontrjaguin generators $(p_1, \ldots, p_{[n/2]}, e_n)$. These generators are related by the only relations

$$\begin{aligned} &e_n = 0 && \text{if } n \text{ is odd,} \\ &e_n^2 = p_{[n/2]} && \text{if } n \text{ is even.} \end{aligned}$$

(iii) From (i) and (ii), we deduce that

$$I(H) \simeq \mathbb{R}[\tilde{c}] \otimes \mathbb{R}[p].$$

(iv) $\wedge P(U(n+k))$ is isomorphic to the algebra

$$\wedge(x) = \wedge(x_1, \ldots, x_{2n+2k-1}),$$

where the index denotes the degree of the generators.

Each (x_{2l-1}) is identified with the bi-invariant differential form θ_{2l-1} on $U(n+k)$, which is the image by the Cartan map of the Chern polynomials (c_l).

(v) We choose the transgression τ, defined by $\tau(x_{2i-1}) = c_i$ for all i.

Consequently, the real cohomology of $U(n+k)/SO(n) \times U(k)$ can be identified with the cohomology of the graded algebra $(\mathbb{R}[\tilde{c}] \otimes \mathbb{R}[p]) \otimes \wedge(x)$, endowed with the differential d given by

$$\begin{aligned} &\mathrm{d}(\mathbb{R}[\tilde{c}] \otimes 1) = 0, \\ &\mathrm{d}(\mathbb{R}[p] \otimes 1) = 0, \\ &\mathrm{d}(1 \otimes x_{2i-1}) = c_{i|\mathfrak{so}(n) \oplus \mathfrak{u}(k)}. \end{aligned} \tag{3}$$

To give an explicit description of the cohomology of $\mathcal{T}G_n(\mathbb{C}^{n+k})$, we need the following

1.5. LEMMA. *Let $(c_l) \subset I(U(n+k))$, $1 \leqslant l \leqslant n+k$, be the Chern polynomials of $U(n+k)$.*

Let $(\tilde{c}_i) \subset I(U(k))$, $1 \leqslant i \leqslant k$, be the Chern polynomials of $U(k)$.

Let $(p_j) \subset I(SO(n))$, $1 \leqslant 2j \leqslant n$, *be the Pontrjaguin polynomials of* $SO(n)$. *Then we have:*

(i) *For* $1 \leqslant 2l+1 \leqslant k$, $c_{2l+1}|_{\mathfrak{so}(n)\oplus \mathfrak{u}(k)} = \tilde{c}_{2l+1} + \Sigma_{i+j=l}(-1)^j \tilde{c}_{2i+1}p_j$,

(ii) *For* $k < 2l+1 \leqslant n+k$, $c_{2l+1}|_{\mathfrak{so}(n)\oplus \mathfrak{u}(k)} = \Sigma_{i+j=l}(-1)^j \tilde{c}_{2i+1}p_j$,

(iii) *If n is odd*, $c_{n+k}|_{\mathfrak{so}(n)\oplus\mathfrak{u}(k)} = 0$.

Proof of the lemma. Let $X \in \mathfrak{u}(n+k)$, such that:

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \qquad X_1 \in \mathfrak{so}(n),\ X_2 \in \mathfrak{u}(k).$$

We have

$$\det(\lambda I_{n+k} + iX) = \det(\lambda I_n + iX_1)\det(\lambda I_k + iX_2).$$

Then,

$$\sum_l (-1)^l c_l \lambda^{n+k-l} = \left(\sum_i (-1)^i \tilde{c}_i \lambda^{k-i}\right)\left(\sum_j (-1)^j p_j \lambda^{n-2j}\right).$$

By identification we deduce immediately the Lemma.

1.6. PROPOSITION. *The differential* d *defined on* $(\mathbb{R}[\tilde{c}] \otimes \mathbb{R}[p]) \otimes \wedge(x)$ *satisfies:*

$$d(x_{4l+1}) = \tilde{c}_{2l+1} + \sum_{i+j=l} (-1)^j \tilde{c}_{2i+1} p_j,$$

where:

$$1 \otimes x_i = x_i,$$
$$p \otimes 1 = p,$$
$$\tilde{c} \otimes 1 = \tilde{c},$$
$$\tilde{c}_{2i+1} = 0 \quad \textit{if} \quad 2i+1 > k.$$

Proof of the proposition. It is a direct consequence of (3) and Lemma 1.5.

Now, we define the following sequence in $(\mathbb{R}[\tilde{c}] \otimes \mathbb{R}[p]) \otimes \wedge(x)$:

$$\begin{aligned} y_1 &= x_1, \\ y_5 &= x_5 + y_1 p_1, \\ &\vdots \\ y_{4l+1} &= x_{4l+1} + y_{4l-3}p_1 \cdots + (-1)^l y_1 p_l, \\ &\vdots \end{aligned} \tag{4}$$

where $(1 \leqslant 4l + 1 \leqslant 2n + 2k - 1)$.

We obtain from (4):

$$\begin{aligned} \mathrm{d}y_{4l+1} &= \tilde{c}_{2l+1}, \quad 1 \leqslant 2l + 1 \leqslant k, \\ \mathrm{d}y_{4l+1} &= 0, \qquad\ \ k < 2l + 1 \leqslant n + k. \end{aligned} \tag{5}$$

Finally, if n is odd, $x_{2n+2k-1}$ is a cocycle.

Moreover, we deduce from (3),

$$(1 + \tilde{C}_1 + \cdots + \tilde{C}_k)(1 + P_1 + \cdots + P_{[n/2]}) = 1,$$

where $\tilde{C}_i$ and P_j are the characteristic classes corresponding to the $\tilde{c}_i$ and p_j.

(Remark that we can write

$$\mathrm{d}x_{4l+1} = \sum_i x_{4i+1}\varphi_{il}, \ k < 2l + 1 \leqslant n + k, \tag{6}$$

where the φ_{il} are products of Pontrjaguin polynomials).

