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The Kodaira dimension of certain moduli spaces of Abelian surfaces

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0. Introduction

In [T] Tai showed that the moduli spaces $\mathscr{A}^{(g)}$ of principally polarized abelian varieties of dimension g are of general type for $g \geqslant 9$. This was strengthened to $g \geqslant 8$ by Freitag [F] and to $g \geqslant 7$ by Mumford [M2]. The space $\mathscr{A}^{(2)}$ of principally polarized abelian surfaces is rational. On the other hand O'Grady [O'G] showed that the space of polarized abelian surfaces with a polarization of type $(1, p^2)$ is of general type for $p \geqslant 17$, (p a prime). Here we consider abelian surfaces X with a $(1, p)$-polarization and a level structure, i.e., a symplectic basis of $\ker(X \to \hat{X})$. These level structures appear naturally when one considers Heisenberg invariant embeddings of X. We denote the corresponding moduli space by $\mathscr{A}_p$. It is quasiprojective and we choose a projective compactification $\bar{\mathscr{A}}_p$. Our main result is

THEOREM. *$\bar{\mathscr{A}}_p$ is of general type for $p \geqslant 41$.*

$\mathscr{A}_p$ is the quotient

$$\mathscr{A}_p = \mathscr{S}_2/\Gamma_{1,p}$$

by the group

$$\Gamma_{1,p} = \left\{ g \in \mathrm{Sp}(4, \mathbf{Z});\, g - \mathbf{1} \in \begin{pmatrix} \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & p\mathbf{Z} \\ p\mathbf{Z} & p\mathbf{Z} & p\mathbf{Z} & p^2\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & p\mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & p\mathbf{Z} \end{pmatrix} \right\}$$

acting on $\mathscr{S}_2$ by

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\colon Z \mapsto (AZ + B)(CZ + D)^{-1}.$$

Let $\bar{\mathscr{A}}_p$ be the toroidal compactification of $\mathscr{A}_p$ corresponding to the Igusa compactification in the principally polarized case. For details see [HKW1], [HKW2].

Our approach is as follows: we use suitable modular forms for $\Gamma_{1,p}$ to get a supply of pluricanonical forms over the open subset $\mathscr{A}_p^0$ of $\mathscr{A}_p$ where the natural projection $\pi: \mathscr{S}_2 \to \mathscr{A}_p$ is unramified. Then (following [T]) we have to estimate the number of conditions imposed by the need to extend the forms over each component of $\bar{\mathscr{A}}_p \backslash \mathscr{A}_p^0$. We have to resolve some of the singularities of $\bar{\mathscr{A}}_p$ in order to do this. Since $\mathscr{A}_p^0 \subset \mathscr{A}_p$ the procedure falls (roughly) into two parts: extension over the boundary divisors $\bar{\mathscr{A}}_p \backslash \mathscr{A}_p$ and extension over the other components, which are two Humbert surfaces.

The space $\mathscr{A}_5$ is rational. More precisely it is birational to the projectivized space of sections of the Horrocks-Mumford bundle [HM]. The cases $7 \leqslant p \leqslant 37$ remain open.

Throughout this paper p denotes a prime number, and we shall always assume $p \geqslant 5$. We use v_∞ for the number of cusps of the modular curve $X(p)$, which is $(p^2-1)/2$, and $\mu = pv_\infty$ for the index of $\bar{\Gamma}(p)$ in $\bar{\Gamma}(1)$ (see [Sh]).

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1. Modular forms and extensions to the boundary

Let F be a modular form of weight $3k$, $k>0$, for $\Gamma_{1,p}$. Let $\omega = d\tau_1 \wedge d\tau_2 \wedge d\tau_3$, a differential 3-form on $\mathscr{S}_2$; then $F\omega^{\otimes k}$ is a k-fold differential form on $\mathscr{S}_2$ which is invariant under $\Gamma_{1,p}$.

We take $\mathscr{A}_p^0$ to be the Zariski-open subset of $\mathscr{A}_p$ where the covering map $\pi: \mathscr{S}_2 \to \mathscr{A}_p$ is unbranched. $F\omega^{\otimes k}$ descends to an k-fold differential 3-form on $\mathscr{A}_p^0$, i.e., a section of $kK_{\mathscr{A}_p^0}$.

The toroidal compactification $\bar{\mathscr{A}}_p$ is defined (see [SC, p. 253] and [T]) by a family of maps

$$\pi_D: (\mathscr{S}_2/U(D)_{\mathbf{Z}})_{\{\sigma_\alpha\}} \to \bar{\mathscr{A}}_p$$

corresponding to boundary components D, which cover $\bar{\mathscr{A}}_p$. The set $\mathscr{A}_p^0$ is the set of points of $\mathscr{A}_p$ where the π_D are unbranched.

Let $\bar{\mathscr{A}}_p^0$ be the set of points where π_D is unbranched. This is a Zariski-open subset of $\bar{\mathscr{A}}_p$: its complement certainly includes the closure of $\mathscr{A}_p \backslash \mathscr{A}_p^0$ and may include some other points in the boundary (in our case it does). However, $\bar{\mathscr{A}}_p^0$ is dense in the boundary $\bar{\mathscr{A}}_p \backslash \mathscr{A}_p$, as may be seen from the results of [HKW1].

Here, and later, we shall need detailed information about the geometry of

$\bar{\mathscr{A}}_p$ and especially of the boundary. We shall quote this as necessary, mainly from [HKW2].

The boundary $\bar{\mathscr{A}}_p \backslash \mathscr{A}_p$ consists of one distinguished component D_0, of codimension 1, called the central component; ν_∞ other components $D_{(a,b)}$ (where $(a, b) \in ((\mathbf{Z}_p)^2 \backslash \{(0, 0)\})/\pm 1$), also of codimension 1, called the peripheral components; and some boundary curves which are contained in the closure of D_0. Full details are in [HKW1].

By [HKW1, Proposition 2.2] we can expand $F(Z)$ in a Fourier-Jacobi series near D_0:

$$F(Z) = \sum_{m \geqslant 0} \theta_m^0(\tau_3, \tau_2) \exp\{2\pi i m \tau_1\}$$

where $m \in \mathbf{Z}$ (see [Ba] for the most general form of this assertion, which implies that in our case we may take $m \geqslant 0$).

Similarly, near $D_{(0,1)}$ we have

$$F(Z) = \sum_{m \geqslant 0} \theta_m^{0,1}(\tau_1, \tau_2) \exp\{2\pi i m \tau_3/p^2\}$$

and there are similar expansions

$$F(Z) = \sum_{m \geqslant 0} \theta_m^{a,b}(\tau_1^{(a,b)}, \tau_2^{(a,b)}) \exp\{2\pi i m \tau_3^{(a,b)}/p^2\}$$

for suitable variables $(\tau_1^{(a,b)}, \tau_2^{(a,b)}, \exp\{2\pi i m \tau_3^{(a,b)}/p^2\})$ near $D_{(a,b)}$.

Note that all the peripheral components are equivalent under the action of the group $\Gamma_{1,p}^0/\Gamma_{1,p}$, where $\Gamma_{1,p}^0$ is the group which preserves $(1, p)$-polarizations but not level structure. ($\Gamma_{1,p}$ is normal in $\Gamma_{1,p}^0$ but not in $\mathrm{Sp}(4, \mathbf{Z})$.) Consequently the number of conditions imposed by each $D_{(a,b)}$ is the same, so we shall be able to work with $D_{(0,1)}$ all the time.

PROPOSITION 1.1. *Suppose F is a modular form of weight $3k$ with Fourier-Jacobi expansions as above. If*

$$\theta_m^0(\tau_3, \tau_2) \equiv 0 \quad \textit{and} \quad \theta_m^{a,b}(\tau_1^{(a,b)}, \tau_2^{(a,b)}) \equiv 0$$

for all $m < k$ and for all (a, b), then the form coming from $F\omega^{\otimes k}$ extends to a section of kK over $\bar{\mathscr{A}}_p^0$.

Proof. This is the same as in [SC, Chapter IV, Theorem 1], except for two minor modifications. In [SC] it is assumed that Γ (which corresponds to our $\Gamma_{1,p}$) is neat, so that π_D is unbranched. The proof is local, however, and goes through without alteration away from the branch locus. Secondly, in [SC] it is necessary to consider the expansions near all boundary components, not just those of codimension 1. In our case, however, $\bar{\mathscr{A}}_p$ is smooth everywhere on the boundary components of codimension $\geqslant 2$ (see [HKW2]), so the pluricanonical forms can always be extended there. □

2. The space of cusp forms

Let $M_k(\Gamma_{1,p})$ be the space of modular forms of weight $3k$ on $\mathscr{S}_2$ with respect to the group $\Gamma_{1,p}$. By $S_k(\Gamma_{1,p})$ we denote the corresponding space of cusp forms.

PROPOSITION 2.1. $\dim S_k(\Gamma_{1,p}) = \frac{p(p^4-1)}{640} k^3 + O(k^2)$.

Proof. Let $M_k(l)$ be the space of modular forms of weight $3k$ with respect to the principal congruence subgroup $\Gamma(l) \subset \mathrm{Sp}(4, \mathbf{Z})$, and $S_k(l)$ be the space of cusp forms. By [T, p. 428] (see also [M1, Corollary 3.5]) we have for l sufficiently large

$$\dim M_k(l) \sim \dim S_k(l) \sim 2^{-5}3^3k^3V_2\pi^{-3}[\Gamma(1):\Gamma(l)]$$

where V_2 is the symplectic volume of $\mathscr{S}_2$. By Siegel's result

$$V_2 = 2^5\pi^3 \prod_{j=1}^{2} \frac{(j-1)!}{(2j)!} B_j$$

where the B_j are the Bernoulli numbers. Straightforward calculation gives

$$\dim M_k(l) \sim S_k(l) \sim \frac{k^3}{320} [\bar{\Gamma}(1): \Gamma(l)].$$

Here $\bar{\Gamma}(1) = \Gamma(1)/(\pm 1)$.
Now let l be such that $p^2 | l$. Then $\Gamma(l) \subset \Gamma_{1,p}$ and

$$S_k(\Gamma_{1,p}) = S_k^{\bar{\Gamma}_{1,p}}(l)$$

i.e., the space of forms in $S_k(\Gamma_{1,p})$ invariant under the group $\bar{\Gamma}_{1,p} = \Gamma_{1,p}/\Gamma(l)$. Just as in Tai [T] we can proceed by Hirzebruch's method [Hir] to compute this space using the Atiyah-Bott fixed point theorem:

$$\dim S_k(\Gamma_{1,p}) \sim \frac{1}{[\Gamma_{1,p}:\Gamma(l)]} \dim S_k(l)$$
$$\sim \frac{[\bar{\Gamma}(1): \Gamma(l)]}{[\Gamma_{1,p}:\Gamma(l)]} \frac{k^3}{320}.$$

the result now follows from [HW, p. 413] since $[\bar{\Gamma}(1): \Gamma_{1,p}] = p(p^4-1)/2$. □

3. Conditions imposed by the boundary components

We have already seen that every element $F \in M_k(\Gamma_{1,p})$ has a Fourier expansion of the form

$$F(Z) = \sum_{m \geq 0} \theta_m^0(\tau_3, \tau_2) \exp\{2\pi i m \tau_1\} \tag{1}$$

with respect to the central boundary component and of the form

$$F(Z) = \sum_{m \geq 0} \theta_m^{a,b}(\tau_1^{(a,b)}, \tau_2^{(a,b)}) \exp\{2\pi i m \tau_3^{(a,b)}/p^2\} \tag{2}$$

with respect to each of the peripheral boundary components. We have also seen that the form $F\omega^{\otimes k}$ can be extended to $\bar{\mathscr{A}}_p^0 \cap D_0$ if the θ_m^0 vanish for $m \leqslant k-1$, and similarly for $D_{(a,b)}$. In order to count the number of conditions which this imposes we want to interpret the coefficients θ_m as Jacobi forms.

