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SHIGEYUKI KONDŌ

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On the Kodaira dimension of the moduli space of K3 surfaces

SHIGEYUKI KONDŌ

Department of Mathematics, Saitama University, Shimo-Okubo 255, Urawa, Saitama 338, Japan

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0. Introduction

A compact complex smooth surface X is called a *K3 surface* if the canonical line bundle K_X is trivial and $\dim H^1(X, \mathcal{O}_X) = 0$. The period space of algebraic K3 surfaces with a primitive polarization of degree $2d$ is of the form $\mathscr{K}_{2d} = \mathscr{D}_{2d}/\Gamma_{2d}$ where $\mathscr{D}_{2d}$ is a 19-dimensional bounded symmetric domain of type IV and Γ_{2d} is an arithmetic subgroup acting properly discontinuously on $\mathscr{D}_{2d}$. The $\mathscr{K}_{2d}$ is an irreducible normal quasi-projective variety (Baily, Borel [4]). It follows from the Torelli theorem for K3 surfaces (Piatetskii-Shapiro, Shafarevich [18]) and the surjectivity of the period map (Kulikov [10]) that $\mathscr{K}_{2d}$ is the coarse moduli space of K3 surfaces with a primitive polarization of degree $2d$. It is known that $\mathscr{K}_{2d}$ is unirational for $1 \leqslant d \leqslant 9$ or $d = 11$ (Mukai [11]). The purpose of this paper is to prove the following:

THEOREM. *Assume that $d = p^2$ where p is a sufficiently large prime number, then $\mathscr{K}_{2d}$ is of general type.*

In case of the moduli space $\mathfrak{S}_g/\mathrm{Sp}(2g, \mathbb{Z})$ of principally polarized abelian varieties, $\mathfrak{S}_g/\mathrm{Sp}(2g, \mathbb{Z})$ is of general type for $g \geqslant 7$ (Tai [23], Freitag [6], Mumford [13]).

Our argument is based on the theory of toroidal compactifications of arithmetic quotient of bounded symmetric domains (Ash, Mumford, Rapoport, Tai [1]) and the extendability of pluri-canonical forms developed by Tai [23]. Let $\mathscr{K}^{\circ}_{2d}$ be the open set in $\mathscr{K}_{2d}$ on which the projection $\pi: \mathscr{D}_{2d} \to \mathscr{K}_{2d}$ is unramified. An automorphic form on $\mathscr{D}_{2d}$ of weight k (with respect to Γ_{2d}) gives a k-th pluri-canonical holomorphic differential form on $\mathscr{K}^{\circ}_{2d}$. The generalized Hirzebruch's proportionality theorem, due to Mumford [12], implies that there are sufficiently many automorphic forms of weight $k \gg 0$. On the other hand,

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if d is a square of integer, then Γ_{2d} is a (non-normal) subgroup of Γ_2 (Lemma 3.2), and hence we get a finite map $\mathcal{K}_{2d} \to \mathcal{K}_2$. Therefore we can compare the data of $\mathcal{K}_{2d}$ with those of $\mathcal{K}_2$. Moreover, Scattone [20] studied the Satake-Baily-Borel compactification of $\mathcal{K}_{2d}$ (or equivalently, Γ_{2d}-inequivalent rational boundary components of $\mathcal{D}_{2d}$). These allow us to study the extension problem of pluri-canonical holomorphic differential forms on $\mathcal{K}^{o}_{2d}$ to a non singular model of a toroidal compactification of $\mathcal{K}_{2d}$ with only quotient singularities (Theorems 6.18, 7.11, 8.4, 9.7).

To obtain explicit prime number p for which $\mathcal{K}_{2d}$ is of general type needs to solve several arithmetic problems, e.g. to count the number of integral points in some cone (see §5). The author does not know the answer of this problem.

The plane of this paper is as follows: In Section 1, we recall some definitions of lattices (= symmetric bilinear forms) and the moduli space of polarized K3 surfaces. Next Section 2 is devoted to the description of $\mathcal{D}_{2d}$ as Siegel domain of the third kind, which is essentially given in [17]. In Section 3, we study the group Γ_{2d}. In this section and Section 9, we use the theory of lattices due to Nikulin [16]. Section 4 is devoted to Tai's criterion and the dimension formula of cusp forms. In Sections 5, 6, we study the extension problem of pluri-canonical holomorphic differential forms on $\mathcal{K}^{o}_{2d}$ to the directions of 0- and 1-dimensional rational boundary components. To do this, we estimate the dimension of the space of Fourier–Jacobi coefficients of automorphic forms. In Section 6 (1-dimensional case), we essentially use the transformation formula of theta functions. Sections 7 and 8 are devoted to the study of singularities of a toroidal compactification of $\mathcal{K}_{2d}$. In particular, we see that, by using Reid-Tai's criterion, these singularities are canonical, and determine the branch divisor. Lastly, in Section 9, we prove that there are sufficiently many cusp forms extended holomorphically to a general point of the branch divisor, and complete the proof of the main theorem.

In this paper, we shall use the following notation: H^+ = the upper half plane, $\Gamma = SL(2, \mathbb{Z})$, $\Gamma(N)$ = the principal N-congruence subgroup of Γ and $\Gamma^1(N) = \{[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \Gamma \mid a \equiv 1, d \equiv 1, b \equiv 0 \pmod N\}$.

1. Moduli space of polarized K3 surfaces of degree 2d

(1.1) A *lattice* L is a free $\mathbb{Z}$-module of finite rank endowed with an integral symmetric bilinear form $\langle\, ,\, \rangle$. If L_1 and L_2 are lattices, then $L_1 \oplus L_2$ denotes the orthogonal direct sum of L_1 and L_2. An isomorphism of lattices preserving the bilinear form is called an *isometry*. For a lattice L, we denote by $O(L)$ the group of self-isometries of L.

A lattice L is *even* if $\langle x, x \rangle$ is even for each $x \in L$. A lattice L is *non-degenerate* if the determinant $\det(L)$ of the matrix of its bilinear form is non zero, and *unimodular* if $\det(L) = \pm 1$. If L is a non-degenerate lattice, the *signature* of L is a pair (t_+, t_-) where $t_\pm$ denotes the multiplicity of the eigenvalue ± 1 for the quadratic form on $L \otimes \mathbb{R}$. A sublattice S of L is *primitive* if L/S is torsion free.

Let L be a non-degenerate even lattice. The bilinear form of L determines a canonical embedding $L \to L^* = \mathrm{Hom}(L, \mathbb{Z})$. The factor group L^*/L, which is denoted by A_L, is an abelian group of order $|\det(L)|$. We denote by $\iota(L)$ the number of minimal generator of A_L. A non-degenerate even lattice L is called *2-elementary* if $A_L \simeq (\mathbb{Z}/2\mathbb{Z})^{\iota(L)}$.

We extend the bilinear form on L to one on L^*, taking value in $\mathbb{Q}$, and define

$$b_L\colon A_L \times A_L \to \mathbb{Q}/\mathbb{Z},\ b_L(x + L, x + L) = \langle x, y\rangle + \mathbb{Z},$$

$$q_L\colon A_L \to \mathbb{Q}/2\mathbb{Z},\ q_L(x + L) = \langle x, x\rangle + 2\mathbb{Z},\ (x, y \in L^*).$$

We call b_L (resp. q_L) the *discriminant bilinear form* (resp. *discriminant quadratic form*) of L. We denote by $O(q_L)$ the group of isomorphisms of A_L preserving the form q_L. Let $\tau\colon O(L) \to O(q_L)$ be a canonical homomorphism. We define $\tilde{O}(L) = \mathrm{Ker}(\tau)$ which is a normal subgroup of $O(L)$ of finite index.

We denote by U the hyperbolic lattice $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ which is an even unimodular lattice of signature (1, 1), and by E_8 an even unimodular negative definite lattice of rank 8 associated to the Dynkin diagram of type E_8. Also we denote by $\langle \mathfrak{n}\mathfrak{t}\rangle$ the lattice of rank 1 with the matrix $(\mathfrak{n}\mathfrak{t})$. For more details, we refer the reader to [16].

(1.2) Let S be a K3 surface. The second cohomology group $H^2(S, \mathbb{Z})$ admits a canonical structure of a lattice induced from the cup product $\langle\ ,\ \rangle$. It is even, unimodular and of signature (3, 19), and hence isometric to $L = U \oplus U \oplus U \oplus E_8 \oplus E_8$ (e.g. [21]). Let h be a primitive vector of L (i.e. $L/\mathbb{Z}h$ is torsion free) with $\langle h, h\rangle = 2d$. Then the orthogonal complement of h in L is isometric to $L_{2d} = U \oplus U \oplus E_8 \oplus E_8 \oplus \langle -2d\rangle$. Put $\Omega_{2d} = \{[\omega] \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) \mid \langle \omega, \omega\rangle = 0$ and $\langle \omega, \bar{\omega}\rangle > 0\}$. Then Ω_{2d} consists of two connected components, which are mapped into each other by complex conjugation. We denote by $\mathscr{D}_{2d}$ either one component of Ω_{2d}, which is a bounded symmetric domain of type IV and of dimension 19. Let $\tilde{\Gamma}_{2d}$ be the group of isometries of L which fix h. It follows from [16], Proposition 1.5.1 that $\tilde{\Gamma}_{2d} = \tilde{O}(L_{2d})$. $\tilde{\Gamma}_{2d}$ acts on Ω_{2d} as automorphisms. We denote by Γ_{2d} the subgroup of $\tilde{\Gamma}_{2d}$ of index 2 which consists of isometries preserving the connected components of Ω_{2d}. Then Γ_{2d} acts on $\mathscr{D}_{2d}$ properly discontinuously, and hence by Cartan's theorem $\mathscr{D}_{2d}/\Gamma_{2d}$ has a canonical structure of normal analytic space. By [4], $\mathscr{D}_{2d}/\Gamma_{2d}$ is a quasi-projective variety.

A *polarized K3 surface of degree 2d* is a pair (X,H) where X is an algebraic K3 surface and H is a primitive (i.e. mC implies $m = \pm 1$), numerically effective divisor on X with $H^2 = 2d$. A nowhere vanishing holomorphic 2-form ω_X on X gives a point $[\omega_X]$ of $\mathscr{D}_{2d}$ modulo Γ_{2d}, which is called the *period* of (X, H). It follows from the global Torelli theorem [18] and the surjectivity of the period map [10] that $\mathscr{D}_{2d}/\Gamma_{2d}$ is the coarse moduli space of polarized K3 surfaces of degree $2d$.

2. Siegel domain of the third kind

(2.1) Let $G_{\mathbb{R}}$ be a connected component of a linear algebraic group $O(L_{2d}\otimes\mathbb{R})$, defined over $\mathbb{Q}$, with associated bounded symmetric domain $\mathscr{D}_{2d} = G_{\mathbb{R}}/K$, where K is a maximal compact subgroup of $G_{\mathbb{R}}$. Let $\check{\mathscr{D}}_{2d}$ be the analytic set $\{[\omega]\in\mathbb{P}(L_{2d}\otimes\mathbb{C})|\langle\omega,\omega\rangle = 0\}$, which is called the *compact dual* of $\mathscr{D}_{2d}$. We denote by $\bar{\mathscr{D}}_{2d}$ the topological closure of $\mathscr{D}_{2d}$ in $\check{\mathscr{D}}_{2d}$. A *boundary component F* of $\mathscr{D}_{2d}$ is a maximal connected complex analytic subset in $\bar{\mathscr{D}}_{2d}\backslash\mathscr{D}_{2d}$. Group theoretically, it can be seen that the stabilizer group $N(F) = \{g\in G_{\mathbb{R}}|g(F)\subset F\}$ is a maximal parabolic subgroup of $G_{\mathbb{R}}$, and conversely. A boundary component F is *rational* if $N(F)$ is defined over $\mathbb{Q}$. For more details we refer the reader to [1], [14], [17], [19]. In our case, we can determine the rational boundary components of $\mathscr{D}_{2d}$ as follows (e.g. [20]):

PROPOSITION 2.2. *The set of all rational boundary components of* $\mathscr{D}_{2d}$ *corresponds to the set of all primitive totally isotopic sublattices of* L_{2d}. *If E is a primitive totally isotropic sublattice of* L_{2d}, *then the corresponding rational boundary component is defined by* $\mathbb{P}(E\otimes\mathbb{C})\cap\bar{\mathscr{D}}_{2d}$.

Since the signature of $L_{2d} = (2,19)$, the dimension of a rational boundary component is either 0 or 1. We need more explicit description of a rational boundary components to deal with a toroidal compactification of $\mathscr{D}_{2d}/\Gamma_{2d}$.

In the following of this section, we assume $d = p^2$ for a prime number p. To describe 0-dimensional rational boundary components, we fix an orthogonal direct decomposition

$$L_{2d} = U_1 \oplus U_2 \oplus E_8 \oplus E_8 \oplus \langle -2d\rangle$$

where U_i is a copy of U $(i = 1, 2)$. Let $\{e, f\}$ be a base of U_1 with $\langle e,e\rangle = \langle f,f\rangle = 0$ and $\langle e,f\rangle = 1$, and u is a base of $\langle -2d\rangle$. Put $v_0 = e$ and

$v_m = mu + p\{e + m^2 f\}$ $(m \in \mathbb{N})$.

PROPOSITION 2.3. ([20], Remark 4.2.3) *The set of primitive isotropic rank* 1 *sublattices* $\{\mathbb{Z}v_m \mid 0 \leqslant m \leqslant (p-1)/2\}$ *corresponds to a complete set of* Γ_{2d}-*inequivalent* 0-*dimensional rational boundary components of* $\mathscr{D}_{2d}$.

PROPOSITION 2.4. ([20]) (i) *Let E be a primitive totally isotropic rank two sublattice of* L_{2d}. *Then there exists a base* $\{e_1, \ldots, e_{21}\}$ *of* L_{2d} *such that* $E = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, $E^\perp = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_{19}$ *and the corresponding matrix of* L_{2d} *is*

$$(\langle e_i, e_j \rangle) = \begin{bmatrix} 0 & 0 & H \\ 0 & K & 0 \\ H & 0 & A \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & 0 \\ 0 & 2\beta \end{bmatrix}$$

where $E^\perp$ *is the orthogonal complement of E in* L_{2d}, α *is either* 1 *or p*, β *is an integer with* $0 \leqslant \beta < \alpha$ *and K is any matrix representing the bilinear form on* $E^\perp/E$.

(ii) *Let r be the number of* Γ_{2d}-*inequivalent* 1-*dimensional rational boundary components of* $\mathscr{D}_{2d}$. *Then* $r \leqslant Md^8$, *where M is a constant not depending on d.*

Proof. The assertion (i) corresponds to [20], Lemma 5.2.1. The second assertion follows from [20], Corollaries 5.4.8, (3), 5.6.10 and the result of §5.3. □

(2.5) *Remark.* Let $(\mathscr{D}_{2d}/\Gamma_{2d})^*$ be the Satake–Baily–Borel compactification of $\mathscr{D}_{2d}/\Gamma_{2d}$ ([4]). Assume that p is odd. Then the configuration of the boundary $(\mathscr{D}_{2d}/\Gamma_{2d})^* \backslash (\mathscr{D}_{2d}/\Gamma_{2d})$ can be described ([20], Figure 5.5.7). There are two types of 1-dimensional component F^* of the boundary corresponding to F as in Proposition 2.4:

$$F^* \simeq \begin{cases} H^+/\Gamma & \text{if } \alpha = 1 \\ H^+/\Gamma^1(p) & \text{if } \alpha = p, \end{cases}$$

where H^+ is the upper half plane, $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ and $\Gamma^1(p) = \{[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] \in \Gamma \mid a \equiv 1, d \equiv 1, b \equiv 0 \pmod p\}$ There are $(p+1)/2$ 0-dimensional components v_i^* corresponding to v_i as in Proposition 2.3. The component v_0^* is a cusp of all modular curves F^* and v_i^* $(1 \leqslant i \leqslant (p-1)/2)$ is a cusp of F^* isomorphic to $H^+/\Gamma^1(p)$.

(2.6) Let F be a rational boundary component of $\mathscr{D}_{2d}$. We denote by $N(F)$,

$W(F)$ or $U(F)$ the stabilizer subgroup of $F(\subset G_{\mathbb{R}})$, the unipotent radical of $N(F)$ or the center of $W(F)$ respectively. Define

$$\mathscr{D}_{2d}(F) = U(F)_{\mathbb{C}} \cdot \mathscr{D}_{2d}(\subset \check{\mathscr{D}}_{2d}).$$

Then there exists a holomorphic isomorphism

$$\mathscr{D}_{2d}(F) \simeq F \times \mathbb{C}^m \times U(F)_{\mathbb{C}}$$

where $m = 0$ if $\dim F = 0$, $m = 17$ if $\dim F = 1$, and

$$\mathscr{D}_{2d} = \{(\tau, w, z) \in F \times \mathbb{C}^m \times U(F)_{\mathbb{C}} \mid \mathrm{Im}(z) - h_\tau(w, w) \in C(F)\}$$

where $C(F)$ is a self-dual homogeneous cone in $U(F)$ with respect to a positive definite bilinear form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $U(F)$ defined over $\mathbb{Q}$ and

$$h_\tau : \mathbb{C}^m \times \mathbb{C}^m \to U(F)$$

is a quasi-hermitian form depending real analytically on $\tau \in F$ ([1], Chap. 3, §4 or [17]). Here *self-dual* means that

$$C(F) = \{x \in U(F) \mid \langle\!\langle x, y \rangle\!\rangle > 0 \text{ for any } y \in \overline{C(F)} \backslash \{0\}\}$$

([19], p. 31). The above representation is called the *Siegel domain of the third kind* of $\mathscr{D}_{2d}$. In the following we shall give a description of the Siegel domain of the third kind of $\mathscr{D}_{2d}$. This is essentially given in [17].

(2.7) *In the case F is of dimension* 0: Assume F corresponds to the isotropic sublattice v_m (Proposition 2.3). Let $\{e_1, e_2\}$ or $\{e_3, \ldots, e_{18}\}$ be a base of U_2 or $E_8 \oplus E_8$ respectively. Put $e_{19} = u + 2mpf$, $e_{20} = f$ and $e_{21} = v_m$. We consider $\{e_1, \ldots, e_{21}\}$ as a $\mathbb{Q}$-base of $L_{2d} \otimes \mathbb{Q}$. Then

$$Q = (\langle e_i, e_j \rangle) = \begin{bmatrix} K & 0 \\ 0 & H \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} \quad \text{and} \quad K = (\langle e_i, e_j \rangle)_{1 \le i,j \le 19}$$

where $\alpha = \begin{cases} 1 & \text{if } m = 0 \\ p & \text{if } m > 0 \end{cases}$. By definition,

$$N(F) = \{g \in G_{\mathbb{R}} \mid g(\mathbb{R}e_{21}) = \mathrm{R}e_{21}\}.$$

Write $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and further $B = (B_1, B_2)$, $C = {}^t(C_1, C_2)$ and $D = (d_{ij})_{1 \geqslant i,j \geqslant 2}$ so that B_i, C_j are column vectors in $\mathbb{R}^{19}$. Then a direct calculation shows:

$$N(F) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G_{\mathbb{R}} \middle| \begin{array}{l} {}^tAKA = K,\ {}^tAKB_1 + \alpha d_{11} C_2 = 0,\ {}^tB_1 K B_1 + 2\alpha d_{11} d_{21} = 0, \\ B_2 = C_1 = 0,\ d_{12} = 0 \text{ and } d_{11} \cdot d_{22} = 1 \end{array} \right\},$$

$$W(F) = U(F) = \left\{ \begin{bmatrix} I_{19} & B_1 & 0 \\ 0 & 1 & 0 \\ {}^tC_2 & d_{21} & 1 \end{bmatrix} \middle| KB_1 + \alpha C_2 = 0 \text{ and } {}^tB_1 K B_1 + 2\alpha d_{21} = 0 \right\}.$$

Let $z = \Sigma_{i=1}^{21} z_i e_i \in \mathbb{P}(L_{2d} \otimes \mathbb{C})$ be a homogeneous coordinate. Then

$$\begin{aligned} \Omega_{2d} = \{(z_1 : \cdots : z_{21}) \mid & 2z_1 z_2 + 2\alpha z_{20} z_{21} + q_0(z_0) = 0, \\ & 2\,\mathrm{Re}(z_1 \bar{z}_2) + 2\alpha\,\mathrm{Re}(z_{20} \bar{z}_{21}) + q_0(z_0, \bar{z}_0) > 0\}, \end{aligned}$$

where q_0 is a symmetric bilinear form defined by the matrix $(\langle e_i, e_j \rangle)_{3 \leqslant i,j \leqslant 19}$ and $z_0 = (z_3, \ldots, z_{19})$. If $z \in \mathscr{D}_{2d}$, then $z_{20} \neq 0$, and hence we may assume $z_{20} = 1$. Then $z \in \Omega_{2d}$ if and only if $2\,\mathrm{Im}(z_1) \cdot \mathrm{Im}(z_2) + q_0(\mathrm{Im}(z_0)) > 0$. Since $q_0(\mathrm{Im}(z_0)) < 0$, a connected component of Ω_{2d} is then defined by the additional condition $\mathrm{Im}(z_1) > 0$ or $\mathrm{Im}(z_1) < 0$. We may assume that $\mathscr{D}_{2d}$ corresponds to the component with $\mathrm{Im}(z_1) > 0$. Put $\mathscr{D}_{2d}(F) = U(F)_{\mathbb{C}} \cdot \mathscr{D}_{2d}$ $(\subset \check{\mathscr{D}}_{2d})$. Then it is easy to see that

$$\mathscr{D}_{2d}(F) \simeq (\mathbb{C}^{19}, (z_1, \ldots, z_{19})).$$

Now we conclude:

PROPOSITION 2.8. *Put* $C(F) = \{(y_i) \in \mathbb{R}^{19} \mid 2y_1 y_2 + q_0(y_3, \ldots, y_{19}) > 0,\ y_1 > 0\}$. *Then* $\mathscr{D}_{2d} = \{(z_i) \in \mathscr{D}_{2d}(F) \mid (\mathrm{Im}(z_i)) \in C(F)\}$.

