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The Fundamental lemma of a relative trace formula for $GL(3)$

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1. Introduction

Let $E = F(\sqrt{\tau})$ be a quadratic extension of a number field F, and ζ the quadratic idele class character of F attached to E. In [H-L-R] there is an argument which shows that an automorphic representation π of $GL(n, E_{\mathbf{A}})$ with central character $\chi \circ N_{E/F}$ is the base change of an automorphic representation of $GL(n, F_{\mathbf{A}})$ with central character χ or $\chi\zeta$ if π is H_η-distinguished for a unitary group H_η with respect to an invertible Hermitian matrix η. Here π is said to be H_η-distinguished if there is a function ϕ in the space of π such that $\int \phi(h)\chi \circ \lambda_\eta(h)\, dh \neq 0$, where $h \in Z_{E_{\mathbf{A}}} H'_\eta(F)\backslash H'_\eta(F_{\mathbf{A}})$, H'_η is the group of unitary similitudes, and λ_η is the similitude ratio. This property of π being distinguished then might imply a possible pole of an L-function attached to the representation π (cf. [H-L-R]).

An interesting question is whether the converse is true. For $GL(2)$ it is answered affirmatively in [H-L-R] and later in [Y] and [J-Y], while for $GL(n)$ it is conjectured to be true by Jacquet and Ye in [J-Y]. The approach in [Y] and [J-Y] is to construct a relative trace formula. For $GL(3)$ the fundamental lemma of the relative trace formula is proved for unit elements of Hecke algebras in [J-Y2]. In this paper we will prove that fundamental lemma for general spherical functions on $GL(3)$.

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2. The fundamental lemma

From now on we will denote by F a local non Archimedean field of odd residual characteristic with the ring of integers R_F, and by $E = F(\sqrt{\tau})$ an unramified quadratic extension of F with the ring of integers R_E, where $\tau \in R_F^\times$. Let ζ be the quadratic character of F attached to E, ϖ_F (resp. ϖ_E)

a prime element in R_F (resp. R_E), and q_F the cardinality of $R_F/\varpi_F R_F$. Select an additive character ψ of F of order zero and set $\psi_E = \psi \circ \mathrm{tr}_{E/F}$. Then ψ_E is a character of E of order zero. For $n \in N_F$ we define $\theta_F(n) = \psi(\Sigma\, n_{i,i+1})$. Define θ_E on N_E likewise. Let χ be an unramified character of $F^\times$.

We will consider spherical functions f (resp. f'), i.e., bi-K_F-invariant (resp. bi-K_E-invariant) functions of compact support of $GL(3, F)$ (resp. $GL(3, E)$). Denote by $\mathscr{H}$ (resp. $\mathscr{H}_E$) the Hecke algebra consisting of the functions f (resp. f'). Write $f(\lambda) = f(m)$ and $f'(\lambda) = f'(m')$ and define $\Phi_f(\lambda) = \int_{N_F} f(mm)\,\mathrm{d}n$, $\Psi_f(\lambda) = \int_{N_F} f(mn)\theta_F(n)\,\mathrm{d}n$, $\Phi'_{f'}(\lambda) = \int_{N_E} f'(m'n)\,\mathrm{d}n$, and $\Psi'_{f'}(\lambda) = \int_{N_E} f'(m'n)\theta_E(n)\,\mathrm{d}n$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbf{Z}^3$, $m = \mathrm{diag}(\varpi_F^{\lambda_1}, \varpi_F^{\lambda_2}, \varpi_F^{\lambda_3})$ and $m' = \mathrm{diag}(\varpi_E^{\lambda_1}, \varpi_E^{\lambda_2}, \varpi_E^{\lambda_3})$. Then the base change map $b : f' \mapsto f$ from $\mathscr{H}_E$ into $\mathscr{H}$ can be characterized by the equations

$$\Phi_f(\lambda) = 0 \qquad \text{if } \lambda \not\equiv (0,0,0) \pmod 2; \tag{1}$$

$$= \Phi'_{f'}(\lambda/2) \qquad \text{if } \lambda \equiv (0,0,0) \pmod 2. \tag{2}$$

Let H_F be the unitary group with respect to the Hermitian matrix $\left(\begin{smallmatrix} & & 1\\ & 1 & \\ 1 & & \end{smallmatrix}\right)$. Then the group of unitry similitudes with respect to $\left(\begin{smallmatrix} & & 1\\ & 1 & \\ 1 & & \end{smallmatrix}\right)$ is $Z_E H_F$. For $f' \in \mathscr{H}_E$ we define a function Ω on the space of Hermitian matrices by $\Omega({}^t\bar{g}\left(\begin{smallmatrix} & & 1\\ & 1 & \\ 1 & & \end{smallmatrix}\right)g) = \int f'(hg)\,\mathrm{d}h$ where $h \in H_F$, and $\Omega(s) = 0$ for Hermitian matrices s not of the form ${}^t\bar{g}\left(\begin{smallmatrix} & & 1\\ & 1 & \\ 1 & & \end{smallmatrix}\right)g$. For $a = \mathrm{diag}(1, q, pq)$ with $p, q \in F^\times$ we define the relative Kloosterman integral $J(f'; p, q) = \int \Omega({}^t\bar{n}zan)\chi(z)\theta_E(n)\,\mathrm{d}n\,\mathrm{d}^\times z$, where $n \in N_E$ and $z \in Z_F$, and a Kloosterman integral $I(f; p, q) = \int f({}^t xzay)\theta_F(x)\theta_F(y)\chi\zeta(z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}^\times z$, where $x, y \in N_F$ and $z \in Z_F$. We also define several singular integrals:

$$J_1(f') = \int_{N_{1_E}\backslash N_E}\int_{Z_F} \Omega\left({}^t\bar{n}z\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix}n\right)\chi(z)\theta_E(n)\,\mathrm{d}n\,\mathrm{d}^\times z,$$

$$J_2(f'; p) = \int_{N_{2_E}\backslash N_E}\int_{Z_F} \Omega\left({}^t\bar{n}z\begin{pmatrix} & p & \\ p & & \\ & & 1\end{pmatrix}n\right)\chi(z)\theta_E(n)\,\mathrm{d}n\,\mathrm{d}^\times z,$$

$$J_3(f'; p) = \int_{N_{3_E}\backslash N_E}\int_{Z_F} \Omega\left({}^t\bar{n}z\begin{pmatrix}1 & & \\ & & p\\ & p & \end{pmatrix}n\right)\chi(z)\theta_E(n)\,\mathrm{d}n\,\mathrm{d}^\times z,$$

$$I_1(f)=\int_{N_F}\int_{Z_F} f(zy)\chi\zeta(z)\theta_F(y)\,\mathrm{d}y\,\mathrm{d}^\times z,$$

$$I_2(f;p)=\int_{N_{4_F}\backslash N_F}\theta_F(x)\,\mathrm{d}x\int_{N_F}\int_{Z_F} f\left({}^txz\begin{pmatrix} & p & \\ p & & \\ & & 1\end{pmatrix}y\right)\chi\zeta(z)\theta_F(y)\,\mathrm{d}y\,\mathrm{d}^\times z,$$

$$I_3(f;p)=\int_{N_{5_F}\backslash N_F}\theta_F(x)\,\mathrm{d}x\int_{N_F}\int_{Z_F} f\left({}^txz\begin{pmatrix} 1 & & \\ & & p \\ & p & \end{pmatrix}y\right)\chi\zeta(z)\theta_F(y)\,\mathrm{d}y\,\mathrm{d}^\times z.$$

Here $N_{1_E}=\{n\in N_E|{}^t\bar{n}(_{1}{}^{1}{}^{1})n=(_{1}{}^{1}{}^{1})\}$, $N_{4_F}=\{x\in N_F|(_{p}{}^{p}{}_{1})^{-1}\,{}^tx(_{p}{}^{p}{}_{1})\in N_F\}$, and N_{2_E}, N_{3_E}, N_{5_F} are defined similarly.