DEFINITION. Let $\Gamma \subset \mathrm{SL}(2, \mathbf{Z})$ be a subgroup of finite index. A *Jacobi form* of *weight* k and *index* m is a holomorphic function

$$\Phi: \mathscr{S}_1 \times \mathbf{C} \to \mathbf{C}$$

with the following properties:

(i) For every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$\Phi\left(\frac{a\tau + b}{c\tau + d}, z\right) = (c\tau + d)^k \exp\{2\pi i m c z^2/(c\tau + d)\}\Phi(\tau, z)$$

(ii) For $n_1, n_2 \in \mathbf{Z}$

$$\Phi(\tau, z + n_1\tau + n_2) = \exp\{-2\pi i m(n_1^2\tau + 2n_1 z)\}\Phi(\tau, z).$$

(iii) At the cusp at infinity (and similarly at all the other cusps of Γ) Φ has a Fourier expansion of the form

$$\Phi(\tau, z) = \sum c(n, r) \exp\{2\pi i(n\tau + rz)\}$$

where $c(n, r) = 0$ unless $n \geqslant r^2/4m$.

Remarks. (i) The numbers n, r in the Fourier expansion are in general rational numbers. Their denominator, however, is bounded.

(ii) For the theory of Jacobi forms see the book [EZ] by Eichler and Zagier. For a geometric approach, which is similar to ours, see [K].

We now return to the Fourier expansion (1).

Let

$$\tilde{\theta}^0_m(\tau_3, \tau_2) := \theta^0_m(p\tau_3, p\tau_2).$$

PROPOSITION 3.1. *The Fourier coefficients $\tilde{\theta}^0_m(\tau_3, \tau_2)$ are Jacobi forms of weight $3k$ and index mp with respect to the principal congruence subgroup $\Gamma_1(p)$.*

Proof. Recall the stabilizer subgroup P_{l_0} of the central boundary component which was described in [HK W1, Proposition 2.2]. It is an extension

$$1 \to P'_{l_0} \to P_{l_0} \to P''_{l_0} \to 1$$

where P'_{l_0} is the lattice

$$P'_{l_0} = \left\{ \left(\begin{array}{cc} \mathbf{1} & \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \\ 0 & \mathbf{1} \end{array} \right) \middle| \, r \in \mathbf{Z} \right\}$$

and

$$P''_{l_0} = \left\{ \begin{pmatrix} 1 & \varepsilon n_1 & \varepsilon n_2 \\ 0 & \varepsilon a & \varepsilon b \\ 0 & \varepsilon c & \varepsilon d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p);\ n_1, n_2 \in \mathbf{Z};\ \varepsilon = \pm 1 \right\}.$$

Consider the map

$$e_{l_0}\colon \quad \mathscr{S}_2 \to \mathbf{C}^* \times \mathbf{C} \times \mathscr{S}_1$$

$$\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \mapsto (t = \exp\{2\pi i \tau_1\}, \tau_2, \tau_3).$$

The natural action of P''_{l_0} on $\mathbf{C}^* \times \mathbf{C} \times \mathscr{S}_1$ extends to $\mathbf{C} \times \mathbf{C} \times \mathscr{S}_1$ where it is given by

$$\begin{pmatrix} 1 & \varepsilon n_1 & \varepsilon n_2 \\ 0 & \varepsilon a & \varepsilon b \\ 0 & \varepsilon c & \varepsilon d \end{pmatrix} : \begin{pmatrix} t \\ \tau_2 \\ \tau_3 \end{pmatrix} \mapsto \begin{pmatrix} t' \\ \tau'_2 \\ \tau'_3 \end{pmatrix} = \begin{pmatrix} t e^{2\pi i \varepsilon [m\tau_2 - \tau'_2(cp^{-1}\tau_2 + c\varepsilon n_1 - d\varepsilon n_2)]} \\ (\varepsilon\tau_2 + n_1\tau_3 + pn_2)(cp^{-1}\tau_3 + d)^{-1} \\ p(ap^{-1}\tau_3 + b)(cp^{-1}\tau_3 + d)^{-1} \end{pmatrix}.$$

Let $F \in M_k(\Gamma_{1,p})$ and recall its transformation behaviour

$$F(\gamma\tau) = \det(C\tau + D)^{3k} F(\tau) \tag{3}$$

for $\tau = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{1,p}$. The elements of P'_{l_0} leave F invariant. Hence we can study the transformation behaviour with respect to the group P''_{l_0}.

(i) We first consider elements of the form

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(p).$$

Using (3) a straightforward, although slightly tedious, calculation shows

$$\theta_m^0\left(p\left(\frac{ap^{-1}\tau_3 + b}{cp^{-1}\tau_3 + d}\right), \frac{\tau_2}{cp^{-1}\tau_3 + d}\right) = \exp\left\{\frac{2\pi imc\tau_2^2}{p(cp^{-1}\tau_3 + d)}\right\}(cp^{-1}\tau_3 + d)^{3k}\theta_m^0(\tau_3, \tau_2).$$

This implies immediately the first transformation law for $\tilde{\theta}_m^0(\tau_3, \tau_2)$:

$$\tilde{\theta}_m^0\left(\frac{a\tau_3 + b}{c\tau_3 + d}, \tau_2\right) = (c\tau_3 + d)^{3k} \exp\{2\pi impc\tau_2^2/(c\tau_3 + d)\}\tilde{\theta}_m^0(\tau_3, \tau_2).$$

(ii) The second transformation law can be checked in exactly the same way using elements of the form

$$\gamma = \begin{pmatrix} 1 & n_1 & n_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iii) It remains to check the Fourier expansion of $\tilde{\theta}_m^0(\tau_3, \tau_2)$. In view of the action of the group $\Gamma_{1,p}^0/\Gamma_{1,p}$ which acts transitively on the cusps it will be sufficient to consider the standard cusp. We look at the lattice given by the symplectic matrices of the form

$$\left(\begin{array}{c|cc} \mathbf{1} & \mathbf{Z} & p\mathbf{Z} \\ & p\mathbf{Z} & p^2\mathbf{Z} \\ \hline 0 & \multicolumn{2}{c}{\mathbf{1}} \end{array}\right)$$

that is,

$$L = \left\{ s \mid s = {}^t s,\ s \in \begin{pmatrix} \mathbf{Z} & p\mathbf{Z} \\ p\mathbf{Z} & p^2\mathbf{Z} \end{pmatrix} \right\} \subset \mathrm{Symm}(2, \mathbf{Q}).$$

On Symm(2, **R**) one has the natural form

$$(s, s^*) = \mathrm{tr}(ss^*).$$

The dual lattice of L with respect to this form is clearly

$$L_* = \left\{ s^* \,\middle|\, s^* = \begin{pmatrix} a^* & \dfrac{b^*}{2p} \\ \dfrac{b^*}{2p} & \dfrac{d^*}{p^2} \end{pmatrix};\ a^*, b^*, c^*, d^* \in \mathbf{Z} \right\}.$$

By [Ba] we have a Fourier expansion

$$\begin{aligned} F(Z) &= \sum_{\substack{s^* \in L_* \\ s^* \geqslant 0}} c(s^*) \exp\{2\pi i\, \mathrm{tr}(s^* t)\} \\ &= \sum_{\substack{s^* \in L_* \\ s^* \geqslant 0}} c(s^*) t^{a^*} \exp\left\{2\pi i \left(\frac{b^*}{p}\tau_2 + \frac{d^*}{p^2}\tau_3\right)\right\} \end{aligned}$$

Hence

$$\theta_m^0(\tau_3, \tau_2) = \sum_{\substack{s^* \in L_* \\ s^* \geqslant 0 \\ a^* = m}} c(s^*) \exp\left\{2\pi i \left(\frac{b^*}{p}\tau_2 + \frac{d^*}{p^2}\tau_3\right)\right\}$$

For given $a^* = m \geqslant 0$ the condition $s^* \geqslant 0$ is equivalent to $4a^*d^* \geqslant (b^*)^2$. Hence we get

$$\theta_m^0(\tau_3, \tau_2) = \sum_{\substack{b^*, d^* \in Z \\ (b^*)^2 \leqslant 4md^*}} c(b^*, d^*) \exp\left\{2\pi \mathrm{i} \left(\frac{b^*}{p}\tau_2 + \frac{d^*}{p^2}\tau_3\right)\right\}.$$

From this the required statement about $\tilde{\theta}_m^0(\tau_3, \tau_2)$ follows immediately. □

REMARK. Using the element $\gamma = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \in P''_{l_0}$ one can show that in addition

$$\tilde{\theta}^0_m(\tau_3, -\tau_2) = (-1)^{3k}\tilde{\theta}^0_m(\tau_3, \tau_2).$$

When dealing with the peripheral boundary components we can restrict ourselves in view of the action of $\Gamma^0_{1,p}/\Gamma_{1,p}$ to the component $D_{(0,1)}$. We set

$$\tilde{\theta}^{0,1}_m(\tau_1, \tau_2) := \theta^{0,1}_m(\tau_1^{(0,1)}, p\tau_2^{(0,1)}).$$

PROPOSITION 3.2. *The Fourier coefficients $\tilde{\theta}^{0,1}_m(\tau_1, \tau_2)$ are Jacobi forms of weight $3k$ and index m with respect to the modular group* $\mathrm{SL}(2, \mathbf{Z})$.

Proof. Identical to the proof of Proposition 3.1. □

We now want to count the number of conditions imposed by the requirement that the form $F\omega^{\otimes k}$ extends to the boundary components. By Proposition 1.1 such a form can be extended if the Fourier coefficients θ_m vanish for $m = 0, \ldots, k-1$. Hence by our previous result our problem reduces to the calculation of certain spaces of Jacobi forms. We shall treat the simpler case first, i.e., the peripheral boundary components.