Put $N(F)_{\mathbb{Z}} = N(F) \cap \Gamma_{2d}$.

PROPOSITION 2.9. *The action of* $g \in N(F)_{\mathbb{Z}}$ *on* $\mathscr{D}_{2d}(F)$ *is as follows*:

$$g = \begin{bmatrix} A & B \\ C & D \end{bmatrix}: z \to \pm(Az + B_1),\ z = {}^t(z_1, \dots, z_{19}),$$

where $\pm A$ *preserves the cone* $C(F)$.

Proof. Since $g \in \Gamma_{2d}$ and $e_{21} = v_m$ is primitive in L_{2d}, $g(e_{21}) = d_{22} \cdot e_{21}$ is also primitive, and hence $d_{22} = \pm 1$. By the relation $d_{11} \cdot d_{22} = 1$, we have $d_{11} = \pm 1$. Now the assertion is obvious. □

(2.10) We introduce a subgroup of $N(F)$ which is used in §5. First note that $N(F)$ acts on $U(F)$ by conjugation and its kernel is $U(F)$. We denote by $G_l(F)$ the image of the induced homomorphism $p: N(F) \to \mathrm{Aut}(U(F))$. Then $G_l(F)$ preserves the cone $C(F)$ and $N(F) = G_l(F) \cdot U(F)$ (semi-direct product). Under the identification

$$U(F) \to \mathbb{R}^{19} \tag{2.11}$$

given by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \to B_1$, $N(F)$ acts on $\mathbb{R}^{19}$ as

$$\begin{bmatrix} A & B_1 & 0 \\ 0 & 1/a & 0 \\ {}^tC_2 & d_{21} & a \end{bmatrix}: B_1 \to aAB_1.$$

We denote by $\bar{\Gamma}_{2d}(F)$ the image $p(\Gamma_{2d})$. Since $a = \pm 1$ for any $\varphi \in \Gamma_{2d}$, we may consider $\bar{\Gamma}_{2d}(F)$ as a subgroup of $O(K \otimes \mathbb{Q}) \times \{\pm 1\}$.

(2.12) *In case F is of dimension* 1: Let $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ be a totally isotropic sublattice of L_{2d} corresponding to F. Let $\{e_1, \dots, e_{21}\}$ be a base of L_{2d} as in Proposition 2.4. We denote by $t = \Sigma_{i=1}^{21} t_i e_i \in \mathbb{P}(L_{2d} \otimes \mathbb{C})$ a homogeneous coordinate. Then

$$\Omega_{2d} = \{(t_1, \dots, t_{21}) \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) \mid 2t_1 t_{20} + 2\alpha t_2 t_{21} + q_0(t_0) + 2\beta t_{21}^2 = 0,$$
$$2\,\mathrm{Re}(t_1 \bar{t}_{20}) + 2\alpha\,\mathrm{Re}(t_2 \bar{t}_{21}) + q_0(t_0, \bar{t}_0) + 2\beta |t_{21}|^2 > 0\}$$

where q_0 is a symmetric bilinear form induced from K and $t_0 = (t_3, \dots, t_{19})$. We may assume $t_{21} = 1$. Then $t \in \Omega_{2d}$ if and only if $2\,\mathrm{Im}(t_1) \cdot \mathrm{Im}(t_0) +$

$q_0(\mathrm{Im}(t_0)) > 0$. We may assume that $\mathscr{D}_{2d}$ is the connected component of Ω_{2d} with $\mathrm{Im}(t_1) > 0$. An elementary calculation shows:

$$N(F) = \left\{ \begin{bmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{bmatrix} \middle| \begin{matrix} {}^tUHZ = H,\ {}^tXKX = K,\ {}^tXKY + {}^tVHZ = 0, \\ {}^tYKY + {}^tZHW + {}^tWHZ + {}^tZAZ = A, \det(U) > 0 \end{matrix} \right\}$$

where U, W, Z are 2 by 2 matrices, X is 17 by 17 matrix and ${}^tV$, Y are 17 by 2 matrices;

$$W(F) = \left\{ \begin{bmatrix} \mathrm{I}_2 & V & W \\ 0 & \mathrm{I}_{17} & Y \\ 0 & 0 & \mathrm{I}_2 \end{bmatrix} \middle| KY + {}^tVH = 0, {}^tYKY + HW + {}^tWH = 0 \right\};$$

$$U(F) = \left\{ \begin{bmatrix} \mathrm{I}_2 & 0 & W \\ 0 & \mathrm{I}_{17} & 0 \\ 0 & 0 & \mathrm{I}_2 \end{bmatrix} \middle| W = \begin{bmatrix} 0 & \alpha c \\ -c & 0 \end{bmatrix}, c \in \mathbb{R} \right\}.$$

Then it is easy to see that:

$$\mathscr{D}_{2d}(F) = U(F)_{\mathbb{C}} \cdot \mathscr{D}_{2d} \simeq (\mathbb{C} \times \mathbb{C}^{17} \times H^+, (t_1, t_3, \ldots, t_{19}, t_{20}))$$

where H^+ is the upper-half plane. Now we define a quasi-hermitian form on $\mathbb{C}^{17}$ as follows: for $\tau \in H^+$, w, $w' \in \mathbb{C}^{17}$,

$$h_\tau(w, w') = 1/4\,\mathrm{Im}(\tau) \cdot \{-q_0(w, \bar{w}') + q_0(w, w')\}.$$

Finally define $C(F) = \{y \in \mathbb{R} \mid y > 0\}$. Then we have:

PROPOSITION 2.13. *Put* $z = t_1$, $w = (t_3, \ldots, t_{19})$ *and* $\tau = t_{20}$. *Then* $\mathscr{D}_{2d} = \{(z, w, \tau) \in \mathscr{D}_{2d}(F) \mid \mathrm{Im}(z) - \mathrm{Re}\{h_\tau(w, w)\} \in C(F)\}$.

PROPOSITION 2.14. *The action of* $N(F)_{\mathbb{Z}} = N(F) \cap \Gamma_{2d}$ *on* $\mathscr{D}_{2d}(F)$ *is as follows*:

$$g = \begin{bmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{bmatrix} : \begin{bmatrix} z \\ w \\ \tau \end{bmatrix} \to$$

$$\begin{bmatrix} z + (c\tau + d)^{-1}\{cq_0(w)/2 + c\beta + v_1 w + w_1\tau + w_2\} \\ (c\tau + d)^{-1}\left\{Xw + Y\begin{bmatrix}\tau \\ 1\end{bmatrix}\right\} \\ (a\tau + b)/(c\tau + d) \end{bmatrix}$$

where $Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma = \mathrm{SL}(2, \mathbb{Z})$, v_1 *is the first row vector of* V *and* (w_1, w_2) *is the first row vector of* W.

Proof. Since $g \in \Gamma_{2d}$, ${}^tUHZ = H$ and $\det(U) > 0$, $U, Z \in \Gamma$. Hence the assertion follows from the relation $2\alpha t_2 = -2t_1t_{20} - q_0(t_0) - 2\beta$. □

(2.15) Let F_0 be a 0-dimensional rational boundary component and F_1 a 1-dimensional rational boundary component such that F_0 is a cusp of F_1. Take a $\mathbb{Q}$-base $\{e_1, \ldots, e_{21}\}$ such that F_1 (resp. F_0) corresponds to the totally isotropic subspace $\mathbb{Q}e_1 \oplus \mathbb{Q}e_2$ (resp. $\mathbb{Q}e_2$) and

$$(\langle e_i, e_j\rangle)_{i,j} = \begin{bmatrix} 0 & 0 & I_2 \\ 0 & K & 0 \\ I_2 & 0 & 0 \end{bmatrix}$$

where K is a negative definite matrix of degree 17. Let $t = \Sigma t_i e_i$ be a homogeneous coordinate of $\mathbb{P}(L_{2d} \otimes \mathbb{C})$. Then by the same way as in (2.7), (2.12), we have:

$$\mathscr{D}_{2d} = \{(t_1, t_3, \ldots, t_{20}) \mid (\mathrm{Im}(t_i)) \in C(F_0)\}$$

where $C(F_0) = \{(y_i) \in \mathbb{R}^{19} \mid 2y_1y_{20} + q_0(y_3, \ldots, y_{19}) > 0,\ y_1 > 0\}$;

$$\mathscr{D}_{2d} = \{(z, w, \tau) \mid \mathrm{Im}(z) - \mathrm{Re}(h_\tau(w, w)) \in C(F_1)\}$$

where $z = t_1$, $w = (t_3, \ldots, t_{19})$, $\tau = t_{20}$ and $C(F_1) = \{y \in \mathbb{R} \mid y > 0\}$. Under the identification (2.11), $U(F_1) = \mathbb{R} \simeq \{(c, 0, \ldots, 0) \in \mathbb{R}^{19}\} \subset U(F_0) = \{B_1 \mid B_1 \in \mathbb{R}^{19}\}$ and $C(F_1) \subset \overline{C(F_0)}$. For $y = (y_i) \in \mathbb{R}^{19}$, put $q(y) = 2y_1y_{20} + q_0(y_3, \ldots, y_{19})$.

Since the signature of q is (1, 18), if $y \in \overline{C(F_0)} \backslash C(F_0)$, then $q(y) = 0$, and hence we have

$$\overline{C(F_0)} \backslash C(F_0) = \bigcup_{y:q(y)=0} \mathbb{R}^+ \cdot y, \qquad \mathbb{R}^+ = \{y \in \mathbb{R} \mid y > 0\}.$$

These half lines $\mathbb{R}^+ \cdot y$ are called *boundary components* of $C(F_0)$. Note that the lattice L_{2d} induces a $\mathbb{Q}$-structure on $\mathbb{R}^{19}$. If y is defined over $\mathbb{Q}$, then the corresponding boundary component $\mathbb{R}^+ \cdot y$ is called *rational*. It is known that for fixed 0-dimensional rational boundary component F_0 of $\mathscr{D}_{2d}$, there is a bijective correspondence between the set of 1-dimensional rational boundary components F of $\mathscr{D}_{2d}$ with $\bar{F} \supset F_0$ and the set of rational boundary components of $C(F_0)$ as

$$\{F \mid \bar{F} \supset F_0\} \to \{\text{rational boundary components of } C(F_0)\}$$
$$F \to C(F)$$

([1], Chap. III, §4, Theorem 3).

3. The group Γ_{2d}

(3.1) In this section we assume $d = p^2$ and compare the groups Γ_{2d} and Γ_2. First we fix a direct sum $L_2 = U_1 \oplus U_2 \oplus E_8 \oplus E_8 \oplus \langle -2 \rangle$, where U_1, U_2 are copies of U. Let u' be a base of $\langle -2 \rangle$. Then L_{2d} is isometric to the sublattice of L_2 generated by $u = pu'$ and $U_1 \oplus U_2 \oplus E_8 \oplus E_8$. In the following, we consider L_{2d} as a sublattice of L_2 under the above isomorphism. The following was suggested by K. G. O'Grady.

LEMMA 3.2. *Γ_{2d} is a subgroup of Γ_2 of finite index. Furthermore if $d = p^2$ for an odd prime number p, then $[\Gamma_2 : \Gamma_{2d}] = p^{20} + p^{10}$.*

Proof. First we shall prove that $O(L_{2d}) \subset O(L_2)$. Consider the subgroup $M = L_2/L_{2d}$ in $A_{L_{2d}} = L_{2d}^*/L_{2d}$ which is totally isotropic with respect to $q_{L_{2d}}$. The lattice L_2 is reconstructed from M as

$$L_2 = \{x \in L_{2d}^* \mid x \bmod L_{2d} \in M\} \qquad \text{([16], Proposition 1.4.1).}$$

Let $\varphi \in O(L_{2d})$. Then φ naturally extends to an isomorphism φ^* of L_{2d}^*. Let $\bar{\varphi}^*$ be the induced isomorphism of $A_{L_{2d}}$. Since $A_{L_{2d}}$ is a cyclic group, $\bar{\varphi}^*$ preserves M, and hence $\varphi^*(L_2) \subset L_2$. Thus φ can be uniquely extended to an isometry of L_2. Hence $\Gamma_{2d} \subset \Gamma_2$.

Next consider the inclusion of lattices: $pL_2 \subset L_{2d} \subset L_2$. Then the affine space

L_2/pL_2 ($\simeq(\mathbb{Z}/p\mathbb{Z})^{21}$) has a structure of non-degenerate finite quadratic form over $\mathbb{Z}/p\mathbb{Z}$ defined by

$$(\bar{x}, \bar{y}) = \langle x, y\rangle \bmod p\mathbb{Z} \quad \text{for } \bar{x} = x \bmod pL_2,\ \bar{y} = y \bmod pL_2.$$

The group $O(L_2)$ naturally acts on L_2/pL_2 as isometries. Note that L_{2d}/pL_2 is a non-degenerate hyperplane of L_2/pL_2. There are two types of nondegenerate quadratic space of dimension $2m$ over $\mathbb{Z}/p\mathbb{Z}$ according to its invariant $(-1)^m\Delta$ in $(\mathbb{Z}/p\mathbb{Z})^*/(\mathbb{Z}/p\mathbb{Z})^{*2}$, where Δ is the discriminant (e.g. [5], p. 63). In our case, we have the following two types:

$$V_+ = P_1 \oplus \cdots \oplus P_{10},$$
$$V_- = P_1 \oplus \cdots \oplus P_9 \oplus \langle u, v\rangle,$$

where P_i is a hyperbolic space and $\langle u, v\rangle$ is a 2-dimensional space generated by $\{u, v\}$ with $(u, u) = 1$, $(u, v) = 0$, $(v, v) = \theta$ $(-\theta \notin (\mathbb{Z}/p\mathbb{Z})^{*2})$. Since L_{2d}/pL_2 is the direct sum of two copies of $(U \oplus E_8)/p(U \oplus E_8)$, the discriminant of L_{2d}/pL_{2d} is a square in $(\mathbb{Z}/p\mathbb{Z})^*$. Hence $L_{2d}/pL_{2d} \simeq V_+$.

Claim: The group $O(L_2)$ acts transitively on the set of all non-degenerate hyperplanes in L_2/pL_2 isometric to V_+.

We shall give the proof of the claim in the latter. Consider the action of $O(L_2)$ on L_2/pL_2 which induces a homomorphism

$$O(L_2) \to O(L_2/pL_2).$$

The group $O(L_{2d})$ (resp. $O(L_{2d}/pL_2) \times \mathbb{Z}/2\mathbb{Z}$) is the stabilizer subgroup of the hyperplane L_{2d}/pL_2 of $O(L_2)$ (resp. $O(L_2/pL_2)$), where $\mathbb{Z}/2\mathbb{Z}$ is generated by a symmetry with respect to the hyperplane L_{2d}/pL_2. It follows from the claim that $O(L_2/pL_2)$ also acts transitively on the set of all nondegenerate hyperplanes isometric to V_+ and

$$[O(L_2){:}\,O(L_{2d})] = [O(L_2/pL_2){:}\,O(L_{2d}/pL_2) \times \mathbb{Z}/2\mathbb{Z}].$$

It is well known that ([5], p. 63):

$$|O(L_2/pL_2)| = 2p^{100} \cdot \prod_{i=1}^{10} (p^{2i} - 1),$$

$$|O(L_{2d}/pL_2)| = |O(V_+)| = 2p^{90}(p^{10}-1) \cdot \prod_{i=1}^{9} (p^{2i}-1).$$

Hence $[O(L_2)\colon O(L_{2d})] = (p^{20}+p^{10})/2$. On the other hand, $[O(L_2)\colon \Gamma_2] = 2$ and$[O(L_{2d})\colon \Gamma_{2d}] = 4$ ([20], Lemma 3.6.1, Example 3.6.2). Thus we have $[\Gamma_2 \colon \Gamma_{2d}] = p^{20} + p^{10}$. □

Proof of Claim. Let V be a non-degenerate hyperplane in L_2/pL_2 isometric to V_+. Define

$$L = \{x \in L_2 \mid x \bmod pL_2 \in V\}.$$

Then L is an even lattice with $[L\colon pL_2] = p^{20}$. Since

$$|\det(L)| \cdot [L\colon pL_2]^2 = |\det(pL_2)|,$$

we have $|\det(L)| = 2p^2$. Moreover V is a totally isotropic subgroup in $A_{pL_2} = (pL_2)^*/pL_2$ with respect to q_{pL_2} and $q_L \simeq (q_{pL_2} \mid V^\perp)/V$ ([16], Proposition 1.4.1). Hence $A_L \simeq \mathbb{Z}/2p^2\mathbb{Z}$. Let q_2 (resp. q_p^θ) be the restriction of q_L on $\mathbb{Z}/2\mathbb{Z}$ (resp. $\mathbb{Z}/p^2\mathbb{Z}$): $q_L = q_2 \oplus q_p^\theta$ ([16], Proposition 1.2.2). Then L_2/L is a totally isotropic subgroup of A_L of order p and $q_{L_2} \simeq (q_L \mid (L_2/L)^\perp)/(L_2/L) \simeq q_2$. Hence

$$q_2\colon \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$$

is given by $q_2(m \bmod 2\mathbb{Z}) = -m^2/2$. On the other hand, q_p^θ is one of the following:

$$q_p^\theta\colon \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z},\ q_p^\theta(m \bmod p^2\mathbb{Z}) = am^2/p^2$$

where, using the Legendre symbol, $\theta = (a/p) = \pm 1$ (see [16], §1.8). Let α be an integer with $(-2\alpha/p) = \theta$. We now consider the following lattice

$$L' = U \oplus E_8 \oplus E_8 \oplus \begin{bmatrix} 2\alpha & p \\ p & 0 \end{bmatrix} \oplus \langle -2 \rangle$$

which has the discriminant form $q_2 \oplus q_p^\theta$. Since $\operatorname{rank}(L) > 2 + l(L)$, the genus of L contains only one isomorphism class ([16], Theorem 1.14.2). Hence $L \simeq L'$. Fix a decomposition

$$L = U \oplus E_8 \oplus E_8 \oplus \begin{bmatrix} 2\alpha & p \\ p & 0 \end{bmatrix} \oplus \langle -2 \rangle$$

and take a base $\{x, y\}$, $\{u\}$ of $\begin{bmatrix} 2\alpha & p \\ p & 0 \end{bmatrix}$, $\langle -2 \rangle$, respectively. Since L_2 is obtained from L by adding $(1/p)y$, $y \in pL_2$. Hence

$$V = L/pL_2 = (U \oplus E_8 \oplus E_8)/p(U \oplus E_8 \oplus E_8) \oplus \langle \bar{x}, \bar{u} \rangle$$

where $\bar{x} = x \bmod pL_2$, $\bar{u} = u \bmod pL_2$. Note that the first factor of the above decomposition is the direct sum of 9 hyperbolic spaces because $(-1)^9\Delta = 1$, where Δ is the discriminant of $U \oplus E_8 \oplus E_8$. Hence $V \simeq V_+$ if and only if $\langle \bar{x}, \bar{u} \rangle$ is isometric to a hyperbolic lattice. Since the discriminant of $\langle \bar{x}, \bar{u} \rangle$ is equal to -4α, $\langle \bar{x}, \bar{u} \rangle$ is isometric to a hyperbolic space if and only if $(4\alpha/p) = (\alpha/p) = 1$. Hence we may assume that $\alpha = 1$. Then

$$L \simeq U \oplus E_8 \oplus E_8 \oplus \begin{bmatrix} 2 & p \\ p & 0 \end{bmatrix} \oplus \langle -2 \rangle.$$

Note that $q_L \simeq q_{L_{2d}}$. Hence, again by [16], Theorem 1.14.2, $L \simeq L_{2d}$. The sublattices L and L_{2d} give two decompositions of L_2 into

$$U \oplus E_8 \oplus E_8 \oplus \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \oplus \langle -2 \rangle.$$

Obviously there is an isometry $\varphi \in O(L_2)$ which sends one decomposition to the other. Then $\varphi(L) \subset L_{2d}$, and hence V is equivalent to L_{2d}/pL_2 modulo $O(L_2)$. □

(3.3) We use the same notation as in (2.7). Let F be a 0-dimensional rational boundary component of $\mathcal{D}_{2d}$ corresponding to $\mathbb{Z}v_m$. Note that v_0 or $v_m/p = mu' + e + m^2 f (m > 0)$ defines a primitive isotropic sublattice of L_2. Hence we may consider F as a rational boundary component of $\mathcal{D}_2$. In Section 5, we need to estimate the indices $[\bar{\Gamma}_2(F) : \bar{\Gamma}_{2d}(F)]$ and $[U(F) \cap \Gamma_2 : U(F) \cap \Gamma_{2d}]$.

