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A problem of D. H. Lehmer and its generalization (II)*

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Abstract. Let $q > 2$ be an odd number. For each integer x with $0 < x < q$ and $(q, x) = 1$, we define \bar{x} by $x\bar{x} \equiv 1 \pmod{q}$ and $0 < \bar{x} < q$. Let $r(q)$ be the number of integers x with $0 < x < q$ for which x and \bar{x} are of opposite parity. The main purpose of this paper is to give a sharper asymptotic formula for $r(q)$ for all odd numbers q .

1. Introduction

Let q be an odd integer > 2 . For each integer x with $0 < x < q$ and $(q, x) = 1$, we know that there exists one and only one \bar{x} with $0 < \bar{x} < q$ such that $x\bar{x} \equiv 1 \pmod{q}$. Let $r(q)$ be the number of cases in which x and \bar{x} are of opposite parity. For example, $r(3) = 0$, $r(5) = 2$, $r(7) = 0$, $r(13) = 6$. For $q = p$ a prime, D. H. Lehmer [1] asks us to find $r(p)$ or at least to say something nontrivial about it. It is known that $r(p) \equiv 2$ or $0 \pmod{4}$ according to $p \equiv \pm 1 \pmod{4}$. About this problem, the author [2] obtained an asymptotic formula for $r(p^2)$ and $r(p_1 p_2)$, where p , p_1 and p_2 are primes. In this paper, as an improvement of [2], we shall give an asymptotic formula for $r(q)$ for all odd numbers q . The constants implied by the O -symbols and the symbols \ll , \gg used in this paper do not depend on any parameter, unless otherwise indicated. By using estimates for character sum and Kloosterman sums, and the properties of Dirichlet L-functions, we prove the following two theorems:

THEOREM 1. *For every prime $p > 2$ we have the asymptotic formula*

$$r(p) = \frac{1}{2} p + \frac{1}{\pi^2} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} \{S(a, b; p) - S(a, -b; p) \\ + 4S(\bar{4}a, b; p) - 4S(\bar{4}a, -b; p) - 4S(\bar{2}a, b; p) \\ + 4S(\bar{2}a, -b; p)\} + O(\ln^3 p)$$

where $d\bar{d} \equiv 1 \pmod{p}$, $S(m, n; p) = \sum'_{d \pmod{p}} e\left(m \frac{d}{p} + n \frac{\bar{d}}{p}\right)$ is the Kloosterman sum, and $e(y) = e^{2\pi iy}$.

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THEOREM 2. For every odd integer $q > 2$ we have

$$r(q) = \frac{1}{2}\phi(q) + O(q^{1/2}d^2(q)\ln^2 q)$$

where $\phi(q)$ is the Euler function and $d(q)$ is the divisor function.

From theorem 1 we can see that if we could get a nontrivial upper bound estimate for the mean value $\sum_{q=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} S(ma, b; p)$, then we may obtain a more accurate asymptotic formula. From theorem 2 we can also deduce the following:

COROLLARY. For every prime $p > 2$ we have the asymptotic formula

$$r(p) = \frac{1}{2}p + O(p^{1/2} \ln^2 p)$$

2. Some lemmas

To complete the proofs of the theorems, we need some lemmas. First we have:

LEMMA 1. Let $q > 2$ be an odd number. Then we have

$$r(q) = \frac{1}{2} \phi(q) - \frac{2}{\phi(q)} \sum_{\chi(-1)=-1} \chi(4) \left(\sum_{a=1}^{(q-1)/2} \chi(a) \right)^2$$

where the summation is over all odd characters mod q .