THEOREM 1 (The fundamental lemma). *Assume $f\in\mathcal{H}$ and $f'\in\mathcal{H}_E$ satisfy $f=b(f')$. Then $I(f;p,q)=\zeta(q)J(f';p,q)$, $I_1(f)=J_1(f')$, $I_2(f;p)=\zeta(p)J_2(f';p)$, and $I_3(f;p)=\zeta(p)J_3(f';p)$ for p, $q\in F^\times$.*

We remark that the above fundamental lemma is purely a local argument. For a version of the global relative trace formula for $GL(3)$ we refer to [J-Y2]. For a discussion of the singular integrals please see [Y]. The rest of this article is devoted to the proof of Theorem 1.

3. Mautner's identities

Mautner in [M] formulated several integral identities of functions on $GL(2)$. Although the proof of these $GL(2)$ identities is rather elementary, let us list them, together with others, in a systematic way for quick references.

Let f (resp. f') be a spherical function on $GL(2,F)$ (resp. $GL(2,E)$). Write $f(\lambda_1,\lambda_2)=f(\mathrm{diag}(\varpi_F^{\lambda_1},\varpi_F^{\lambda_2}))$, $f'(\lambda_1,\lambda_2)=f'(\mathrm{diag}(\varpi_E^{\lambda_1},\varpi_E^{\lambda_2}))$ and define Φ_f, $\Phi'_{f'}$, Ψ_f and $\Psi'_{f'}$ in a way similar to the $GL(3)$ case. Then

$$\begin{aligned}\Psi_f(\lambda_1,\lambda_2)&=0, \qquad \text{if } \lambda_1>\lambda_2;\\ &=f(\lambda_1,\lambda_2)-f(\lambda_1-1,\lambda_2+1)\\ &=\Phi_f(\lambda_1,\lambda_2)-q_F\Phi_f(\lambda_1-1,\lambda_2+1) \qquad \text{if } \lambda_1\leqslant\lambda_2.\end{aligned}$$

Assume $f=b(f')$. Then when $\lambda_1\leqslant\lambda_2$ we have

$$\Psi_f(\lambda_1,\lambda_2)=\Phi'_{f'}(\lambda_1/2,\lambda_2/2) \text{ if } (\lambda_1,\lambda_2)\equiv(0,0)(\mathrm{mod}\,2);$$

$$= -q_F\Phi'_{f'}((\lambda_1 - 1)/2, (\lambda_2 + 1)/2) \text{ if } (\lambda_1, \lambda_2) \equiv (1,1)(\mathrm{mod}\, 2);$$
$$= 0, \qquad \text{if } (\lambda_1, \lambda_2) \not\equiv (0,0), (1,1)(\mathrm{mod}\, 2).$$

Finally by $\Psi'_{f'}(\lambda_1, \lambda_2) = \Phi'_{f'}(\lambda_1, \lambda_2) - q_F^2\Phi'_{f'}(\lambda_1 - 1, \lambda_2 + 1)$ for $\lambda_1 \leqslant \lambda_2$, we can get various relationships between Ψ_f and $\Psi'_{f'}$, under the base change map $b: f' \mapsto f$.

The above identities played an important role, although sometimes implicitly, in the author's paper [Y]. In this section we will generalize these identities to the case of $GL(3)$ over quadratic extensions. The generalization to $GL(n)$ is the subject matter of a separate paper [Y2] of the author.

Now let us go back to the $GL(3)$ case. We will generally not assume that $f = b(f')$, unless otherwise mentioned. Since the orders of the characters ψ and ψ_E are zero, we know $\Psi_f(\lambda_1, \lambda_2, \lambda_3) = 0$ and $\Psi'_{f'}(\lambda_1, \lambda_2, \lambda_3) = 0$ unless $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$.

LEMMA 1. *For* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *with* $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ *we have*

$$\Psi_f(\lambda) = \sum_{e_1,e_2,e_3=0,1} (-1)^{e_1+e_2+e_3} q_F^{e_3} f(\lambda + (-e_1 - e_3, e_1 - e_2, e_2 + e_3)).$$

Proof. Trivial.

Define

$$\Lambda(\lambda) = \int_{(F)^2} f\begin{pmatrix} \varpi_F^{\lambda_1} & x_1 & x_3 \\ & \varpi_F^{\lambda_2} & 0 \\ & & \varpi_F^{\lambda_3} \end{pmatrix} dx_1\, dx_3.$$

For a spherical function f on $GL(2, F)$ we note that

$$\Phi_f(\lambda_2, \lambda_3) = f(\lambda_2, \lambda_3) + (1 - q_F^{-1}) \sum_{k<0} q^{-k} f(\lambda_2 + k, \lambda_3 - k)$$

if $\lambda_2 \leqslant \lambda_3$. Applying this equality to the integral with respect to x_2 in

$$\Phi_f(\lambda) = q_F^{2\lambda_1+\lambda_2} \int_{(F)^3} f\begin{pmatrix} \varpi_F^{\lambda_1} & x_1 & x_3 \\ & \varpi_F^{\lambda_2} & x_2 \\ & & \varpi_F^{\lambda_3} \end{pmatrix} dx_1\, dx_2\, dx_3$$

we get

$$\Phi_f(\lambda) = q_F^{2\lambda_1}\Lambda(\lambda) + (1 - q_F^{-1})q_F^{2\lambda_1} \sum_{k<0} q_F^{-k}\Lambda(\lambda_1, \lambda_2 + k, \lambda_3 - k) \quad (3)$$

if $\lambda_2 \leqslant \lambda_3$.