First note that $\{e_1, \ldots, e_{21}\}$ generates a sublattice in L_{2d} of index α and $\{e_1, \ldots, e_{18}, e_{19}/p, e_{20}, e_{21}/\alpha\}$ is a base of L_2. Let M_{2d} (resp. M_2) be a primitive sublattice of L_{2d} (resp. L_2) generated by $\{e_1, \ldots, e_{19}\}$ (resp. $\{e_1, \ldots, e_{18}, e_{19}/p\}$).

LEMMA 3.4. $[\bar{\Gamma}_2(F) : \bar{\Gamma}_{2d}(F)] \leqslant O(p^{18})$.

Proof. First consider the case that F corresponds to v_0. Then $\{e_1, \ldots, e_{21}\}$ is a base of L_{2d}. Hence, for any

$$\varphi = \begin{bmatrix} A & B_1 & 0 \\ 0 & \pm 1 & 0 \\ {}^tC_2 & d_{21} & \pm 1 \end{bmatrix} \in N(F) \cap \Gamma_{2d},$$

$A \in O(M_{2d})$. Therefore $\bar{\Gamma}_{2d}(F)$ is a subgroup of $O(M_{2d}) \times \{\pm 1\}$. Since $L_{2d} = U \oplus M_{2d}$, any $\psi \in \tilde{O}(M_{2d})$ can be extended to an isometry $\varphi \in \tilde{O}(L_{2d})$ with $\varphi | M_{2d}^{\perp} = 1$. Hence $\psi \in \bar{\Gamma}_{2d}(F)$ if $\psi(C(F)) \subset C(F)$. By [20], Lemma 3.6.1, $[O(M_{2d}) \colon \tilde{O}(M_{2d})] = 2$, and hence the index of $\bar{\Gamma}_{2d}(F)$ in $O(M_{2d}) \times \{\pm 1\}$ is at most 8. Similarly, $\bar{\Gamma}_2(F)$ is a subgroup of $O(M_2) \times \{\pm 1\}$ of index at most 8. By the same argument as in the proof of Lemma 3.2, $[O(M_2) \colon O(M_{2d})] = O(p^{18})$. Thus we have $[\bar{\Gamma}_2(F) \colon \bar{\Gamma}_{2d}(F)] = O(p^{18})$.

Next consider the case that F corresponds to v_m $(m > 0)$. Let $\psi \in \tilde{O}(M_{2d})$. Then, by [16], Proposition 1.5.1, ψ can be extended to an isometry $\varphi \in \tilde{O}(L_{2d})$ with $\varphi | M_{2d}^{\perp} = 1$. Hence $\tilde{O}(M_{2d}) \subset p(\tilde{O}(L_{2d}))$ where $p \colon N(F) \to \mathrm{Aut}(U(F))$ is the projection. On the other hand, $[O(M_2) \colon O(M_{2d})] = O(p^{18})$ and $\bar{\Gamma}_2(F)$ is a subgroup of $O(M_2) \times \{\pm 1\}$ of index at most 8. Hence the assertion follows. □

For $\varphi = \begin{bmatrix} I_{19} & B_1 & 0 \\ 0 & 1 & 0 \\ {}^tC_2 & d_{21} & 1 \end{bmatrix} \in U(F)$, put ${}^tB_1$

$= ({}^tB'_1, b)$ and ${}^tC_2 = ({}^tC'_2, c)$.

LEMMA 3.5. *Assume that F corresponds to $\mathbb{Z}v_0 = \mathbb{Z}e$. Then*

$$U(F) \cap \Gamma_{2d} \simeq \{B_1 \mid B_1 \in \mathbb{Z}^{19}\},$$
$$U(F) \cap \Gamma_2 \simeq \{({}^tB'_1, b) \mid B'_1 \in \mathbb{Z}^{18} \text{ and } b \in (1/p)\mathbb{Z}\}.$$

Proof. First note that $\{e_1, \ldots, e_{21}\}$ is a base of L_{2d}. Hence B_1, C_2 and d_{21} are integral. Conversely if $B_1 \in \mathbb{Z}^{19}$, then $C_2 = -KB_1$ and $d_{21} = -{}^tB_1KB_1/2$ are also integral. Hence we have

$$U(F) \cap O(L_{2d}) \simeq \{B_1 \mid B_1 \in \mathbb{Z}^{19}\}.$$

Since $\{e_1, \ldots, e_{18}, (1/p)e_{19}, e_{20}, e_{21}\}$ is a base of L_2, we can see that

$$U(F) \cap O(L_2) \simeq \{(B'_1, b) \mid B'_1 \in \mathbb{Z}^{18},\ b \in (1/p)\mathbb{Z}\}.$$

If $\varphi \in U(F) \cap O(L_{2d})$, then $\varphi(u) = u + ce_{21} = u + 2p^2be_{21}$. Hence $\varphi(u/2p^2) \equiv u/2p^2 \pmod{L_{2d}}$, i.e. $\varphi \in \tilde{O}(L_{2d})$. Obviously $U(F)$ preserves $\mathcal{D}_{2d}$. Therefore $U(F) \cap \Gamma_{2d} = U(F) \cap O(L_{2d})$ and we have the desired result. □

LEMMA 3.6. *Assume that F corresponds to* $\mathbb{Z}v_m$ $(m > 0)$. *Then*

$$U(F) \cap \Gamma_{2d} \simeq \{({}^tB'_1, b) \mid B'_1 \in (p\mathbb{Z})^{18} \text{ and } b \in \mathbb{Z}\},$$

$$U(F) \cap \Gamma_2 \simeq \{({}^tB'_1, b) \mid B'_1 \in \mathbb{Z}^{18} \text{ and } b \in (1/p)\mathbb{Z}\}.$$

Proof. If $\varphi \in U(F) \cap \Gamma_{2d} = U(F) \cap \tilde{O}(L_{2d})$, then $C'_2 \in \mathbb{Z}^{18}$ by considering $\varphi(e_i)$ $(1 \leqslant i \leqslant 18)$ and using the primitiveness of e_{21} in L_{2d}. Since $\varphi \in \tilde{O}(L_{2d})$ and $e_{19}/2p = u/2p + mf \in L^*_{2d}$, we have

$$e_{19}/2p \equiv \varphi(e_{19}/2p) = e_{19}/2p + (c/2p)e_{21} \pmod{L_{2d}}.$$

Since e_{21} is primitive in L_{2d}, $c \in 2p\mathbb{Z}$. Recall that

$$K'B'_1 + pC'_2 = 0,\ 2p^2b = pc \text{ and } {}^tB'_1K'B'_1 - 2p^2b^2 + 2pd_{21} = 0$$

where $K = \begin{bmatrix} K' & 0 \\ 0 & -2d \end{bmatrix}$ and K' is an unimodular matrix. By these equations, $B'_1 \in (p\mathbb{Z})^{18}$ and $b \in \mathbb{Z}$, $d_{21} \in p\mathbb{Z}$. Conversely if $\varphi \in U(F)$ with $B'_1 \in (p\mathbb{Z})^{18}$, $b \in \mathbb{Z}$, then it is easy to see that $\varphi(e) \in L_{2d}$. Hence $\varphi \in O(L_{2d})$ because $\{e_1, \ldots, e_{20}, e\}$ is a base of L_{2d}. Moreover an easy calculation shows that $\varphi(u/2p^2) \equiv u/2p^2 \pmod{L_{2d}}$, i.e. $\varphi \in \tilde{O}(L_{2d})$. Hence we have the first assertion. For $U(F) \cap \Gamma_2$, we can see the assertion by using the fact that $\{e_1, \ldots, e_{18}, (1/p)e_{19}, e_{20}, (1/p)e_{21}\}$ is a base of L_2. □

4. Tai's criterion and dimension formula

(4.1) Let $\mathcal{D}_{2d} \subset \mathbb{C}^{19}$ be a realization of $\mathcal{D}_{2d}$ as a bounded domain. Let $u = (u_1, \ldots, u_{19})$ be a coordinate of $\mathbb{C}^{19}$ and put $\omega = du_1 \wedge \cdots \wedge du_{19}$. A holomorphic function $f: \mathcal{D}_{2d} \to \mathbb{C}$ is called a *Γ_{2d}-automorphic form of weight k* if for any $\gamma \in \Gamma_{2d}$, $f(\gamma u) = J(\gamma, u)^{-k} \cdot f(u)$ where $J(\gamma, u)$ is the Jacobian of γ. We denote by $A_k(\Gamma_{2d})$ the space of Γ_{2d}-automorphic forms of weight k. For $f \in A_k(\Gamma_{2d})$, $f \cdot \omega^{\otimes k}$ is invariant under Γ_{2d}, hence can be considered as a form in $H^0(\mathcal{K}^\circ_{2d}, \Omega^{\otimes k})$ where $\mathcal{K}^\circ_{2d}$ is the open set of $\mathcal{K}_{2d}$ such that the projection

$\pi: \mathscr{D}_{2d} \to \mathscr{D}_{2d}/\Gamma_{2d}$ is unramified over $\mathscr{K}^{\circ}_{2d}$ and Ω is the sheaf of top-dimensional holomorphic forms on $\mathscr{K}^{\circ}_{2d}$.

(4.2) Recall the construction of a toroidal compactification of $\mathscr{D}_{2d}/\Gamma_{2d}$ ([1], §5). Let F be a rational boundary component of $\mathscr{D}_{2d}$. Let $U(F)$, $C(F)$ and $\mathscr{D}(F)$ be the same as in §2. Then $\mathscr{D}_{2d}$ is realized in $\mathscr{D}(F) \simeq F \times \mathbb{C}^m \times U(F)_{\mathbb{C}}$ as Siegel domain of the third kind. Put $T(F) = U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}}$. Let $\{\sigma_\alpha\}$ be an admissible polyhedral decomposition of $C(F) \subset U(F)$ such that all cones σ_α are regular ([14], Theorem 7.20). The collection $\{\sigma_\alpha\}$ defines a torus *embedding* $T(F) \subset T(F)_{\{\sigma_\alpha\}}$. Finally define $(\mathscr{D}_{2d}/U(F)_{\mathbb{Z}})_{\{\sigma_\alpha\}}$ = the interior of the closure of $\mathscr{D}_{2d}/U(F)_{\mathbb{Z}}$ in $(\mathscr{D}(F)/U(F)_{\mathbb{Z}}) \times_{T(F)} T(F)_{\{\sigma_\alpha\}}$, which is smooth by our assumption on $\{\sigma_\alpha\}$. Then by the main theorem in [1] (also see [14], Theorem 7.20), there exists a compact analytic space $\tilde{\mathscr{K}}_{2d} = \overline{\mathscr{D}_{2d}/\Gamma_{2d}}$ with only quotient singularities and a morphism

$$\pi_F: (\mathscr{D}_{2d}/U(F)_{\mathbb{Z}})_{\{\sigma_\alpha\}} \to \tilde{\mathscr{K}}_{2d}$$

such that $\mathscr{K}_{2d}$ is a Zariski open set in $\tilde{\mathscr{K}}_{2d}$ and every point of $\tilde{\mathscr{K}}_{2d}$ is in the image of π_F or π. We denote by $\tilde{\mathscr{K}}^{\circ}_{2d}$ the open set of $\tilde{\mathscr{K}}_{2d}$ such that π and π_F are unramified for any F.

In the following we recall the Tai's criterion of extendable pluri-canonical differential forms to $\tilde{\mathscr{K}}^{\circ}_{2d}$. Let F be a rational boundary component of $\mathscr{D}_{2d}$. Let (z, w, τ) be a coordinate of $U(F)_{\mathbb{C}} \times \mathbb{C}^m \times F$ (see §2). Put $U(F)_{\mathbb{Z}} = U(F) \cap \Gamma_{2d}$. For $f \in A_k(\Gamma_{2d})$, f is $U(F)_{\mathbb{Z}}$-invariant, and hence we have a Fourier–Jacobi expansion

$$f = f(z, w, \tau) = \sum_{\rho \in U(F)^*_{\mathbb{Z}}} \xi_\rho(\tau, w)\mathbf{e}(\langle\langle \rho, z\rangle\rangle)$$

where $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$; $\langle\langle\ ,\ \rangle\rangle$ is a positive definite bilinear form on $U(F)$ define over $\mathbb{Q}$ and $U(F)^*_{\mathbb{Z}}$ is the dual $U(F)_{\mathbb{Z}}$ with respect to $\langle\langle\ ,\ \rangle\rangle$. By Koecher's theorem ([3], [17]), $\xi_\rho \neq 0$ only for $\rho \in \overline{C(F)} \cap U(F)^*_{\mathbb{Z}}$. In §5 and §6, we shall need the following criterion due to Tai.

THEOREM 4.3. ([1], Chap. IV, §1). *Suppose $f \in A_k(\Gamma_{2d})$, then $f \cdot \omega^{\otimes k}$ defines a k-fold canonical differential form on $\tilde{\mathscr{K}}^{\circ}_{2d}$ if f satisfies the following condition: for any rational boundary component F, $\xi_\rho \neq 0$ implies $\langle\langle \rho, x\rangle\rangle \geqslant k$ for all non-zero $x \in \overline{C(F)} \cap U(F)_{\mathbb{Z}}$.*

Let $f \in A_k(\Gamma_{2d})$. We call f a *cusp form* if the Fourier–Jacobi coefficient $\xi_0(\tau, w) = 0$ for any rational boundary components. We denote by $S_k(\Gamma_{2d})$ the vector space of cusp forms of weight k.

PROPOSITION 4.4. $\dim S_k(\Gamma_{2d}) = (2\cdot 19^{18}/18!)\cdot \mathrm{vol}(\mathscr{D}_{2d}/\Gamma_{2d})\cdot k^{19} + O(k^{18})$ *where* $\mathrm{vol}(\mathscr{D}_{2d}/\Gamma_{2d})$ *is the volume with respect to a metric on* $\mathscr{D}_{2d}$.

Proof. Let Γ'_{2d} be a normal neat subgroup of Γ_{2d}. By the Proportionality theorem [12],

$$\dim S_k(\Gamma'_{2d}) = \mathrm{vol}(\mathscr{D}_{2d}/\Gamma'_{2d})\cdot \dim H^0(\check{\mathscr{D}}_{2d}, \Omega^{\otimes(-k+1)}) + O(k^{18})$$

where Ω is the sheaf of holomorphic 19-forms on the compact dual $\check{\mathscr{D}}_{2d}$. Recall that $\check{\mathscr{D}}_{2d}$ is a smooth quadric in $\mathbb{P}^{20}$. Hence

$$\begin{aligned}\dim H^0(\check{\mathscr{D}}_{2d}, \Omega^{\otimes(-k+1)}) &= (19k+1)!/20!(19k-19)! - (19k-1)!/20!(19k-21)!\\ &= (2\cdot 19^{18}/18!)k^{19} + O(k^{18}).\end{aligned}$$

By the Tai's argument in the proof of [23], Proposition 2.1, we have

$$\begin{aligned}\dim S_k(\Gamma_{2d}) &= \dim S_k(\Gamma'_{2d})^{\Gamma_{2d}/\Gamma'_{2d}}\\ &= 1/[\Gamma_{2d}:\Gamma'_{2d}]\cdot \sum_{\gamma\in\Gamma_{2d}/\Gamma'_{2d}} \mathrm{trace}(\gamma \mid S_k(\Gamma'_{2d}))\\ &= 1/[\Gamma_{2d}:\Gamma'_{2d}]\cdot \dim S_k(\Gamma'_{2d}) + O(k^{18})\\ &= 2\cdot 19^{18}\cdot k^{19}\cdot \mathrm{vol}(\mathscr{D}_{2d}/\Gamma'_{2d})/18![\Gamma_{2d}:\Gamma'_{2d}] + O(k^{18}).\end{aligned}$$

Since $\mathrm{vol}(\mathscr{D}_{2d}/\Gamma'_{2d})/[\Gamma_{2d}:\Gamma'_{2d}] = \mathrm{vol}(\mathscr{D}_{2d}/\Gamma_{2d})$, we now have proved the assertion. □

COROLLARY 4.5. *Assume* $d = p^2$. *Then*

$$\dim S_k(\Gamma_{2d}) = (2\cdot 19^{18}/18!)\cdot \mathrm{vol}(\mathscr{D}_{2d}/\Gamma_2)\cdot [\Gamma_2:\Gamma_{2d}]\cdot k^{19} + O(k^{18}).$$

Proof. This follows from the facts that $\Gamma_{2d} \subset \Gamma_2$ (Lemma 3.2) and $\mathrm{vol}(\mathscr{D}_{2d}/\Gamma_{2d}) = [\Gamma_2:\Gamma_{2d}]\cdot \mathrm{vol}(\mathscr{D}_2/\Gamma_2)$. □

5. Coefficients of Fourier–Jacobi series I

(5.1) Let F be a rational boundary component of $\mathscr{D}_{2d}$. For $f\in A_k(\Gamma_{2d})$, consider the Fourier–Jacobi series $f = \Sigma_\rho \xi_\rho(\tau, w)\mathrm{e}(\langle\langle \rho, z\rangle\rangle)$ with respect to F, where $(z, w, \tau)\in U(F)_{\mathbb{C}} \times \mathbb{C}^m \times F$ (see §4). Put

$$A'_k(\Gamma_{2d})_F = \{f \in A_k(\Gamma_{2d}) \mid \xi_\rho(\tau, w) \equiv 0 \quad \text{if } \langle\langle \rho, x \rangle\rangle < k$$

for some non-zero $x \in \overline{C(F)} \cap U(F)_{\mathbb{Z}}\}$,

$$A'_k(\Gamma_{2d}) = f \in A_k(\Gamma_{2d}) \mid f \in A'_k(\Gamma_{2d})_F$$

for any rational boundary components F of $\mathscr{D}_{2d}\}$.

The purpose of this section and the next is to estimate the dimension of $A'_k(\Gamma_{2d})$ (Theorem 6.18). In this section, we shall study the coefficients of Fourier–Jacobi series with respect to 0-dimensional rational boundary components.