Proof. From the definition of $r(q)$ and the orthogonality of characters we get

$$\begin{aligned} r(q) &= \frac{1}{2} \sum_{a=1}^{q-1} \sum_{\substack{b=1 \\ ab \equiv 1(q)}}^{q-1} \{1 - (-1)^{a+b}\} \\ &= \frac{1}{2} \phi(q) - \frac{1}{2} \sum_{a=1}^{q-1} \sum_{\substack{b=1 \\ ab \equiv 1(q)}}^{q-1} (-1)^{a+b} \\ &= \frac{1}{2} \phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \bmod q} \left(\sum_{a=1}^{q-1} (-1)^a \chi(a) \right)^2 \\ &= \frac{1}{2} \phi(q) - \frac{1}{2\phi(q)} \sum_{\chi \neq \chi^0} \left(\sum_{a=1}^{q-1} (-1)^a \chi(a) \right)^2 \end{aligned} \tag{1}$$

where $\sum_{\chi \neq \chi^0}$ denotes the summation over all nonprincipal characters mod q .

Now if $\chi(-1) = 1$ and $\chi \neq \chi^0$, then we have

$$\sum_{a=1}^{q-1} (-1)^a \chi(a) = 0 \tag{2}$$

while if $\chi(-1) = -1$, then

$$\sum_{a=1}^{q-1} (-1)^a \chi(a) = 2 \sum_{a=1}^{(q-1)/2} \chi(2a) \tag{3}$$

Combining (1), (2) and (3) we may immediately deduce lemma 1. □

LEMMA 2. *Let $q > 1$ be any odd number and let χ be any Dirichlet character modulo q , not necessarily primitive. Then*

$$(1 - 2\chi(2)) \sum_{\gamma=1}^{q-1} \gamma \chi(\gamma) = \chi(2)q \sum_{\gamma=1}^{(q-1)/2} \chi(\gamma)$$

Proof. (See reference [3]). □

LEMMA 3. *Let m, n, q be integers with $q > 1$. Then*

$$S(m, n; q) = \sum_{d=1}^{q-1} e \left(m \frac{d}{q} + n \frac{\bar{d}}{q} \right) \ll (m, n, q)^{1/2} q^{1/2} d(q)$$

where $d\bar{d} \equiv 1 \pmod{q}$, $d(q)$ is the divisor function, and (m, n, q) denotes the greatest common factor of m, n and q . \sum'_a denotes the summation over a such that $(a, q) = 1$.

Proof. (See reference [4]). □

LEMMA 4. *Let q be an odd integer > 2 . Then for any integer b we have the estimate*

$$\sum_{\chi(-1)=-1} \tau^2(\chi) \chi(b) L^2(1, \bar{\chi}) \ll \phi(q) q^{1/2} d(q) \ln^2 q$$

where $L(s, \chi)$ is the Dirichlet L -function and $\tau(\chi)$ is the Gauss sum corresponding to χ .

Proof. First for any integer r with $(r, q) = 1$ we have

$$\sum_{\chi(-1)=-1} \chi(r) = \begin{cases} \frac{1}{2} \phi(q), & \text{if } r \equiv 1 \pmod{q}, \\ -\frac{1}{2} \phi(q), & \text{if } r \equiv -1 \pmod{q}, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

$$\begin{aligned} \frac{2}{p-1} \sum_{\chi_p(-1)=-1} \tau^2(\chi)\chi(b)L^2(1, \bar{\chi}) &= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ac} S(\bar{b}, ac; p) \\ &- \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \frac{1}{ac} S(\bar{b}, -ac; p) + O(\ln^3 p) \end{aligned}$$

Proof. Let $K(r, q) = \sum_{a=1}^{p-1} a^q e\left(\frac{ra}{p}\right)$, $p \nmid r$, then we have the estimate

$$K(r, q) \ll \frac{p^q}{\left| \sin\left(\frac{\pi r}{p}\right) \right|} \tag{7}$$