To show that $(U(n + k), SO(n) \times U(k))$ is a Cartan pair, we must compare the dimension of $P(U(n + k), SO(n) \times U(k))$ with

$$\mathrm{d} = \operatorname{rank}(U(n + k)) - \operatorname{rank}(SO(n) \times U(k)).$$

We have the following board:

n	k	rank of $U(n+k)$	rank of $SO(n)$	rank of $U(k)$	d	number of ℓ such that $k < 2\ell + 1$	dimension of Samelson Space
$2m+1$	$2k_1$	$2m+2k_1+1$	m	$2k_1$	$m+1$	$m+1$	$m+1$
$2m+1$	$2k_1+1$	$2m+2k_1+2$	m	$2k_1+1$	$m+1$	m	$m+1$
$2m$	$2k_1$	$2m+2k_1$	m	$2k_1$	m	m	m
$2m$	$2k_1+1$	$2m+2k_1+1$	m	$2k_1+1$	m	m	m

(Remark that the degrees of x_{4l+1} are different, and that $x_{2n+2k-1}$ is a generator when n and k are odd).

We can summarise the previous results in the following

1.7. THEOREM. (i) *$\mathcal{T}G_n(\mathbb{C}^{n+k})$ is a homogeneous space isomorphic to $U(n + k)/SO(n) \times U(k)$.*

(ii) *$(U(n + k), SO(n) \times U(k))$ is a Cartan pair.*

(iii) *$H^*(\mathcal{T}G_n(\mathbb{C}^{n+k})), \mathbb{R})$ is isomorphic to $H^*[(\mathbb{R}[c] \otimes \mathbb{R}[p]) \otimes \wedge(x), \mathbb{R}]$,* *where* $(\mathbb{R}[c] \otimes \mathbb{R}[p] \otimes \wedge(x)$ *is endowed with the differential* d *given by*

$$\mathrm{d}(\mathbb{R}[\tilde{c}] \otimes 1) = 0,$$
$$\mathrm{d}(\mathbb{R}[p] \otimes 1) = 0,$$
$$\mathrm{d}(1 \otimes x_{2i-1}) = c_{i|so(n) \oplus u(k)}.$$

(iv) $H^*((\mathbb{R}[c] \otimes \mathbb{R}[p]) \otimes \wedge(x), \mathbb{R})$ *is spanned, as a ring,*
–*In even dimension, by the following generators,*

$$(P_1, \dots, P_j, \dots, P_{[n/2]}, E_n, \tilde{C}_1, \dots, \tilde{C}_k),$$

with the only relations,

$$(1 + P_1 + \cdots + P_{[n/2]})(1 + \tilde{C}_1 + \cdots + \tilde{C}_k) = 1,$$
$$E_n = 0 \qquad \text{if } n \text{ is odd,}$$
$$E_n^2 = P_{[n/2]} \qquad \text{is } n \text{ is even.}$$

–*In odd dimension, by the following generators,* (*the index denotes the degree*),

$$(y_{4[(k+1)/2]+1}, \dots, y_{4t+1}, \dots, y_{2n+2k-1}), \qquad \text{if} \quad n \text{ is even and } k \text{ odd,}$$
$$(y_{4[(k+1)/2]+1}, \dots, y_{4t+1}, \dots, y_{2n+2k-3}), \qquad \text{if} \quad n \text{ is even and } k \text{ even,}$$
$$(y_{4[(k+1)/2]+1}, \dots, y_{4t+1}, \dots, y_{2n+2k-3}, x_{2n+2k-1}), \quad \text{if} \quad n \text{ is odd.}$$

(v) *Consider the principal bundle*

$$\pi\colon U(n+k) \to U(n+k)/U(k) \times SO(n).$$

$H^*[\mathcal{T}G_n(\mathbb{C}^{n+k}), \mathbb{R}]$ *is isomorphic* (*as a ring*), *to the tensor product*

$$\pi^*[H^*(U(n+k)/U(k) \times SO(n), \mathbb{R})] \otimes \mathcal{A},$$

where $\mathcal{A}$ *denotes the ring of characteristic classes of the previous principal bundle.*

(vi) $\pi^*[H^*(U(n+k)/U(k) \times SO(n), \mathbb{R})]$, *identified with a subspace of* $\wedge(x)$, *is spanned by the set*

$$\{x_{4l+1}\},\ k < 2l + 1 \leqslant n + k, \quad \text{and} \quad x_{2(n+k)-1} \ \text{if } n \text{ is odd.}$$

The even cohomology of $\mathcal{T}G_n(\mathbb{C}^{n+k})$ is easy to describe in terms of differential forms: Geometrically, the P_j are the Pontrjaguin classes of the tautological bundle v over $\mathscr{T}G_n(\mathbb{C}^{n+k})$. Using the identification of $\mathscr{T}G_n(\mathbb{C}^{n+k})$ with $U(n+k)/SO(n) \times U(k)$, we can express each generator by standard formulas:

Let Ω be the curvature tensor of the space $U(n+k)/SO(n) \times U(k)$. We know that

$$\Omega(X, Y) = -\frac{1}{2}[X, Y]_{\mathfrak{so}(n)\oplus\mathfrak{u}(k)}, \quad \forall X, Y \in \mathfrak{m}.$$

where $[\ ,\ ]_{\mathfrak{so}(n)\oplus\mathfrak{u}(k)}$ denotes the projection of the bracket on $\mathfrak{so}(n) \oplus \mathfrak{u}(k)$. Then the Pontryaguin class P_j is the cohomology class of the closed form Π_j defined by

$$\pi^*(\Pi_j) = \frac{1}{(2j)!}\sum \delta^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}\Omega^{i_1}_{j_1} \wedge \cdots \wedge \Omega^{i_k}_{j_k}, \tag{7}$$

(the Ω^i_j are the components of Ω and $\delta^{j_1,\ldots,j_k}_{i_1,\ldots,i_k}$ the Kronecker symbol).