(5.2) First we give a relation of 0- and 1-dimensional cases. We use the same notation as in (2.15). Then the group $U(F_1)_{\mathbb{Z}} = U(F_1) \cap \Gamma_{2d}$ ($\simeq \mathbb{Z}$) acts on $\mathscr{D}_{2d}$ as follows (see Proposition 2.14):

$$\begin{bmatrix} I_2 & 0 & W \\ 0 & I_{17} & 0 \\ 0 & 0 & I_2 \end{bmatrix} : (z, w, \tau) \to (z + c, w, \tau),$$

where

$$W = \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}.$$

Let $f \in A_k(\Gamma_{2d})$. Then we have a Fourier–Jacobi expansion of f:

$$f = \sum_{\substack{\langle\langle \rho, \xi \rangle\rangle = m \geq 0 \\ \rho \in U(F_1)^*_{\mathbb{Z}} \cap \overline{C(F_1)}}} \theta_m(\tau, w) \mathbf{e}(\langle\langle \rho, z \rangle\rangle) \tag{5.3}$$

where ξ is a base of $U(F_1)_{\mathbb{Z}}$ with $\xi \in C(F_1)$. Also the group $U(F_0)_{\mathbb{Z}}$ acts on $\mathscr{D}_{2d}$ as translations (see Proposition 2.9). Hence we have a Fourier–Jacobi expansion of f:

$$f = \sum_{\rho \in \overline{C(F_0)} \cap U(F_0)^*_{\mathbb{Z}}} c_\rho \mathbf{e}(\langle\langle \rho, (t_i) \rangle\rangle) \quad (c_\rho \in \mathbb{C}). \tag{5.4}$$

Here recall that $C(F_1) \subset \overline{C(F_0)}$ and $\xi \in \overline{C(F_0)} \cap U(F_0)_{\mathbb{Z}}$ (see (2.15)). It follows from (5.3), (5.4) and the uniqueness of Fourier–Jacobi coefficients that

$$\theta_m(\tau, w) = \sum_{\substack{\rho \in \overline{C(F_0)} \cap U(F_0)^*_{\mathbb{Z}} \\ \langle\langle \rho, \xi \rangle\rangle = m}} c_\rho \cdot \mathbf{e}(\langle\langle \rho, (0, t_3, \ldots, t_{20}) \rangle\rangle). \tag{5.5}$$

Assume $\theta_m(t, w) \equiv 0$ for some m. Then it follows from (5.5) that $c_\rho = 0$ for any $\rho \in \overline{C(F_0)} \cap U(F_0)^*_{\mathbb{Z}}$ with $\langle\langle \rho, \xi \rangle\rangle = m$. Conversely, each half line in $\overline{C(F_0)} \backslash C(F_0)$ defined over $\mathbb{Q}$ corresponds to $C(F'_1)$ for a rational 1-dimensional boundary component F'_1 ((2.15)). Thus we have:

LEMMA 5.6. *Let $f \in A_k(\Gamma_{2d})$ and assume that $f \in A'_k(\Gamma_{2d})_{F_1}$ for any 1-dimensional rational boundary components F_1 of $\mathscr{D}_{2d}$. Then $F \in A'_k(\Gamma_{2d})$ if f satisfies the following: for any 0-dimensional rational boundary components F_0 of $\mathscr{D}_{2d}$, $c_\rho = 0$ for any $\rho \in \overline{C(F_0)} \cap U(F_0)^*_{\mathbb{Z}}$ with $\langle\langle \rho, x \rangle\rangle < k$ for some $x \in C(F_0) \cap U(F_0)_{\mathbb{Z}}$.*

(5.7) In the following we use the same notation as in (2.7), (3.3). Let F be a 0-dimensional rational boundary component of $\mathscr{D}_{2d}$. Put $U(F)_{\mathbb{Z},\delta} = U(F) \cap \Gamma_{2\delta}$ ($\delta = 1$ or d) and

$$H(\delta, F, k) = \{\rho \in \overline{C(F)} \cap U(F)^*_{\mathbb{Z},\delta} | \langle\langle \rho, x \rangle\rangle < k$$

for some $x \in C(F) \cap U(F)_{\mathbb{Z},\delta}\}$.

Recall that $\bar{\Gamma}_{2\delta}(F)$ acts on $\overline{C(F)} \cap U(F)_{\mathbb{Z},\delta}$ ((2.10)). This action induces the action of $\bar{\Gamma}_{2\delta}(F)$ on $H(\delta, F, k)$ satisfying the condition $\langle\langle \varphi^*(\rho), x \rangle\rangle = \langle\langle \rho, \varphi(x) \rangle\rangle$ for $\varphi \in \bar{\Gamma}_{2\delta}(F)$, $\rho \in U(F)^*$ and all $x \in U(F)$. Since $\bar{\Gamma}_2(F) \supset \bar{\Gamma}_{2d}(F)$, $\bar{\Gamma}_{2\delta}(F)$ also acts on $H(1, F, k)$. For $f \in A_k(\Gamma_{2d})$, Γ_{2d}-automorphicity of f and the uniqueness of Fourier–Jacobi coefficients imply that $c_\rho = \pm c_{\varphi^*(\rho)}$ for any $\varphi \in \bar{\Gamma}_{2d}(F)$ where c_ρ is a Fourier–Jacobi coefficient of f with respect to F. Hence to estimate the dimension $A'_k(\Gamma_{2d})_F$, it suffices to count the order of the set $H(d, F, k)$ modulo $\bar{\Gamma}_{2d}(F)$.

LEMMA 5.8. *$H(\delta, F, k)$ modulo $\bar{\Gamma}_{2\delta}(F)$ is a finite set.*

Proof. Let $\{\sigma_\nu\}$ be a $\bar{\Gamma}_{2\delta}(F)$-admissible polyhedral decomposition of $\overline{C(F)}$ ([1], Chap. III, §5). By [14], Theorem 7.20, we may assume that each σ_ν is regular, that is, $\sigma_\nu = \Sigma_{i=1}^r \mathbb{R}_{\geqslant 0} \cdot e_i^\nu$ where $\{e_i^\nu\}$ is a part of a base of $U(F)_{\mathbb{Z},\delta}$ and $\mathbb{R}_{\geqslant 0} = \{y \in \mathbb{R} \,|\, y \geqslant 0\}$. The admissibility of $\{\sigma_\nu\}$ implies that the number of classes of cones modulo $\bar{\Gamma}_{2\delta}(F)$ is finite. Therefore it suffices to see that, for each top dimensional polyhedral cone σ_ν, the set

$$H_{\nu,\delta} = \{\rho \in \sigma_\nu \cap U(F)^*_{\mathbb{Z},\delta} \,|\, \langle\langle \rho, x \rangle\rangle < k \text{ for some } x \in C(F) \cap U(F)_{\mathbb{Z},\delta}\}$$

is finite. We may assume that $\langle\langle\ ,\ \rangle\rangle$ is integral on $U(F)_{\mathbb{Z},\delta}$ if necessary replacing $\langle\!\langle\ ,\ \rangle\!\rangle$ by some multiple of $\langle\!\langle\ ,\ \rangle\!\rangle$ because the order $|H_{v,\delta}|$ does not depend on $\langle\!\langle\ ,\ \rangle\!\rangle$. For $\rho \in H_{v,\delta}$, write $\rho = \Sigma_{i=1}^{19} a_i \cdot e_i$, $a_i \in \mathbb{Q}$, $a_i \geqslant 0$. Then $k > \langle\!\langle \rho, x \rangle\!\rangle = \Sigma a_i \langle\!\langle e_i, x \rangle\!\rangle$ for some $x \in C(F) \cap (F)_{\mathbb{Z},\delta}$. The self-duality of $C(F)$ implies that $\langle\!\langle e_i, x \rangle\!\rangle$ is a positive integer. Therefore $|H_{v,\delta}| \leqslant [U(F)^*_{\mathbb{Z},\delta} : U(F)_{\mathbb{Z},\delta}] \cdot k^{19}$. □

We denote by $h(\delta, F, k)$ the order of the set $H(\delta, F, k)$ modulo $\bar{\Gamma}_{2\delta}(F)$. Recall that the set $\{v_m \mid 0 \leqslant m \leqslant (p-1)/2\}$ is a complete set of Γ_{2d}-inequivalent 0-dimensional rational boundary components of $\mathscr{D}_{2d}$.

LEMMA 5.9. *Assume that F corresponds to* v_0. *Then*

$$h(d, F, k) \leqslant cp[\bar{\Gamma}_2(F) : \bar{\Gamma}_{2d}(F)]k^{19}$$

where c is a constant not depending on p and k.

Proof. Let $\{\sigma_v\}$ be a $\Gamma_2(F)$-admissible polyhedral decomposition of $\overline{C(F)}$. We may assume that each σ_v is regular. Let σ_v be a top dimensional polyhedral cone and let $\{e_i\}$ be a base of $U(F)_{\mathbb{Z},1}$ with $\sigma_v = \Sigma_{i=1}^{19} \mathbb{R}_{\geqslant 0} \cdot e_i$. Put

$$H'_{v,d} = \{\rho \in \sigma_v \cap U(F)^*_{\mathbb{Z},d} \mid \langle\!\langle \rho, x \rangle\!\rangle < k \text{ for some } x \in C(F) \cap U(F)_{\mathbb{Z},1}\}.$$

Then $H_{v,d} \subset H'_{v,d}$. By Lemma 3.5 and the proof of Lemma 5.8, we have

$$|H'_{v,d}| \leqslant p \cdot [U(F)^*_{\mathbb{Z},1} : U(F)_{\mathbb{Z},1}] \cdot k^{19}.$$

Therefore

$$h(d, F, k) \leqslant c'p(U(F)^*_{\mathbb{Z},1} : U(F)_{\mathbb{Z},1}][\bar{\Gamma}_2(F) : \bar{\Gamma}_{2d}(F)]k^{19}$$

where c' is the number of $\bar{\Gamma}_2(F)$-inequivalent cones σ_v. Since c' and $[U(F)^*_{\mathbb{Z},1} : U(F)_{\mathbb{Z},1}]$ are not dependent on p, we have the desired result. □

LEMMA 5.10. *Assume that F corresponds to* v_m $(m > 0)$. *Then*

$$h(d, F, k) \leqslant [\bar{\Gamma}_2(F) : \bar{\Gamma}_{2d}(F)] \cdot h(1, F, k)$$

Proof. By Lemma 3.6, we have bijections

$$U(F)_{\mathbb{Z},1} \xrightarrow{\simeq} U(F)_{\mathbb{Z},d} \quad \text{and} \quad U(F)^*_{\mathbb{Z},d} \xrightarrow{\simeq} U(F)^*_{\mathbb{Z},1}$$

given by $B_1 \to pB_1$, $B_1^* \to pB_1^*$ respectively which induce a $\bar{\Gamma}_{2d}(F)$-equivariant bijection $H(1, F, k) \simeq H(d, F, k)$. Therefore $h(d, F, k) = \#\{H(1, F, k)$ modulo $\bar{\Gamma}_{2d}(F)\}$, and hence

$$h(d, F, k) \leqslant [\bar{\Gamma}_2(F):\bar{\Gamma}_{2d}(F)]\cdot h(1, F, k). \qquad \square$$

THEOREM 5.11. *$\Sigma_F h(d, F, k) \leqslant c''\cdot p^{19}\cdot k^{19}$, where the summation in the left hand side means that F moves on a set of all Γ_{2d}-inequivalent 0-dimensional rational boundary components of $\mathscr{D}_{2d}$ and c'' is a constant not depending on p and k.*

Proof. We denote by F_0 (resp. F_m, $1 \leqslant m \leqslant (p-1)/2$) the rational boundary component corresponding to $\mathbb{Z}e$ (resp. $\mathbb{Z}v_m$). Then it follows from Lemmas 5.9, 5.10 that

$$\sum_F h(d, F, k) \leqslant$$

$$cp[\bar{\Gamma}_2(F_0):\bar{\Gamma}_{2d}(F_0)]k^{19} + \sum_{m=1}^{(p-1)/2} [\bar{\Gamma}_2(F_m):\bar{\Gamma}_{2d}(F_m)]\cdot h(1, F_m, k)$$

Note that $h(1, F_m, k) \leqslant c'k^{19}$ where c' is a constant not depending on p. Now the assertion follows from Lemma 3.4. $\square$

6. Coefficients of Fourier–Jacobi series II

In this section we consider the case of 1-dimensional boundary component. We assume $d = p^2$ for some prime number p.

LEMMA 6.1. *We use the same notation as in Proposition 2.4. Let F be a 1-dimensional rational boundary component. Let $f \in A_k(\Gamma_{2d})$ and let $f = \Sigma_{m\geqslant 0}\,\theta_m(\tau, w)\mathbf{e}(mz/\alpha)$ be the Fourier–Jacobi series of f. Then*

(i) $\theta_m(\tau, w + \alpha n) = \theta_m(\tau, w)$ *for* $n \in \mathbb{Z}^{17}$,

(ii) $\theta_m(\tau, w + \tau n) = \theta_m(\tau, w)\mathbf{e}[m/\alpha\{{}^t nKw + (\tau/2){}^t nKn\}]$ *for* $n \in \mathbb{Z}^{17}$,

(iii) $\theta_m(\gamma\tau, \gamma w) = \theta_m(\tau, w)(c\tau + d)^{19k}\mathbf{e}[-m/\alpha(U - \beta c)]$ *for*

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^1(\alpha),$$

where $\gamma\tau = (a\tau + b)/(c\tau + d)$, $\gamma w = w/(c\tau + d)$ and $U = (c/2)(c\tau + d)^{-1}$ $({}^t wKw)$.

Proof. Consider the following transformations in $N(F)$ (see (2.12)):

$$\gamma_1 = \begin{bmatrix} I_2 & V_1 & W_1 \\ 0 & I_{17} & Y_1 \\ 0 & 0 & I_2 \end{bmatrix} : (z, w, \tau) \to (z, w + \alpha n, \tau)$$

where

$$V_1 = \begin{bmatrix} 0 & \cdots & 0 \\ & -{}^t nK & \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\alpha {}^t nKn/2 \end{bmatrix}, \quad Y_1 = {}^t\begin{bmatrix} 0 & \cdots & 0 \\ & \alpha {}^t n & \end{bmatrix}$$

and $n \in \mathbb{Z}^{17}$ is a column vector;

$$\gamma_2 = \begin{bmatrix} I_2 & V_2 & W_2 \\ 0 & I_{17} & Y_2 \\ 0 & 0 & I_2 \end{bmatrix} : (z, w, \tau) \to (z - {}^t nKw - (\tau/2){}^t nKn, w + \tau n, \tau)$$

where

$$V_2 = \begin{bmatrix} & -{}^t nK & \\ 0 & \cdots & 0 \end{bmatrix}, \quad W_2 = \begin{bmatrix} -{}^t nKn/2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y_2 = {}^t\begin{bmatrix} & {}^t n & \\ 0 & \cdots & 0 \end{bmatrix}.$$

Note that K is an even lattice (i.e. ${}^t nKn \in 2\mathbb{Z}$), and hence $\gamma_1, \gamma_2 \in \Gamma_{2d}$. Since $J(\gamma_1) = J(\gamma_2) = 1$, the relations (i) and (ii) follows from the Γ_{2d}-automorphicity of f and the uniqueness of coefficients of Fourier–Jacobi series.

Next, in case $\alpha = 1$, consider

$$\gamma_3 = \begin{bmatrix} U & 0 & 0 \\ 0 & I_{17} & 0 \\ 0 & 0 & Z \end{bmatrix}, \qquad \text{where } Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$$

and $U = {}^t Z^{-1}$. Then by Proposition 2.14,

$$\gamma_3(z, w, \tau) = (z + (c/2(c\tau + d)){}^t wKw, w/(c\tau + d), (a\tau + b)/(c\tau + d))$$

and $J(\gamma_3) = (c\tau + d)^{-19}$. Hence the relation (iii) holds.

Finally, in case $\alpha = p$, consider

$$\gamma_3' = \begin{bmatrix} U & 0 & W \\ 0 & I_{17} & 0 \\ 0 & 0 & Z \end{bmatrix} \qquad \text{where } Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^1(p),\ U = \begin{bmatrix} d & -cp \\ -b/p & a \end{bmatrix},$$

$$W = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix} = \begin{bmatrix} -\beta c^2 & -\beta c(d+1) \\ \beta c(a-1)/p & \beta(a-1)(d+1)/p \end{bmatrix} \in M(2, \mathbb{Z}).$$

Then it is easy to see that $\gamma_3' \in N(F)_{\mathbb{Z}}$ (see (2.12)). By Proposition 2.14

$$\begin{aligned} \gamma_3'(z, w, \tau) &= (z + (c\tau + d)^{-1}\{c/2 \cdot {}^t wKw + c\beta + \tau w_1 + w_2\}, \\ &\qquad w/(c\tau + d), (a\tau + d)/(c\tau + d)) \\ &= (z + (c\tau + d)^{-1}(c/2)^t wKw - \beta c, w/(c\tau + d), \\ &\qquad (a\tau + d)/(c\tau + d)). \end{aligned}$$

Hence the assertion (iii) follows. □

(6.2) In case $\alpha = 1$, we need to introduce the following: we use the same notation as in Proposition 2.4. Denote by N_{2d} the sublattice of L_{2d} generated by $\{e_3, \ldots, e_{19}\}$. Then its discriminant quadratic form $q_{N_{2d}}$ is equal to $q_{L_{2d}}$, and hence $q_{N_{2d}} \simeq (\mathbb{Z}/2d\mathbb{Z}, \ -1/2d \bmod 2\mathbb{Z})$. Note that the subgroup of $\mathbb{Z}/2d\mathbb{Z}$ of order p is totally isotropic with respect to $q_{N_{2d}}$. Hence it follows from [16], Proposition 1.4.1 that there exists an even lattice N_2 with $\mathrm{rank}(N_2) = \mathrm{rank}(N_{2d})$, $N_2 \supset N_{2d}$ and $q_{N_2} \simeq (\mathbb{Z}/2\mathbb{Z}, \ -1/2 \bmod 2\mathbb{Z})$. Let $\{\tilde{e}_1, \ldots, \tilde{e}_{17}\}$ be a base of N_2. Put $(\tilde{e}_1, \ldots, \tilde{e}_{17}) = (e_3, \ldots, e_{19})Q$ where $Q \in M(17, \mathbb{Q})$. Then $|\det(Q)| = 1/p$, Q^{-1} is integral and $\tilde{K} = (\langle \tilde{e}_i, \tilde{e}_j \rangle) = {}^tQ(\langle e_i, e_j \rangle)Q = {}^tQKQ$. We note here that pQ is integral because $pN_2 \subset N_{2d}$.

(6.3) Let k be a positive integer and l a positive integer with $p \mid l$, $l > 2$. Let $\theta(\tau, w)$ be a holomorphic function on $H^+ \times \mathbb{C}^{17}$ which satisfies the relation

$$\theta(\gamma\tau, \gamma w) = \theta(\tau, w)(c\tau + d)^{19k}\mathbf{e}[(-m/\alpha)U]$$
$$\text{for } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(l). \tag{6.4}$$

For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Q})$, put

$$\theta \mid \gamma = \theta(\gamma\tau, \gamma w)(c\tau + d)^{-19k}\mathbf{e}[(m/\alpha)U].$$

Then $(\theta \mid \gamma) \mid \gamma' = \theta \mid \gamma\gamma'$ for $\gamma, \gamma' = \mathrm{SL}(2, \mathbb{Q})$ (see Lemma 6.7). Let s be a cusp of $\Gamma(l)$ and take $\gamma \in \mathrm{SL}(2, \mathbb{Q})$ with $\gamma(\infty) = s$. Then by the above relation and (6.4), $\theta|\gamma$ is invariant under the action of $\gamma^{-1}\Gamma(l)_s\gamma = \left\{\begin{bmatrix} 1 & nh \\ 0 & 1 \end{bmatrix} \middle| n \in \mathbb{Z}\right\}$, where $\Gamma(l)_s$ is the stabilizer subgroup of s and h is a positive rational number. We say $\theta(\tau, w)$ *holomorphic* at s if $\theta|\gamma$ is holomorphic as a function of $u = \mathbf{e}(\tau/h)$ at $u = 0$.

Let $H_{m,\alpha}(l)$ be a vector space of holomorphic functions $\theta(\tau, w)$ satisfying the following conditions:

(θ_1): $\theta(\tau, w + \alpha n_1 + \tau n_2) = \theta(\tau, w)\mathbf{e}[m/\alpha\{{}^t n_2 K w + (\tau/2){}^t n_2 K n_2\}]$ for $n_1, n_2 \in \mathbb{Z}^{17}$,

(θ_2): $\theta(\gamma\tau, \gamma w) = \theta(\tau, w)(c\tau + d)^{19k}\mathbf{e}[((-m/\alpha)U]$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(l)$,

(θ_3): In case $\alpha = p$, as a function of τ, $\theta(\tau, w)$ is holomorphic at each cusp of $\Gamma(l)$. In case $\alpha = 1$, for any $n_1, n_2 \in \mathbb{Z}^{17}$,

$$\begin{aligned}\theta^{(n_1,n_2)}(\tau, w) = \theta(\tau, w + Qn_1 + \tau Qn_2)\mathbf{e}[-m\{{}^t Qn_2)Kw \\ + (\tau/2){}^t(Qn_2)K(Qn_2)\}]\end{aligned}$$

is holomorphic at each cusp of $\Gamma(l)$.