Namely we have

$$\begin{aligned} K(\gamma, q) \left(1 - e\left(\frac{\gamma}{p}\right)\right) &= \sum_{a=1}^{p-1} a^q \left(e\left(\frac{\gamma a}{p}\right) - e\left(\frac{(a+1)\gamma}{p}\right) \right) \\ &= \sum_{a=1}^{p-2} \left((a+1)^q - a^q \right) e\left(\frac{(a+1)\gamma}{p}\right) + e\left(\frac{\gamma}{p}\right) - (p-1)^q e\left(\frac{p\gamma}{p}\right) \\ &\ll p^q + \sum_{a=1}^{p-2} \left((a+1)^q - a^q \right) \ll p^q. \end{aligned}$$

Thus

$$|K(\gamma, q)| \ll \frac{p^q}{\left| 1 - e\left(\frac{\gamma}{p}\right) \right|} \ll \frac{p^q}{\left| \sin\left(\frac{\pi\gamma}{p}\right) \right|}.$$

This completes the proof of (7). □

By applying (6) with $q = p$, $N = p^2 + p$ we get

$$\begin{aligned} \frac{2}{p-1} \sum_{\chi_p(-1)=-1} \tau^2(\chi)\chi(b)L^2(1, \bar{\chi}) &= \sum'_{1 \leq m, n \leq p^2+p} \frac{1}{mn} \{S(1, \bar{b}mn; p) - S(1, -\bar{b}mn; p)\} + O\left(\frac{\ln^2 p}{\sqrt{p}}\right) \\ &= \sum_{a=1}^{p-1} \sum_{n=0}^p \sum_{c=1}^{p-1} \sum_{m=0}^p \frac{1}{(np+a)(mp+a)} \{S(1, \bar{b}ac; p) - S(1, -\bar{b}ac; p)\} \\ &+ O\left(\frac{\ln^2 p}{\sqrt{p}}\right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \frac{1}{ac} \{S(\bar{b}, ac; p) - S(\bar{b}, -ac; p)\} + O\left(\frac{\ln^2 p}{\sqrt{p}}\right) \\
&\quad + \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^p \frac{1}{a(mp+c)} \{S(1, \bar{b}ac; p) - S(1, -\bar{b}ac; p)\} \\
&\quad + \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{n=1}^p \frac{1}{c(np+a)} \{S(1, \bar{b}ac; p) - S(1, -\bar{b}ac; p)\} \\
&\quad + \sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^p \sum_{n=1}^p \frac{1}{(np+a)(mp+c)} \{S(1, \bar{b}ac; p) - S(1, -\bar{b}ac; p)\} \\
&\equiv M_1 + O\left(\frac{\ln^2 p}{\sqrt{p}}\right) + M_2 + M_3 + M_4 \tag{8}
\end{aligned}$$

It is clear that M_1 is the main term of lemma 5. Now we shall estimate the other three terms M_2 , M_3 and M_4 . Using the power series expansion of $(1-x)^{-1}$ we get

$$\begin{aligned}
&\sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^p \frac{1}{a(mp+c)} S(1, \bar{b}ac; p) \\
&= \sum_{a=1}^{p-1} \frac{1}{a} \sum_{c=1}^{p-1} \sum_{m=1}^p \frac{1}{(m+1)p - (p-c)} S(1, \bar{b}ac; p) \\
&= \sum_{a=1}^{p-1} \frac{1}{a} \sum_{k=0}^{\infty} \frac{1}{p^{k+1}} \sum_{m=1}^p \frac{1}{(m+1)^{k+1}} \sum_{c=1}^{p-1} (p-c)^k S(1, \bar{b}ac; p) \\
&= \sum_{a=1}^{p-1} \frac{1}{a} \sum_{k=0}^{\infty} \frac{1}{p^{k+1}} \sum_{m=1}^p \frac{1}{(m+1)^{k+1}} \sum_{c=1}^{p-1} c^k S(1, -\bar{b}ac; p) \tag{9}
\end{aligned}$$