For our purpose, the odd cohomology is more interesting. We will study it carefully.

We recall the following Lemma [G-H-V]:

1.8. LEMMA. *The bi-invariant forms* θ_{2l-1}, $1 \leqslant l \leqslant n+1$, *defined on* $U(n+k)$ *by*

$$\theta_{2l-1}(X_1,\ldots,X_{2l-1}) = \frac{1}{i^l}\frac{((l-1)!)^2}{(2l-1)!}\sum_{\sigma\in\mathscr{S}^{2l-1}} \varepsilon^\sigma \operatorname{trace}(X_{\sigma_1},\ldots,X_{\sigma_{2l-1}}), \qquad \square$$

$(\forall X_1,\ldots,X_{2l-1} \in \mathfrak{u}(n+k))$, *span the real cohomology of* $U(u+k)$.

Using 1.6 and 1.7, we get the following

1.9. PROPOSITION. *The bi-invariant forms on* $U(n+k)$

$\theta_{4[(k+1)/2]+1},\ldots,\theta_{4l+1},\ldots,\theta_{2n+2k-1}$, *if n is even and k odd*

$\theta_{4[(k+1)/2]+1},\ldots,\theta_{4l+1},\ldots,\theta_{2n+2k-3}$, *if n is even and k even*

$\theta_{4[(k+1)/2]+1},\ldots,\theta_{4l+1},\ldots,\theta_{2n+2k-3}, \theta_{2n+2k-1}$, *if n is odd,*

define a system of generators of $\pi^*(H^*(U(n+k)/U(k) \times SO(n)), \mathbb{R})$.

1.10. An important remark

These forms, up to $\theta_{2n+2k-1}$, if n is even, are not projectable. The following theorem gives explicit closed forms on $U(n+k)/U(k) \times SO(n)$ whose cohomology classes span $H^*(U(n+k)/U(k) \times SO(n), \mathbb{R})$.

1.11. THEOREM. *Let* Ψ_{4l+1} *be the left invariant forms defined by induction on* $T_e(\mathscr{T}G_n(\mathbb{C}^{n+k})) = \mathfrak{m} \subset \mathfrak{u}(n+k)$, *by the formula*

$$\Psi_{4l+1} = \theta_{4l+1} - \sum_{i+j=1} \Psi_{4i+1} \wedge \Pi_j, \qquad \square$$

Then Ψ_{4l+1}, *for* $k < 2l+1 \leqslant n+k$, *and* $\theta_{2n+2k-1}$ *if* n *is odd, are closed on* $\mathscr{T}G_n(\mathbb{C}^{n+k})$.

Their cohomology classes span the odd cohomology of $\mathscr{T}G_n(\mathbb{C}^{n+k})$.

2. Isotropic submanifolds in $\mathbb{C}^{n+k}$

In Theorem (1.11), we gave explicit formulas for the cohomology classes of the isotropic Grassmannian, and their associated closed forms. We shall now use these forms to build characteristic forms and classes on isotropic submanifolds of $\mathbb{C}^{n+k}$.

2.1. *The general geometric frame*

Let $i: M^n \hookrightarrow \mathbb{E}^{n+p}$ be an isometric immersion of an n dimentional oriented Riemannian manifold M^n into $\mathbb{E}^{n+p}$. Let ∇, (resp. $\tilde{\nabla}$) be the Levi-Civita connection of M, (resp. $\mathbb{E}^{n+p}$). We can write:

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y), \\ \tilde{\nabla}_X \xi &= -A_\xi X + \nabla^\perp_X \xi, \end{aligned} \tag{10}$$

for all X, Y in TM and ξ in $T^\perp M$. Let h be the second fundamental form, which takes its values in the normal bundle $T^\perp M$, A is the adjoint of h and $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$.

The Gauss-Codazzi-Ricci equations relate the curvature R of M and the normal curvature $R^\perp$ to the second fundamental form h:

$$\begin{aligned} \langle R(X, Y)Z, W\rangle &= \langle h(X, Z), h(Y, W)\rangle - \langle h(X, W), h(Y, Z)\rangle, \\ (\bar{\nabla}_X h)(Y, Z) &= (\bar{\nabla}_Y h)(X, Z), \\ \langle R^\perp(X, Y)\xi, \eta\rangle &= \langle A_\xi X, A_\eta Y\rangle - \langle A_\xi Y, A_\eta X\rangle, \end{aligned} \tag{11}$$

where $\bar{\nabla} h$ is defined by:

$$\bar{\nabla}_X h(Y, Z) = \nabla^\perp_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(X_1, \nabla_Y Z).$$

Let G be the Gauss map of $i.G$ assigns to each point $m \in M$ the subspace of

$\mathbb{E}^{n+p}$ parallel to $T_mM.G$ takes its values in the Grassmannian $G_n(\mathbb{E}^{n+p})$, identified with:

$$SO(n+p)/SO(n) \times SO(p).$$

Let us fix m. Up to isometry, we may suppose that $G(m) = \mathbb{E}^n$. A classical result asserts that $\mathrm{d}G$ can be identified with h in the following way:

Let $X \in T_mM$, and define

$$h_X\colon T_mM \to T_m^{\perp}M$$

by

$$h_X(Y) = h(X, Y).$$

Identifying $T_m(M)$ with $\mathbb{E}^n$ and $T_m^{\perp}M$ with $\mathbb{E}^p$, we get a ($\mathbb{R}$-linear) map h_X which belongs to the space $\mathrm{End}(\mathbb{E}^n, \mathbb{E}^p)$. Using these identifications, we can write:

$$\mathrm{d}G_m(X) = (h_X)_m,$$

or in a simplified notation:

$$\mathrm{d}G = h.$$

2.2. *Geometry of isotropic submanifolds*

Let us consider now that $i\colon M^n \hookrightarrow \mathbb{C}^{n+k}$ is an isotropic immersion. The Gauss map G can be factorised through $\mathscr{T}G_n(\mathbb{C}^{n+k})$, and we get the following diagram:

$$\begin{tikzcd} M \arrow[rr, "G"] \arrow[dr, "\tilde{G}"'] & & G_n(\mathbb{E}^{2(n+k)}) \\ & \mathscr{T}G_n(\mathbb{C}^{n+k}) \arrow[ur, "j"'] & \end{tikzcd}$$