LEMMA 2. *For* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *with* $\lambda_1 - 1 \leqslant \lambda_2 \leqslant \lambda_3$ *we have*

$$\begin{aligned} &q_F^{2\lambda_1}[\Lambda(\lambda) - q_F^{-1}\Lambda(\lambda_1 - 1, \lambda_2 + 1, \lambda_3) - \Lambda(\lambda_1 - 1, \lambda_2, \lambda_3 + 1) \\ &\qquad + q_F^{-1}\Lambda(\lambda_1 - 2, \lambda_2 + 1, \lambda_3 + 1)] \\ &\quad = f(\lambda) - f(\lambda_1 - 1, \lambda_2 + 1, \lambda_3) - q_F f(\lambda_1 - 1, \lambda_2, \lambda_3 + 1) \\ &\qquad + q_F f(\lambda_1 - 2, \lambda_2 + 1, \lambda_3 + 1). \end{aligned}$$

Proof. We first calculate the integral $\Lambda(\lambda_1, \lambda_2, \lambda_3)$ once for $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ and once for $\lambda_1 - 1 = \lambda_2 \leqslant \lambda_2$ and express the results in terms of $f(\lambda)$, $\lambda \in \mathbf{Z}^3$. Since $\Lambda(\lambda_1, \lambda_3, \lambda_2) = \Lambda(\lambda_1, \lambda_2, \lambda_3)$, we may then apply these computations to the linear combination of Λ in the lemma. After cancellation we prove the lemma.

LEMMA 3. *For* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *with* $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ *we have*

$$\begin{aligned} &\sum_{e_1,e_2,e_3=0,1} (-1)^{e_1+e_2+e_3} q_F^{e_1+e_2+2e_3} \Phi_f(\lambda + (-e_1 - e_3, e_1 - e_2, e_2 + e_3)) \\ &\quad = \sum_{e_1,e_2,e_3=0,1} (-1)^{e_1+e_2+e_3} q_F^{e_3} f(\lambda + (-e_1 - e_3, e_1 - e_2, e_2 + e_3)). \end{aligned}$$

Proof. Applying (3) to the left side above and using Lemma 2 to simplify the result, we prove the lemma.

COROLLARY. *For* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *with* $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ *we have*

$$\begin{aligned} \Psi_f(\lambda) = &\sum_{e_1,e_2,e_3=0,1} (-1)^{e_1+e_2+e_3} q_F^{e_1+e_2+2e_3} \\ &\times \Phi_f(\lambda + (-e_1 - e_3, e_1 - e_2, e_2 + e_3)). \end{aligned} \quad (4)$$

Proof. By Lemmas 1 and 3.

We remark that we also have

$$\begin{aligned} \Psi'_{f'}(\lambda) = &\sum_{e_1,e_2,e_3=0,1} (-1)^{e_1+e_2+e_3} q_F^{2e_1+2e_2+4e_3} \\ &\times \Phi'_{f'}(\lambda + (-e_1 - e_3, e_1 - e_2, e_2 + e_3)). \end{aligned} \quad (5)$$

Since the base change map b can be characterized by Equations (1) and (2) between Φ_f and $\Phi'_{f'}$, we obtain a set of relationships between Ψ_f and $\Phi'_{f'}$ for spherical functions f and f' with $f = b(f')$. More precisely

THEOREM 2. *Let $f \in \mathcal{H}$ and $f' \in \mathcal{H}_E$ be spherical functions with $f = b(f')$. For $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ we have*

$$\Psi_f(\lambda) = \Phi'_{f'}\left(\frac{\lambda}{2}\right) - q_F^4\Phi'_{f'}\left(\frac{\lambda}{2} + (-1, 0, 1)\right) \qquad \text{if } \lambda \equiv (0, 0, 0) (\text{mod } 2);$$

$$= -q_F\Phi'_{f'}\left(\frac{\lambda_1 - 1}{2}, \frac{\lambda_2 + 1}{2}, \frac{\lambda_3}{2}\right) + q_F^3\Phi'_{f'}\left(\frac{\lambda_1 - 1}{2}, \frac{\lambda_2 - 1}{2}, \frac{\lambda_3}{2} + 1\right)$$

$$\text{if } \lambda \equiv (1, 1, 0) (\text{mod } 2);$$

$$= -q_F\Phi'_{f'}\left(\frac{\lambda_1}{2}, \frac{\lambda_2 - 1}{2}, \frac{\lambda_3 + 1}{2}\right) + q_F^3\Phi'_{f'}\left(\frac{\lambda_1}{2} - 1, \frac{\lambda_2 + 1}{2}, \frac{\lambda_3 + 1}{2}\right)$$

$$\text{if } \lambda \equiv (0, 1, 1) (\text{mod } 2);$$

$$= 0 \text{ if } \lambda \not\equiv (0, 0, 0), (1, 1, 0), (0, 1, 1) (\text{mod } 2).$$

Proof. Apply (1) and (2) to the right side of (4).

4. The identity between $I_1(f)$ and $J_1(f')$

We recall from Section 2 that $I_1(f) = \int_{N_F}\int_{Z_F} f(zn)\zeta\chi(z)\psi(n)\, \mathrm{d}n\, \mathrm{d}^\times z$. Since the characters ζ and χ are unramified, $I_1(f) = \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z, z, z)$. In the sequel we will always write $z = (z, z, z)$. Note that $\zeta(\varpi_F^z) = (-1)^z$. By Theorem 2, $\Psi_f(z) = 0$ when z is odd, if we assume $f = b(f')$. Consequently

$$I_1(f) = \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})\Psi_f(2z)$$

$$= \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})[\Phi'_{f'}(z) - q_F^4\Phi'_{f'}(z + (-1, 0, 1))]. \tag{6}$$

On the other hand

$$J_1(f') = \int_{H_F/H_1(F)} \int_{N_E} \int_{Z_E} f'\left(zh\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}n\right) \chi \circ N(z)\theta_E(n)\, \mathrm{d}^\times z\, \mathrm{d}n\, \mathrm{d}h$$

where

$$H_1(F) = \left\{ \begin{pmatrix} 1 & & \\ -(x\bar{x}/2) + \mu\sqrt{\tau} & 1 & x \\ -\bar{x} & & 1 \end{pmatrix} \middle|\, \mu \in F, x \in E \right\}.$$

To determine a measure on $H_F/H_1(F)$ we use the Iwasawa decomposition $H_F = K_1A_1H_1$ where $K_1 = K_E \cap H_F$ and $A_1 = \{(^{a}\ {}_{1/a}\ {}_{1}) \,|\, a \in F^\times\}$. Since the right measure on A_1H_1 is given by

$$\mathrm{d}\left[\begin{pmatrix} a & & \\ & 1/a & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ -(x\bar{x}/2) + \mu\sqrt{\tau} & 1 & x \\ -\bar{x} & & 1 \end{pmatrix}\right] = |a|_F^{-4}\, \mathrm{d}^\times a\, \mathrm{d}x\, \mathrm{d}\mu$$

for $a \in F^\times$, $x \in E$ and $\mu \in F$, we can set $h = (^{a}\ {}_{1/a}\ {}_{1})$, $\mathrm{d}h = |a|_F^{-4}\, \mathrm{d}^\times a$ for any $h \in K_1\backslash H_F/H_1(F)$. Hence

$$J_1(f') = \int_{F^\times} \int_{N_E} \int_{Z_E} f'\left(z\begin{pmatrix} a & & \\ & 1/a & \\ & & 1 \end{pmatrix}\right.$$

$$\left. \times \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}n\right) \chi \circ N(z)\theta_E(n)|a|_F^{-4}\, \mathrm{d}^\times z\, \mathrm{d}n\, \mathrm{d}^\times a.$$

Since f' is bi-invariant under K_E, the function f' in the integrand above equals $f'(z(^{1/a}\ {}_{1}\ {}_{a})n)$, and hence $J_1(f') = \sum_{z\in\mathbf{Z}} \sum_{k\leq 0} \chi(\varpi_F^{2z}) \times \Psi'_{f'}(z + (k, 0, -k))q_F^{-4k}$ where $k = -\mathrm{ord}(a)$. The condition $k \leq 0$ is necessarybecause otherwise $\Psi'_{f'}(z + (k, 0, -k)) = 0$.