Applying (7) we get the estimate

$$\begin{aligned}
\sum_{c=1}^{p-1} c^k S(1, -\bar{b}ac; p) &= \sum_{d=1}^{p-1} e\left(\frac{d}{p}\right) \sum_{c=1}^{p-1} c^k e\left(\frac{-\bar{b}acd}{p}\right) \\
&\ll \sum_{d=1}^{p-1} \frac{p^k}{\left|\sin\left(\frac{\pi \bar{b}ad}{p}\right)\right|} \ll p^{k+1} \ln p \tag{10}
\end{aligned}$$

From (9) and (10) we immediately deduce the estimate

$$\sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^p \frac{1}{a(mp+c)} S(1, \bar{b}ac; p)$$

$$\begin{aligned} &\ll \sum_{a=1}^{p-1} \frac{1}{a} \sum_{m=1}^p \sum_{k=0}^{\infty} \frac{1}{(m+1)^{k+1} p^{k+1}} \cdot p^{k+1} \ln p \\ &\ll \sum_{a=1}^{p-1} \frac{1}{a} \sum_{m=1}^p \frac{1}{m+1} \ln p \ll \ln^3 p \end{aligned} \tag{11}$$

Similarly we can deduce that

$$\sum_{a=1}^{p-1} \sum_{c=1}^{p-1} \sum_{m=1}^p \frac{1}{a(mp+c)} S(1, -\bar{b}ac; p) \ll \ln^3 p \tag{12}$$

Combining (11) and (12) we get

$$M_2 \ll \ln^3 p \tag{13}$$

In the same way we get the estimates

$$M_3 \ll \ln^3 p \tag{14}$$

$$M_4 \ll \ln^3 p \tag{15}$$

Now lemma 5 follows at once from (8), (13), (14) and (15). □

LEMMA 6. *Suppose χ is an odd character mod q , generated by the primitive character χ_m mod m . Then we have*

$$\sum_{a=1}^q a\chi(a) = \frac{q}{m} \left(\prod_{\substack{p|q \\ p \nmid m}} (1 - \chi_m(p)) \right) \left(\sum_{a=1}^m a\chi_m(a) \right)$$

Proof. Let l be the largest divisor of q that is coprime with m . Then we have

$$\begin{aligned} \sum_{a=1}^q a\chi(a) &= \sum_{i=0}^{(q/ml)-1} \sum_{j=1}^{ml} (iml+j)\chi(iml+j) \\ &= \sum_{i=0}^{(q/ml)-1} \sum_{j=1}^{ml} (iml+j)\chi(j) = \frac{q}{ml} \sum_{j=1}^{lm} j\chi(j) \\ &= \frac{q}{lm} \sum_{a=1}^{lm} a\chi_m(a) \sum_{\substack{d|a \\ d|l}} \mu(d) \\ &= \frac{q}{lm} \sum_{d|l} \mu(d) \sum_{\substack{a=1 \\ d|a}}^{lm} a\chi_m(a) \end{aligned}$$

$$\begin{aligned}
&= \frac{q}{lm} \sum_{d|l} \mu(d) d \chi_m(d) \sum_{b=1}^{lm/d} b \chi_m(b) \\
&= \frac{q}{lm} \sum_{d|l} \mu(d) d \chi_m(d) \sum_{i=0}^{(l/d)-1} \sum_{j=1}^m (im+j) \chi_m(im+j) \\
&= \frac{q}{lm} \sum_{d|l} \mu(d) d \chi_m(d) \cdot \frac{l}{d} \sum_{a=1}^m a \chi_m(a) \\
&= \frac{q}{m} \left(\prod_{\substack{p|q \\ p \nmid m}} (1 - \chi_m(p)) \right) \left(\sum_{a=1}^m a \chi_m(a) \right)
\end{aligned}$$

This proves lemma 6. □

LEMMA 7. Let χ be a primitive character mod m with $\chi(-1) = -1$. Then we have

$$\frac{1}{m} \sum_{\gamma=1}^m \gamma \chi(\gamma) = -\frac{i}{\pi} \tau(\chi) L(1, \bar{\chi})$$