PROPOSITION 3.3. *The number of conditions imposed by the peripheral boundary components is at most*

$$\frac{11}{144}(p^2 - 1)k^3 + O(k^2).$$

Proof. We shall consider the boundary component $D_{(0,1)}$. By $J_{k,m}$ we denote the space of Jacobi forms of weight k and index m with respect to the modular group $\mathrm{SL}(2, \mathbf{Z})$. We have to determine the dimension of the space

$$J^{(0,1)} := \bigoplus_{m=0}^{k-1} J_{3k,m}.$$

Here we shall treat the case k even. The case k odd is analogous. By [EZ, p. 37] we have for k even

$$\dim J_{k,m} \leqslant \dim M_k + \dim S_{k+2} + \cdots + \dim S_{k+2m}$$

where M_k is the space of modular forms of weight k and S_k is the corresponding space of cusp forms. Hence,

$$\begin{aligned} J^{(0,1)} &= \sum_{m=0}^{k-1} \dim J_{3k,m} \\ &\leqslant \sum_{i=0}^{k-1} (k-i)\dim M_{3k+2i} \\ &\leqslant \sum_{i=0}^{k-1} (k-i)\left(\frac{3k+2i}{12}+1\right) \\ &= \frac{11}{72}k^3 + O(k^2) \end{aligned}$$

The assertion of the proposition now follows since the number of peripheral boundary components is $\frac{1}{2}(p^2-1)$. □

Let us finally return to the central boundary component D_0. By Proposition 3.1 the Fourier coefficients θ_m^o are Jacobi forms of weight $3k$ and index mp with respect to the principal congruence subgroup $\Gamma_1(p)$ of level p of $\mathrm{SL}(2,\mathbf{Z})$. There are formulae in [EZ] bounding the dimensions of spaces of Jacobi forms also in the case of groups different from $\mathrm{SL}(2,\mathbf{Z})$. It is not easy to apply these formulae directly to our situation. Our strategy is to relate the Jacobi forms in question to certain line bundles on the Shioda modular surface $S(p)$: compare [K].

As usual let the semi-direct product $\mathbf{Z}^2 \rtimes \Gamma_1(p)$ act on $\mathscr{S}_1 \times \mathbf{C}$ by

$$(n_1, n_2; \gamma):(\tau, z) \mapsto \left(\frac{a\tau+b}{c\tau+d}, \frac{z+n_1\tau+n_2}{c\tau+d}\right)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The open Shioda modular surface $S^0(p)$ is the quotient

$$S^0(p) = (\mathscr{S}_1 \times \mathbf{C})/(\mathbf{Z}^2 \rtimes \Gamma_1(p)).$$

$S^0(p)$ has a natural projection to the (open) modular curve $X^0(p) = \mathscr{S}_1/\Gamma_1(p)$. Shioda's modular surface $S(p)$ is a natural compactification of $S^0(p)$ over $X(p)$. For details see [Shi], [BH].

For fixed weight k and index mp the transformation formulae (i), (ii) in the definition of Jacobi forms define a holomorphic vector bundle $\mathscr{L}^0 = \mathscr{L}^0(k, mp)$ on $S^0(p)$. The Jacobi forms can be interpreted in a natural way as sections of $\mathscr{L}^0$.

PROPOSITION 3.4. *For given weight k and index mp one can extend the line bundle* $\mathscr{L}^0 = \mathscr{L}^0(k, mp)$ *to a line bundle* $\mathscr{L} = \mathscr{L}(k, mp)$ *on* $S(p)$ *in such a way that the Jacobi forms of weight k and index mp extend to global sections of* $\mathscr{L}(k, mp)$.

Proof. Once again it will be enough to consider the standard cusp of $X(p)$. We have to recall briefly how $S(p)$ is constructed near this cusp. Let $\Gamma_1^\infty(p)$ be the stabilizer of $i\infty$ in $\Gamma_1(p)$, i.e.,

$$\Gamma_1^\infty(p) = \left\{ \begin{pmatrix} 1 & bp \\ 0 & 1 \end{pmatrix};\ b \in \mathbf{Z} \right\}.$$

Let

$$P_\infty := \mathbf{Z}^2 \rtimes \Gamma_1^\infty(p).$$

Then we have an exact sequence

$$0 \to P' \to P_\infty \to P'' \to 0$$

where

$$P' = (\{0\} \times \mathbf{Z}) \rtimes \Gamma_1^\infty(p)$$

and $P'' \cong \mathbf{Z}$ can be identified with

$$P'' = (\mathbf{Z} \times \{0\}) \rtimes \{1\}.$$

For a suitable neighbourhood of $i\infty$:

$$U = \{\tau \in S_1;\ \operatorname{Im} \tau > N\}$$

one has

$$U \times \mathbf{C}/P' = \Delta^* \times \mathbf{C}^*$$

where

$$\Delta^* = \{t \in \mathbf{C};\ 0 < |t| < \varepsilon = e^{-2\pi(N/p)}\} \quad (t = e^{2\pi i\tau/p}).$$

We denote the coordinate on $\mathbf{C}^*$ by $u = e^{2\pi i z}$. The induced action of P'' on $\Delta^* \times \mathbf{C}^*$ is given by

$$(t, u) \mapsto (t, t^p u).$$

Let $\Delta = \Delta^* \cup \{0\}$. In order to extend the above action over the origin we consider

$$B_a := \Delta \times \mathbf{C}^* \quad (a \in \mathbf{Z}).$$

On the disjoint union

$$B := \coprod_{a \in \mathbf{Z}} B_a$$

we define an equivalence relation by

$$B_a \ni (t, u) \sim (t', u') \in B_{a'}$$

if and only if

$$0 \neq t = t'; \; t^a u = t^{a'} u'.$$

The map

$$\begin{aligned} B_a &\to B_{a+p} \\ (t, u) &\mapsto (t, u) \end{aligned}$$

gives an action of $\mathbf{Z}$ on B which descends to the quotient $B' = B/\sim$. Then

$$S^{\#}(p) := B'/\mathbf{Z}$$

contains $\Delta^* \times \mathbf{C}^*/P''$ as an open set. In fact $S^{\#}(p)$ is Shioda's modular surface $S(p)$ near the cusp at infinity minus the p singular points of the fibre over the cusp.

The Jacobi functions of weight k and index mp are invariant under P'. Hence we have to consider the trivial bundle over $\Delta^* \times \mathbf{C}^*$. For $n_1 = 1$, $n_2 = 0$ in the transformation law (ii) we get

$$\Phi(t, u^p t) = t^{-mp^2} u^{-2mp} \Phi(t, u). \tag{4}$$

The map

$$\begin{aligned} (\Delta^* \times \mathbf{C}^*) \times \mathbf{C} &\to (\Delta^* \times \mathbf{C}^*) \times \mathbf{C} \\ ((t, u), w) &\mapsto ((t, t^p u), t^{-mp^2} u^{-2mp} w) \end{aligned}$$

generates an action of P'' on the trivial bundle on $\Delta^* \times \mathbf{C}^*$ compatible with the transformation formula (4). We now proceed in a way very similar to the above construction. We set

$$C_\alpha := (\Delta \times \mathbf{C}^*) \times \mathbf{C} \quad (a \in \mathbf{Z}).$$

On the disjoint union

$$C = \coprod_{a \in \mathbf{Z}} C_a$$

we introduce an equivalence relation by

$$C_a \ni (t, u, w) \sim (t', u', w') \in C_{a'}$$

if and only if the two points are equal or

$$0 \neq t = t', \quad t^a u = t^{a'} u', \quad t^{-ma^2} u^{-2ma} w = t^{-ma'^2} u'^{-2ma'} w'. \tag{5}$$

As before the map

$$\begin{aligned} C_a &\to C_{a+p} \\ (t, u, w) &\mapsto (t, u, w) \end{aligned}$$

induces an action of $P'' = \mathbf{Z}$ on the quotient $C' = C/\sim$ and we get the desired line bundle on $S^\#(p)$ as

$$\mathscr{L}^\# = C'/\mathbf{Z}.$$

By the transformation laws (i), (ii) and by our construction every Jacobi form Φ of weight k and index mp defines a holomorphic section of $\mathscr{L}^\#$ outside $t = 0$. We now have to see that these sections extend holomorphically to sections of $\mathscr{L}^\#$ on $S^\#(p)$. To see this we consider the Fourier expansion

$$\Phi(\tau, z) = \sum c(n, r) \exp\{2\pi i n\tau + rz\}$$

where $c(n, r) = 0$ unless $n \geqslant r^2/4mp$. By the transformation laws (i), (ii) it follows that $n = n'/p$ with $n' \in \mathbf{Z}$ and $r \in \mathbf{Z}$. Hence

$$\Phi(\tau, z) = \Phi(t, u) = \sum_{r^2 \leqslant 4mn'} c(n', r) t^{n'} u^r.$$

To check that the sections can be extended holomorphically to $S^{\#}(p)$ we have to look at the functions

$$t^{ma^2}u^{2ma}\Phi(t, t^a u) = \sum_{r^2 \leqslant 4mn'} c(n', r)t^{ma^2+ra+n'}u^{2ma+r}.$$

We have to check that

$$ma^2 + ra + n' \geqslant 0.$$

This follows from $4mn' - r^2 \geqslant 0$.

Extending $\mathscr{L}^{\#}$ to $\mathscr{L}$ on $S(p)$ can be done in several ways. If one wants to work in the analytic category one can argue as follows. By construction $\mathscr{L}^{\#}$ has global sections, i.e., is of the form $\mathcal{O}_{S^{\#}}(D)$ for some effective divisor D. The divisor D can be extended to $S(p)$ by the Remmert-Stein extension theorem. Hence $\mathscr{L}^{\#}$ can be extended too, and the extension of the sections is a consequence of the second Riemann removable singularity theorem.

Our next task is to compute the space of sections of the line bundles $\mathscr{L} = \mathscr{L}(k, mp)$. Before we can do this, we have to recall a few basic facts about the Shioda modular surfaces $S(p)$. For a reference see, e.g., [BH, p. 78]. Recall that $\nu_\infty = \nu_\infty(p)$ is the number of cusps of $X(p)$ and that $\mu = \mu(p) = p\nu_\infty$ is the order of $\mathrm{PSL}(2, \mathbf{Z}_p)$. The basic invariants of $S(p)$ are

$$e(S(p)) = c_2(S(p)) = \mu$$

$$\chi(\mathcal{O}_{S(p)}) = \frac{1}{12}\mu$$

$$K_{S(p)} = \pi^*\mathscr{M}$$

where $\pi: S(p) \to X(p)$ and $\mathscr{M}$ is a line bundle on the base curve $X(p)$ with

$$\deg \mathscr{M} = \frac{p-4}{4}\nu_\infty.$$

The elliptic surface $S(p) \to X(p)$ has exactly p^2 sections L_{ij}, $(i, j) \in \mathbf{Z}_p^2$. Their self-intersection is given by

$$L_{ij}^2 = -\chi(\mathcal{O}_{S(p)}) = -\frac{1}{12}\mu.$$

Inoue and Livné showed the existence of a divisor $I \in \operatorname{Pic} S(p)$ such that up to numerical equivalence

$$I \equiv \frac{1}{p}\sum L_{ij}$$

(see also [BH, Proposition 2]). Finally, let F denote the class of a fibre.