For $X \in T_mM$, $\mathrm{d}\tilde{G}(X)$ is tangent at $\mathbb{E}^n$ to $\mathscr{T}G_n(\mathbb{C}^{n+k})$, that is to say, $\mathrm{d}\tilde{G}(X)$ belongs to $\mathfrak{m} \subset \mathfrak{u}(n+k)$ (cf. (1.1)). We can write:

$$\mathrm{d}\tilde{G}(X) = \tilde{N} = \begin{pmatrix} iA & -{}^t\bar{B} \\ B & O \end{pmatrix} \tag{12}$$

where B is a complex (n, k) matrix, and A is a real (n, n) symmetric matrix. The data of this matrix are equivalent to the data of the matrix

$$N = \begin{pmatrix} A \\ B \end{pmatrix}$$

The correspondence $\tilde{N}: \to N$ describes the map j above. In the sequel we shall identify $d\tilde{G}(X)$, $dG(X)$ and h_X. In this context, for $X_1, X_2 \in T_m M$, the composition $h_{X_1} \circ h_{X_2}$ has a clear meaning, and corresponds to a matrices product in $\mathfrak{u}(n+k)$. Of course, such a product does not belong to $\mathfrak{m}$.

Since $\tilde{\nabla} J = 0$, h satisfies the following property:

$$\langle h(X, Y), JZ \rangle = \langle h(Z, Y), JX \rangle = \langle h(X, Z), JY \rangle, \tag{13}$$

for all X, Y, Z in TM^n.

2.3. *Characteristic forms and classes of isotropic submanifolds of* $\mathbb{C}^{n+k}$

The previous diagram induces the following ones:

$$\begin{tikzcd}
\wedge G_n(\mathbb{E}^{2n+2k}) \arrow[rr, "G^*"] \arrow[dr, "j^*"'] & & \wedge M \\
& \wedge \mathscr{T} G_n(\mathbb{C}^{n+k}) \arrow[ur, "\tilde{G}^*"'] &
\end{tikzcd}$$

and in cohomology,

$$\begin{tikzcd}
H^*(G_n(\mathbb{E}^{2n+2k}, \mathbb{R}) \arrow[rr, "G^*"] \arrow[dr, "j^*"'] & & H^*(M, \mathbb{R}) \\
& H^*(\mathscr{T} G_n(\mathbb{C}^{n+k}), \mathbb{R}) \arrow[ur, "\tilde{G}^*"'] &
\end{tikzcd}$$

By pulling back the even generators of the cohomology of $\mathscr{T} G_n(\mathbb{C}^{n+k})$, we classically obtain the Pontrjaguin classes P_r of TM^n and the Chern classes C_s of ν. Since $T(\mathbb{C}^{n+k})|_M$ is trivial, these classes are related by the relation

$$(1 + P_1 + \cdots + P_r + \cdots)(1 + C_1 + \cdots + C_s + \cdots) = 1.$$

We shall restrict our attention to the odd cohomology, to get secondary-cohomology classes.

2.4. *Notations*

(i) $\bar{\theta}_{4l+1}$ denotes the $(4l+1)$-differential form defined on M by

$$\bar{\theta}_{4l+1}(X_1, \ldots, X_{4l+1}) = (-1)^l i \frac{(2l-2)!)^2}{(4l+1)!} \sum_{s \in \mathscr{S}^{4l+1}} \varepsilon^{\sigma} \operatorname{trace}(h_{X_{s_1}} \circ \cdots \circ h_{X_{s_{4l+1}}}). \quad (14)$$

(ii) $\bar{\Pi}_j$ denotes the jth Pontrjaguin form defined on M, by using the Riemmannian metric. We state our main

2.5. *Theorem and definition*

Let $i\colon M^n \to \mathbb{C}^{n+k}$ be an isometric isotropic immersion of a Riemannian manifold M^n into $\mathbb{C}^{n+k}$. Let C be a fixed coisotropic $(n+2k)$-subpsace of $\mathbb{C}^{n+k}$. Let α_{4l+1} be the $(4l+1)$-forms defined by induction on M^n by:

$$\alpha_{4l+1} = \bar{\theta}_{4l+1} - \alpha_{4l-3} \wedge \bar{\Pi}_1 - \cdots - \alpha_1 \wedge \bar{\Pi}_l. \quad (15)$$

Then,

(i) *For every l such that $k < 2l+1 \leqslant n+k$, these forms are closed.*

(ii) *If there exists l, $(k < 2l+1 \leqslant n+k)$, such that the cohomology class $[\alpha_{4l+1}]$ is not null, then there does not exist any deformation of M^n through isotropic submanifolds of $\mathbb{C}^{n+k}$ onto a submanifold M_1 which is transversal to the fixed coisotropic $(n+2k)$-subspace C.*

The cohomology classes $[\alpha_{4l+1}]$, $k < 2l+1 \leqslant n+k$, define a system of generators of isotropic classes of M^n.

Proof. (i) We take the pull back of the differential forms defined on $\mathscr{T}G_n(\mathbb{C}^{n+k})$ by the Gauss map of the immersion i. We obtain, from (9),

$$\alpha_{4l+1} = G^*(\Psi_{4l+1}) = G^*(\bar{\theta}_{4l+1}) - G^*(\Psi_{4l-3} \wedge \Pi_1) - \cdots - G^*(\Psi_1 \wedge \Pi_l). \quad (16)$$

The end of the proof is clear.

(ii) Is a simple consequence of a result of F. Lalonde: In $\mathscr{T}G_n(\mathbb{C}^{n+k})$ the set of isotropic n-dimensional subspaces of $\mathbb{C}^{n+k}$ which are transverse to a fixed coisotropic subspace is contractile [La].