Proof. (See Theorem 12.11 and Theorem 12.20 of [5]). □

3. Proof of the theorems

In this section, we shall complete the proofs of the theorems. First we prove theorem 2. From lemmas 2, 4, 6 and lemma 7 we know that for every odd integer $q > 2$,

$$\begin{aligned}
&\sum_{\chi(-1)=-1} \chi(4) \left(\sum_{a=1}^{(q-1)/2} \chi(a) \right)^2 \\
&= \frac{1}{q^2} \sum_{\chi(-1)=-1} (1 - 2\chi(2))^2 \left(\sum_{\gamma=1}^{q-1} \gamma \chi(\gamma) \right)^2 \\
&= \sum_{m|q} \sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}}^* \frac{1}{m^2} (1 - 2\chi(2))^2 \left(\prod_{\substack{p|q \\ p \nmid m}} (1 - \chi_m(p)) \right)^2 \left(\sum_{a=1}^m a \chi_m(a) \right)^2 \\
&= -\frac{1}{\pi^2} \sum_{m|q} \sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}}^* (1 - 2\chi_m(2))^2 \left(\prod_{\substack{p|q \\ p \nmid m}} (1 - \chi_m(p)) \right)^2 \tau^2(\chi_m) L^2(1, \bar{\chi}_m) \\
&= -\frac{1}{\pi^2} \sum_{m|q} \sum_{d|m} \mu\left(\frac{m}{d}\right) \sum_{\substack{\chi \bmod m \\ \chi(-1)=-1}} (1 - 2\chi(2))^2 \left(\prod_{\substack{p|q \\ p \nmid m}} (1 - \chi(p)) \right)^2 \tau^2(\chi) L^2(1, \bar{\chi})
\end{aligned}$$

$$\begin{aligned} &\ll \sum_{m|q} \sum_{k|m} 2^{\omega(q)-\omega(m)} k^{1/2} d(k) \phi(k) \ln^2 k \\ &\ll \phi(q) q^{1/2} d^2(q) \ln^2 q \end{aligned} \tag{16}$$

where $\omega(n)$ denotes the number of all distinct prime divisor of n . Now lemma 1 and (16) imply that

$$r(q) = \frac{1}{2} \phi(q) + O(q^{1/2} d^2(q) \ln^2 q)$$

This completes the proof of theorem 2. □

Now we prove theorem 1. Let p be a prime > 2 . Note that every odd character $\chi \pmod p$ is a primitive character $\pmod p$. Thus, from lemma 2 and lemma 7 we derive

$$\begin{aligned} &\sum_{\chi_{p(-1)} = -1} \chi(4) \left(\sum_{a=1}^{(p-1)/2} \chi(a) \right)^2 \\ &= -\frac{1}{\pi^2} \sum_{\chi_{p(-1)} = -1} (1 - 2\chi(2))^2 \tau^2(\chi) L^2(1, \bar{\chi}) \\ &= -\frac{1}{\pi^2} \sum_{\chi_{p(-1)} = -1} (1 - 4\chi(2) + 4\chi(4)) \tau^2(\chi) L^2(1, \bar{\chi}) \end{aligned} \tag{17}$$

From (17), lemma 1 and lemma 5 we deduce that

$$\begin{aligned} r(p) &= \frac{1}{2} p + \frac{1}{\pi^2} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} \{S(1, ab; p) - S(1, -ab; p) \\ &\quad + 4S(\bar{4}, ab; p) - 4S(\bar{4}, -ab; p) - 4S(\bar{2}, ab; p) \\ &\quad + 4S(\bar{2}, -ab; p)\} + O(\ln^3 p) \end{aligned} \tag{18}$$

Notice that

$$\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} S(m, ab; p) = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \frac{1}{ab} S(ma, b; p).$$

Together with (18) this implies theorem 1.

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