LEMMA 4. *For* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *with* $\lambda_1 \leq \lambda_2 \leq \lambda_3$ *we have*

$$\sum_{k\leq 0} \Psi'_{f'}(\lambda + (k, 0, -k))q_F^{-4k}$$
$$= \Phi'_{f'}(\lambda) - q_F^2\Phi'_{f'}(\lambda + (-1, 1, 0)) - q_F^2\Phi'_{f'}(\lambda + (0, -1, 1))$$
$$+ q_F^4\Phi'_{f'}(\lambda + (-1, 0, 1)).$$

Proof. Applying (5) in Section 3 to the right side, we get a sum of $\Phi'_{f'}$. After cancellation the four terms above are the only terms left.

By Lemma 4

$$J_1(f') = \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})[\Phi'_{f'}(z) - q_F^2\Phi'_{f'}(z + (-1, 1, 0))$$
$$- q_F^2\Phi'_{f'}(z + (0, -1, 1)) + q_F^4\Phi'_{f'}(z + (-1, 0, 1))].$$

We point out that $q_F^2\Phi'_{f'}(z + (-1, 1, 0)) = q_F^2\Phi'_{f'}(z + (0, -1, 1)) = q_F^4\Phi'_{f'}(z + (-1, 0, 1))$ because they are essentially Satake coefficients of f' which are independent of the order of their entries. Therefore

$$J_1(f') = \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})[\Phi'_{f'}(z) - q_F^4\Phi'_{f'}(z + (-1, 0, 1))]. \tag{7}$$

Comparing (6) and (7) we prove that $I_1(f) = J_1(f')$ for $f = b(f')$.

5. The identity between $I_2(f)$ and $J_2(f')$

By similar computation we can show that

$$I_2(f; p) = \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)[\Psi_f(z + (P, P, 0)) - q_F\Psi_f(z + (P - 1, P, 1))$$
$$+ q_F\Psi_f(z + (P - 1, P - 1, 2))] \qquad \text{if } P = \mathrm{ord}_F(p) \leq 2;$$
$$= \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f\left(z + \left(\frac{2P}{3}, \frac{2P}{3}, \frac{2P}{3}\right)\right)$$
$$\times \int_{\substack{x_1\in\varpi_F^{-P/3}R_F^\times,\\ x_3\in\varpi_F^{-2P/3}R_F^\times}} \psi\left(x_1 + \frac{x_3}{x_1} + \frac{1}{px_3}\right) dx_1\, dx_3$$
$$\text{if } P > 2,\ P \equiv 0 (\mathrm{mod}\ 3);$$
$$= 0 \qquad \text{if } P > 2,\ P \not\equiv 0 (\mathrm{mod}\ 3).$$

We proceed to rewrite $I_2(f)$ in terms of Φ_f by applying Theorem 2 to the above formula. For instance when $P=2$, the only nonzero term above is $q_F\Psi_f(z+(P-1,P-1,2))$, and hence $I_2(f;p)=q_F\sum_{z\in\mathbf{Z}}\zeta\chi(\varpi_F^z)\Psi_f(z+(1,1,2))$. Since $\Psi_f(z+(1,1,2))=0$ unless z is even, we get $I_2(f;p)=q_F\sum_{z\in\mathbf{Z}}\chi(\varpi_F^{2z})\Psi_f(2z+(1,1,2))$. The expression of $I_2(f;p)$ in terms of Φ_f for $P=2$ then follows from Theorem 2. By the same argument, we can prove the following results:

$$\begin{aligned}
I_2(f;p) &= \sum_{z\in\mathbf{Z}}\chi(\varpi_F^{2z})[-q_F^2\Phi'_{f'}(z+(0,1,1)) \\
&\qquad + q_F^4\Phi'_{f'}(z+(0,0,2))] \qquad \text{if } P=2; \\
&= \sum_{z\in\mathbf{Z}}\chi(\varpi_F^{2z})[(1+q_F^{-1})q_F^2\Phi'_{f'}(z+(0,0,1)) - q_F^4\Phi'_{f'}(z+(-1,1,1)) \\
&\qquad - q_F^5\Phi'_{f'}(z+(-1,0,2))] \qquad \text{if } P=1; \\
&= \sum_{z\in\mathbf{Z}}\chi(\varpi_F^{2z})\left[\Phi'_{f'}\left(z+\left(\frac{P}{2},\frac{P}{2},0\right)\right) - (q_F^2+q_F^4)\right. \\
&\qquad \left.\times\Phi'_{f'}\left(z+\left(\frac{P}{2}-1,\frac{P}{2},1\right)\right) + q_F^4\Phi'_{f'}\left(z+\left(\frac{P}{2}-1,\frac{P}{2}-1,2\right)\right)\right] \\
&\qquad \text{if } P\leqslant 0,\ P\equiv 0 \pmod 2; \\
&= \sum_{z\in\mathbf{Z}}\chi(\varpi_F^{2z})\left[-q_F\Phi'_{f'}\left(z+\left(\frac{P-1}{2},\frac{P+1}{2},0\right)\right)\right. \\
&\qquad + (q_F+q_F^2+q_F^3)\Phi'_{f'}\left(z+\left(\frac{P-1}{2},\frac{P-1}{2},1\right)\right) \\
&\qquad \left.- q_F^4\Phi'_{f'}\left(z+\left(\frac{P-3}{2},\frac{P+1}{2},1\right)\right) - q_F^5\Phi'_{f'}\left(z+\left(\frac{P-3}{2},\frac{P-1}{2},2\right)\right)\right] \\
&\qquad \text{if } P<0,\ P\equiv 1 \pmod 2; \\
&= \sum_{z\in\mathbf{Z}}\chi(\varpi_F^{2z-(2P/3)})[\Phi'_{f'}(z) - q_F^4\Phi'_{f'}(z+(-1,0,1))] \\
&\qquad \times\int_{\substack{x_1\in\varpi_F^{-P/3}R_F^\times,\\ x_3\in\varpi_F^{-2P/3}R_F^\times}}\psi\left(x_1+\frac{x_3}{x_1}+\frac{1}{px_3}\right)\mathrm{d}x_1\,\mathrm{d}x_3 \\
&\qquad \text{if } P>2,\ P\equiv 0 \pmod 3; \\
&= 0 \qquad \text{if } P>2,\ P\not\equiv 0 \pmod 3.
\end{aligned}$$