2.6. *Remarks*

(i) If $k = 0$, we obtain the classical Maslov classes and Maslov forms described in [M-N 1].

(ii) For obvious reasons of dimension, the pull-back of $\theta_{2n+2k-1}$ is always null.

2.7. Examples

EXAMPLE 1. Although the first form $x_1 = y_1$ (in the notations of 1.6), does not span any cohomology class in $\mathcal{T}G_n(\mathbb{C}^{n+k})$ as soon as $k \geqslant 1$, we can observe the following phenomena, (which can be extended in the context of fiber bundles without difficulties). In $\mathcal{T}G_n(\mathbb{C}^{n+k})$, we have:

$$dx_1 = c_1.$$

The form θ_1 corresponding to x_1 has the following explicit expression:

$$\theta_1(X) = \frac{1}{i}\,\mathrm{trace}(X) \quad \forall X \in T\mathcal{T}G_n(\mathbb{C}^{n+k}). \tag{17}$$

The pull-back on M^n of the form θ_1 by the Gauss map is the form α_1, defined by

$$\alpha_1(X) = \mathrm{trace}\, Jh(X,.) \quad \forall X \in TM^n,$$

that is, using (13),

$$\alpha_1(X) = \langle JH, X\rangle, \forall X \in TM^n, \tag{18}$$

where H is the mean curvature vector field of M^n.

Using Codazzi equation, we see that α_1 is not closed in general, and satisfies, on M^n,

$$d\alpha_1 = S^{\perp}, \tag{19}$$

where $S^{\perp}$ is the Ricci tensor of the normal subbundle ν, defined by

$$S^{\perp}(X, Y) = \sum_{\alpha=1}^{k} \langle R^{\perp}(X, Y)\xi_\alpha, J\xi_\alpha\rangle, \tag{20}$$

$(\xi_\alpha)_{\alpha=1,\dots,k}$ being an orthonormal frame of ν.

We shall say that an *isotropic submanifold is* ν*-flat if* $S^{\perp}$ *is null everywhere.* The simplest way to build a ν-flat isotropic submanifold is to consider a Lagrangian submanifold L of $\mathbb{C}^n$,

$$L \overset{i}{\hookrightarrow} \mathbb{C}^n,$$

and any isotropic immersion of $\mathbb{C}^n$ into $\mathbb{C}^{n+k}$ with flat normal connection, (for instance, the standard totally geodesic one),

$$\mathbb{C}^n \xrightarrow{j} \mathbb{C}^{n+k}.$$

Then $j \circ i$ gives a ν-flat isotropic immersion of L into $\mathbb{C}^{n+k}$.

It is clear that any ν-flat isotropic submanifold is endowed with a real cohomology class of degree one, $[\alpha_1]$. It corresponds to the classical Maslov class for Lagrangian submanifolds, [Mo 1].

The following example shows that, under ν-flat deformations, this cohomology class may vary:

EXAMPLE 2. Let γ_0 be a circle in $\mathbb{C}$, and γ_1 be a curve describing an "height" in $\mathbb{C}$. The Maslov class m_0 of γ_0 is non zero, and the Maslov class m_1 of γ_1 is zero. Up to a constant, the Maslov class is spanned by the 1-form kds, where k is the curvature of γ_o (resp. γ_1) and ds the arc-length. Consider the standard imbedding of $\mathbb{C}$ into $\mathbb{C}^2$. The curves γ_0 and γ_1 are (of course) is isotropic in $\mathbb{C}^2$, and we can deform γ_0 into γ_1 through closed (isotropic) curves γ_t. At each step, $[k_t ds]$ defines a cohomology class which varies form m_o to m_1.

EXAMPLE 3. This example is a generalisation of the previous one. Consider i_0 the standard (Lagrangian) embedding of the torus T^2 into $\mathbb{C}^2$. Its Maslov class m_1 is non zero. Consider the product of two "eight" in $\mathbb{C}$. This gives an Lagrangian immersion of T^2 into $\mathbb{C}^2$, with null Maslov class $\tilde{m}_1$. Let $\mathbb{C}^3 \simeq (\mathbb{C} \times \mathbb{R}) \times (\mathbb{C} \times \mathbb{R})$. These two tori can be considered as isotropic surfaces in $\mathbb{C}^3$, with different Maslov classes. It is clear that we can deform each "eight" in $\mathbb{R}^3\mathbb{C} \times \mathbb{R}$ to get a circle in $\mathbb{C}$. This gives deformation i_t of i_0 into i_1, which is isotropic in $\mathbb{C}^3$ for each t. It is also clearly ν-flat for each t. This shows that the one dimensional cohomology class (given by the mean curvature vector fields of i_t), varies continuously with t. Of course, this phenomena cannot occur for the classes α_{4l+1}, $l \geqslant 2k + 1$, $(2k = \mathrm{rank}(\nu))$.

EXAMPLE 4. This first interesting dimension is $k = 1$, $n = 5$. So, we must consider a 5-dimensonal isotropic submanifold in $\mathbb{C}^6$.

$$M^5 \hookrightarrow \mathbb{C}^6.$$

With the notations of (1.6), the only generator of the odd cohomology in $H^*(\mathscr{T}G_5(\mathbb{C}^6), \mathbb{R})$ is

$$y_5 = x_5 - x_1 p_1,$$

and the corresponding 5-form is given by Φ_5. Consequently, the only isotropic characteristic form which appears in M^5 is

$$\alpha_5 = \tilde{G}^*((\Psi).$$

This situation occurs in the following example: Consider the (complex) vector space V of symmetric complex (3×3)-matrices. V can be identified with $\mathbb{C}^6$. The map

$$\begin{aligned} U(3) &\to V \\ A &\to A^tA \end{aligned}$$

induces a map

$$U(3)/SO(3) \xrightarrow{f} C^6.$$

A simple computation shows that f is a Lagrangian embedding, (see [M-N 1]). Now, the standard inclusion i:

$$SU(3)/SO(3) \overset{i}{\hookrightarrow} U(3)/SO(3)$$

gives rise to an isotropic embedding $f\circ i$:

$$SU(3)/SO(3) \overset{f\circ i}{} C^6.$$

A simple computation shows that the 5-form α_5 defined on $SU(3)/SO(3)$ is nothing but the restriction of the Maslov form of degree 5, defined on $U(3)/SO(3)$ and is a volume form on $SU(3)/SO(3)$. So, $a_5=[\alpha_5]$ is not null.