PROPOSITION 3.5. *The numerical equivalence class* L *of the line bundle* $\mathscr{L} = \mathscr{L}(k, mp)$ *is given by*

$$L \equiv 2mI + \frac{1}{12} \nu_\infty(kp + 2m)F$$

Proof. We first show that L is of the form

$$L \equiv aI + bF.$$

Since every Jacobi form of weight k and index mp defines a theta function of degree $2mp$ on every smooth fibre of $S(p)$ one finds $L \cdot F = 2mp$. It follows that

$$L - 2mI \equiv \sum_k \sum_{i=0}^{p-1} c_i^k C_i^k$$

where k runs over all cusps in $X(p)$ and where $C_0^k, \ldots, C_{p-1}^k$ are the (-2)-curves in the p-gon over such a cusp. We have to show that $c_0^k = \cdots = c_{p-1}^k$ for every k. Let us fix some k. From the construction in Proposition 3.4 it follows that the degree of $\mathscr{L}|_{C_j^k}$ is independent of j and hence must be $2m$. (This can also be seen directly.) It follows that

$$0 = (L - 2mI)C_j^k = \sum_{i=0}^{p-1} c_i^k C_i^k C_j^k.$$

Hence

$$(c_0^k, \ldots, c_{p-1}^k)Q = 0$$

where

$$Q = (C_i^k \cdot C_j^k) = \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix}.$$

Since corank $Q = 1$ and

$$\ker Q = \mathbf{C}(1, \ldots, 1)$$

we are done. So

$$L \equiv 2mI + bF$$

and it remains to determine b. It follows from (i) and (iii) in the definition of Jacobi forms that every such form of weight k defines an entire modular form of weight k on $\mathscr{S}_1$ by setting $z = 0$. Using [Sch, Theorem V.8] this shows

$$L \cdot L_{00} = k \frac{\mu}{12}.$$

Hence

$$b = (L - 2mI) \cdot L_{00} = k \frac{\mu}{12} - 2mI \cdot L_{00}$$

and the assertion follows since

$$I \cdot L_{00} = \frac{1}{p} L_{00}^2 = -\frac{1}{12} v_\infty. \qquad \square$$

PROPOSITION 3.6. *Assume $p \geqslant 5$ and $k \geqslant 3$. Then*

$$h^0(\mathscr{L}(k, mp)) = \frac{\mu}{12}(2mpk + 2m^2 - 3m(p-4) + 1).$$

Proof. We first note that $h^2(\mathscr{L}) = h^0(\mathscr{L}^\vee \otimes K) = 0$ since $(-L + K) \cdot F = -2mp < 0$. To show that $h^1(\mathscr{L}) = 0$ we use [BH, Proposition 6(ii)]. This applies provided

$$\frac{1}{12} v_\infty(kp + 2m) > \frac{1}{12} v_\infty(2m + 3p - 12)$$

or equivalently

$$p(k-3) + 12 > 0$$

which holds for $k \geqslant 3$. (Note that we need $p \geqslant 5$ to apply [BH].) The assertion is now a straightforward calculation using Riemann-Roch. $\square$

PROPOSITION 3.7. *The number of conditions imposed by the central boundary component is at most*

$$\frac{\mu}{24}(3p + \tfrac{2}{3})k^3 + O(k^2).$$

Proof. By Theorem 1.1 and Proposition 3.4 the number of conditions is bounded by

$$\sum_{m=0}^{k-1} h^0(\mathscr{L}(3k, mp)).$$

Computing this number gives

$$\frac{\mu}{12}\left(3p + \frac{2}{3}\right)k^3 + O(k^2).$$

To see that in fact the number of conditions imposed is only half this number, we recall from the remark following Proposition 3.1 that the functions $\tilde{\theta}^0_m(\tau_3, \tau_2)$ are even, resp. odd functions with respect to the involution $\tau_2 \mapsto -\tau_2$, depending on the parity of k. This involution induces an involution ι on $S(p)$ which is the standard involution $x \mapsto -x$ on all smooth fibres. Let

$$K(p) = S(p)/\iota$$

be the corresponding Kummer surface. Then there exist line bundles $\bar{\mathscr{L}}(3k, mp)$ on $K(p)$ with $\pi^*\bar{\mathscr{L}}(3k, mp) = \mathscr{L}(3k, mp)$, where $\pi: S(p) \to K(p)$ is the quotient map. One has

$$\pi_*\mathscr{L}(3k, mp) \cong \bar{\mathscr{L}}(3k, mp) \oplus (\bar{\mathscr{L}}(3k, mp) \otimes \mathcal{O}(-B))$$

where $2B$ is the class of the branching divisor on $K(p)$. The even, resp. odd sections of $\mathscr{L}(3k, mp)$ can be identified with the sections of $\bar{\mathscr{L}}(3k, mp)$, resp. $\bar{\mathscr{L}}(3k, mp) \otimes \mathcal{O}(-B)$. Since the higher cohomology of $\mathscr{L}(3k, mp)$ and hence also of $\pi_*\mathscr{L}(3k, mp)$ vanishes, we can use Riemann-Roch on $K(p)$ to compute the space of even, resp. odd sections. The only term which contributes to k^3 comes from

$$\tfrac{1}{2}(\bar{\mathscr{L}}(3k, mp))^2 = \tfrac{1}{2}\cdot\tfrac{1}{2}(\mathscr{L}(3k, mp))^2.$$

This gives the desired factor 2. $\square$

4. Conditions imposed by the branch locus

The branch locus of the maps π_D (see section 1) consists of the singular locus of $\bar{\mathscr{A}}_p$ together with the two Humbert surfaces described in [HKW1]. We shall call these Humbert surfaces H'_1 and H'_2. As is shown in [HKW1], the singular locus consists of two curves C_1 and C_2, both contained in H'_1, and two isolated points Q'_1 and Q'_2 on each peripheral boundary component; Q'_1 lies on H'_2.

The transverse singularity at any point of C_1 is an ordinary double point. It is resolved by blowing up the singular point, and the exceptional curve is a (-2)-curve. So we can resolve all the singularities along C_1 by simply blowing up $\bar{\mathscr{A}}_p$ along C_1. When we do this we get an exceptional divisor E which is a geometrically ruled surface over C_1, and the fibre of $E \to C_1$ has normal bundle $\mathcal{O} \oplus \mathcal{O}(-2)$.

Similarly, the transverse singularity at any point of C_2 is the cone on the twisted cubic, so blowing up $\bar{\mathscr{A}}_p$ along C_2 resolves the singularities. The exceptional divisor E' is a geometrically ruled surface over C_2, and the fibre of $E' \to C_2$ has normal bundle $\mathcal{O} \oplus \mathcal{O}(-3)$.

The singularity at each point Q'_1 is the cone on the Veronese and is also resolved by a single blow-up. The exceptional divisor $E''_{(a,b)}$ over $Q'_1 \in D_{(a,b)}$ is isomorphic to $\mathbf{P}^2$ and has $\mathcal{O}_{E''}(-E'') \cong \mathcal{O}(2)$.

DEFINITION. Let $\phi\colon \mathbf{A} \to \bar{\mathscr{A}}_p$ be the blow-up of $\bar{\mathscr{A}}_p$ along C_1 and C_2 and at each point $Q'_1 \in D_{(a,b)}$, together with a resolution of each point Q'_2.

We let H_1, H_2 be the strict transforms in $\mathbf{A}$ of H'_1, H'_2 respectively.

PROPOSITION 4.1. *$\phi^*H'_1 = H_1 + \frac{1}{2}E + \frac{1}{3}E'$ and $\phi^*H'_2 = H_2 + \frac{1}{2}\Sigma E''_{(a,b)}$.*

Proof. Note that $\phi^*H'_1$ and $\phi^*H'_2$ make sense because H'_1 and H'_2 are $\mathbf{Q}$-Cartier divisors on $\bar{\mathscr{A}}_p$: in fact $6H'_1$ and $2H'_2$ are Cartier.

It follows from [HKW1] that, near a point of C_1, $\bar{\mathscr{A}}_p$ is isomorphic to $\mathbf{C}^3/\alpha$, where $\alpha\colon (x, y, z) \mapsto (-x, -y, z)$ and $2H'_1$ is the image of $(x^2 = 0)$. Similarly, near a point of C_2 we take $\alpha\colon (x, y, z) \mapsto (\rho x, \rho y, z)$ (ρ is a primitive cube root of unity) and $3H'_1 = (x^3 = 0)$ and near Q'_1 we take $\alpha\colon (x, y, z) \mapsto (-x, -y, -z)$ and $2H'_2 = (x^2 = 0)$. (So H'_1 is smooth but H'_2 has an ordinary double point at Q'_1.) From this a simple calculation (e.g., by toric methods) shows that the coefficients of the exceptional components are in $\phi^*H'_1$ are as stated. □

Definition. Let $\mathscr{K} = K_{\mathbf{A}} + \frac{1}{2}H_1 + \frac{1}{2}H_2 + \frac{1}{4}E + \frac{1}{2}E'$ in $\operatorname{Pic} \mathbf{A} \otimes \mathbf{Q}$.

PROPOSITION 4.2. *$12\mathscr{K}$ is a bundle, and if F is a modular form of weight $36n$ for $\Gamma_{1,p}$ which satisfies the conditions* (1.4) *of Theorem* 1.1 *then $F\omega^{\otimes 12n}$ defines a section in $12n\mathscr{K}$.*

Proof. By Theorem 1.1, $F\omega^{\otimes 12n}$ defines an element of $H^0(\bar{\mathscr{A}}^0_p, 12nK_{\bar{\mathscr{A}}_p})$: note that $6nK_{\bar{\mathscr{A}}_p}$ is a bundle because the singularities of $\bar{\mathscr{A}}_p$ have Gorenstein index 2 or 3 (see [YPG]). Above H'_1 and H'_2 the maps π_D are ramified with index 2 (H_1 and H_2 come from the fixed point sets of elliptic elements of order 2 acting locally by reflection), so $F\omega^{\otimes 12n}$ acquires poles of order $6n$ along H'_1 and H'_2. Consequently

$$F\omega^{\otimes 12n} \in H^0(\mathbf{A};\ 12n\phi^*(K_{\bar{\mathscr{A}}_p} + \tfrac{1}{2}H'_1 + \tfrac{1}{2}H'_2)).$$

and therefore

$$F\omega^{\otimes 12n} \in H^0(\mathbf{A};\ 12n\phi^*(K_{\bar{\mathscr{A}}_p} + \tfrac{1}{2}H'_1 + \tfrac{1}{2}H'_2)).$$

Now we calculate the discrepancy $K_A - \phi^*K_{\bar{\mathscr{A}}_p}$. It is supported on the exceptional locus of ϕ and in fact

$$K_{\mathbf{A}} - \phi^*K_{\bar{\mathscr{A}}_p} = -\tfrac{1}{3}E' + \tfrac{1}{2}\sum E''_{(a,b)} + Z$$

where Z is an effective divisor coming from Q'_2. The contributions from C_1, C_2 and Q'_1 are easy to calculate (and are all done in [YPG]). All we need to know about the contribution from Q'_2 is that it is effective, i.e., that the singularities at Q'_2 are canonical. This follows from the description in [HKW1], using the criterion of Reid, Shepherd-Barron and Tai (see [YPG], [T]) for cyclic quotient singularities to be canonical. It would be easy to calculate Z precisely if we needed to.