(6.5) *Remark.* The condition (θ_3) will be used in the proof of Lemma 6.15. We remark here that under the assumption $p|l$, $\theta^{(n_1,n_2)}(\tau, w)$ satisfies the relation (6.4). Recall that pQ and ${}^tQKQ$ are integral matrices. Then by using the relation (θ_2), we have

$$\begin{aligned}\theta^{(n_1,n_2)}(\gamma\tau, \gamma w) &= \theta(\gamma\tau, \gamma(w + (c\tau + d)Qn_1 + (a\tau + b)Qn_2)) \\ &\quad \times \mathbf{e}[-m\{{}^t(Qn_2)K(\gamma w) + \gamma\tau/2 \cdot {}^t(Qn_2)KQn_2\}] \\ &= (c\tau + d)^{19k}\theta(\tau, w + (c\tau + d)Qn_1 + (a\tau + b)Qn_2) \\ &\quad \cdot \mathbf{e}[-m\{{}^t(Qn_2)K(\gamma w) + \gamma\tau/2 \cdot {}^t(Qn_2)KQn_2\}] \\ &\quad \times \mathbf{e}[-mc/2 \cdot (c\tau + d)^{-1} \cdot {}^t(w + (c\tau + d)Qn_1 \\ &\quad + (a\tau + b)Qn_2)K(w + (c\tau + d)Qn_1 + (\alpha\tau + b)Qn_2)].\end{aligned}$$

Since

$$\begin{aligned}w + (c\tau + d)Qn_1 + (a\tau + b)Qn_2 &= w + Qn_1 + \tau Qn_2 + bQn_2 \\ &\quad + (d - 1)Qn_1 + c\tau Qn_1 + (a - 1)\tau Qn_2,\end{aligned}$$

pQ is integral and $\gamma \in \Gamma(l)$, it follows from (θ_1) that

$$\theta(\tau, w + (c\tau + d)Qn_1 + (a\tau + b)Qn_2) = \theta(\tau, w + Qn_1 + \tau Qn_2)$$
$$\times \mathbf{e}[m\{{}^t(cQn_1 + (a-1)Qn_2)K(w + Qn_1 + \tau Qn_2)$$
$$+ \tau/2 \cdot {}^t(cQn_1 + (a-1)Qn_2)K(cQn_1 + (a-1)Qn_2)\}].$$

Now the assertion follows from a direct calculation by using the fact ${}^tQKQ$ are even integral. Moreover we can see that the group $\mathscr{G} = (Q\mathbb{Z}^{17} \oplus \tau Q\mathbb{Z}^{17})/(\mathbb{Z}^{17} \oplus \tau\mathbb{Z}^{17})$ acts on $H_{m,1}(l)$ as $(Qn_1, Qn_2)(\theta(\tau, w)) = \theta^{(n_1,n_2)}(\tau, w)$ and the invariant subspace under this action corresponds to the space of functions satisfying the similar relations as $(\theta_1) - (\theta_3)$, which contains Fourier–Jacobi coefficients of Γ_2-automorphic forms.

LEMMA 6.6. *Assume $p \mid l$. Let $f \in A_k(\Gamma_{2d})$ and $f = \Sigma\, \theta_m(\tau, w)\mathbf{e}(m/\alpha \cdot z)$ its Fourier–Jacobi series. Then $\theta_m(\tau, w) \in H_{m,\alpha}(l)$.*

Proof. Since $\alpha \mid l$, Lemma 6.1 implies that $\theta_m(\tau, w)$ satisfies the conditions (θ_1), (θ_2). For (θ_3), first, we shall see that $\theta_m(\tau, w)$ is holomorphic at ∞. Let $e_1, \ldots, e_{21}$ be a base of L_{2d} as in Proposition 2.4. Denote by F_0 the boundary component corresponding to $\mathbb{Z}e_2$. Then by the same way as in (5.2) we have

$$\theta_m(\tau, w) = \sum_{\rho} c_\rho \mathbf{e}(\langle\langle \rho, (\tau, w, 0)\rangle\rangle)$$

where the summation on the right hand side means that ρ runs on the set $\{\rho \in U(F_0)^*_{\mathbb{Z}} \cap \overline{C(F_0)} \mid \langle\langle \rho, \xi\rangle\rangle = m\}$ (see (5.5)). Note that $\mathbf{e}(\tau/\alpha)$ is a coordinate around ∞. Since ρ and $(1, 0, 0) \in \overline{C(F_0)}$, $\langle\langle \rho, (1, 0, 0)\rangle\rangle \geqslant 0$ and hence $\theta_m(\tau, w)$ is holomorphic at ∞.

In case $\alpha = p$, we need to estimate $\theta_m(\tau, w)$ at cusps s of $\Gamma^1(p)$. For $f \in A_k(\Gamma_{2d})$ and $\varphi \in \Gamma_{2d} \otimes \mathbb{Q}$, put

$$f \mid \varphi = f(\varphi x)J(\varphi, x)^k.$$

Then $f \mid \varphi$ is a $\varphi^{-1}\Gamma_{2d}\varphi$-automorphic form of weight k. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{Q})$ with $\gamma(\infty) = s$. Consider the following transformation in $N(F) \cap \Gamma_{2d} \otimes \mathbb{Q}$ (see the proof of Lemma 6.1):

$$\varphi = \begin{bmatrix} U & 0 & W \\ 0 & \mathrm{I}_{17} & 0 \\ 0 & 0 & Z \end{bmatrix}$$

where

$$Z = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad U = \begin{bmatrix} d & -cp \\ -b/p & a \end{bmatrix}$$

and

$$W = \begin{bmatrix} w_1 & w_2 \\ w_3 & w_4 \end{bmatrix} \in M(2, \mathbb{Q}), \quad w_1 = -\beta c^2, \quad w_2 = -\beta c(d+1),$$

$$w_3 = \beta c(a-1)/p, \quad w_4 = \beta(a-1)(d+1)/p.$$

An elementary calculation shows $U(F) \cap \varphi^{-1}\Gamma_{2d}\varphi = U(F)_{\mathbb{Z}}$. Thus we have the Fourier–Jacobi expansion $f \mid \varphi = \Sigma\, \bar{\theta}_m(\tau, w)\mathbf{e}(mz/p)$. On the other hand, we have

$$f \mid \varphi = \sum \theta_m(\gamma\tau, \gamma w)(c\tau + d)^{-19k}\mathbf{e}[m/p(U - \beta c)]\mathbf{e}(mz/p).$$

Hence $\theta_m(\gamma\tau, \gamma w)(c\tau + d)^{-19k}\mathbf{e}[(m/p)U] = \bar{\theta}_m(\tau, w)\mathbf{e}(m\beta c/p)$. By the same proof as in the case $\alpha = 1$, $\bar{\theta}_m(\tau, w)$ is holomorphic at ∞, and hence $\theta_m(\tau, w)$ is so at s.

Lastly we shall see the property (θ_3) for $\alpha = 1$. Consider

$$\gamma = \begin{bmatrix} \mathrm{I}_2 & V & W \\ 0 & \mathrm{I}_{17} & Y \\ 0 & 0 & \mathrm{I}_2 \end{bmatrix} \in W(F) \cap \Gamma_{2d} \otimes \mathbb{Q}$$

where

$$Y = [Qn_2, Qn_1], \quad V = -{}^tYK,$$

$$W = \begin{bmatrix} -1/2 \cdot {}^t(Qn_2)KQn_2 & 0 \\ -{}^t(Qn_2)KQn_1 & -1/2 \cdot {}^t(Qn_1)KQn_1 \end{bmatrix}.$$

Put $f \mid \gamma = f(\gamma x)J(\gamma, x)^k = f(\gamma x)$. Then $f \mid \gamma$ is a $\gamma^{-1}\Gamma_{2d}\gamma$-automorphic form of weight k. Note that $U(F) \cap \gamma^{-1}\Gamma_{2d}\gamma = U(F)_{\mathbb{Z}}$ because $U(F)$ is the center of $W(F)$, and hence $\theta^{(n_1, n_2)}(\tau, w)$ is nothing but the Fourier–Jacobi coefficient of $f \mid \gamma$. Now the assertion follows from the same way as above. □

LEMMA 6.7. *Let l be a positive integer with $p \mid l$. Then the group $\Gamma^1(\alpha)/\Gamma(l)$ acts on $H_{m,\alpha}(l)$ as follows: for* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma^1(\alpha)$,

$$\gamma: \theta(\tau, w) \to \theta|\gamma = \theta(\gamma\tau, \gamma w)(c\tau + d)^{-19k}\mathbf{e}\{(m/\alpha)(U - \beta c)\}.$$

Proof. Let $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\gamma' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in \Gamma^1(\alpha)$. Then we can easily see that

$$(c \cdot \gamma'\tau + d)(c'\tau + d') = (ca' + dc')\tau + (cb' + dd')$$

and

$$(c'/2)(c'\tau + d')^{-1} + (c/2)(c \cdot \gamma'\tau + d)^{-1}(c'\tau + d')^{-2}$$
$$= ((ca' + dc')/2)((ca' + dc')\tau + (cb' + dd'))^{-1}.$$

Moreover in case $\alpha = p$, $c + c' \equiv ca' + dc'$ (mod p) since γ, $\gamma' \in \Gamma^1(p)$. The property (θ_1), (θ_2) of $\theta|\gamma$ follows from those of $\theta(\tau, w)$.

In case $\alpha = 1$, by using the fact $1/2 \cdot {}^t(Qn_i)KQn_i$, ${}^t(Qn_1)KQn_2 \in \mathbb{Z}$, we can see that

$$(c\tau + d)^{-1}(Qn_1 + \tau Qn_2) = Q(an_1 - bn_2) + \gamma(\tau)Q(-cn_1 + dn_2),$$
$$(c\tau + d)^{-1} \cdot {}^t(Q(-cn_1 + dn_2)) = {}^t(Qn_2) - c(c\tau + d)^{-1} \cdot {}^t(Qn_1 + \tau Qn_2),$$
$$c(c\tau + d)^{-1} \cdot {}^t(Qn_1 + \tau Qn_2)K(Qn_1 + \tau Qn_2) - \tau \cdot {}^t(Qn_2)KQn_2$$
$$\equiv -\gamma(\tau) \cdot {}^t(Q(-cn_1 + dn_2))KQ(-cn_1 + dn_2) (\mathrm{mod}\, 2\mathbb{Z}).$$

By these equations, we can directly see that

$$(\theta|\gamma)^{(n_1,n_2)}(\tau, w) = \theta^{(an_1 - bn_2, -cn_1 + dn_2)}(\tau, w)|\gamma \quad \text{for } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma.$$

Hence the property (θ_3) also holds. □

(6.8) In the following, we shall compute the dimension of $H_{m,\alpha}(l)$ by using the transformation formula of theta functions. First we recall the theory of theta functions (e.g. [8], Chap. 2). For a fixed τ, we denote by $R^{\tau}_{m,\alpha}$ ($\tau \in H^+$, $\alpha = 1$ or p) the space of holomorphic functions $\theta(w)$ on $\mathbb{C}^{17}$ satisfying the conditions (θ_1). Put

$$L_\tau(w, w') = m/(\alpha \mathrm{Im}(\tau)) \cdot \{-{}^t wK\bar{w}' + {}^t wKw'\},$$

$$A_\tau(w, w') = m/(2\sqrt{-1} \cdot \alpha \mathrm{Im}(\tau)) \cdot \{{}^t w'K\bar{w} - {}^t wK\bar{w}'\}$$

$$= 1/2\sqrt{-1}\cdot\{L_\tau(w, w') - L_\tau(w', w)\}.$$

Then L_τ is a quasi-hermitian form and A_τ is an alternating form on $\mathbb{C}^{17} \times \mathbb{C}^{17}$. Put $L = \alpha\mathbb{Z}^{17} \oplus \tau\mathbb{Z}^{17}$ which is a lattice in $\mathbb{C}^{17}$ in the usual sense. Then $A_\tau(\xi, \xi') \in \mathbb{Z}$ for any $\xi, \xi' \in L$. Put $e_i = {}^t(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^{17}$ and take a base $\{\tau e_i, \alpha e_j \mid 1 \leqslant i, j \leqslant 17\}$ of L. Then the matrix of A_τ with respect to this base is

$$\begin{bmatrix} 0 & -mK \\ mK & 0 \end{bmatrix}.$$

Let B_τ denote the $\mathbb{R}$-bilinear form on $\mathbb{C}^{17} \times \mathbb{C}^{17}$ defined by the matrix

$$\begin{bmatrix} 0 & -mK \\ 0 & 0 \end{bmatrix}.$$

Then $A_\tau(w, w') = B_\tau(w, w') - B_\tau(w', w)$, B_τ is $\mathbb{Z}$-valued on $L \times L$ and $B_\tau(\xi, \xi) = -m{}^tn_1Kn_2$ for any $\xi = \alpha n_1 + \tau n_2 \in L$. We remark here that $R^\tau_{m,\alpha}$ coincides with the vector space of theta functions relative to L with

$$\begin{aligned} &\mathbf{e}\{1/2\sqrt{-1}\cdot L_\tau(w, \xi) + 1/4\sqrt{-1}\cdot L_\tau(\xi, \xi) + 1/2\cdot B_\tau(\xi, \xi)\} \\ &\quad = \mathbf{e}[m/\alpha\{{}^tn_2Kw + (\tau/2){}^tn_2Kn_2\}] \end{aligned}$$

as an automorphy factor ([8], Chap. II, Theorem 3). By a theorem of elementary divisor, there exist matrices P, R in $\mathrm{GL}(17, \mathbb{Z})$ satisfying

$$-PKR = \varepsilon = \begin{bmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_{17} \end{bmatrix}$$

where ε_i is a positive integer with $\varepsilon_i \mid \varepsilon_{i+1}$ $(i = 1, \dots, 16)$. Note that $\varepsilon_1 \cdots \varepsilon_{17} = |\det(K)| = 2p^2/\alpha^2$. Now we take a base $\{\tau Re_i, \alpha{}^tPe_j\}$ of L. Then the corresponding matrix of A_τ is

$$\begin{bmatrix} 0 & m\varepsilon \\ -m\varepsilon & 0 \end{bmatrix}.$$

Take $\{m^{-1}\varepsilon_i^{-1}\alpha{}^tPe_i\}_{i\leqslant i\leqslant 17}$ as a base of $\mathbb{C}^{17}$ and denote by z the coordinate with respect to this base. Then $z = -(m/\alpha){}^tRKw$. Put $\Omega = -(m\tau/\alpha){}^tRKR =$

$(-(1/2\sqrt{-1})\cdot L_\tau(\tau Re_i, \tau Re_j))_{i,j}$. Then Ω is contained in the Siegel upper half-plane $\mathfrak{S}_{17}$ of degree 17 because K is negative definite. For $q \in (m\varepsilon)^{-1}\mathbb{Z}^{17}/\mathbb{Z}^{17}$, put

$$\Theta_q(\Omega, z) = \sum_{p \in \mathbb{Z}^{17}} \mathbf{e}\{(1/2)^t(p+q)\Omega(p+q) + {}^t(p+q)z)\}.$$

Since $R \in \mathrm{GL}(17, \mathbb{Z})$, we have

$$\Theta_q(\Omega, z) = \sum_{n \in \mathbb{Z}^{17}} \mathbf{e}[(m/\alpha)\{-(\tau/2)^t(n+Rq)K(n+Rq) - {}^t(n+Rq)Kw\}].$$

We denote by $\Theta_q(\tau, w)$ the right hand side of the above equation. Then by the theory of theta functions ([18, Chap. 2, p. 75), we have

PROPOSITION 6.9. *For fixed* $\tau \in H^+$, *the set* $\{\Theta_q(\tau, w) \mid q \in (m\varepsilon)^{-1}\mathbb{Z}^{17}/\mathbb{Z}^{17}\}$ *is a base of* $R^\tau_{m,\alpha}$.

Define

$$\bar{\Gamma} = \left\{M \in M(34, \mathbb{Z}) \mid M \begin{bmatrix} 0 & m\varepsilon \\ -m\varepsilon & 0 \end{bmatrix} {}^tM = \begin{bmatrix} 0 & m\varepsilon \\ -m\varepsilon & 0 \end{bmatrix}\right\}.$$

For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{R})$, put

$$\tilde{\gamma} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A = aI_{17}$, $B = b/\alpha \cdot {}^tRP^{-1}$, $C = \alpha cP^tT^{-1}$ and $D = d\mathrm{I}_{17}$. Then

$$\tilde{\gamma} \begin{bmatrix} 0 & m\varepsilon \\ -m\varepsilon & 0 \end{bmatrix} {}^t\tilde{\gamma} = \begin{bmatrix} 0 & m\varepsilon \\ -m\varepsilon & 0 \end{bmatrix}$$

because $PKR = -\varepsilon = -{}^t\varepsilon = {}^tRK^tP$. Put

$$\vec{\Theta}(\tau, w) = \{\Theta_q(\tau, w)\}_{q \in (m\varepsilon)^{-1}\mathbb{Z}^{17}/\mathbb{Z}^{17}}.$$

PROPOSITION 6.10. *For* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Q})$ *with* $\tilde{\gamma} \in \tilde{\Gamma}$,

$$\vec{\Theta}(\gamma\tau, \gamma w) = \mathbf{e}(-(m/\alpha)U)(c\tau + d)^{17/2}\rho(\gamma)\vec{\Theta}(\tau, w)$$

where $\rho(\gamma)$ *is a constant unitary matrix.*

Proof. Put $z^{\#} = m\varepsilon \cdot {}^t(C\Omega + mD\varepsilon)^{-1}z$ and $\Omega^{\#} = (A\Omega + mB\varepsilon) \times (C\Omega + mD\varepsilon)^{-1}m\varepsilon$. Then $\Omega^{\#} = -(m/\alpha)\gamma\tau \cdot {}^tRKR$, $z^{\#} = z/(c\tau + d)$, $(C(m\varepsilon){}^tD)_0 = (-\alpha cdmPK{}^tP)_0 \equiv 0$ and $(A(m\varepsilon){}^tB)_0 = (-(abm/\alpha){}^tRKR)_0 \equiv 0 \pmod 2$, where for a matrix T, $(T)_0$ denotes the vector with i-th diagonal coefficient of T as its i-th coefficient. Now by the transformation formula for $\vec{\Theta}(\Omega, z)$ ([8], Ch. 2, Theorem 6), we have

$$\begin{aligned} \vec{\Theta}(\gamma\tau, \gamma w) &= \vec{\Theta}(\Omega^{\#}, z^{\#}) \\ &= \mathbf{e}[1/2\{{}^tz(C\Omega + mD\varepsilon)^{-1}Cz\}] \cdot \det((m\varepsilon)^{-1} \\ &\quad \times (C\Omega + mD\varepsilon))^{1/2} \cdot \tilde{\rho}(\tilde{\gamma}) \cdot \vec{\Theta}(\Omega, z) \end{aligned}$$

where $\tilde{\rho}(\tilde{\gamma})$ is a constant unitary matrix. Put $\rho(\gamma) = \tilde{\rho}(\tilde{\gamma})$. An easy calculation shows that ${}^tz(C\Omega + mD\varepsilon)^{-1}Cz = -(mc/\alpha)(c\tau + d)^{-1} \cdot {}^twKw = -(2m/\alpha)U$ and $\det((m\varepsilon)^{-1}(C\Omega + mD\varepsilon)) = (c\tau + d)^{17}$. Thus we have the desired result. □

(6.11) *Remark.* In case $\alpha = 1$, $\tilde{\gamma} \in \tilde{\Gamma}$ for any $\gamma \in SL(2, \mathbb{Z})$, and in case $\alpha = p$, $\tilde{\gamma} \in \tilde{\Gamma}$ for any $\gamma = \begin{bmatrix} a & pb \\ c/p & d \end{bmatrix}$ with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Also it follows from Proposition 6.10 that $(c\tau + d)^{17/2} \cdot \rho(\gamma)$ satisfies the cocycle condition. Since $(c\tau + d)$ satisfies the cocycle condition, $\rho \otimes \rho$ is so, i.e. $\rho \otimes \rho$ is a unitary representation.

LEMMA 6.12. *For any cusps s of* $\Gamma(l)$, *there exists a* $\gamma \in SL(2, \mathbb{Q})$ *with* $\tilde{\gamma} \in \tilde{\Gamma}$ *and* $\gamma(s) = \infty$.