3. Isotropic subbundles of a symplectic bundle

3.1. *Generalities*

Let $E\to M$ be a symplectic vector-bundle of rank $2(n+k)$. This means that each fiber is a real vector space of dimension $2(n+k)$, endowed with a symplectic form σ. Let $\ll,\gg$ be a hermitian structure on E, adapted to the symplectic form. We denote by $<,>$ the associated Riemannian metric, and by J the complex structure $(J^2=-\mathrm{Id})$ defined by

$$\sigma(.,.) = <J.,.>.$$

(We know that two such hermitian structures are homotopic).

We denote by $P \to M$ the $U(n+k)$-principal bundle of ortho-normal frames on E.

Now, we assume that E admits an (oriented) isotropic subbundle I of rank n. We remark first of all that this assumption implies restrictions on the (primary) characteristic classes of E.

In fact, we can write

$$E = I \oplus JI \oplus \nu,$$

where ν is the orthogonal complement of $I \oplus JI$ (for $<, >$). Consequently, the Chern classes C_i of E, $\tilde{C}_i$ of ν and the Pontrjaguin classes of I are related by the relation:

$$(1 + \tilde{C}_1 + \cdots \tilde{C}_k)(1 + P_1 + \cdots P_{[n/2]}) = 1 + C_1 + \cdots + C_{n+k} \tag{21}$$

3.2. *Remarks*

(i) We deduce from this relation various conclusions on E. For instance, if C_{n+k} is non zero, then, E does not admit any isotropic subbundle of odd rank.

(ii) We can build particular connections on E adapted to the isotropic subbundle of E: Let ω_1 be a (Riemannian) connection on I. We complexify ω_1 to get a connection $\tilde{\omega}_1$ on $I \oplus JI$. We take any complex connection ω_2 on ν. We set

$$\omega = \tilde{\omega}_1 \oplus \omega_2.$$

ω is a connection on E (for which I, JI and ν are parallel). Let Ω the curvature of ω. Then, the classes $\tilde{C}_i$, P_j, C_k can be expressed in terms of Ω (see 1.7) and we get an analogous formula for (21) in terms of Ω.

3.3. From now, we shall assume that E admits two isotropic oriented subbundles (I_0, I) of rank n. We deal with the problem of deformation of I_0 onto I in E, and with the problem of transversality of I_0 and I with a common coisotropic subbundle of E. It is clear that the difference of the Pontrjaguin classes of I_0 and I are obstructions to a deformation of I_0 onto I. We also remark that if I_0 and I are transversal to a common coisotropic subbundle C, then they are isomorphic. (In fact they are isomorphic to $\mathrm{Ker}(\sigma|_C)^*$, where $\sigma|_C$ denotes the restriction of σ to C). So their Pontrjaguin classes are equal.

In the following, we shall go further, and construct explicitly secondary characteristic forms and classes which are deeper obstructions. The reason is that if two isotropic subbundles I_0 and I have a common transversal coisotropic subbundle, then there exists an isotropic deformation which sends I onto

I_0. (See [La] for a proof). Unfortunately, we need to assume that Q_0 is flat, which is not true in general. However, remark that if there exists a trivialisation of E which sends every fibre of I_0 on the standard isotropic subspace of $\mathbb{C}^{n+k}$, then the flatness of Q_0 is clear. If it is not the case, we can increase $I_0^{\perp}$, I and E into $I_0'^{\perp}$, I' and E' such that such a trivialisation exits. (See [La]). Then we can endow Q_0' (with obvious notations) with a flat connection. These secondary classes are *symplectic* in the sense that they are obstructions to:

–Deformations of I onto I_0 through *isotropic* (oriented) subbundles.

–Defomations of I through *isotropic* (oriented) subbundles onto an isotropic (oriented) subbundle $\tilde{I}$ which is transversal to the coisotropic subbundle $I_0^{\perp}$ (where $\perp$ is the orthogonality relative to any adapted metric).

To get these classes, we apply the Chern-Weil construction.

3.4 *The theory of Chern-Weil*

We shall recall here the basic facts on the theory of Chern-Weil.

Let G a Lie group, with Lie algebra $\mathfrak{g}$. Let $I(G)$ be the algebra of invariant polynomials on $\mathfrak{g}$. Let $P \to M$ be a G-principal bundle.

Let ω be any connection on P, with curvature Ω. If $f \in I(G)$ is a homogeneous polynomial of degree l, we define

(i) the $2l-$differential form $\Delta_\omega f$ defined on M by its lift on P:

$$\pi^*\Delta_\omega f = f(\Omega, \dots, \Omega)$$

(ii) the $(2l-1)$-differential form $T_\omega(f)$ defined on P by

$$Tf(\omega) = l\int_0^1 f(\omega, \Omega_t, \dots, \Omega_t)\,\mathrm{d}t \quad \left(\text{where } \Omega_t = t\Omega + \frac{1}{2}(t^2 - t)[\omega, \omega]\right).$$

With these notations, we have the following relations:

$$\mathrm{d}Tf(\omega) = \pi^*\Delta_\omega f, \tag{22}$$

$$\begin{aligned} T(fg)(\omega) &= Tf(\omega) \wedge \pi^*\Delta_\omega(g) + \text{exact form}, \\ &= Tg(\omega \wedge \pi^*\Delta_\omega(f) + \text{exact form}. \end{aligned} \tag{23}$$