Now, by Proposition 4.1,

$$12n\mathscr{K} = 12n\phi^*(K_{\bar{\mathscr{A}}_p} + \tfrac{1}{2}H'_1 + \tfrac{1}{2}H'_2) + 6n\sum E''_{(a,b)} + 12nZ$$

so $F\omega^{\otimes 12n}$ can be thought of as a section in $12n\mathscr{K}$. □

A. Obstruction from E and E′

Put $\mathscr{K}_1 = \mathscr{K} - \frac{1}{2}H_1 - \frac{1}{2}H_2E'$, so $K_A = \mathscr{K}_1 - \frac{1}{4}E$. The next step is to compare $h^0(12nK_{\mathbf{A}})$ with $h^0(12nK_1)$.

PROPOSITION 4.3. $h^0(12nK_{\mathbf{A}}) \geqslant h^0(12nK_1) - \sum_{j=1}^{3n} h^0((12n\mathscr{K}_1 - (3n-j)E)|_E)$.

Proof. We use an idea from [O'G]. We have taken $k = 12n$ to ensure that everything we write is a Cartier divisor.

There is an exact sequence

$$0 \to \mathcal{O}_{\mathbf{A}}(-E) \to \mathcal{O}_{\mathbf{A}} \to \mathcal{O}_E \to 0$$

which we twist by $12n\mathscr{K}_1 - (3n-1)E$ to get

$$0 \to \mathcal{O}_A(12nK_A) \to \mathcal{O}_A(12n\mathscr{K}_1 - (3n-1)E) \to \mathcal{O}_E(12n\mathscr{K}_1 - (3n-1)E) \to 0.$$

Hence

$$0 \to H^0(12nK_A) \to H^0(12n\mathscr{K}_1 - (3n-1)E) \to H^0((12n\mathscr{K}_1 - (3n-1)E)|_E) \to \cdots$$

and similarly, using $12n\mathscr{K}_1 - (3n-j)E$

$$0 \to H^0(12n\mathscr{K}_1 - (3n-j+1)E) \to H^0(12n\mathscr{K}_1 - (3n-j)E) \to H^0((12n\mathscr{K}_1 - (3n-j)E)|_E) \to \cdots$$

From this it follows immediately that

$$h^0(12nK_A) \geqslant h^0(12n\mathscr{K}_1) - \sum_{j=1}^{3n} h^0((12n\mathscr{K}_1 - (3n-j)E)|_E)$$

as required. □

In order to estimate this obstruction we need to understand the geometry of the ruled surface E. The facts about ruled surfaces that we need are given in [H, Chapter V.2].

Since by [HKW1] (see also above) the Humbert surface H'_1 in $\bar{\mathscr{A}}_p$ is smooth, the intersection of H_1 and E is transversal. We put $\Sigma = E \cap H_1$.

PROPOSITION 4.4. *Σ is a section of $\phi: E \to C_1$ and if Φ is a fibre*

(i) $(\Sigma \cdot \Sigma)_E = -\mu/6$;
(ii) $(\Sigma \cdot E)_A = 0$;
(iii) $(\Phi \cdot E)_A = -2$

Proof. (i) H_1 comes from the surface

$$\mathscr{H}_1 = \left\{ Z \in \mathscr{S}_2 \,\middle|\, Z = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}, \tau_1, \tau_3 \in \mathscr{S}_1 \right\}$$

in $\mathscr{S}_2$, on which $\Gamma_{1,p}$ acts via elements

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & p\beta \\ 0 & 0 & 1 & 0 \\ 0 & p^{-1}\gamma & 0 & \delta \end{pmatrix}$$

with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_1(p)$. Write $g' = \begin{pmatrix} \alpha & p\beta \\ p^{-1}\gamma & \delta \end{pmatrix}$, and let $\Gamma'_1(p)$ be the subgroup of $SL_2(\mathbf{R})$ consisting of such elements. C_1 is then $\mathscr{S}_1/\Gamma'_1(p)$ compactified in the usual way and comes from

$$\mathscr{C}_1 = \left\{ Z \in \mathscr{H}_1 \,\middle|\, Z = \begin{pmatrix} i & 0 \\ 0 & \tau_3 \end{pmatrix}, \tau_3 \in \mathscr{S}_1 \right\}.$$

To make the blow-up we consider $Z = \begin{pmatrix} i+z & w \\ w & \tau_3 \end{pmatrix}$. With g as above we have

$$g(Z) = \begin{pmatrix} i + z - p^{-1}\gamma w^2(p^{-1}\gamma\tau_3 + \delta)^{-1} & w(p^{-1}\gamma\tau_3 + \delta)^{-1} \\ w(p^{-1}\gamma\tau_3 + \delta)^{-1} & g'(\tau_3) \end{pmatrix}$$

and we can take $(z : w^2)$ as homogeneous coordinates in a fibre of E. Then Σ is given, even over a cusp, by $w^2 = 0$, so Σ is a section and the normal bundle $\mathscr{N}_{\Sigma/E}$ is given by w^2.

Over the open part $X^0(p)$ (i.e., away from the cusps) this bundle is given by an action of $\Gamma'_1(p)$ on $\mathbf{C} \times \mathscr{S}_1$, namely

$$g' : (w^2, \tau_3) \mapsto (w^2(p^{-1}\gamma\tau_3 + \delta)^{-2}, g'(\tau_3))$$

and this extends to the cusps so the meromorphic sections are just modular forms of weight -1 for $\Gamma'_1(p)$. Since $\Gamma'_1(p)$ is conjugate to $\Gamma_1(p)$ in $SL_2(\mathbf{R})$ it has index μ and ν_∞ cusps: also $-I \notin \Gamma'_1(p)$ and $\Gamma'_1(p)$ has no elliptic elements. So by [Sh, Proposition 2.16]

$$\deg \mathscr{N}_{\Sigma/E} = -(2g(X(p)) - 2 + \nu_\infty).$$

By [Sh, Proposition 1.40]

$$2g(X(p)) - 2 = \frac{\mu}{6} - \nu_\infty$$

so $\Sigma^2 = -\mu/6$.

(ii) By [HKW2], H_1 is isomorphic to $\Sigma \times X(1)$, so $(\Sigma \cdot \Sigma)_{H_1} = 0$. Therefore $(\Sigma \cdot E|_E)_{\mathbf{A}} = 0$.

(iii) $(\Phi \cdot E|_E)_{\mathbf{A}} = \deg \mathscr{N}_{E/\mathbf{A}}|_\Phi$ and $\mathscr{N}_{E/\mathbf{A}}|_\Phi = \mathscr{N}_{\Phi/\mathbf{A}}/\mathscr{N}_{\Phi/E}$. But $\mathscr{N}_{\Phi/\mathbf{A}} \cong \mathscr{O}_\Phi \oplus \mathscr{O}_\Phi(-2)$, so $(\Phi \cdot E)_{\mathbf{A}} = -2$. □

Remark. Another proof of (i) can be given by using the geometry of **A**, from 4.14, 4.17 and 4.18 below.

COROLLARY 4.5. *Num E is generated by Σ and Φ, and $K_E \equiv -2\Sigma - \nu_\infty \Phi$.*

Proof. Σ and Φ generate Num E by [H], Proposition V.2.3. Φ is a smooth rational curve and $\Phi^2 = 0$, so $K_E \cdot \Phi = -2$. Similarly $K_E \cdot \Sigma = 2g(X(p)) - 2 - \Sigma^2$, and these two equations give the result. □

PROPOSITION 4.6.

$$(12n\mathscr{K}_1 - (3n-j)E)|_E \equiv -2j\Sigma + [12n((\mu/4) - \nu_\infty) + (3n-j)\mu/3]\Phi.$$

Proof. We work in Num $E \otimes \mathbf{Q}$. Put $E|_E = a\Sigma + b\Phi$: by 4.4(iii) we have $a = -2$ and then by 4.4(ii)

$$0 = \Sigma \cdot E|_E = -2\Sigma^2 + b$$

so $b = -\mu/3$ and $E|_E = -2\Sigma - (\mu/3)\Phi$.

$K_A = \mathscr{K}_1 - \frac{1}{4}E$ and $K_E = (K_A + E)|_E = (\mathscr{K}_1 + \frac{3}{4}E)|_E$ so

$$\begin{aligned}\mathscr{K}_1|_E &\equiv K_E - \frac{3}{4}E|_E \\ &\equiv -2\Sigma - \nu_\infty\Phi + \frac{3}{2}\Sigma + \frac{\mu}{4}\Phi \\ &\equiv -\frac{1}{2}\Sigma + \left(\frac{\mu}{4} - \nu_\infty\right)\Phi\end{aligned}$$

by 4.4 and 4.5. Hence

$$\begin{aligned}12n\mathscr{K}_1|_E - (3n-j)E|_E &\equiv 12n\left[-\frac{1}{2}\Sigma + \left(\frac{\mu}{4} - \nu_\infty\right)\Phi\right] - (3n-j)\left(-2\Sigma - \frac{\mu}{3}\Phi\right) \\ &\equiv -2j\Sigma + \left[12n\left(\frac{\mu}{4} - \nu_\infty\right) + (3n-j)\frac{\mu}{3}\right]\Phi.\end{aligned}$$ □

COROLLARY 4.7. *The obstruction coming from E (that is, the difference between $h^0(12nK_A)$ and $h^0(12n\mathscr{K}_1)$) is zero.*

Proof. $(-2j\Sigma + [12n(\mu/4 - \nu_\infty) + (3n-j)\mu/3]\Phi) \cdot \Phi = -2j < 0$, so there are no sections. □

Now put $\mathscr{K}_2 = \mathscr{K} - \frac{1}{2}H_1 - \frac{1}{2}H_2 = \mathscr{K}_1 + \frac{1}{2}E'$, so $K_A = \mathscr{K}_2 - \frac{1}{4}E - \frac{1}{2}E'$. We compare $h^0(12n\mathscr{K}_2)$ with $h^0(12n\mathscr{K}_1)$.

PROPOSITION 4.8. $h^0(12n\mathscr{K}_1) \geq h^0(12n\mathscr{K}_2) - \sum_{j=1}^{6n} h^0((12n\mathscr{K}_2 - (6n-j)E')|_{E'})$.

Proof. Exactly as for Proposition 4.3. □

Put $\Sigma' = E' \cap H_1$ (the intersection is transversal, as before).

PROPOSITION 4.9. *Σ' is a section of $\phi: E' \to C_2$ and if Φ' is a fibre*

(i) $(\Sigma' \cdot \Sigma')_{E'} = -\mu/6$;
(ii) $(\Sigma' \cdot E')_{\mathrm{A}} = 0$;
(iii) $(\Phi' \cdot E')_{\mathrm{A}} = -3$.

Proof. Exactly as for Proposition 4.4. □

COROLLARY 4.10. Num E' *is generated by Σ' and Φ', and one has* $K_E \equiv -2\Sigma' - \nu_\infty \Phi'$.