Proof. In case $\alpha = 1$, the assertion is obvious. In case $\alpha = p$, by Definition of $\tilde{\Gamma}$, it suffices to see the assertion for cusps of $\Gamma_0(p) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| b \equiv 0 \pmod p \right\}$. Recall that $\{0, \infty\}$ is a complete set of $\Gamma_0(p)$-inequivalent cusps (cf. [22], p. 26). Then $\gamma = \begin{bmatrix} 0 & -p \\ 1/p & 0 \end{bmatrix}$ sends 0 to ∞. By the Remark 6.11, $\tilde{\gamma} \in \tilde{\Gamma}$. □

Let $\theta(\tau, w) \in H_{m,\alpha}(l)$. Then by Proposition 6.9,

$$\theta(\tau, w) = \sum_q f_q(\tau) \cdot \Theta_q(\tau, w) = {}^t\vec{F}(\tau)\vec{\Theta}(\tau, w)$$

where $f_q(\tau)$ is a holomorphic function of τ and $\vec{F}(\tau) = \{f_q(\tau)\}_{q \in (m\varepsilon)^{-1}\mathbb{Z}^{17}/\mathbb{Z}^{17}}$ (For the holomorphicity of $f_q(\tau)$, see the proof of Lemma 6.15). It follows from the condition (θ_2) and Proposition 6.10 that

$$\vec{F}(\gamma\tau) = (c\tau + d)^{19k - 17/2} \cdot {}^t\rho(\gamma)^{-1} \cdot \vec{F}(\tau) \quad \text{for } \gamma \in \Gamma(l). \tag{6.13}$$

Also, by Definition of $\Theta_q(\tau, w)$ and the relation ${}^tq{}^tRKRq = -{}^tq{}^tRP^{-1}\varepsilon q$, we can easily see that $\Theta_q(\tau + l, w) = \Theta_q(\tau, w)$ if $(4mp^2/\alpha) \mid l$. Hence we have

$$\rho\left(\begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}\right) = 1 \qquad \text{if } (4mp^2/\alpha) \mid l. \tag{6.14}$$

Put

$$\vec{F} \mid \gamma = (c\tau + d)^{-19k + 17/2} \cdot {}^t\rho(\gamma)\vec{F}(\gamma\tau)$$

for $\gamma \in \mathrm{SL}(2, \mathbb{Q})$ with $\tilde{\gamma} \in \tilde{\Gamma}$. Then $(\vec{F} \mid \gamma) \mid \sigma = \vec{F} \mid \gamma\sigma$ for γ, $\sigma \in \mathrm{SL}(2, \mathbb{Q})$ with $\tilde{\gamma}$, $\tilde{\sigma} \in \tilde{\Gamma}$ (Remark 6.11). Let s be a cusp of $\Gamma(l)$ and take $\gamma \in \mathrm{SL}(2, \mathbb{Q})$ with $\gamma(\infty) = s$ and $\tilde{\gamma} \in \tilde{\Gamma}$ (Lemma 6.12). Denote by $\Gamma(l)_s$ the stabilizer subgroup of s. Then $\gamma^{-1}\Gamma(l)_s\gamma = \left\{\begin{bmatrix} 1 & nh \\ 0 & 1 \end{bmatrix} \middle| n \in \mathbb{Z}\right\}$, where h is a positive rational number. By (6.13), $\vec{F} \mid \gamma$ is invariant under the action of $\gamma^{-1}\Gamma(l)_s\gamma$, and hence there exists a vector valued function $\vec{G}(u) = \{g_q(u)\}$ in $0 < |u| < \delta$ with $\vec{F} \mid \gamma = \vec{G}(\mathbf{e}(\tau/h))$. We call $\vec{F}(\tau)$ *meromorphic* at s if $g_q(u)$ is meromorphic at $u = 0$ for all $q \in (m\varepsilon)^{-1}\mathbb{Z}^{17}/\mathbb{Z}^{17}$ and define the *order* of pole of $\vec{F}$ at s by the maximum of those of $\{g_q(u)\}$. Also put

$$\vec{\Theta} \mid \gamma = \mathbf{e}(m/\alpha \cdot U)(c\tau + d)^{-17/2} \cdot \rho^{-1}(\gamma)\vec{\Theta}(\gamma\tau, \gamma w).$$

Then we can similarly define the order of zero of $\vec{\Theta}(\tau, w)$ at cusps. By Proposition 6.10, $\vec{\Theta} \mid \gamma = \vec{\Theta}$ for $\gamma \in \mathrm{SL}(2, \mathbb{Q})$ with $\tilde{\gamma} \in \tilde{\Gamma}$. Therefore, by Lemma 6.12, the behaviors of $\vec{\Theta}(\tau, w)$ at any cusps are the same as that of $\vec{\Theta}(\tau, w)$ at ∞.

LEMMA 6.15. *$\vec{F}(\tau)$ is meromorphic at any cusps of $\Gamma(l)$ and its order of pole is at most cml/α, where c is a constant depending only on the matrix $\tilde{K} = {}^tQKQ$ if $\alpha = 1$, K if $\alpha = p$, in particular, does not depend on p, m and k.*

Proof. First we shall prove the meromorphicity of $\vec{F}(\tau)$ at any cusps. Put $N = |m^{-1}\varepsilon^{-1}\mathbb{Z}^{17}/\mathbb{Z}^{17}|$. Since for fixed $\tau_0 \in H^+$, $\{\Theta_q(\tau_0, w)\}$ is a base of $R^{\tau}_{m,\alpha}$, there exist N elements $x_1, \ldots, x_N$ in $\mathbb{C}^{17}$ satisfying

$$\det(\Theta_q(\tau_0, x_i))_{q,i} \neq 0 \quad \text{([8], Chap. II, §5, Lemma 9).}$$

We take τ, $w_1, \ldots, w_N$ in some neighborhoods of τ_0, $x_1, \ldots, x_N$ and consider N linear equations

$$\sum_{q=1}^{N} f_q(\tau)\Theta_q(\tau, w_i) = \theta(\tau, w_i) \qquad (i = 1, \ldots, N).$$

We can uniquely solve this equation in the unknown $f_q(\tau)$. Thus $f_q(\tau)$ is holomorphic on H^+. Moreover by definition of Θ_q and the property (θ_3), as a function of $u = \mathbf{e}(\tau/\alpha)$, Θ_q and θ are holomorphic on $\{u \mid |u| < \delta\}$. Therefore $\vec{F}(\tau)$ is meromorphic at cusps.

Next we shall estimate the order of poles of $\vec{F}(\tau)$ at cusps. We shall see the assertion for $\alpha = 1$. For $\alpha = p$, the proof is similar and simpler. Since Q^{-1} and ${}^tQKQ$ are integral, for $n_1, n_2 \in \mathbb{Z}^{17}$,

$$\begin{aligned}
&\Theta_q(\tau, w + Qn_1 + \tau Qn_2)\mathbf{e}[-m\{{}^t(Qn_2)Kw + (\tau/2){}^t(Qn_2)KQn_2\}] \\
&\quad = \mathbf{e}[-m\cdot{}^t(Rq)KQn_1] \sum_n \mathbf{e}[-m\{(\tau/2){}^t(n + Rq + Qn_2) \\
&\qquad \times K(n + Rq + Qn_2) + {}^t(n + Rq + Qn_2)Kw\}] \\
&\quad = \mathbf{e}[-m\cdot{}^t(Rq)KQn_1]\cdot\Theta_{q+R^{-1}Qn_2}(\tau, w)
\end{aligned}$$

and hence

$$\theta^{(n_1,n_2)}(\tau, w) = \sum_q \mathbf{e}(-m\cdot{}^t(Rq)KQn_1)\cdot f_q(\tau)\cdot\Theta_{q+R^{-1}Qn_2}(\tau, w).$$

By the property (θ_3), it suffices to see that for fixed q, there is a n_2 such that $\Theta_{q+R^{-1}Qn_2}(\tau, w)$ vanishes at ∞ of order at most cml. Since $\mathbf{e}(\tau/l)$ is a co-ordinate around ∞, the order of zero of $\Theta_{q+R^{-1}Qn_2}$ at ∞ is at most $ml/2\cdot\{-{}^t(Rq + Qn_2)K(Rq + Qn_2)\}$. Take $L \in \mathrm{GL}(17, \mathbb{Z})$ such that $-{}^tL\tilde{K}L$ is Min-kovski-reduced ([6], Chap. I, §2 or [8], p. 191). Then there exists a constant c' depending only on the degree of the matrix $\tilde{K}$ such that any entries of ${}^tL\tilde{K}L$ are bounded by $c'\cdot|\det(\tilde{K})| = 2c'$ ([6], Satz 2.5). Now by replacing n_2, we may assume that the absolute values of any entries of $(L^{-1}Q^{-1}Rq + L^{-1}n_2)$ are less than 1. Then

$$-{}^t(Rq + Qn_2)K(Rq + Qn_2) = -{}^t(Q^{-1}Rq + n_2)\tilde{K}(Q^{-1}Rq + n_2)$$
$$= -{}^t(L^{-1}Q^{-1}Rq + L^{-1}n_2){}^tL\tilde{K}L$$
$$\times (L^{-1}Q^{-1}Rq + L^{-1}n_2) \leqslant c. \qquad \square$$

LEMMA 6.16. *Assume that k is even. For any m with* $0 < m < k$,

$$\dim H_{m,\alpha}(1) \leqslant c_1 kp^2 m^{17} + c_2 p^2 m^{18}/\alpha + c_3 p^3 m^{17}\sqrt{m} \qquad (k \gg 0)$$

where c_1, c_2 *and* c_3 *are constants not depending on k, m, p.*

Proof. Put $l = 4mp^2/\alpha$ and consider the following action of $\Gamma(l)$ on $H^+ \times \mathbb{C}^{2p^2m^{17}/\alpha^2}$: for $\gamma \in \Gamma(l)$,

$$\gamma\colon (\tau, w) \to (\gamma\tau, (c\tau + d)^{-17/2} \cdot {}^t\rho(\gamma)^{-1} w).$$

It defines a vector bundle on $H^+/\Gamma(l)$. Let s be a cusp of $\Gamma(l)$ and take $\gamma \in \mathrm{SL}(2, \mathbb{Q})$ with $\tilde{\gamma} \in \tilde{\Gamma}$ and $\gamma(\infty) = s$. Then

$$\gamma^{-1}\Gamma(l)_s\gamma = \begin{cases} \left\{\begin{bmatrix} 1 & nl \\ 0 & 1 \end{bmatrix} \middle| n \in \mathbb{Z}\right\} & \text{if } \gamma \in \Gamma = \mathrm{SL}(2, \mathbb{Z}) \\ \left\{\begin{bmatrix} 1 & nlp^2 \\ 0 & 1 \end{bmatrix} \middle| n \in \mathbb{Z}\right\} & \text{if } \gamma = \begin{bmatrix} 0 & p \\ -1/p & 0 \end{bmatrix} \end{cases}.$$

Hence, by (6.14), the above vector bundle can be extended to a vector bundle W of rank $2p^2m^{17}/\alpha^2$ on the Satake compactification C of $H^+/\Gamma(l)$. Let $\{s_i \mid 1 \leqslant i \leqslant r\}$ be the set of boundary points of C. It is known that $r = [\Gamma : \Gamma(l)]/2l$ ([22], p. 23). Put

$$L_k = K_C^{\otimes(19k/2)} \otimes \mathcal{O}_C\left(\sum_{i=1}^{r} (19k/2 + v)s_i\right)$$

which corresponds to $\Gamma(l)$-automorphic forms on H^+ of weight $19k/2$ meromorphic at each cusp with the order of pole $\leqslant v$, where $v = cml/\alpha$ (see Lemma 6.15). Then by identifying $\theta(\tau, w)$ and the corresponding $\vec{F}(\tau)$,

$$H_{m,\alpha}(l) \subset H^0(C, W \otimes L_k).$$

Since $L_k \otimes \mathcal{O}(-\Sigma_i vs_i)$ is ample, by a vanishing theorem,

$$H^i(C, W \otimes L_k) = H^i(C, L_k) = 0 \qquad \text{for } i > 0,\ k \gg 0.$$

By Remark 6.11, $W \otimes W \simeq L \otimes F$ where L is a line bundle defined by the cocycle $(c\tau + d)^{-17}$ and F is a flat vector bundle defined by $\rho \otimes \rho$. Then $ch(W \otimes W) = (\mathrm{rk}(W))^2 + 2\mathrm{rk}(W)c_1(W)$ and $ch(L \otimes F) = ch(L) \cdot ch(F) = (\mathrm{rk}(F))(1 + c_1(L))$, where $ch(V)$ (resp. $\mathrm{rk}(V)$) is the Chern character of a vector bundle V (resp. the rank of V). Hence $c_1(W) = \mathrm{rk}(W) \cdot c_1(L)/2$. By Riemann–Roch theorem,

$$\begin{aligned}\dim H^0(C, W \otimes L_k) &= \sum (-1)^i \dim H^i(C, W \otimes L_k)\\ &= \deg\{ch(W \otimes L_k) \cdot Td(C)\}_1\\ &= \deg\{(\mathrm{rk}(W) + \mathrm{rk}(W)c_1(L_k)\\ &\quad + \mathrm{rk}(W)c_1(L)/2)(1 - c_1(K_C)/2)\}_1\\ &= \mathrm{rk}(W) \cdot \deg\{c_1(L_k) + c_1(L)/2 - c_1(K_C)/2\}\\ &= \mathrm{rk}(W)\{(19k - 19/2)(g(C) - 1)\\ &\quad + (19k/2 + v - 17/4)r\}\end{aligned}$$

because $L^{\otimes 2} = (K_C \otimes \mathcal{O}_C(\Sigma_{i=1}^r s_i))^{\otimes(-17)}$. By [22], Proposition 1.40, $g(C) - 1 = [\Gamma : \Gamma(l)]/24 - [\Gamma : \Gamma(l)]/4l$ since $\Gamma(l)$ has no elliptic points. Thus we have

$$\begin{aligned}&\dim H^0(C, L_k \otimes W)\\ &\quad = (2p^2 m^{17}/\alpha^2)[\Gamma : \Gamma(l)]\{19k/24 + (2cml/\alpha + 1)/4l - 19/48\} \qquad (k \gg 0).\end{aligned}$$

Put $G = \{\pm 1\}\Gamma^1(\alpha)/\{\pm 1\}\Gamma(l)$. The finite group G acts on the pair $(C, W \otimes L_k)$ as automorphisms which is induced from the action mentioned in Lemma 6.7. Then

$$\begin{aligned}\dim H_{m,\alpha}(1) &= \dim H_{m,\alpha}(l)^G \leqslant \dim H^0(C, W \otimes L_k)^G\\ &= 1/|G| \cdot \sum_{\gamma \in G} \mathrm{trace}\ \gamma^* | H^0(C, W \otimes L_k).\end{aligned}$$

Since $[\Gamma : \Gamma^1(\alpha)] = O(\alpha^2)$ and $\mathrm{rank}(W \otimes L_k) = 2p^2 m^{17}/\alpha^2$, the assertion follows from the following:

LEMMA 6.17.

$$\sum_{\gamma \neq 1 \in G} \mathrm{trace}\ \gamma^* \,|\, \mathrm{H}^0(C, W \otimes L_k) \leqslant c[\Gamma : \Gamma(l)] \cdot \mathrm{rank}(W \otimes L_k) \cdot p\sqrt{m}$$

where c is a constant independent on p, m, k.

Proof. Let $\gamma \in G$ and let $C^{\gamma} = \{x_1, \ldots, x_s\}$ be the set of fixed point of γ. Assume that γ acts on the tangent space of x_i as $\mathbf{e}(\theta_i)$. Then by the holomorphic Lefschetz fixed point formula [2], Theorem 4.6,

$$\operatorname{trace} \gamma^* \mid H^0(C, W \otimes L_k) = \sum_{i=1}^{s} \left\{ \frac{ch(W \otimes L_k \mid x_i)(\gamma)}{1 - \mathbf{e}(-\theta_i)} \right\} [x_i].$$

Note that $|ch(W \otimes L_k|x_i)(\gamma)| \leqslant \operatorname{rank}(W \otimes L_k)$.

Now assume $\alpha = 1$. Recall that $[\Gamma : \Gamma(l)]/2l$, $[\Gamma : \Gamma(l)]/4$ or $[\Gamma : \Gamma(l)]/6$ is the number of $\Gamma(l)$-inequivalent cusps, elliptic points of order 2 or elliptic points of order 3 respectively ([22], p. 22). Since $G = \Gamma/\{\pm 1\}\Gamma(l)$ acts on these sets transitively, $|G_x| = l$, 2 or 3 where x is a cusp, elliptic point of order 2 or elliptic point of order 3 respectively and G_x is the stabilizer subgroup of x. We first consider the contribution of cusps. Denote by $\{\gamma_1, \ldots, \gamma_t\}$ a maximal subset of $\{\gamma \in G \mid \gamma$ is of order l and fixes a cusp$\}$ satisfying $C^{\gamma_i} \cap C^{\gamma_j} = \varnothing$ for $i \neq j$. Let $\{x_{i,1}, \ldots, x_{i,s_i}\}$ denote the set of fixed cusps of γ_i on C. Then $\Sigma_{i=1}^{t} s_i = [\Gamma : \Gamma(l)]/2l$. Assume that γ_i acts on the tangent space of $x_{i,j}$ as $\mathbf{e}(l_{i,j}/l)$ where $l_{i,j}$ is a positive integer with $(l, l_{i,j}) = 1$ and $l_{i,j} < l$. Then the contribution of cusps to $\Sigma_{\gamma \neq 1} \operatorname{trace} \gamma^* | H^0(C, W \otimes L_k)$ is bounded by

$$\sum_{q=1}^{l-1} \sum_{i=1}^{t} \sum_{j=1}^{s_i} \left| \left\{ \frac{ch(W \otimes L_k \mid x_{i,j})(\gamma_i^q)}{1 - \mathbf{e}(-q l_{i,j}/l)} \right\} [x_{i,j}] \right|. \tag{*}$$

Note that $1/|1 - \mathbf{e}(-\theta)| \leqslant c'/\theta$ for $0 < \theta \leqslant 1/2$ where c' is a constant independent on p, m, k. Hence

$$\begin{aligned}\sum_q 1/|1 - \mathbf{e}(-q l_{i,j}/l)| &= \sum_q 1/|1 - \mathbf{e}(-q/l)| \\ &\leqslant 2 \sum_{q=1}^{l/2} 1/|1 - \mathbf{e}(-q/l)| \\ &\leqslant 2c' l \cdot \sum_q 1/q \leqslant 4c' l \sqrt{l}.\end{aligned}$$

Therefore (*) is bounded by

$$c'' \cdot \operatorname{rank}(W \otimes L_k) l \sqrt{l} \cdot \sum_i s_i = c'' \cdot \operatorname{rank}(W \otimes L_k)[\Gamma : \Gamma(l)] p \sqrt{m}/2$$

where c'' is a constant independent on p, k, m. Similarly we can see that the contribution of elliptic points is bounded by

$$c\cdot[\Gamma:\Gamma(l)]\cdot\mathrm{rank}(W\otimes L_k)\quad (c:\text{constant})$$

by using the fact that the term $1/(1-\mathbf{e}(-\theta))$ is independent on p, m. Thus we have proved the assertion for $\alpha=1$. The above proof also implies the assertion for $\alpha=p$. Thus we have proved Lemma 6.17 and Lemma 6.16. □

THEOREM 6.18. *If $d=p^2$ for some prime p, and p, k are sufficiently large, then* $\dim H^0(\bar{\mathscr{K}}^0_{2d},\Omega^{\otimes k})\geqslant c\cdot p^{20}\cdot k^{19}$, *where c is a positive constant.*

Proof. By Proposition 2.4, (ii), Corollary 4.5, Lemma 5.6, Theorem 5.11 and Lemma 6.16, for sufficiently large k,

$$\begin{aligned}\dim A_k'(\Gamma_{2d}) &\geqslant \dim S_k(\Gamma_{2d}) - \sum_{F:0-\dim} h(d,F,k) - \sum_{F:1-\dim}\sum_{m=1}^{k-1}\dim H_{m,\alpha}(1)\\ &\geqslant (c_1[\Gamma_2:\Gamma_{2d}] - c_2\cdot p^{19})k^{19}\end{aligned}$$

where c_1, c_2 are positive constants not depending on p and k. It now follows from Lemma 3.2 and Theorem 4.3 that

$$\dim H^0(\bar{\mathscr{K}}^0_{2d},\Omega^{\otimes k})\geqslant c\cdot p^{20}\cdot k^{19}$$

for sufficiently large p. □

7. Extension over the singular points in $\mathscr{K}_{2d}$

Let $\bar{\mathscr{K}}_{2d}$ be a toroidal compactification of $\mathscr{K}_{2d}$ with only quotient singularities and let $\tilde{\mathscr{K}}_{2d}\to\bar{\mathscr{K}}_{2d}$ be a non-singular model of $\bar{\mathscr{K}}_{2d}$. In this section and the next, we shall see that every element in $H^0(\bar{\mathscr{K}}^{\mathrm{reg}}_{2d},\Omega^{\otimes k})$ extends to $\tilde{\mathscr{K}}_{2d}$, where $\bar{\mathscr{K}}^{\mathrm{reg}}_{2d}$ is the smooth locus of $\bar{\mathscr{K}}_{2d}$. Our argument is based on the following:

THEOREM 7.1 (Reid–Tai's criterion, [7], [23]). *Let V be a vector space of dimension N, $G\subset \mathrm{GL}(V)$ a finite group. For all $g\in G$ of order m, let the eigenvalues of g be $\xi^{a_1},\dots,\xi^{a_N}$, where ξ is a primitive m-th root of* 1 *and $0\leqslant a_i<m$. Let $V^{\circ}\subset V$ be the open set where G acts freely and let ω be a G-invariant m-th root of* 1 *and $0\leqslant a_i<m$. Let $V^{\circ}\subset V$ be the open set where G acts freely and let ω be a G-invariant m-th canonical form on V which can be considered as a form on V°/G. Then ω extends holomorphically to a resolution of $\tilde{V}/G$ of V/G if $\Sigma a_i\geqslant m$ for all $g\in G$.*

PROPOSITION 7.2 ([23]). *We keep the same notation as in 7.1. Then ω extends to $\tilde{V}/G$ iff ω extends to a non-singular model of $V/\langle g\rangle$ for all $g\in G$.*

(7.3) Let (X, H) be a polarized K3 surface of degree $2d$ and $[\omega_X] \in \mathscr{D}_{2d}$ the period of (X, H). Recall that the tangent space of $\mathscr{D}_{2d}$ at $[\omega_X]$ is canonically isomorphic to $V = \mathrm{Hom}(H^{2,0}(X), H^{1,1}_{\mathrm{prim}}(X))$ where $H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ is the Hodge decomposition and $H^{1,1}_{\mathrm{prim}}(X) = H^{\perp}$ in $H^{1,1}(X)$. Let $G \subset \Gamma_{2d}$ be the stabilizer group of $[\omega_X]$. Then G is a finite group. Let $\delta \in H^{1,1}_{\mathrm{prim}}(X) \cap H^2(X, \mathbb{Z})$ with $\delta^2 = -2$. Then

$$s_\delta : x \to x + \langle x, \delta \rangle \delta$$

is a reflection which is contained in G because $\langle \delta, \omega_X \rangle = 0$. Denote by $W(X, H)$ the normal subgroup of G generated by $\{s_\delta \mid \delta^2 = -2 \text{ and } \delta \in H^{1,1}_{\mathrm{prim}}(X) \cap H^2(X, \mathbb{Z})\}$. It follows from the Torelli theorem for K3 surfaces [18] that G is the semi-direct product $\mathrm{A}(X, H) \cdot W(X, H)$, where $\mathrm{A}(X, H)$ is the image of the group of automorphisms of the pair (X, H). For $\gamma \in G$, put $\gamma(\omega_X) = \alpha(\gamma)\omega_X$, then $\alpha(\gamma)$ is a root of unity. We remark that if $\gamma = g \cdot w$ for $g \in \mathrm{A}(X, H)$, $w \in W(X, H)$, then $a(\gamma) = \alpha(g)$ because $W(X, H)$ acts on $H^{2,0}(X)$ as identity. Let S_X denote the *Picard Lattie* of X, i.e. $S_X = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$, and let T_X be the orthogonal complement of S_X in $H^2(X, \mathbb{Z})$ which is called the *transcendental lattice* of X. We remark that $W(X, H)$ acts on T_X as identity.