If ω_0, ω are two connections on $P \to M$, with curvature Ω_0, Ω, we define on M the $(2l-1)$-differential form $\Delta_{\omega\omega_0} f$ such that

$$\pi^*\Delta_{\omega\omega_0} f = l\int_0^1 f(\omega - \omega_0, t\Omega + (1-t)\Omega_0, \dots, t\Omega + (1-t)\Omega_0)\,\mathrm{d}t.$$

This form satisfies the following relations:

$$d(\pi^* \Delta f) = \pi^* \Delta_{\omega}(f) - \pi^* \Delta_{\omega_0}(f), \tag{24}$$

$$\pi^*(\Delta f) = Tf(\omega) - Tf(\omega_0) + \text{exact form.} \tag{25}$$

We apply the previous theory in the following context: Let (I_0, I) a couple of isotropic oriented subbundles of rank n, of the symplectic bundle E. Let P be the $U(n+k)$-principal bundle of orthonormal frames of E. Let Q_0, (resp. Q), be the principal subbundle of P, constituted by orthonormal rames $(e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+k})$, such that $(e_1, \ldots, e_n)$ is an oriented $\mathbb{R}$-orthonormal frame of I_0, (resp. I). Q_0, (resp. Q) is principal bundle with structural group $SO(n) \times U(k)$. We say that we obtain Q_0, (resp. Q), by reduction of $U(n+k)$ to $SO(n) \times U(k)$. Let $\bar{\omega}_0$, (resp. $\bar{\omega}$), be a connection on Q_0, (resp. Q). We can extend canonically $\bar{\omega}_0$, (resp. $\bar{\omega}$), to a connecion ω_0, (resp. ω), on P. Finally, we get on P two different connections ω_0, ω.

3.5. *Isotropic characteristic classes*

We suppose E endowed with an adapted hermitian structure. In the sequel, the polynomials φ_{il} are those defined in (6) Section 1.6.

THEOREM. *Let (I, I_0) be a couple of isotropic subbundles. If Q_0 admits a flat connection $\bar{\omega}_0$, then:*

(i) *The differential form of degree $4l+1$, defined on M by*

$$\phi_{4l+1}(\bar{\omega}) = \Delta_{\omega\omega_0}(c_{2l+1}) - \sum_{2i+1 \leqslant k} \Delta_{\omega\omega_0}(c_{2i+1}) \wedge \Delta_{\bar{\omega}}(\varphi_{il})$$

$$(k < 2l+1 \leqslant n+k),$$

and $\Delta_{\omega\omega_0}(c_{n+k})$ if n is odd are closed.

(ii) *Their comology classes does not depend on the adapted hermitian structure nor on the connection $\bar{\omega}$.*

(iii) *If I can be deformed into I_0 through an isotropic deformation, then these classes are null.*

(iv) *If I is transversal to the coisotropic subbundle $I_0^{\perp}$, then these classes are null.*

Proof. (i) We will compute $d\pi^* \phi_{2l+1}(\bar{\omega})$. It suffices to compute its values on Q. Then we have, using $\bar{\Omega}_0 = 0$:

$$\pi^* d(\phi_{4l+1}(\bar{\omega})) = \pi^* \Delta_{\omega}(c_{2l+1}) - \sum_{2i+1 \leqslant k} \pi^* \Delta_{\omega}(c_{2i+1}) * \Delta_{\bar{\omega}}(\varphi_{il}),$$

$$\pi^* d(\phi_{4l+1}(\bar{\omega}))|_Q = \pi^* \Delta_{\bar{\omega}}\left(c_{2l+1} - \sum_{2i+1 \leqslant k} c_{2i+1} \cdot \varphi_{il} \right) = \pi \wedge 0 = 0,$$

since $\bar{\Omega}$ takes its values in $\mathfrak{so}(n) \oplus u(k)$ (see 1.5). The same proof is also valid for $\Delta_{\omega\omega_0}(c_{n+k})$.

(ii) and (iii). We have seen above that two hermitian structures compatible with the symplectic structure are homotopic. For the sequel, let Q^s be a differentiable family of $SO(n) \times U(k)$ reductions of $P \to M$, that is to say a $SO(n) \times U(k)$ reduction $\hat{Q} \to M \times I$ of $P \times I \to M \times I$ (where $I = [0, 1]$), such that $Q^s = \hat{Q}|_{s=cte}$. We suppose that every Q^s is endowed with a connection $\bar{\omega}^s$ varying differentiably with s. Every $\bar{\omega}^s$ can be extended in ω^s to P. The collection of $\bar{\omega}^s$ define a connection $\bar{\gamma}$ on $\hat{Q}$, and the collection of ω^s define a connection γ on $P \times I$ which is the extension of $\bar{\gamma}$. Let $\hat{Q}_0 = Q_0 \times I$ be endowed with the connection $\bar{\gamma}_0$ obtained from $\bar{\omega}_0$ by the previous construction. Let i_0 and i_1 be the canonical injections of M into $M \times I$. We use fiber integration ([Le], [Va]) with the differential form $\phi_{4l+1}(\bar{\gamma})$:

$$i_1^*(\phi_{4l+1}(\bar{\gamma})) - i_0^*(\phi_{4l+1}(\bar{\gamma})) = \fint_I \mathrm{d}(\phi_{4l+1}(\bar{\gamma})) - \mathrm{d} \fint_I \phi_{4l+1}(\bar{\gamma}).$$

Since $\phi_{4l+1}(\bar{\gamma})$ is closed (by (i)), we obtain:

$$\phi_{4l+1}(\bar{\omega}^1) - (\phi_{4l+1}(\bar{\omega}^0) = \text{exact form on } M.$$

(iv) This is a trivial consequence of Section 2.2.

3.6. *Definition*

The classes $\alpha_{4l+1}(I, I_0) = [\phi_{4l+1}]$ *and* $\alpha_{2n+2k-1}(I, I_0) = [\Delta_{\omega\omega_0}(c_{n+k})]$ *if* n *is odd are called isotropic (secondary characteristic) classes of the couple* (I, I_0).