On the other hand

$$J_2(f';p) = \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})[-q_F^2\Phi'_{f'}(z+(0,1,1))$$

$$+ q_F^4\Phi'_{f'}(z+(0,0,2))] \qquad \text{if } P=2;$$

$$= \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})[-(1+q_F^{-1})q_F^2\Phi'_{f'}(z+(0,0,1))$$

$$+ q_F^4\Phi'_{f'}(z+(-1,1,1)) + q_F^5\Phi'_{f'}(z+(-1,0,2))] \qquad \text{if } P=1;$$

$$= \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})\left[\Phi'_{f'}\left(z+\left(\frac{P}{2},\frac{P}{2},0\right)\right) - (q_F^2+q_F^4)\right.$$

$$\left.\times\Phi'_{f'}\left(z+\left(\frac{P}{2}-1,\frac{P}{2},1\right)\right) + q_F^4\Phi'_{f'}\left(z+\left(\frac{P}{2}-1,\frac{P}{2}-1,2\right)\right)\right]$$

$$\text{if } P\leqslant 0,\ P\equiv 0 \pmod 2;$$

$$= \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})\left[q_F\Phi'_{f'}\left(z+\left(\frac{P-1}{2},\frac{P+1}{2},0\right)\right) - (q_F+q_F^2+q_F^3)\right.$$

$$\times\Phi'_{f'}\left(z+\left(\frac{P-1}{2},\frac{P-1}{2},1\right)\right) + q_F^4\Phi'_{f'}\left(z+\left(\frac{P-3}{2},\frac{P+1}{2},1\right)\right)$$

$$\left.+ q_F^5\Phi'_{f'}\left(z+\left(\frac{P-3}{2},\frac{P-1}{2},2\right)\right)\right] \qquad \text{if } P<0,\ P\equiv 1 \pmod 2;$$

$$= \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z-(2P/3)})[\Phi'_{f'}(z) - q_F^4\Phi'_{f'}(z+(-1,0,1))]q_F^{5P/3}$$

$$\times \int_{\varpi_E^{P/3}R_E^\times} \psi_E\left(\frac{x\bar{x}}{2p}+\frac{1}{x}\right)\mathrm{d}x \qquad \text{if } P>2,\ P\equiv 0 \pmod 3;$$

$$= 0 \qquad \text{if } P>2,\ P\not\equiv 0 \pmod 3.$$

Comparing the above results, we conclude that $I_2(f;p) = \zeta(p)J_2(f';p)$ for any $p\in F^\times$ if we can prove the following lemma.

LEMMA 5. *Assume the orders of* ψ *and* $\psi_E = \psi\circ\mathrm{tr}$ *to be zero. When* $P = \mathrm{ord}(p) \leqslant -3$, $P\equiv 0 \pmod 3$, *we have*

$$q_F^{-2P/3}\int_{\varpi_E^{-P/3}R_E^\times} \psi_E\left(\frac{x\bar{x}p}{2}+\frac{1}{x}\right)\mathrm{d}x$$

$$= (-1)^P q_F^P \int_{\substack{x_1 \in \varpi_F^{-P/3} R_F^\times, \\ x_2 \in \varpi_F^{-2P/3} R_F^\times}} \psi\left(x_1 + \frac{x_2}{x_1} + \frac{p}{x_2}\right) \mathrm{d}x_1 \, \mathrm{d}x_2 .$$

Proof. Let σ be a character of $F^\times$. Then

$$\int_{\varpi_F^P R_F^\times} \sigma^{-1}(p) \, \mathrm{d}p q_F^{-2P/3} \int_{\varpi_E^{-P/3} R_E^\times} \psi_E\left(\frac{x\bar{x}p}{2} + \frac{1}{x}\right) \mathrm{d}x$$

$$= \int_{\varpi_F^{P/3} R_F^\times} \sigma^{-1}(p) \psi(p) \, \mathrm{d}p \int_{\varpi_E^{P/3} R_E^\times} \sigma(x\bar{x}) \psi_E\left(\frac{1}{x}\right) \mathrm{d}x q_F^{-4P/3} .$$

It is known that the integral with respect to p on the right side equals $\epsilon(\sigma, \psi)$ when the conductor of σ is $-P/3$, and vanishes otherwise. When the conductor of σ is $-P/3$, the integral with respect to x becomes

$$\int_{\varpi_E^{-P/3} R_E^\times} \sigma(x\bar{x}) \psi_E\left(\frac{1}{x}\right) \mathrm{d}x$$

$$= q_F^{4P/3} \int_{\varpi_E^{P/3} R_E^\times} \sigma^{-1}(x\bar{x}) \psi_E(x) \, \mathrm{d}x = q_F^{4P/3} \epsilon(\sigma \circ N, \psi_E).$$

Therefore

$$\int_{\varpi_F^P R_F^\times} \sigma^{-1}(p) \, \mathrm{d}p q_F^{-2P/3} \int_{\varpi_E^{-P/3} R_E^\times} \psi_E\left(\frac{x\bar{x}p}{2} + \frac{1}{x}\right) \mathrm{d}x$$

$$= \epsilon(\sigma, \psi) \epsilon(\sigma \circ N, \psi_E)$$

if the conductor of σ is $-P/3$, and vanishes otherwise.

By the same reason

$$\int_{\varpi_F^P R_F^\times} \sigma^{-1}(p) \, \mathrm{d}p (-1)^P q_F^P \int_{\substack{x_2 \in \varpi_F^{P/3} R_F^\times, \\ x_2 \in \varpi_F^{2P/3} R_F^\times}} \psi\left(x_1 + \frac{x_2}{x_1} + \frac{p}{x_2}\right) \mathrm{d}x_1 \, \mathrm{d}x_2$$

$$= \epsilon^2(\sigma, \psi) \epsilon(\sigma\zeta, \psi)$$

if the conductor of σ is $-P/3$, and vanishes otherwise.

Since $\epsilon(\sigma \circ N, \psi_E) = \epsilon(\sigma, \psi)\epsilon(\sigma\zeta, \psi)$, the lemma follows from the Fourier inversion formula.

We observe that this lemma and Lemmas 12 and 14 in Section 12 are all of the type of the identity between finite exponential sums proved by Zagier in [Z].

6. The identity between $I_3(f)$ and $J_3(f')$

The proof of $I_3(f;p)=\zeta(p)J_3(f';p)$ is based on the similarity between $I_2(f)$ and $I_3(f)$, and between $J_2(f')$ and $J_3(f')$. We will denote by $I_2(f;p;\chi,\psi)$ the integral $I_2(f;p)$ involving characters χ and ψ. Denote I_3, J_2 and J_3 similarly. Define $\tilde f(g)=f({}^tg^{-1})$ and $\tilde f'(g)=f'({}^tg^{-1})$. Write $w=\left(\begin{smallmatrix} & & 1\\ & 1 & \\ 1 & & \end{smallmatrix}\right)$.