PROPOSITION 4.11. *The following holds:*

$$(12n\mathscr{K}_2 - (6n - j)E')|_{E'} \equiv (12n - 3j)\Sigma' + [6n(\mu - 2\nu_\infty) - j(\mu/2)]\Phi'.$$

Proof. As in Proposition 4.6, $E'|_{E'} = a'\Sigma' + b'\Phi'$. In this case $a' = -3$ and so $b' = -\mu/2$ and $E'|_{E'} = -3\Sigma' - (\mu/2)\Phi'$.

$K_{\mathrm{A}} = \mathscr{K}_2 - \tfrac{1}{4}E - \tfrac{1}{2}E'$ and

$$\begin{aligned} K_{E'} &= (K_{\mathrm{A}} + E')|_{E'} \\ &= (\mathscr{K}_2 - \tfrac{1}{4}E + \tfrac{1}{2}E')|_{E'} \\ &= (\mathscr{K}_2 + \tfrac{1}{2}E'|_{E'}) \end{aligned}$$

(since E and E' are disjoint); so

$$\begin{aligned} \mathscr{K}_2|_{E'} &\equiv K_{E'} - \frac{1}{2}E'|_{E'} \\ &\equiv -2\Sigma' - \nu_\infty\Phi' + \frac{3}{2}\Sigma' + \frac{\mu}{4}\Phi' \\ &\equiv -\frac{1}{2}\Sigma' + \left(\frac{\mu}{4} - \nu_\infty\right)\Phi' \end{aligned}$$

by 4.9 and 4.10. Hence

$$\begin{aligned} &(12n\mathscr{K}_2 - (6n - j)E')|_{E'} \\ &\equiv 12n\left[-\frac{1}{2}\Sigma' + \left(\frac{\mu}{4} - \nu_\infty\right)\Phi'\right] - (6n - j)\left(-3\Sigma' - \frac{\mu}{2}\Phi'\right) \\ &\equiv (12n - 3j)\Sigma' + \left[6n(\mu - 2\nu_\infty) - j\frac{\mu}{2}\right]\Phi'. \end{aligned}$$

□

THEOREM 4.12. *The obstruction coming from E' for modular forms of weight $3k$ is*

$$\frac{1}{12}(p^2-1)\left(\frac{7}{18}p-1\right)k^3+O(k^2)$$

if k is a multiple of 12 and $p \geqslant 5$.

Proof. Put

$$L_j = 12n\mathscr{K}_2|_{E'} - (6n-j)E'|_{E'}.$$

In view of 4.8, we want to calculate

$$\sum_{j=1}^{6n} h^0(L_j).$$

It follows from 4.11 that $L_j \cdot \Phi' < 0$ for $j > 4n$, hence $h^0(L_j) = 0$ for $j > 4n$. Hence it remains to calculate

$$\sum_{j=1}^{4n} h^0(L_j).$$

By 4.10 and 4.11

$$L_j - K_{E'} \equiv (12n-3j+2)\Sigma' + \left[\frac{\mu}{2}(12n-j) - v_\infty(12n-1)\right]\Phi'$$

Since $j \leqslant 4n$ we can use [H, Proposition V.2.20] to conclude that $L_j - K_{E'}$ is ample, provided

$$\frac{\mu}{2}(12n-j) - v_\infty(12n-1) > (12n-3j+2)\frac{\mu}{6}.$$

Since $p \geqslant 5$ this is true.

Now apply Riemann-Roch to L_j. Since $L_j - K_{E'}$ is ample, Kodaira vanishing gives $\chi(L_j) = h^0(L_j)$, so

$$\begin{aligned} h^0(L_j) &= \frac{1}{2}L_j(L_j - K_{E'}) + 1 - g(X(p)) \\ &= \frac{1}{2}\left((12n-3j)\Sigma' + \left[6n(\mu - 2v_\infty) - j\frac{\mu}{2}\right]\Phi'\right)^2 \\ &\quad + \left[-\frac{1}{2}L_j\cdot K_{E'} + 1 - g(X(p))\right]. \end{aligned}$$

We are only interested in the coefficients of n^3 in $\sum_{j=1}^{4n} h^0(L_j)$. We may therefore neglect the term $-\frac{1}{2}L_jK_{E'} + 1 - g(X(p))$, which does not contribute to this.

$$\begin{aligned}\frac{1}{2}L_j^2 &= \frac{1}{2}\left((12n-3j)\Sigma' + \left[6n(\mu - 2v_\infty) - j\frac{\mu}{2}\right]\right)\Phi'\Big)^2 \\ &= n^2(72(\Sigma')^2 + 72(\mu - 2v_\infty)) + nj(-36(\Sigma')^2 - 6\mu - 18(\mu - 2v_\infty)) \\ &\quad + j^2\left(\frac{3}{2}(\Sigma')^2 + \frac{3}{2}\mu\right) \\ &= n^2(60\mu - 144v_\infty) - 18nj(\mu - 2v_\infty) + \frac{3}{4}j^2\mu.\end{aligned}$$

So

$$\begin{aligned}\sum_{j=1}^{4n} h^0(L_j) &= (240\mu - 576v_\infty - 144\mu + 288v_\infty + 16\mu)n^3 + O(n^2) \\ &= (112\mu - 288v_\infty)n^3 + O(n^2).\end{aligned}$$

Since $n = k/12$, $\mu = pv_\infty$ and $v_\infty = (p^2-1)/2$ we get

$$\begin{aligned}\sum_{j=1}^{4n} h^0(L_j) &= \left(\frac{7}{108}\mu - \frac{1}{6}v_\infty\right)k^3 + O(k^2) \\ &= \frac{1}{12}(p^2-1)\left(\frac{7}{18}p - 1\right)k^3 + O(k^2)\end{aligned}$$

as claimed. □

B. The obstructions from H_1 and H_2

The estimation of the obstructions from H_1 and H_2 proceeds along similar lines. We shall need to calculate $(C \cdot H_i)_{\mathbf{A}}$ for certain curves C, and shall do this by arranging for C to lie in the boundary components D_0 or $D_{(0,1)}$. First, therefore, we study the geometry of these components.

PROPOSITION 4.13. *The closed boundary component $D_{(0,1)}$ in $\mathbf{A}$ is isomorphic to a resolution $\widetilde{K(1)}$ of the Kummer modular surface $K(1)$. This resolution is the minimal resolution except possibly over Q_2'. The normalization of D_0 is isomorphic to $K(p)$, if $p \geqslant 5$.*

Proof. All of this comes from [HKW2] except for the remark that the modification of $K(1)$ that occurs is the minimal resolution. This is an immediate consequence of the choice of $\phi: \mathbf{A} \to \bar{\mathscr{A}}_p$ to be simple blow-ups at Q_1, Q_2 and Q_1'. ($Q_i = C_i \cap D_{(0,1)}$ in $\bar{\mathscr{A}}_p$—see [HKW1].) At Q_2' we have not specified the choice of ϕ at all, since we shall not need to consider what happens there in detail. □

$K(p)$ is the (natural) toroidal compactification of a certain quotient of $\mathscr{S}_1 \times \mathbf{C}$. Let $\Gamma_\pm(p) = \Gamma_1(p) \cup -\Gamma_1(p) \lhd \mathrm{SL}_2(\mathbf{Z})$: then the natural extension $\mathbf{Z}^2 \rtimes \Gamma_\pm(p)$ acts on $\mathscr{S}_1 \times \mathbf{C}$ by

$$(n_1, n_2; \gamma):(\tau, z) \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{z + n_1\tau + n_2}{c\tau + d}\right)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\pm(p)$. The quotient is a complex analytic space with at most isolated singularities, which can be compactified in a natural way to give $K(p)$.

If $p \geqslant 3$ then $-I \notin \Gamma_1(p)$ and there is a double cover $S(p) \to K(p)$ from the Shioda modular surface $S(p)$ ([BH] and section 3 above). In this case $K(p)$ is smooth. In any case $K(p)$ is birationally a ruled surface over the modular curve $X(p)$.

We shall be interested in the zero sections $\tilde{\Delta}_1$, Δ_p of $\widetilde{K(1)}$ and $K(p)$; the Bring curves $\tilde{B}_1$, B_p (described below); the exceptional curves of the resolution $\widetilde{K(1)} \to K(1)$; and the fibre of $\widetilde{K(1)}$ over the unique cusp of $X(1)$.

$\Delta_l \subset K(l)$ and $B_l \subset K(l)$ are by definition the closures of the images of $\mathscr{S}_1 \times \{0\}$ and $\mathscr{S}_1 \times \{1/2\}$ respectively. In the case $l = 1$ we use the notation $\tilde{\Delta}_1$, $\tilde{B}_1$ for the strict transforms in $\widetilde{K(1)}$ of these curves. Δ_l is a section of $K(l) \to X(l)$, and B_l is a 3-section. In $S(l)$, which is the universal elliptic curve with level l structure, B_l is the curve of non-zero 2-torsion points.

PROPOSITION 4.14. *If $p \geqslant 3$ then*

$$\Delta_p^2 = -\frac{\mu}{6} \quad \textit{and} \quad B_p^2 = -\frac{\mu}{2} + 2\nu_\infty$$

in $K(p)$.

Proof. We use the double cover $S(p) \xrightarrow{\psi} K(p)$, which is branched along Δ_p and B_p. Let $\hat{\Delta}_p$, $\hat{B}_p$ be the zero-section and the Bring curve in $S(p)$, so $\psi^*\Delta_p = 2\hat{\Delta}_p$ and $\psi^* B_p = 2\hat{B}_p$. Let F be a general fibre of $S(p) \to X(p)$. In [BH] it is shown that

$$\hat{\Delta}_p^2 = -\frac{\mu}{12} \quad \text{and} \quad K_{S(p)} \equiv \left(\frac{\mu}{4} - \nu_\infty\right)F.$$

From the first of these it follows at once that $\Delta_p^2 = -\mu/6$. (See also section 3, above.)

A point on $\hat{B}_p$ is given, away from the cusps of $X(p)$, by a point of $X(p)$ and a non-zero 2-torsion point in the corresponding elliptic curve. Thus $\hat{B}_p$ is isomorphic to the modular curve $X_0(p)$ (one has to check that the behaviour at the cusps is as expected, which is easy). It is well known that $X_0(p)$ has genus given by

$$2g(X_0(p)) - 2 = 3(2g(X(p)) - 2) + \nu_\infty = 3\left(\frac{\mu}{6} - \nu_\infty\right) = \frac{\mu}{2} - 2\nu_\infty$$

where μ and ν_∞ are the index and number of cusps for $\Gamma(p)$. As remarked above, $\hat{B}_p$ is a 3-section, so

$$\begin{aligned}\hat{B}_p^2 &= \frac{\mu}{2} - 2\nu_\infty - K_{S(p)} \cdot \hat{B}_p \\ &= \frac{\mu}{2} - 2\nu_\infty - 3\left(\frac{\mu}{4} - \nu_\infty\right) \\ &= -\frac{\mu}{4} + \nu_\infty\end{aligned}$$

whence it follows that $B_p^2 = -\mu/2 + 2\nu_\infty$. □

PROPOSITION 4.15. *In* $\widetilde{K(1)}$, $\tilde{\Delta}_1^2 = -1$ *and* $\tilde{B}_1^2 = 1$.