3.7. *Isotropic characteristic classes of a couple of isotropic subbundles*

The previous definition of isotropic characteristic classes can be extended to any couple of isotropic subbundles of a symplectic bundle in the following way:

DEFINITION. *Let* I_0 *an isotropic subbundle of* $E \to M$ *such that* Q_0 *admits a flat connection. Let* I *and* I' *be two oriented isotropic subbundles of the symplectic bundle* $E \to M$. *The cohomology classes:*

$$\alpha_{4l+1}(I, I') = \alpha_{4l+1}(I, I_0) - \alpha_{4l+1}(I', I_0),$$

and

$$\alpha_{2n+2k-1}(I, I') = \alpha_{2n+2k-1}(I, I_0) - \alpha_{2n+2k-1}(I', I_0), \textit{ if } n \textit{ is odd,}$$

are called isotropic characteristic classes of the couple (I, I').

It is clear that these classes are obstructions to deformations of I onto I', through oriented isotropic subbundles of E, and that if one of these classes in not null, then I is not transverse to $I'^{\perp}$.

3.8. *Remark*

Let $E \to M$ be a *trivial* symplectic bundle, endowed with a trivial isotropic subbundle I_0. Let I be another isotropic subbundle of E. Then I gives rise to a "generalised Gauss map" $\tilde{G}$ defined as follows: we put an hermitian metric on E, compatible with the symplectic structure, and consider a frame

$$\{e_1, \dots, e_n, e_{n+1}, \dots, e_{n+k}\},$$

such that $(e_1, \dots, e_n)$ is a real frame of I_o. For $m \in M$, we set $\tilde{G}(m) = \dot{u}$, where u is any unitary matrix which sends I_o into I, ($\dot{u}$ denotes the equivalence class of u). We get a map $\tilde{G}$ which takes its values into

$$\mathscr{T}G_n(\mathbb{C}^{n+k}) \simeq U(n+k)/SO(n) \times U(k).$$

Then, the isotropic characteristic classes of (I, I_0) defined in Section 3.6 are the pull back by $\tilde{G}$ of the odd generators of $\mathscr{T}G_n(\mathbb{C}^{n+k})$.

3.9. An example

The simplest example of a symplectic bundle endowed with two isotropic subbundles with non trivial isotropic characteristic classes is the following:

Let $\mathbb{C}^{n+k} = \mathbb{E}^{n+k} \otimes \mathbb{C} = (\mathbb{E}^n \times \mathbb{E}^k) \otimes \mathbb{C}$; consider the trivial bundle

$$\mathscr{T}G_n(\mathbb{C}^{n+k}) \times \mathbb{C}^{n+k} \mathscr{T}G_n(\mathbb{C}^{n+k}),$$

and let $I_0 = \mathbb{E}^n \times 0$, be the first (fixed) isotropic subbundle.

Let I be the canonical real n-bundle on $\mathscr{T}G_n(\mathbb{C}^{n+k})$.

It is clear that the isotropic characteristic classes of (I, I_o) coincide with the odd generators of the cohomology of $\mathscr{T}G_n(\mathbb{C}^{n+k})$.

4. A particular property of the last isotropic class

In this paragraph, we shall deal with the isotropic class of maximum degree, when n is odd. We consider a symplectic bundle E endowed with two oriented isotropic subbundles I and I_0 of odd rank n. As in Section 3, we assume that the $SO(n) \times U(k)$ reduction corresponding to I_0 admits a flat connection $\tilde{\omega}_0$. Under this assumption, we have the following:

4.1. *Theorem*

The isotropic class of maximum degree $\alpha_{2n+2k-1}$ *associated to the couple* (I, I_0) *is an obstruction to the deformation of* I *onto* I_0 *through any deformation of (oriented) subbundles of* E, *(the deformation does not need to be isotropic).*

To prove this theorem, we need the following: 4.2. Lemma.

Let $j\colon U(n+k) \to SO(2(n+k))$ *be the standard inclusion. Let* pf *be the Pfaffian in* $I(SO(2(n+k)))$ *defined by:*

$$pf(X) = \sqrt{\det X} \quad \textit{for} \quad X \in \mathfrak{so}(2(n+k)).$$

Then the restriction to $\mathfrak{u}(n+k)$ *of* pf *is the Chern polynomial of degree* $2n+2k$ *in* $I(U(n+k))$.

Proof of the Lemma. We can check this proposition on maximal tori of each group [K-N]. j is then the map defined in the following way:

If $A = \mathrm{Diag}(i\lambda_1, \ldots, i\lambda_{n+k})$, then

$$j(A) = \mathrm{Diag}\left(\begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} (0 & -\lambda_{n+k} \\ \lambda_{n+k} & 0) \end{pmatrix}\right)$$

whose determinant is $\lambda_1^2, \ldots, \lambda_{n+k}^2$. Since $(-1)^{n+k} c_{n+k}(A) = (-1)^{n+k} \mathrm{Det}(iA) = \lambda_1, \ldots, \lambda_{n+k}$ we obtain the lemma.

We can now prove the theorem. We consider I_0 and I as real subbundles of E. This means that we forget the hermitian structure, and we extend the $U(n+k)$-principal bundle $P \to M$ to the $SO(2(n+k))$-principal bundle of all oriented orthonormal $2(n+k)$-frames of E. We extend the connections ω_0 and ω to ω'_0 and ω' on this new principal bundle. With this notation, $[_{\omega'\omega'_0}(pf)$ define a cohomology class on M, invariant by any deformation. In particular, we have with obvious notations:

$$\pi'^*(\Delta_{\omega'\omega'_0}(pf)) = \pi^*(\Delta_{\omega\omega_0}(c_{2n+2k})).$$

4.3. *Remark*

If n is even, this construction is valid, but the class of $\Delta_{\omega'\omega'_0}(pf))$ is null.

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