LEMMA 6. *For $p\in F^\times$, we have $I_3(f;p;\chi,\psi)=I_2(\tilde f;1/p;\chi^{-1},\bar\psi)$.*

Proof. Since

$$f\left(z\begin{pmatrix}1&0&x_3\\ &1&x_2\\ & &1\end{pmatrix}\begin{pmatrix} & p & \\ & & p\\ 1 & & \end{pmatrix}y\right)=\tilde f\left(z^{-1}w\left(w\,{}^t\begin{pmatrix}1&0&x_3\\ &1&x_2\\ & &1\end{pmatrix}^{-1}w\right)\right.$$

$$\left.\times\left(w\,{}^t\begin{pmatrix} & p & \\ & & p\\ 1 & & \end{pmatrix}^{-1}w\right)(w\,{}^ty^{-1}w)w\right)$$

and $\tilde f$ is bi-invariant under K_F, we have

$$I_3(f;p;\chi,\psi)$$

$$=\int_{(F)^2}\psi(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_3\int_{(F)^3}\int_{Z_F}\tilde f\left(z^{-1}\begin{pmatrix}1&-x_2&-x_3\\ &1&0\\ & &1\end{pmatrix}\begin{pmatrix} & & 1\\ 1/p & & \\ & 1/p & \end{pmatrix}\right.$$

$$\left.\times\begin{pmatrix}1&-y_2&y_1y_2-y_3\\ &1&-y_1\\ & &1\end{pmatrix}\right)\psi(y_1+y_2)\zeta\chi(z)\,\mathrm{d}y\,\mathrm{d}^\times z.$$

Changing variables we get the lemma.

LEMMA 7. *For $p\in F^\times$, we have $J_3(f';p;\chi,\psi_E)=J_2(\tilde f';1/p;\chi^{-1},\bar\psi_E)$.*

Proof. Similar to the proof of Lemma 6.

From $\Phi_{\tilde f}(\lambda_1,\lambda_2,\lambda_3)=\Phi_f(-\lambda_3,-\lambda_2,-\lambda_1)$ and $\Phi_{\tilde f'}(\lambda_1,\lambda_2,\lambda_3)=\Phi_{f'}(-\lambda_3,-\lambda_2,-\lambda_1)$, we know that $\tilde f=b(\tilde f')$ if and only if $f=b(f')$, by (1) and (2) in Section 2. Consequently $I_2(\tilde f;1/p;\chi^{-1},\bar\psi)=$

$\zeta(p)J_2(\tilde{f}'; 1/p; \chi^{-1}, \bar{\psi}_E)$ for $p \in F^\times$, according to Section 5. By Lemmas 6 and 7 we get $I_3(f; p; \chi, \psi) = \zeta(p)J_3(f'; p; \chi, \psi_E)$.

7. Reduction formulas for $I(f)$ and $J(f')$

Recall that the principal orbital integrals $I(f)$ and $J(f')$ are matched in [J-Y2] for unit elements f and f' of Hecke algebras. Although it is not clear whether we can use the techniques in [J-Y2] to match $I(f)$ and $J(f')$ for general functions f and f', we may use the results in [J-Y2] to simplify our calculation of $I(f)$ and $J(f')$ for spherical functions.

Let f_0' be a bi-K_E-invariant function of compact support in $Z_E K_E$. By Lemma 1, $\Psi_{f_0'}(\lambda) = f_0'(z)$ if $\lambda = z$ for some $z \in \mathbf{Z}$, and $\Psi_{f_0'}(\lambda) = 0$ otherwise. By the corollary of Lemma 3

$$\sum_{e_1,e_2,e_3=0,1} (-1)^{e_1+e_2+e_3} q_F^{e_1+e_2+2e_3} \Phi_{f_0'}(\lambda + (-e_1 - e_3, e_1 - e_2, e_2 + e_3))$$

$$= f_0'(z) \text{ if } \lambda = z \text{ for some } z \in \mathbf{Z},$$

$$= 0 \quad \text{otherwise.}$$

Because of the compactness of support of f_0', these equations can be solved uniquely for $\Phi_{f_0'} : \Phi_{f_0'}(\lambda) = f_0'(z)$ if $\lambda = z$ for some $z \in \mathbf{Z}$, and $\Phi_{f'}(\lambda) = 0$ otherwise.

Now denote the image of f_0' under the base change map b by $f_0 = b(f_0')$. Then by (1) and (2) in Section 2, $\Phi_{f_0}(\lambda) = f_0'(z)$ if $\lambda = 2z$ for some $z \in \mathbf{Z}$, and $\Phi_{f_0}(\lambda) = 0$ otherwise. By the corollary of Lemma 3 again, $\Psi_{f_0}(\lambda) = f_0'(z)$ if $\lambda = 2z$ for some $z \in \mathbf{Z}$, and $\Psi_{f_0}(\lambda) = 0$ otherwise. Hence we get a set of equations from Lemma 1:

$$\sum_{e_1,e_2,e_3=0,1} (-1)^{e_1+e_2+e_3} q_F^{e_3} f_0(\lambda + (-e_1 - e_3, e_1 - e_2, e_2 + e_3))$$

$$= f_0'(z) \text{ if } \lambda = 2z \text{ for some } z \in \mathbf{Z},$$

$$= 0 \quad \text{otherwise.}$$

Consequently $f_0(\lambda) = f_0'(z)$ if $\lambda = 2z$ for some $z \in \mathbf{Z}$, and $f_0(\lambda) = 0$ otherwise, because f_0 is also compactly supported.

The argument in [J-Y2] actually implies that $I(f_0; p, q) = \zeta(q)J(f_0'; p, q)$ for the above functions f_0 and f_0'. Thus when we match $I(f)$ and $J(f')$ for spherical functions f and f' with $f = b(f')$, we may select such functions f_0 and f_0' so that $f_0 = b(f_0')$ and $\Psi_{f-f_0}(2z) = 0$ for every $z \in \mathbf{Z}$. Since $f = b(f')$

and $f_0 = b(f_0')$ imply that $f - f_0 = b(f' - f_0')$, we have $\Psi_{f-f_0}(z) = 0$ for any $z \in \mathbf{Z}$ and it is necessary to match $I(f - f_0)$ and $J(f' - f_0')$ in order to match $I(f)$ and $J(f')$.

According to Theorem 2, $\Psi_{f-f_0}(2z) = 0$ if and only if $\Phi'_{f'-f_0'}(z) - q_F^4 \times \Phi'_{f'-f_0'}(z + (-1, 0, 1)) = 0$. Therefore we will assume without loss of generality that the spherical functions $f \in \mathscr{H}$ and $f' \in \mathscr{H}_E$ satisfy $f = b(f')$, $\Psi_f(z) = 0$, and $\Phi'_{f'}(z) = q_F^4 \Phi'_{f'}(z + (-1, 0, 1))$, for all $z \in \mathbf{Z}$, and devote the rest of the paper to the identity between principal orbital integrals $I(f)$ and $J(f')$ for such functions f and f'.

8. The orbital integral $I(f)$

First let us look at a theorem which gives the orbital integral $I(f)$ in terms of Ψ_f. Since its proof is quite long, we will prove it in Section 11.