Proof. (i) For Δ_1 we will work directly on $K(1)$. There $6\Delta_1$ is a Cartier divisor, because the singularities of $K(1)$ are finite quotient singularities of index 2 or 3. From the construction of $K(1)$ it follows that the function z^{12} on $\mathcal{S}_1 \times \mathbf{C}$ defines the pullbak of the line bundle $\mathcal{O}_{K(1)}(-6\Delta_1)$ to $\mathcal{S}_1 \times \mathbf{C}$. Since $z^{12} \mapsto z^{12}(c\tau + d)^{-12}$ it follows from [Sh, Proposition 2.16] that the degree of the line bundle $\mathcal{O}_{K(1)}(-6\Delta_1)|_{\Delta_1}$ is 1. So $(6\Delta_1)\cdot\Delta_1 = -1$ in the sense of [Fu, p. 33]. Since the intersection numbers defined there agree with those defined on normal surfaces ([Fu, p. 125]) in the case of Cartier divisors, they must agree (by linearity) for $\mathbf{Q}$-Cartier divisors also. Hence $\Delta_1^2 = -\frac{1}{6}$ on $K(1)$, in the sense of [Fu, p. 125]. There are two singular points of $K(1)$ on Δ_1, corresponding to $\tau = i$ and $\tau = e^{2\pi i/3}$. Blowing them up produces a (-2)-curve E_1 and a (-3)-curve E_2 in $\widetilde{K(1)}$ (they are the points Q_1 and Q_2 of [HKW1, Proposition 2.8], if we identify $K(1)$ with $D_{(0,1)}$). Now, as in [Fu], there are rational numbers λ_1, λ_2 such that for $i = 1, 2$

$$(\tilde{\Delta}_1 \cdot E_i)_{\widetilde{K(1)}} + \sum \lambda_j (E_j \cdot E_i) = 0$$

and since $\tilde{\Delta}_1 \cdot E_i = 1$ we have $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$. According to [Fu, p. 142]

(Example 8.3.11) (which is just the global version),

$$(\Delta_1 \cdot \Delta_1)_{K(1)} = \left(\tilde{\Delta}_1 + \frac{1}{2} E_1 + \frac{1}{3} E_2\right)^2_{\widetilde{K(1)}}$$

which gives $\tilde{\Delta}_1^2 = -1$.

(ii) Num $\widetilde{K(1)} \otimes \mathbf{Q}$ is generated by $\tilde{\Delta}_1$, the general fibre F of $\widetilde{K(1)} \to X(1)$, and the exceptional curves of $\widetilde{K(1)} \to K(1)$. These are E_1 and E_2 as above, a (-2)-curve E_3 coming from Q_1' where $K(1)$ has an A_1 singularity, and some other curves coming from the A_2 singularity at Q_2'. They will not concern us, but we can arrange for them to be two (-2)-curves, E_4 and E_5, if we choose ϕ to resolve Q_2' by blowing up twice. The fibre over the cusp turns out to be smooth in this case ([HKW2]). All seven curves are smooth and rational. So is $\tilde{B}_1$, as one can seen by checking the ramification or by realizing it as $X_0(1)$ (or by considering it as a curve in H_2—see below). We have

$$\tilde{\Delta}_1^2 = -1;\ F^2 = 0;\ E_i^2 = -2\ (i \neq 2);\ E_2^2 = -3;$$
$$F \cdot E_i = 0;\ \tilde{\Delta}_1 \cdot E_1 = \tilde{\Delta}_1 \cdot E_2 = 1;\ \tilde{\Delta}_1 \cdot E_i = 0,\ i > 2;\ \tilde{\Delta}_1 \cdot F = 1.$$

From this it follows that $K_{\widetilde{K(1)}} \equiv -2\tilde{\Delta}_1 - F - E_1 - E_2$. Since $\tilde{B}_1$ does not meet $\tilde{\Delta}_1$, E_1 or E_2, we have $K_{\widetilde{K(1)}} \cdot \tilde{B}_1 = -3$ and, since $\tilde{B}_1$ is a smooth rational curve, $\tilde{B}_1^2 = 1$. □

REMARK. There are other ways of calculating these intersection numbers. One is to use modular forms and intersection theory on normal surfaces throughout, thinking of $\tilde{B}_1$ as a modular curve (but the behaviour at the cusps is no longer trivial). Another is to use the existence of a covering map $S(p) \to K(1)$. To show that such a map really exists, however, involves a detailed and complicated examination of the fibres of $S(p)$ over the cusps ([HKW2]).

Put $\mathscr{K}_3 = \mathscr{K} - \frac{1}{2}H_2 = \mathscr{K}_2 + \frac{1}{2}H_1$, so $K_A = \mathscr{K}_3 - \frac{1}{2}H_1 - \frac{1}{4}E - \frac{1}{2}E'$.

PROPOSITION 4.16.

$$h^0(12n\mathscr{K}_2) \geqslant h^0(12n\mathscr{K}_3) - \sum_{j=1}^{6n} h^0((12n\mathscr{K}_3 - (6n-j)H_1)|_{H_1}).$$

Proof. The same, mutatis mutandis, as Proposition 4.3. □

Now we need to study the geometry of H_1, which is encouragingly simple.

PROPOSITION 4.17. *H_1' and H_1 are both isomorphic to $X(p) \times X(1)$.*

Proof. In [HKW2]. Although the result is simple the proof is a little complicated. □

PROPOSITION 4.18. *Let* Σ_1, Φ_1 *be fibres of* $H_1 \to X(1)$ *and* $H_1 \to X(p)$ *respectively. Then*

(i) $(\Sigma_1 \cdot \Sigma_1)_{H_1} = (\Phi_1 \cdot \Phi_1)_{H_1} = 0$
(ii) $(\Sigma_1 \cdot H_1)_{\mathrm{A}} = -\mu/6$
(iii) $(\Phi_1 \cdot H_1)_{\mathrm{A}} = -1$.

Proof. (i) Obvious.

(ii) We can take Σ_1 to be the fibre over $\rho = e^{2\pi i/3}$, which is $\Sigma' = E' \cap H_1$. Since the intersection of E and H_1 is transversal, $(\Sigma_1 \cdot H_1)_{\mathrm{A}} = (\Sigma' \cdot \Sigma')_{E'} = -\mu/6$ by Proposition 4.9(i).

(iii) We can take Φ_1 to be the fibre over a cusp, say $i\infty$, of $X(p)$. Then Φ_1 is a curve in $D_{(0,1)} \cong \widetilde{K(1)}$. It is easy to see that this curve is $\tilde{\Delta}_1$. Hence $(\Phi_1 \cdot H_1)_{\mathrm{A}} = (\tilde{\Delta}_1 \cdot \tilde{\Delta}_1)K_{\widetilde{K(1)}} = -1$ by Proposition 4.15. □

REMARK. We could also calculate $(\Sigma_1 \cdot H_1)_{\mathrm{A}}$ by thinking of Σ_1 as the fibre over the cusp and using Proposition 4.14. Note that H_1 does meet D_0 transversely.

THEOREM 4.19. *The obstruction coming from* H_1 *is zero for modular forms of weight* $3k$ *if* k *is a multiple of* 12 *and* $p \geq 5$.

Proof. We want to estimate $h^0((12n\mathscr{K}_3 - (6n-j)H_1)|_{H_1})$. We have

$$K_{H_1} \equiv -2\Sigma_1 + (2g(X(p)) - 2 + \Sigma_1^2)\Phi_1 = -2\Sigma_1 + \left(\frac{\mu}{6} - \nu_\infty\right)\Phi_1$$

and from 4.18 it follows easily that

$$H_1|_{H_1} \equiv -\Sigma_1 - \frac{\mu}{6}\Phi_1.$$

Also we know that

$$E|_{H_1} \equiv E'|_{H_1} \equiv \Sigma_1.$$

So in $\operatorname{Num} H_1 \otimes \mathbf{Q}$ we have

$$\begin{aligned}\mathscr{K}_3|_{H_1} &\equiv K_{\mathrm{A}}|_{H_1} + \tfrac{1}{2}H_1|_{H_1} + \tfrac{1}{4}E|_{H_1} + \tfrac{1}{2}E'|_{H_1} \\ &\equiv K_{H_1} - \tfrac{1}{2}H_1|_{H_1} + \tfrac{1}{4}E|_{H_1} + \tfrac{1}{2}E'|_{H_1} \\ &\equiv -\tfrac{3}{4}\Sigma_1 + \left(\frac{\mu}{4} - \nu_\infty\right)\Phi_1\end{aligned}$$

and

$$12n\mathscr{K}_3|_{H_1} - (6n-j)H_1|_{H_1} \equiv -(3n+j)\Sigma_1 + \left[(4\mu - 12v_\infty)n - j\frac{\mu}{6}\right]\Phi_1.$$

But $-(3n+j) < 0$, so this bundle has no sections. □

Now we come to H_2, which is more complicated.

PROPOSITION 4.20. $h^0(12n\mathscr{K}_3) \geqslant h^0(12n\mathscr{K}) - \sum_{j=1}^{6n} h^0((12n\mathscr{K}-(6n-j)H_2)|_{H_2})$.

Proof. As for 4.3 and 4.8. □

According to [HKW2] there are maps

$$X(2p) \times X(2) \xrightarrow{\psi_2} \bar{H}_2 \xrightarrow{\psi_1} X(p) \times X(1)$$

where both ψ_1 and ψ_2 are Galois covers with group $SL_2(\mathbf{Z}_2) \cong S_3$. The fibres $\psi_1^{-1}(\text{cusp}, \infty)$ and $\psi_1^{-1}(i, \infty)$ each consist of two points, one of them an ordinary double point. In this way $\bar{H}_2$ acquires $2v_\infty$ ordinary double points. It is smooth outside these points.

H_2 is obtained from $\bar{H}_2$ by blowing up (and hence resolving minimally) these $2v_\infty$ singular points. (H_2' is obtained by blowing up only those in $\psi_1^{-1}(\text{cusp}, \infty)$: H_2' retains v_∞ ordinary double points at $Q_1' \in \mathscr{A}_p$.) There is therefore a map $\tilde{\psi}_1: H_2 \to X(p) \times X(1)$. There are v_∞ exceptional (-2)-curves $R_{(a,b)}$ in H_2 corresponding to $\psi_1^{-1}((a,b),\infty)$ (where (a, b) is a cusp of $X(p)$) and another v_∞ such curves $R'_{(a,b)}$ corresponding to $\psi_1^{-1}((a,b), i)$.