In the following, we denote by m the order of γ and by r the order of the root of unity $\alpha(\gamma)$. Note that $\gamma | T_X \otimes \mathbb{Q}$ is a direct sum of irreducible representations of the cyclic group $\mathbb{Z}/r\mathbb{Z}$ defined over $\mathbb{Q}$ with maximal degree $\varphi(r)$, where φ is the Euler function ([15], Theorem 3.1). In particular $\varphi(r) \leqslant 20$ because rank $T_X \leqslant 21$. Let the action of $\gamma \in G$ on V be given by

$$\begin{bmatrix} \zeta^{a_1} & & \\ & \ddots & \\ & & \zeta^{a_{19}} \end{bmatrix}$$

where ζ is a primitive m-th root of unity and $0 \leqslant a_i < m$. Put

$$\{\gamma, [\omega_X]\} = \sum_i a_i/m.$$

The main result of this section is as follows:

PROPOSITION 7.4. (i) *Under the same notation as above,*

$$\{\gamma, [\omega_X]\} \geqslant 1$$

except γ acts on V as a reflection. In case that γ acts on V as a reflection, a general point $[\omega_X]$ in the fixed point set of γ is one of the following:

(α) $[\omega_X]$ *is the period of a polarized* K3 *surface* (X, H) *with* $\operatorname{rank} T_X = 20$ *such that X contains exactly one smooth rational curve R orthogonal to* $[H]$ *and* γ *is a reflection in* $W(X, H)$ *induced from the class* $[R]$, *i.e.* $\gamma = s_{[R]}$.

(β) $[\omega_X]$ *is the period of a polarized* K3 *surface* (X, H) *with an automorphism* σ *of order* 2 *such that* $\operatorname{rank} T_X = 20$, $\sigma^*|S_X = 1$, $\sigma^*|T_X = -1$ *and* $\gamma = \sigma^*$.

(ii) *Assume that* γ^n *acts on* V *as a reflection* $(m = 2n)$. *Let* $\langle\bar{\gamma}\rangle = \langle\gamma\rangle/\langle\gamma^n\rangle$, $\bar{V} = V/\langle\gamma^n\rangle$ *and let* $[\overline{\omega_X}]$ *be the image of* $[\omega_X]$ *in* $\mathscr{D}_{2d}/\langle\gamma^n\rangle$. *Then the natural map* $\bar{V} \to \bar{V}/\langle\bar{\gamma}\rangle$ *has no ramification divisors and* $\{\bar{\gamma}, [\overline{\omega_X}]\} \geqslant 1$.

For an integer k, we denote by $[k]_m$ the integer satisfying $0 \leqslant [k]_m < m$ and $[k]_m \equiv k \pmod{m}$. Also denote by V_r an irreducible representation of the cyclic group of order r which is defined over $\mathbb{Q}$ and is of maximal degree $\varphi(r)$.

LEMMA 7.5. *If* $\varphi(r) \geqslant 6$, *then* $\{\gamma, [\omega_X]\} \geqslant 1$.

Proof. The restriction $\gamma|T_X \otimes \mathbb{Q}$ is a direct sum of irreducible representations of a cyclic group of order r with maximal degree $\varphi(r)$ ([15], Theorem 3.1, *c*). Let V_r be a component of $\gamma|T_X \otimes \mathbb{Q}$ and put $U = \operatorname{Hom}(H^{2,0}(X), V_r \otimes \mathbb{C} \cap H^{1,1}_{\text{prim}}(X)) \subset V$. Let $k_1, \dots, k_t$ $(t = \varphi(r))$ be all integers with $0 < k_i < r$, $(k_i, r) = 1$ $(1 \leqslant i \leqslant t)$. Note that for fixed i, $[k_i + k_j]_r \neq [k_i + k_l]_r$ $(j \neq l)$. Assume that $\alpha(\gamma) = \zeta^{nk_2}$ and $\overline{\alpha(\gamma)} = \zeta^{nk_1}$ where $m = nr$. By considering the action of γ on U, we have a rough estimate

$$\{\gamma, [\omega_X]\} \geqslant \sum_{j=3}^{t} \frac{[nk_1 + nk_j]_m}{m} = \sum_{j=3}^{t} \frac{[k_1 + k_j]_r}{r} \geqslant \frac{(t-1)(t-2)}{2r}.$$

By a calculation, we check $[(t-1)(t-2)]/2r \geqslant 1$ for all r with $\varphi(r) \leqslant 20$ except $\varphi(r) \leqslant 4$ or $r = 14, 18, 24, 30$. But for $r = 14, 18, 24, 30$, we can directly see $\Sigma_{j=3}^{t} [k_1 + k_j]_r/r \geqslant 1$.

LEMMA 7.6. *If* $r = 1$ *or* 2, *then* $\{\gamma, [\omega_X]\} \geqslant 1$ *except* γ *acts on* V *as a reflection.*

Proof. In case $m = 2$, obviously $\{\gamma, [\omega_X]\} \geqslant 1$ except γ acts on V as a reflection. In case $m > 2$, there is a component V_n of $\gamma|S_X \otimes \mathbb{Q}$ with $n > 2$. By considering the contribution of V_n, we have the assertion. □

(7.7) *Remark.* In Lemma 7.6, consider the case that γ acts on V as a reflection.

(i) Assume $r = 2$. Then the number of irreducible components V_1 of $(\gamma, H^2(X, \mathbb{Q}))$ is 2 and other components are V_2. Since the period of a polarized K3 surface (X, H) with rank $T_X \leqslant 19$ is contained in a subvariety of $\mathscr{D}_{2d}$ of codimension 2, a general point of the fixed point set of γ is the period of a polarized K3 surface (X, H) with rank $T_X = 20$. Since $\gamma|T_X \otimes \mathbb{Q}$ is a direct sum of V_2, $\gamma|T_X = -1$ and $\gamma|S_X = 1$. Hence γ preserves effective divisors on X. Therefore, by Torelli theorem for K3 surface [18], γ is induced from an automorphism σ or order $2 : \sigma^* = \gamma$.

(ii) In case $r = 1$, the number of components V_2 of $(\gamma, H^2(X, \mathbb{Q}))$ is 1 and other components are V_1. As above, a general point of the fixed point set of γ is the period of a polarized K3 surface (X, H) with rank $T_X = 20$. Consider the following primitive sublattices in $H^2(X, \mathbb{Z})$:

$$S = \{x \in H^2(X, \mathbb{Z}) \mid \gamma(x) = x\} \text{ and } N = \{x \in H^2(X, \mathbb{Z}) \mid \gamma(x) = -x\}.$$

Since $\gamma|T_X = 1$ and $\gamma([H]) = [H]$, N is a negative definite sublattice in S_X of rank 1. Obviously $N^\perp = S$. Note that S and N are even 2-elementary lattices. In fact, by [16], Proposition 1.6.1, $A_S \simeq A_N$. It now easily follows from the definition of S and N that $A_N \simeq \mathbb{Z}/2\mathbb{Z}$. Hence $N \simeq \langle -2 \rangle$. Let δ be a base of N. Then by Riemann-Roch theorem, δ or $-\delta$ is represented by an effective divisor R. Since $\mathrm{rank}(S_X) = 2$, H is numerically effective and $H \cdot R = 0$, R is irreducible, and hence a smooth rational curve. Obviously $\gamma = s_{[R]}$.

LEMMA 7.8. *Assume that there is a component V_n $(n > 2)$ of $\gamma|S_X \otimes \mathbb{Q}$ with $n > r$ or $n \nmid r$. Then the contribution of V_n to $\{\gamma, [\omega_X]\}$ is more than $t(t-1)/2n$ where $t = \varphi(n)$.*

Proof. Let $k_1, \ldots, k_t$ be all integers with $0 < k_i < n$ and $(k_i, n) = 1$. Put $m = nl = rk$ and assume $\overline{\alpha(\gamma)} = \zeta^{kc}$, $0 < c < r$, $(c, r) = 1$. Then the action of γ on $\mathrm{Hom}(H^{2,0}(X), V_n \otimes \mathbb{C})$ is given by

$$(\zeta^{lk_i + kc})_{i=1,\ldots,t}.$$

Note that $[lk_i + kc]_m \neq 0$, $[lk_i + kc]_m \neq [lk_j + kc]_m$ $(i \neq j)$ and $[lk_i + kc]_m \equiv [lk_j + kc]_m$ (mod l). Hence the contribution of V_n is more than $lt(t-1)/2m = t(t-1)/2n$. □

(7.9) *Proof of Proposition* 7.4, (i): By Lemmas 7.5, 7.6, we may assume $r = 3$, 4, 5, 6, 8, 10 or 12. We shall prove the assertion for $r = 12$ and omit the proof for other cases. In Lemma 7.8, $t(t-1)/2n \geqslant 1$ except $n = 8$, 10, 18, 30. Hence we may assume that components of $(\gamma, L_{2d} \otimes \mathbb{Q})$ are (V_n ($n = 1$, 2, 3, 4, 6, 8, 10, 12, 18, 30). A direct calculation shows that the contribution of V_1 or V_2 (resp. V_3, V_4, V_6 or V_{12}) is more than 1/12 (resp. 5/6). Denote by r_n the number of components of V_n appeared in $(\gamma, L_{2d} \otimes \mathbb{Q})$. Then $\Sigma_n \varphi(n) r_n = \mathrm{rank}(L_{2d}) = 21$. We now have:

$$\{\gamma, [\omega_X]\} \geqslant (r_1 + r_2)/12 + 5(r_3 + r_4 + r_6 + r_{12} + r_{18})/6 + 3r_8/4$$

$$+3r_{10}/5 + 14r_{30}/15 \geqslant \left(\sum_n \varphi(n)r_n\right)\Big/12 = 21/12. \qquad \square$$

(7.10) *Proof of Proposition* 7.4, (ii): By Proposition 7.4 (i), for any $\gamma' \in \langle\gamma\rangle\backslash\langle\gamma^n\rangle$, the set of fixed points of γ' has codimension $\geqslant 2$. Hence the map $\bar{V} \to \bar{V}/\langle\bar{\gamma}\rangle$ has no ramification divisors. Next consider the action of γ on $L_{2d} \otimes \mathbb{Q}$. If the representation $(\gamma'', L_{2d} \otimes \mathbb{Q}) = V_2 \oplus V_1 \oplus \cdots \oplus V_1$ (see Remark 7.7, (ii)), then n is odd and $(\gamma, L_{2d} \otimes \mathbb{Q}) = V_2 \oplus V_{n_1} \oplus \cdots \oplus V_{n_t}$ (n_i is a divisor of n). If $(\gamma^n, L_{2d} \otimes \mathbb{Q}) = V_1 \oplus V_2 \oplus \cdots \oplus V_2$ (see Remark 7.7, (i)), then $(\gamma, L_{2d} \otimes \mathbb{Q}) = V_1 \oplus V_{n_1} \oplus \cdots \oplus \mathrm{v}_{n_t}$ (n_i is even and $2n/n_i$ is odd) or $V_2 \oplus V_{n_1} \oplus \cdots \oplus V_{n_t}$ (n is even and $2n/n_i$ is odd). In any case, the contribution of $V_{n_1} \oplus \cdots \oplus \mathrm{v}_{n_t}$ to $\{\bar{\gamma}, [\overline{\omega_X}]\}$ is the same as that of $V_{n_1} \oplus \cdots \oplus V_{n_t}$ to $\{\gamma, [\omega_X]\}$. Since $\dim V_{n_1} \oplus \cdots \oplus V_{n_t} = 20$, the same proof as that of Proposition 7.4, (i) holds.

$\square$

Now by Theorem 7.1, Propositions 7.2 and 7.4, we have:

THEOREM 7.11. *Let $\mathscr{K}_{2d}^{\mathrm{reg}}$ be a smooth locus of $\mathscr{K}_{2d} = \mathscr{D}_{2d}/\Gamma_{2d}$. Let ω be a pluri-canonical differential form on $\mathscr{K}_{2d}^{\mathrm{reg}}$. Then ω extends holomorphically to a non-singular model of $\mathscr{K}_{2d}$.*

8. Extension over the singularities on the boundary

We use the same notation as in (4.2). Let $y_0 \in \bar{\mathscr{K}}_{2d}\backslash\mathscr{K}_{2d}$. Then for some F, $y_0 = \pi_F(\tau_0, w_0, \bar{z}_0 + \sigma^\infty)$ where $\tau_0 \in F$, $w_0 \in \mathbb{C}^m$ $z_0 \in U(F)_{\mathbb{C}}$ and $\bar{z}_0 =$ its image in $T(F)$, $\bar{z}_0 + \sigma^\infty$ is the ideal point in the torus embedding $T(F)_\sigma$ (associated to σ) obtained by starting at $\bar{z}_0$ and moving the imaginary part to infinity in the direction of σ, and σ is a face of one of the σ_α's. If y_0 is a singular point, then for some $\gamma \in N(F)_{\mathbb{Z}}$ with $\gamma \not\equiv \mathrm{id} \bmod U(F)_{\mathbb{Z}}$ such that $\gamma(\tau_0, w_0, \bar{z}_0 + \sigma^\infty) = (\tau_0, w_0, \bar{z}_0 + \sigma^\infty)$. The main result of this section is as follows:

PROPOSITION 8.1. *Under the above notation,*

$$\{\gamma, (\tau_0, w_0, \bar{z}_0 + \sigma^\infty)\} \geqslant 1$$

except γ acts on $F \times \mathbb{C}^m \times T(F)_\sigma$ as a reflection. Moreover there are no branch divisors of π_F contained in $\bar{\mathscr{K}}_{2d}\backslash\mathscr{K}_{2d}$.

(8.2) *In case F is of dimension* 1: We use the same notation as in Proposition

2.14. In this case, $C(F) = \{y \mid y > 0\}$ and the polyhedral cone decomposition is unique: $\{\sigma_\alpha\} = \{\{0\},\ \sigma = \{y \mid y \geqslant 0\}\}$. It defines a torus embedding $\mathbb{C}^* \subset T(F)_\sigma \simeq \mathbb{C}$. Put $u = \mathbf{e}(z/\alpha)$. Then (τ, w, u) is a coordinate of $H^+ \times \mathbb{C}^{17} \times T(F)_\sigma$. By Proposition 2.14, the action of γ on the tangent space of $H^+ \times \mathbb{C}^{17} \times T(F)_\sigma$ at $(\tau_0, w_0, 0)$ is

$$\begin{bmatrix} (c\tau_0 + d)^{-2} & * & * \\ 0 & (c\tau_0 + d)^{-1}X & * \\ 0 & 0 & \mathbf{e}(T) \end{bmatrix}$$

where $T = 1/\alpha \cdot (c\tau_0 + d)^{-1}\{cq_0)w_0/2 + c\beta + v_1 w_0 + w_1\tau_0 + w_2\}$.

If $\gamma|H^+ = 1$, then $c = 0$, $d = \pm 1$. We may assume that $d = 1$ if necessary by replacing γ by $-\gamma$. Denote by m the order of X and assume $m > 2$. We consider X as a representation of a cyclic group of order m defined over $\mathbb{Q}$. Let V_n be an irreducible component of X with $n > 2$. Let k, k' be integers with $(k, n) = (k', n) = 1$, $k + k' = n$ and $0 < k, k' < n$. Then we have

$$\{\gamma, (\tau_0, w_0, 0)\} \geqslant \frac{k + k'}{n} = 1.$$

If $m = 1$, the condition $\gamma(\tau_0, w_0, 0) = (\tau_0, w_0, 0)$ implies $Y = 0$, $V = 0$ and hence $\gamma \in U(F)_{\mathbb{Z}}$. If $m = 2$, then $\gamma^2 \in U(F)_{\mathbb{Z}}$ and $VX = -V$, $XY = -Y$ and $2W \equiv -VY \pmod{U(F)_{\mathbb{Z}}}$. It follows from this fact and the equation

$$w_0 = Xw_0 + Y\begin{bmatrix}\tau_0 \\ 1\end{bmatrix} \quad \text{that} \quad 2(v_1 w_0 + w_1\tau_0 + w_2) \equiv 2v_1 w_0 - v_1 Y\begin{bmatrix}\tau_0 \\ 1\end{bmatrix}$$

$$= v_1 w_0 + v_1\left(w_0 - Y\begin{bmatrix}\tau_0 \\ 1\end{bmatrix}\right) = v_1 w_0 + v_1 X w_0 = 0 \pmod{\alpha\mathbb{Z}}.$$

Thus the action of γ at the tangent space is

$$\begin{bmatrix} 1 & * & * \\ 0 & X & * \\ 0 & 0 & \pm 1 \end{bmatrix}.$$

Therefore $\{\gamma, (\tau_0, w_0, 0)\} \geqslant 1$ except γ acts as a reflection. Moreover in case $m = 2$, there are no reflections which fix the locus $\{u = 0\}$. Since $\{u = 0\}$ corresponds to a divisor in $\bar{\mathscr{K}}_{2d} \backslash \mathscr{K}_{2d}$, in this case, there are no branch divisors of π_F contained in $\bar{\mathscr{K}}_{2d} \backslash \mathscr{K}_{2d}$.

If $\gamma|H^+ \neq 1$. then we may assume that τ_0 is equal to $\sqrt{-1}$ or $\mathbf{e}(1/3)$, if necessary by replacing γ by its conjugate. Recall that if $\tau_0 = \sqrt{-1}$ (resp. $\mathbf{e}(1/3)$), then $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\left(\text{resp. } \pm\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \text{ or } \pm\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\right)$ and $c\tau_0 + d = \pm\sqrt{-1}$ (resp. $\pm\mathbf{e}(1/3)$ or $\pm\mathbf{e}(1/6)$). By considering X as a representation of a cyclic group, the action of γ on the tangent space of $\mathbb{C}^{17}$ is similar as that of γ with $\alpha(\gamma) = 3$, 4 or 6 in §7. Since degree $X = 17$, the same argument holds in this case and hence $\{\gamma, (\tau_0, w_0, 0)\} \geqslant 1$.