THEOREM 3. *Suppose $f \in \mathscr{H}$ satisfies $\Psi_f(z) = 0$ for $z \in \mathbf{Z}$. Then*

$$
\begin{aligned}
&I(f; p, q) \\
&= \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z)[\Psi_f(z + (0, Q, P + Q)) \\
&\quad - \Psi_f(z + (-1, Q + 1, P + Q)) - \Psi_f(z + (0, Q - 1, P + Q + 1)) \\
&\quad + \Psi_f(z + (-1, Q, P + Q + 1))(1 - q_F + q_F|_{\text{if}Q=-1} + q_F|_{\text{if}P=-1}) \\
&\quad + q_F\Psi_f(z + (-2, Q + 1, P + Q + 1)) \\
&\quad + q_F\Psi_f(z + (-1, Q - 1, P + Q + 2)) \\
&\quad - q_F\Psi_f(z + (-2, Q, P + Q + 2))] \quad \text{if } P = \mathrm{ord}_F(p) \geqslant -1, Q = \mathrm{ord}_F(q) \geqslant -1; \\
&= \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z)\left[\Psi_f\left(z + \left(\frac{Q}{2}, \frac{Q}{2}, P + Q\right)\right)\right. \\
&\quad - q_F\Psi_f\left(z + \left(\frac{Q}{2} - 1, \frac{Q}{2}, P + Q + 1\right)\right) \\
&\quad \left. + q_F\Psi_f\left(z + \left(\frac{Q}{2} - 1, \frac{Q}{2} - 1, P + Q + 2\right)\right)\right] \int_{\varpi_F^{Q/2} R_F^\times} \psi\left(x - \frac{q}{x}\right) \mathrm{d}x \\
&\quad \text{if } Q < -1, Q \equiv 0 (\mathrm{mod}\, 2), P + \frac{Q}{2} \geqslant -2; \\
&= \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z)\left[\Psi_f\left(z + \left(0, \frac{P}{2} + Q, \frac{P}{2} + Q\right)\right)\right.
\end{aligned}
$$

$$- q_F\Psi_f\left(z + \left(-1, \frac{P}{2} + Q, \frac{P}{2} + Q + 1\right)\right)$$

$$+ q_F\Psi_f\left(z + \left(-2, \frac{P}{2} + Q + 1, \frac{P}{2} + Q + 1\right)\right)\Bigg]$$

$$\times \int_{\varpi_F^{P/2} R_F^\times} \psi\left(x - \frac{p}{x}\right) \mathrm{d}x$$

if $P < -1, P \equiv 0 (\mathrm{mod}\, 2), \dfrac{P}{2} + Q \geqslant -2$;

$= 0$ otherwise.

We note that in Theorem 3 we do not assume $f = b(f')$, but from now on we will assume $f = b(f')$ and use Theorem 2 to rewrite $I(f; p, q)$ in terms of Φ_f.

THEOREM 4. *Suppose* $f \in \mathscr{H}$ *satisfies* $\Psi_f(z) = 0$ *for every* $z \in \mathbf{Z}$ *and* $f = b(f')$ *for some* $f' \in \mathscr{H}_E$. *Then*

$$I(f; p, q) = \zeta\chi(\varpi_F^{-P-Q}) \sum_{z \in \mathbf{Z}} \chi(\varpi_F^{2z}) \Bigg[\Phi'_{f'}\left(z + \left(-\frac{P+Q}{2}, -\frac{P}{2}, 0\right)\right)$$

$$+ q_F\Phi'_{f'}\left(z + \left(-\frac{P+Q}{2} - 1, -\frac{P}{2} + 1, 0\right)\right)$$

$$+ q_F\Phi'_{f'}\left(z + \left(-\frac{P+Q}{2}, -\frac{P}{2} - 1, 1\right)\right)$$

$$- q_F^4(1 + 2q_F^{-1} + 2q_F^{-2} + q_F^{-3})\Phi'_{f'}\left(z + \left(-\frac{P+Q}{2} - 1, -\frac{P}{2}, 1\right)\right)$$

$$+ q_F^4\Phi'_{f'}\left(z + \left(-\frac{P+Q}{2} - 2, -\frac{P}{2} + 1, 1\right)\right)$$

$$+ q_F^4\Phi'_{f'}\left(z + \left(-\frac{P+Q}{2} - 1, -\frac{P}{2} - 1, 2\right)\right)$$

$$+ q_F^5\Phi'_{f'}\left(z + \left(-\frac{P+Q}{2} - 2, -\frac{P}{2}, 2\right)\right)\Bigg]$$

if $P \geqslant 0, Q \geqslant 0, P \equiv Q \equiv 0 (\mathrm{mod}\, 2)$;

$$= \zeta\chi(\varpi_F^{-P-Q}) \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})$$

$$\times\Big[-(1+q_F)\Phi'_{f'}\Big(z+\Big(-\frac{P+Q+1}{2},-\frac{P-1}{2},0\Big)\Big)$$

$$+(1+q_F)q_F^2\Phi'_{f'}\Big(z+\Big(-\frac{P+Q+1}{2},-\frac{P+1}{2},1\Big)\Big)$$

$$+(1+q_F)q_F^2\Phi'_{f'}\Big(z+\Big(-\frac{P+Q+3}{2},-\frac{P-1}{2},1\Big)\Big)$$

$$-(1+q_F)q_F^4\Phi'_{f'}\Big(z+\Big(-\frac{P+Q+3}{2},-\frac{P+1}{2},2\Big)\Big)\Big]$$

if $P\geqslant -1$, $P\equiv 1(\mathrm{mod}\,2)$, $Q\geqslant 0$, $Q\equiv 0(\mathrm{mod}\,2)$;

$$= \zeta\chi(\varpi_F^{-P-Q}) \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})\Big[\Phi'_{f'}\Big(z+\Big(-\frac{P+Q}{2},-\frac{P}{4},-\frac{P}{4}\Big)\Big)$$

$$-(q_F^2+q_F^4)\Phi'_{f'}\Big(z+\Big(-\frac{P+Q}{2}-1,-\frac{P}{4},-\frac{P}{4}+1\Big)\Big)$$

$$+q_F^4\Phi'_{f'}\Big(z+\Big(-\frac{P+Q}{2}-2,-\frac{P}{4}+1,-\frac{P}{4}+1\Big)\Big)\Big]$$

$$\times\int_{\varpi_F^{P/2}R_F^\times}\psi\Big(x-\frac{p}{x}\Big)\,\mathrm{d}x$$

if $P<0$, $P\equiv 0(\mathrm{mod}\,4)$, $Q\equiv 0(\mathrm{mod}\,2)$, $\dfrac{P}{2}+Q\geqslant -2$;

$$= \zeta\chi(\varpi_F^{-P-Q}) \sum_{z\in\mathbf{Z}} \chi(\varpi_F^{2z})$$

$$\times\Big[-q_F\Phi'_{f'}\Big(z+\Big(-\frac{P+Q}{2},-\frac{P+2}{4},-\frac{P-2}{4}\Big)\Big)$$

$$+(q_F+q_F^2+q_F^3)\Phi'_{f'}\Big(z+\Big(-\frac{P+Q}{2}-1,-\frac{P-2}{4},-\frac{P-2}{4}\Big)\Big)$$