Let Σ_2 and Φ_2 be general fibres of $\text{pr}_2 \circ \tilde{\psi}_1: H_2 \to X(1)$ and $\text{pr}_1 \circ \tilde{\psi}_1: H_2 \to X(p)$ respectively. If we need to refer to just one of the $R_{(a,b)}$ we shall choose $R_{(0,1)}$ and call it R_∞ (similarly for R'_∞). R_∞ and R'_∞ are components of the fibre of $\text{pr}_1 \circ \tilde{\psi}_1$ over $(0,1) \in X(p)$: the other component of the same fibre over $X(1)$ will be called Φ_∞. Similarly the component of the fibre over $\infty \in X(1)$ that is not R_∞ will be called Σ_∞.

Num $H_2 \otimes \mathbf{Q}$ is generated by Σ_2, Φ_2 and the $R_{(a,b)}$ and $R'_{(a,b)}$.

We can identify $D_{(0,1)}$ with $\widetilde{K(1)}$. D_0 is non-normal but its normalization is $K(p)$, and we shall be able to calculate everything we need on $K(p)$.

PROPOSITION 4.21. *In H_2 we have the following intersection numbers:*

(i) $\Sigma_2^2 = \Phi_2^2 = \Sigma_2 \cdot R_\infty = \Phi_2 \cdot R_\infty = R_\infty \cdot R_{(a,b)} = \Phi_\infty \cdot R_{(a,b)} = 0$ *for* $(a, b) \neq \infty$, *and similarly for* $R'_{(a,b)}$

(ii) $\Sigma_2 \cdot \Phi_2 = 6$; $R_{(a,b)}^2 = R_{(a,b)}'^2 = -2$; $\Sigma_\infty \cdot \Phi_2 = \Phi_\infty \cdot \Sigma_2 = 3$

(iii) $\Sigma_\infty \cdot R_{(a,b)} = \Phi_\infty \cdot R_\infty = \Phi_\infty \cdot R'_\infty = 1$; $\Sigma_\infty \cdot R'_{(a,b)} = 0$.

(iv) $R_\infty \cdot R'_\infty = R_\infty \cdot R'_{(a,b)} = R'_\infty \cdot R_{(a,b)} = 0$

Proof. Immediate from the description of the fibres of H_2. □

PROPOSITION 4.22. (i) $\Phi_2 \equiv 2\Phi_\infty + R_\infty + R'_\infty$ *and* $\Phi_\infty^2 = -1$

(ii) $\Sigma_2 \equiv 2\Sigma_\infty + \sum_{(a,b)} R_{(a,b)}$ *and* $\Sigma_\infty^2 = -\frac{\nu_\infty}{2}$

Proof. Straightforward calculation. □

PROPOSITION 4.23. *In* Num $H_2 \otimes \mathbf{Q}$, $K_{H_2} \equiv -\frac{1}{3}\Sigma_2 + \left(\frac{\mu}{6} - \frac{\nu_\infty}{2}\right)\Phi_2$.

Proof. $R_{(a,b)}$, $R'_{(a,b)}$ and Φ_2 are smooth rational curves (ψ_2 induces a covering map $X(2) \to \Phi_2$). Σ_∞ is also a smooth curve; if we identify the normalization of D_0 with $K(p)$ then Σ_∞ is identified with B_p. So, by 4.14, $2g(\Sigma_\infty) - 2 = \mu/2 - 2\nu_\infty$. From this the result follows by a routine calculation. □

We could also use the 6-to-1 map $\Sigma_2 \to X(p)$ induced by $\mathrm{pr}_2 \circ \psi_1$ (branched over the cusps of $X(p)$) to calculate $2g(\Sigma_2) - 2$ and use that to calculate K_{H_2}.

$\frac{1}{3}\Sigma_2$ is actually an integral divisor, because there is a multiple fibre over $\rho = e^{2\pi i/3} \in X(1)$ and $\Sigma_\rho = \frac{1}{3}\Sigma_2$.

LEMMA 4.24. (i) $(R_\infty \cdot H_2)_{\mathbf{A}} = -4$;

(ii) $(R'_\infty \cdot H_2)_{\mathbf{A}} = 1$;
(iii) $(\Sigma_2 \cdot H_2)_{\mathbf{A}} = -\mu$;
(iv) $(\Phi_2 \cdot H_2)_{\mathbf{A}} = -1$.

Proof. (i) R_∞ is identified with an adjacent cc-curve in D_0 (see [HKW2]); that is, a component of the singular fibre of $K(p)$ over the cusp $(0,1)$ of $X(p)$ that becomes a curve in D_0 meeting $D_{(0,1)}$ but not contained in it. If $\nu: K(p) \to D_0$ is the normalization then $\nu^*(H_2 \cap D_0) = B_p + 2\Sigma_{(a,b)} R_{(a,b)}$; see [HKW2]. This can also be deduced from [HKW1]. Therefore

$$(R_\infty \cdot H_2)_{\mathbf{A}} = \left(R_\infty \cdot \left(B_p + 2\sum_{(a,b)} R_{(a,b)}\right)\right)_{K(p)}$$

and, since the self-intersection of a cc-curve in $K(p)$ is -2 and $R_\infty \cdot B_p = 0$ in $K(p)$, we get $(R_\infty \cdot H_2)_{\mathbf{A}} = -4$.

(ii) R'_∞ lies in the exceptional surface over Q'_1 in $\mathbf{A}$, which was called $E''_{(0,1)}$ at the beginning of this section. $E''_{(0,1)} \cong \mathbf{P}^2$ and it is easy to check by a local calculation that $\mathcal{O}(R'_\infty) = \mathcal{O}(1)$ (again toric methods provide a simple way of seeing this). $H_2 \cap E''_{(0,1)} = R'_\infty$ (the intersection is transverse) so $(R'_\infty \cdot H_2)_{\mathbf{A}} = (R'_\infty \cdot R'_\infty)_{E''_{(0,1)}} = 1$.

(iii) As in (i),

$$\begin{aligned}(\Sigma_\infty \cdot H_2)_{\mathbf{A}} &= \left(B_p \cdot \left(B_p + 2\sum_{(a,b)} R_{(a,b)}\right)\right)_{K(p)} \\ &= -\frac{\mu}{2} + 2\nu_\infty\end{aligned}$$

by 4.14, so

$$(\Sigma_2 \cdot H_2)_{\mathrm{A}} = \left(\left(2\Sigma_\infty + \sum_{(a,b)} R_{(a,b)}\right) \cdot H_2\right)_{\mathrm{A}}$$
$$= -\mu.$$

(iv) As in (ii), $H_2 \cap D_{(0,1)} = \tilde{B}_1$. So $(\Phi_\infty \cdot H_2)_{\mathrm{A}} = (\tilde{B}_1 \cdot \tilde{B}_1)_{\overline{K(1)}} = 1$, by 4.14, and $(\Phi_2 \cdot H_2)_{\mathrm{A}} = ((2\Phi_\infty + R_\infty + R'_\infty) \cdot H_2)_{\mathrm{A}} = -1$. □

THEOREM 4.25. *The obstruction coming from H_2 is zero for modular forms of weight $3k$ if k is a multiple of 12 and $p \geqslant 5$.*

Proof. We want to estimate $h^0((12n\mathscr{K} - (6n-j)H_2)|_{H_2})$. Using 4.24 a straightforward calculation shows that

$$H_2|_{H_2} \equiv -\frac{1}{6}\Sigma_2 - \frac{\mu}{6}\Phi_2 + 2\sum_{(a,b)} R_{(a,b)} - \frac{1}{2}\sum_{(a,b)} R'_{(a,b)}.$$

Since neither H_1 nor E or E' intersect H_2 we find that

$$12n\mathscr{K}|_{H_2} \equiv 12nK_{\mathrm{A}}|_{H_2} + 6nH_2|_{H_2}$$
$$\equiv 12nK_{H_2} - 6nH_2|_{H_2}.$$

Hence

$$(12n\mathscr{K} - (6n-j)H_2)|_{H_2} \equiv 12nK_{H_2} - (12n-j)H_2|_{H_2}.$$

By 4.23 this gives

$$(12n\mathscr{K} - (6n-j)H_2)|_{H_2}$$
$$\equiv \left(-2n - \frac{j}{6}\right)\Sigma_2 - (12n-j)\left[-\frac{\mu}{6}\Phi_2 + 2\sum_{(a,b)} R_{(a,b)} - \frac{1}{2}\sum_{(a,b)} R'_{(a,b)}\right].$$

Hence

$$(12n\mathscr{K} - (6n-j)H_2)|_{H_2} \cdot \Phi_2 < 0$$

and hence this bundle has no sections. □

Note that $H_2|_{H_2}$ is indeed an integral divisor. This is because we have an equation

$$\Sigma_2 = 2\Sigma_i + \sum R'_{(a,b)} = 3\Sigma_\rho$$

where Σ_ρ is the (reduced) fibre over $\rho \in X(1)$ and where Σ_i and the $R'_{(a,b)}$'s are the (reduced) components of the fibre over $i \in X(1)$. So

$$H_2|_{H_2} \equiv \Sigma_\rho - \Sigma_i - \frac{\mu}{6}\Phi_2 + 2\sum_{(a,b)} R_{(a,b)} - \sum_{(a,b)} R'_{(a,b)}$$

and $-\mu/6$ is an integer.

5. Final calculation

THEOREM 5.1. *$\bar{\mathscr{A}}_p$ is of general type for $p \geqslant 41$.*

Proof. We must calculate the leading term of $\dim M_{3k}$ less all the obstructions.

From Proposition 2.1 we have

$$\dim M_{3k} = \frac{p(p^4-1)}{640}k^3 + O(k^2).$$

The obstructions from the central boundary component are bounded, in view of Proposition 3.7, by

$$\frac{\mu}{24}\left(3p + \frac{2}{3}\right)k^3 + O(k^2).$$

From the peripheral components we have

$$\frac{11}{144}(p^2-1)k^3 + O(k^2)$$

by Proposition 3.3.

From the divisor E' we have by 4.12

$$\frac{1}{12}(p^2-1)\left(\frac{7}{18}p - 1\right)k^3 + O(k^2)$$

The other obstructions are zero. So we have

$$h^0(kK_{\mathrm{A}}) \geqslant (p^2-1)\left(\frac{p^3}{640} + \frac{p}{640} - \frac{p^2}{16} - \frac{p}{72} - \frac{11}{144} - \frac{7}{216}p + \frac{1}{12}\right)k^3 + O(k^2).$$

Hence $\bar{\mathscr{A}}_p$ is of general type if

$$\frac{p^3}{640} - \frac{p^2}{16} - p\left(\frac{-1}{640} + \frac{1}{36} + \frac{7}{216}\right) + \frac{1}{144} > 0.$$

This is true for $p \geqslant 41$. □

Our estimates do not settle the cases $7 \leqslant p \leqslant 37$. The above expression is negative for $p = 37$.

We conclude with an immediate corollary of the main result.

COROLLARY 5.2. *If G is a subgroup of finite index in some $\Gamma_{1,p}$ with $p \geqslant 41$ and $A = \mathscr{S}_2/G$, then any compactification of A is of general type.*

Proof. The Satake compactification of A covers the Satake compactification of $\mathscr{A}_p$. □

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Note added in proof: Recently Manolache and Schreyer have shown that $\mathscr{A}_{1,7}$ is rational.