(8.3) *In case F is of dimension* 0: Recall that $\mathscr{D}(F) = U(F)_{\mathbb{C}}$ and the action of γ on $\mathscr{D}(F)$ is as follows (see Proposition 2.9):

$$\gamma(z) = Az + B_1.$$

Since $\gamma(\bar{z}_0 + \sigma^\infty) = \bar{z}_0 + \sigma^\infty$, $z_0 \equiv Az_0 + B_1 + c \bmod U(F)_{\mathbb{Z}}$, where $c \in L(\sigma) \otimes \mathbb{C}$ and $L(\sigma)$ is the linear span of σ. Since $\gamma^n \equiv 1 \pmod{U(F)_{\mathbb{Z}}}$ for some n, $A^n = 1$. Note that $A(\sigma) \subset \sigma$. By taking a further decomposition of $C(F)$, we may assume that $A|\sigma = 1$. In fact, if $\dim(\sigma) = 1$, then $A|\sigma = 1$ because A and σ are defined over $\mathbb{Q}$ and $A(\sigma) \subset \sigma$. It follows from [9] that there is a unique closed orbit O_σ and a subtorus T'_σ in $T(F)_\sigma$ with $T(F)_\sigma/T'_\sigma \simeq O_\sigma$:

O_σ is defined by $\mathscr{X}^\rho = 0$, $\sigma \geqslant 0$ on σ, $\rho > 0$ on $\mathrm{Int}(\sigma)$.
T'_σ is defined by $\mathrm{Spec}(\mathbb{C}[\mathscr{X}^\rho])_{\rho \in L(\sigma) \cap U(F)^*_{\mathbb{Z}}}$.

Since we assume $A|\sigma =$ identity, A acts on T'_σ trivially, and A fixes $e_\sigma =$ the identity of O_σ. Let t_{z_0} denote the translation by z_0. Then $t_{z_0} \circ A = \gamma \circ t_{z_0}$ on O_σ because $\gamma \circ t_{z_0}(z) - t_{z_0} \circ A(z) \equiv c \bmod U(F)_{\mathbb{Z}}$ for any $z \in U(F)_{\mathbb{C}}$. Therefore, in the O_σ-directions, the eigenvalues of γ on the tangent space of $T(F)_\sigma$ at $\bar{z}_0 + \sigma^\infty$ coincides with those of A on the tangent space of O_σ at e_σ. The latter eigenvalues coincide those of $A \,|\, U(F)_{\mathbb{C}}/L(\sigma)_{\mathbb{C}}$. By the similar way as in (8.2), we have

$$\{\gamma, \bar{z}_0 + \sigma^\infty\} \geqslant 1$$

except A acts on O_σ as a reflection. Moreover if for some 1-dimensional polyhedral cone σ, the orbit O_σ is the fixed points set of γ, then $A = 1$ and hence $\gamma \in U(F)_{\mathbb{Z}}$. Therefore there are no branch divisors in $\bar{\mathscr{K}}_{2d} \backslash \mathscr{K}_{2d}$. Thus we have now proved Proposition 8.1.

Now by the same proof as that of Theorem 7.11, we have:

THEOREM 8.4. *Let ω be a holomorphic pluri-canonical differential form on*

$\bar{\mathscr{K}}_{2d}^{\mathrm{reg}}$. *Then, for any* $d = p^2$ (p: *prime*), ω *extends to a holomorphic pluri-canonical differential form on a non-singular model of* $\bar{\mathscr{K}}_{2d}$.

9. Ramifications

In this section, we shall complete the proof of the main theorem. We assume that $d = p^2$ (p: odd prime). First we shall study the branch divisor of the projection $\pi: \mathscr{D}_{2d} \to \mathscr{D}_{2d}/\Gamma_{2d}$, and see that, for $p \gg 0$, there are sufficiently many cusp forms $f \in S_k(\Gamma_{2d})$ such that $f \cdot \omega^{\otimes k}$ extends to a general point of the branch locus of π. Recall that this branch locus corresponds to the fixed point set of the following involutions in Γ_{2d} (Proposition 7.4): the first one is a reflection s_δ ($\delta \in L_{2d}$ with $\langle \delta, \delta \rangle = -2$), and the second one is an involution induced from an automorphism σ of a K3 surface X with rank $T_X = 20$ such that $\sigma^* | S_X = 1$ and $\sigma^* | T_X = -1$.

LEMMA 9.1. *Assume that p is an odd prime number.*

(i) *Let* δ *be a vector in* L_{2d} *with* $\langle \delta, \delta \rangle = -2$. *Then the orthogonal complement of* δ *in* L_{2d} *is isometric to either*

$$U \oplus E_8 \oplus E_8 \oplus \langle \bar{2} \rangle \oplus \langle -2p^2 \rangle \text{ or } U \oplus E_8 \oplus E_8 \oplus \begin{bmatrix} 2 & p \\ p & 0 \end{bmatrix}.$$

(ii) *Two vectors in* L_{2d} *with length* -2 *are equivalent modulo* $O(L_{2d})$ *if and only if their orthogonal complements in* L_{2d} *are isometric.*

Proof. Let S denote the primitive sublattice of L_{2d} generated by δ. Let K be the orthogonal complement of S in L_{2d}. The primitive embedding of S into L_{2d} is determined by the following sets:

$(H_S, H, \gamma, K, \gamma_K)$ where $H_S \subset A_S \simeq \mathbb{Z}/2\mathbb{Z}$ and $H \subset A_{L_{2d}} \simeq \mathbb{Z}/2p^2\mathbb{Z}$ *are subgroups*, $\gamma: q_S | H_S \to q_{L_{2d}} | H$ is an isomorphism of forms, and $\gamma_K: q_K \to q := -(q_S \oplus (-q_{L_{2d}}) | \Gamma_\gamma^\perp)/\Gamma_\gamma$ is an isomorphism of forms where Γ_γ is the "graph" of γ in $A_S \oplus A_{L_{2d}}$.

Two such sets, $(H_S, H, \gamma, K, \gamma_K)$ and $(H'_S, H', \gamma', K', \gamma'_{K'})$, determine isomorphic primitive embedding if and only if $H_S = H'_S$ and there exists $\xi \in O(q_{L_{2d}})$ and an isometry ψ from K to K' for which $\gamma' = \xi \circ \gamma$ and $\bar{\xi} \circ \gamma_K = \gamma'_{K'} \circ \bar{\psi}$, where $\bar{\xi}$ is the isomorphism of discriminant forms q and q' induced from ξ ([16], Proposition 1.15.1).

Note that $H_S \simeq \{0\}$ or $\mathbb{Z}/2\mathbb{Z}$. In case $H_S \simeq \{0\}$, $q_K \simeq (-q_S) \oplus q_{L_{2d}}$. In case $H_s \simeq \mathbb{Z}/2\mathbb{Z}$, it is easy to see that $q_K \simeq q_{\left[\begin{smallmatrix} 2 & p \\ p & 0 \end{smallmatrix}\right]}$. Since K is an even indefinite lattice with $\mathrm{rank}(K) > l(K) + 2$, it follows from [16], Theorem 1.14.2 that the genus of K contains only one isomorphism class and the homomorphism

$O(K) \to O(q_K)$ is surjective. Hence $K \simeq U \oplus E_8 \oplus E_8 \oplus \langle 2 \rangle \oplus \langle -2p^2 \rangle$ if $H_S \simeq \{0\}$ and $K \simeq U \oplus E_8 \oplus E_8 \oplus \begin{bmatrix} 2 & p \\ p & 0 \end{bmatrix}$ if $H_S \simeq \mathbb{Z}/2\mathbb{Z}$, and the assertion (ii) follows. □

LEMMA 9.2. *Assume that p is an odd prime number. Let (X, H) be a* K3 *surface with* rank $T_X = 20$. *Assume that (X, H) has an automorphism σ of order* 2 *such that $\sigma^*|S_X = 1$ and $\sigma^*|T_X = -1$. Then*

(i) (X, H) *is one of the following*: (a) $S_X \simeq U$, $T_X \simeq U \oplus U \oplus E_8 \oplus E_8$ *and* $[H] = p^2 e + f$, *where* $\{e, f\}$ *is a base of* S_X *with* $\langle e, e\rangle = \langle f, f\rangle = 0$, $\langle e, f\rangle = 1$; (b) $S_X \simeq \langle 2 \rangle \oplus \langle -2 \rangle$, $T_X \simeq U \oplus E_8 \oplus E_8 \oplus \langle 2 \rangle \oplus \langle -2 \rangle$ *and* $[H] = (p^2+1)/2 \cdot x - (p^2-1)/2 \cdot y$, *where* $\{x, y\}$ *is a base of* S_X *with* $\langle x, x\rangle = 2$, $\langle y, y\rangle = -2$, $\langle x, y\rangle = 0$.

(ii) *Let* $(X, H), (X', H')$ *be two polarized* K3 *surfaces as above. Assume that* $S_X \simeq S_{X'}$. *Then there exists an isometry*

$$\varphi: H^2(X, \mathbb{Z}) \to H^2(X', \mathbb{Z})$$

with $\varphi(S_X) = S_{X'}$ *and* $\varphi([H]) = [H']$.

Proof. First note that S_X and T_X are 2-elementary lattices with $l(S_X) = l(T_X) \leqslant \mathrm{rank}(S_X) = 2$ (see Remark (7.7), (ii)). Such S_X is classified as follows ([16], Theorem 3.6.2): $S_X \simeq \langle 2 \rangle \oplus \langle -2 \rangle$, $U(2)$ or U, where $U(2)$ is the lattice defined by the matrix $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$. Then $T_X \simeq U \oplus \langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8 \oplus E_8$, $U \oplus U(2) \oplus E_8 \oplus E_8$ or $U \oplus U \oplus E_8 \oplus E_8$ respectively. Note that $[H]$ is a primitive vector in S_X with length $2p^2$. Since the length of each vector in $U(2)$ is divided by 4, the case $S_X \simeq U(2)$ does not occur. Also it is easy to see that any primitive vector in U (resp. in $\langle 2 \rangle \oplus \langle -2 \rangle$) with length $2p^2$ is of the form $p^2 e + f$ for a base $\{e, f\}$ (resp. $(p^2+1)/2 \cdot x - (p^2-1)/2 \cdot y$ for a base $\{x, y\}$). Thus we have proved the assertion (i).

Next, taking any isometries

$$H^2(X, \mathbb{Z}) \xrightarrow{\simeq} U \oplus U \oplus U \oplus E_8 \oplus E_8 \xleftarrow{\simeq} H^2(X', \mathbb{Z}),$$

we consider S_X, $S_{X'}$ as primitive sublattices in $U \oplus U \oplus U \oplus E_8 \oplus E_8$. Then their orthogonal complement T_X, $T_{X'}$ are mutually isometric and, by [16], Theorem 1.14.2, the map $O(T_X) \to O(q_{T_X})$ is surjective. Let $\psi: S_X \to S_{X'}$ be an isometry with $\psi([H]) = [H']$. Then it follows from [16], Corollary 1.5.2 that ψ can be extended to an isometry of $U \oplus U \oplus U \oplus E_8 \oplus E_8$. Hence the assertion (ii) follows. □

(9.3) Let δ be a vector in L_{2d} with $\langle \delta, \delta \rangle = -2$. Put

$$H_\delta = \{[\omega] \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) \mid \langle \omega, \delta \rangle = 0\}.$$

We denote by $\mathcal{N}$ the union of all hyperplane sections $H_\delta \cap \mathcal{D}_{2d}$ with δ being of length -2. Then $\mathcal{N}$ is invariant under the action of Γ_{2d} and $\mathcal{N}/\Gamma_{2d}$ is a divisor of $\mathcal{D}_{2d}/\Gamma_{2d}$. Recall that $[O(L_{2d}):\Gamma_{2d}] = 4$. Hence Lemma 9.1 implies that $\mathcal{N}/\Gamma_{2d}$ consists of at most 8 irreducible components. We remark here that each component is covered by a 18-dimensional bounded symmetric domain $H_\delta \cap \mathcal{D}_{2d}$ of type IV. Next consider the divisor of $\mathcal{D}_{2d}/\Gamma_{2d}$ consisting the periods of polarized K3 surface as in Lemma 9.2. Let $\mathcal{M}$ denote the union of all hyperplanes

$$\mathcal{D}_{2d} \cap \mathbb{P}(T \otimes \mathbb{C})$$

where T are primitive sublattices in $L = U \oplus U \oplus U \oplus E_8 \oplus E_8$ which are contained in L_{2d} and isometric to $U \oplus U \oplus E_8 \oplus E_8$ or $U \oplus E_8 \oplus E_8 \oplus \langle 2 \rangle \oplus \langle -2 \rangle$. Then $\mathcal{M}$ is invariant under the action of Γ_{2d}. Here note that if $T^\perp$ denotes the orthogonal complement of T in L, then the involution ι on $T \oplus T^\perp$ defined by $\iota | T = -1$ and $\iota | T^\perp = 1$ extends to an involution on L because T is 2-elementary. Hence $\mathcal{M}$ is the union of ramification sets of all involutions appeared in Remark 7.7, (i). By Lemma 9.2, the number of irreducible components of $\mathcal{M}/\Gamma_{2d}$ is at most 8. We denote by $\Delta_1, \ldots, \Delta_r$ (resp. $\Delta_{r+1}, \ldots, \Delta_{r+s}$) the components of $\mathcal{N}/\Gamma_{2d}$ (resp. $\mathcal{M}/\Gamma_{2d}$), r, $s \leqslant 8$. Now we conclude:

COROLLARY 9.4. *Assume that $d = p^2$ for an odd prime number p. Then the branch divisor of the projection $\mathcal{D}_{2d} \to \mathcal{D}_{2d}/\Gamma_{2d}$ is*

$$\Delta_1 + \cdots + \Delta_r + \Delta_{r+1} + \cdots + \Delta_{r+s}, \qquad r \leqslant 8, s \leqslant 8.$$

Let $\mathcal{D}_i$ denote the hyperplane section $H_{\delta_i} \cap \mathcal{D}_{2d}$ corresponding to Δ_i, $1 \leqslant i \leqslant r$, and $\mathcal{D}_j$ the hyperplane section $\mathbb{P}(T_j \otimes \mathbb{C}) \cap \mathcal{D}_{2d}$ corresponding to Δ_j, $r+1 \leqslant j \leqslant r+s$, where δ_i is a vector in L_{2d} with $\langle \delta_i, \delta_i \rangle = -2$ and T_j is a primitive sublattice in L_{2d} isometric to $U \oplus U \oplus E_8 \oplus E_8$ or $U \oplus E_8 \oplus E_8 \oplus \langle 2 \rangle \oplus \langle -2 \rangle$. Put

$$\Gamma_i = \{\varphi \in \Gamma_{2d} \mid \varphi(\delta_i) = \delta_i\}$$

and

$$\Gamma_j = \{\varphi \in \Gamma_{2d} \mid \varphi(T_j) = T_j\}.$$

Then Γ_i (resp. Γ_j) is an arithmetic subgroup of $SO(2, 18)_{\mathbb{R}}$ acting properly discontinuously on $\mathcal{D}_i$ (resp. $\mathcal{D}_j$). Let K_i be the orthogonal complement of δ_i in L_{2d}. Then Γ_i is a subgroup of $\tilde{O}(K_i)$ of index 2 and Γ_j is a subgroup of $\tilde{O}(T_j)$ of index 2.

In case $K_i \simeq U \oplus E_8 \oplus E_8 \oplus \langle 2 \rangle \oplus \langle -2p^2 \rangle$, by the same way as in §3, K_i is a sublattice of $K_i^{\prime} \simeq U \oplus E_8 \oplus E_8 \oplus \langle 2 \rangle \oplus \langle -2 \rangle$ of index p.

In case $K_i \simeq U \oplus E_8 \oplus E_8 \oplus \begin{bmatrix} 2 & p \\ p & 0 \end{bmatrix}$, there is a lattice $K_i^{\prime}$ isometric to $U \oplus U \oplus E_8 \oplus E_8$ which contains K_i as a sublattice of index p. In fact, let $\{x, y\}$ be a base of $\begin{bmatrix} 2 & p \\ p & 0 \end{bmatrix}$ with $\langle x, x \rangle = 2$, $\langle x, y \rangle = p$, $\langle y, y \rangle = 0$. Then $\{x, (1/p)y\}$ generates a lattice isometric to $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \simeq U$.

(9.5) *Remark.* By the same proof as that of Lemma 3.2, Γ_i $(1 \leqslant i \leqslant r)$ is a subgroup of the group $\Gamma_i^{\prime} = \{\varphi \in O(K_i^{\prime}) \mid \varphi(\mathcal{D}_i) = \mathcal{D}_i\}$. Moreover $[\Gamma_i^{\prime} : \Gamma_i] \leqslant O(p^{19})$ because the number of hyperplanes of the quadratic space of dimension 2 is equal to $O(p^{19})$ (see the proof of Lemma 3.2). Also note that $\Gamma_j (r + 1 \leqslant j \leqslant r + s)$ is independent on p.

For a positive integer v and $i = 1, \ldots, r + s$, define

$$S_k(\Gamma_{2d})(-v\mathcal{D}_i) = \{f \in S_k(\Gamma_{2d}) \mid f \text{ vanishes on } \mathcal{D}_i \text{ at least of order } v\}.$$

Note that $f \cdot \omega^{\otimes k}$ extends holomorphically to a general point of Δ_i if $f \in S_k(\Gamma_{2d})(-k\mathcal{D}_i)$.

LEMMA 9.6.

$$\dim \bigcap_{i=1}^{r+s} S_k(\Gamma_{2d})(-k\mathcal{D}_i) \geqslant c \cdot p^{20} \cdot k^{19} \qquad (p \gg 0,\ k \gg 0)$$

where c is a positive constant.

Proof. Let $(z_0 : \cdots : z_{20})$ be a homogeneous coordinate of $\mathbb{P}(L_{2d} \otimes \mathbb{C})$ such that $\mathcal{D}_i = \mathcal{D}_{2d} \cap \{z_0 = 0\}$. Consider the following exact sequence:

$$0 \longrightarrow S_k(\Gamma_{2d})(-(v+1)\mathscr{D}_i) \longrightarrow$$

$$\longrightarrow S_k(\Gamma_{2d})(-v\mathscr{D}_i) \xrightarrow{\alpha_v} S_k(\Gamma_i) \qquad (0 \leqslant v_i \leqslant k-1)$$

where $S_k(\Gamma_i)$ is the vector space of cusp forms on $\mathscr{D}_i$ with respect to Γ_i and α_v is a homomorphism defined by

$$\alpha_v(f) = (f/z_0^v)\,|\,\mathscr{D}_i \qquad \text{for } f \in S_k(\Gamma_{2d})(-v\mathscr{D}_i).$$

Then

$$\dim \bigcap_{i=1}^{r+s} S_k(\Gamma_{2d})(-k\mathscr{D}_i) \geqslant \dim S_k(\Gamma_{2d}) - \sum_{i=1}^{r+s} k\cdot\dim S_k(\Gamma_i).$$

On the other hand, it follows from Corollary 4.5 and Lemma 3.2 that

$$\dim S_k(\Gamma_{2d}) \geqslant c'\cdot p^{20}\cdot k^{19} \qquad (p \gg 0,\ k \gg 0)$$

where c' is a positive constant. By the same proof as that of Proposition 4.4, we have

$$\dim S_k(\Gamma_i) = c_i\cdot \mathrm{vol}(\mathscr{D}_i/\Gamma_i)k^{18} + \mathrm{O}(k^{17}),$$

where c_i is a constant independent on p and k $(1 \leqslant i \leqslant r+s)$. Since Γ_i' $(1 \leqslant i \leqslant r)$, Γ_j $(r+1 \leqslant j \leqslant r+s)$ are independent on p (Remark 9.5), $\mathrm{vol}(\mathscr{D}_i/\Gamma_i')$ and $\mathrm{vol}(\mathscr{D}_j/\Gamma_j)$ are so. Again by Remark 9.5,

$$\mathrm{vol}(\mathscr{D}_i/\Gamma_i) = [\Gamma_i':\Gamma_i]\mathrm{vol}(\mathscr{D}_i/\Gamma_i') = \mathrm{O}(p^{19}) \qquad (1 \leqslant i \leqslant r).$$

Hence the assertion follows. □

THEOREM 9.7. *Let $\tilde{\mathscr{K}}_{2d}$ be a non-singular model of $\bar{\mathscr{K}}_{2d}$. Assume that $d = p^2$ where p is a sufficiently large prime number. Then*

$$\dim \mathrm{H}^0(\tilde{\mathscr{K}}_{2d}, \Omega^{\otimes k}) \geqslant c\cdot k^{19} \qquad (k \gg 0)$$

where c is a positive constant. In particular, $\mathscr{K}_{2d}$ is of general type.

Proof. In the proof of Theorem 6.18, replacing $\dim S_k(\Gamma_{2d})$ by $\dim \cap_{i=1}^{r+s} S_k(\Gamma_{2d})(-k\mathscr{D}_i)$, we have

$$\dim \mathrm{H}^0(\bar{\mathcal{K}}_{2d}^{\mathrm{reg}}, \Omega^{\otimes k}) \geqslant c \cdot k^{19}.$$

Hence the assertion follows from Theorem 8.4. □

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