$$-q_F^4\Phi'_{f'}\Big(z+\Big(-\frac{P+Q}{2}-1,-\frac{P+2}{4},-\frac{P-6}{4}\Big)\Big)$$

$$- q_F^5\Phi'_{f'}\left(z + \left(-\frac{P+Q}{2} - 2, -\frac{P-2}{4}, -\frac{P-6}{4}\right)\right)\Bigg]$$

$$\times \int_{\varpi_F^{P/2}R_F^\times} \psi\left(x - \frac{p}{x}\right) \mathrm{d}x$$

if $P < 0$, $P \equiv 2 (\mathrm{mod}\, 4)$, $Q \equiv 0 (\mathrm{mod}\, 2)$, $\dfrac{P}{2} + Q \geqslant -1$;

$$= \zeta\chi(\varpi_F^{-P-Q}) \sum_{z \in \mathbf{Z}} \chi(\varpi_F^{2z}) \Bigg[\Phi'_{f'}\left(z + \left(-\frac{2P+Q}{4}, -\frac{2P+Q}{4}, 0\right)\right)$$

$$- (q_F^2 + q_F^4)\Phi'_{f'}\left(z + \left(-\frac{2P+Q}{4} - 1, -\frac{2P+Q}{4}, 1\right)\right)$$

$$+ q_F^4\Phi'_{f'}\left(z + \left(-\frac{2P+Q}{4} - 1, -\frac{2P+Q}{4} - 1, 2\right)\right)\Bigg]$$

$$\times \int_{\varpi_F^{Q/2}R_F^\times} \psi\left(x - \frac{q}{x}\right) \mathrm{d}x$$

if $Q < 0$, $Q \equiv 0 (\mathrm{mod}\, 2)$, $P + \dfrac{Q}{2} \geqslant -2$, $P + \dfrac{Q}{2} \equiv 0 (\mathrm{mod}\, 2)$;

$$= \zeta\chi(\varpi_F^{-P-Q}) \sum_{z \in \mathbf{Z}} \chi(\varpi_F^{2z})$$

$$\times \Bigg[- q_F\Phi'_{f'}\left(z + \left(-\frac{2P+Q+2}{4}, -\frac{2P+Q-2}{4}, 0\right)\right)$$

$$+ (q_F + q_F^2 + q_F^3)\Phi'_{f'}\left(z + \left(-\frac{2P+Q+2}{4}, -\frac{2P+Q+2}{4}, 1\right)\right)$$

$$- q_F^4\Phi'_{f'}\left(z + \left(-\frac{2P+Q+6}{4}, -\frac{2P+Q-2}{4}, 1\right)\right)$$

$$- q_F^5\Phi'_{f'}\left(z + \left(-\frac{2P+Q+6}{4}, -\frac{2P+Q+2}{4}, 2\right)\right)\Bigg]$$

$$\times \int_{\varpi_F^{Q/2}R_F^\times} \psi\left(x - \frac{q}{x}\right) \mathrm{d}x$$

$$\text{if } Q < 0,\ Q \equiv 0 (\mathrm{mod}\ 2),\ P + \frac{Q}{2} \geqslant -1,\ P + \frac{Q}{2} \equiv 1 (\mathrm{mod}\ 2);$$

$$= 0 \quad \text{for any other } P, Q \text{ with } Q \equiv 0 (\mathrm{mod}\ 2);$$

$$= -\chi(\varpi_F^{-Q}) \sum_{z \in \mathbf{Z}} \chi(\varpi_F^{2z}) \Big[(1 + q_F) \Phi'_{f'}\Big(z + \Big(-\frac{Q+1}{2}, 0, \frac{P+1}{2} \Big) \Big)$$

$$- (1 + q_F) q_F^2 \Phi'_{f'}\Big(z + \Big(-\frac{Q+1}{2}, -1, \frac{P+3}{2} \Big) \Big)$$

$$- (1 + q_F) q_F^2 \Phi'_{f'}\Big(z + \Big(-\frac{Q+3}{2}, 1, \frac{P+1}{2} \Big) \Big)$$

$$+ (1 + q_F) q_F^4 \Phi'_{f'}\Big(z + \Big(-\frac{Q+3}{2}, 0, \frac{P+3}{2} \Big) \Big) \Big]$$

$$\text{if } P \geqslant -1,\ Q \geqslant -1,\ P \equiv Q \equiv 1 (\mathrm{mod}\ 2);$$

$$= 0 \quad \text{for any other } P, Q \text{ with } P \equiv Q \equiv 1 (\mathrm{mod}\ 2).$$

Note that Theorem 4 does not cover the case of $P \equiv 0 (\mathrm{mod}\ 2)$, $Q \equiv 1 (\mathrm{mod}\ 2)$, because it can be deduced from other cases. Please see the remark before Theorem 5 in Section 9.

Proof. We will only prove the first and the last non-zero cases. The rest are all similar.

(i) Let $P \geqslant 0$, $Q \geqslant 0$, $P \equiv Q \equiv 0 (\mathrm{mod}\ 2)$. Since $(-1, Q, P + Q + 1) \equiv (1, 0, 1) (\mathrm{mod}\ 2)$, we have $\Psi_f(z + (-1, Q, P + Q + 1)) = 0$ for any $z \in \mathbf{Z}$.Thus

$$I(f; p, q) = \sum_{z \in \mathbf{Z}} \zeta \chi(\varpi_F^z) [\Psi_f(z + (0, Q, P + Q))$$

$$- \Psi_f(z + (0, Q - 1, P + Q + 1))|_{\text{if } Q > 0}$$

$$- \Psi_f(z + (-1, Q + 1, P + Q))|_{\text{if } P > 0}$$

$$+ q_F \Psi_f(z + (-2, Q + 1, P + Q + 1))$$

$$+ q_F \Psi_f(z + (-1, Q - 1, P + Q + 2))$$

$$- q_F \Psi_f(z + (-2, Q, P + Q + 2))].$$

Since $(0, Q, P + Q) \equiv (0, 0, 0) (\mathrm{mod}\ 2)$, we know $\Psi_f(z + (0, Q, P + Q)) = 0$ unless z is even. The same conclusion is true for the other five terms.

Therefore the sum above is actually taken over even $z \in \mathbf{Z}$. Changing the index from z to $2z - P - Q$, we get the formula in the theorem for $P \geqslant 0$, $Q \geqslant 0$, $P \equiv Q \equiv 0 (\mathrm{mod}\ 2)$ from Theorem 2.

(ii) Let $P \geqslant -1$, $Q \geqslant -1$, and $P \equiv Q \equiv 1 (\mathrm{mod}\ 2)$. Since $(0, Q, P+Q) \equiv (0, 1, 0) (\mathrm{mod}\ 2)$, we know $\Psi_f(z + (0, Q, P+Q)) = 0$ for every $z \in \mathbf{Z}$. Similarly $\Psi_f(z + (-2, Q, P+Q+2)) = 0$ for every $z \in \mathbf{Z}$. Thus

$$\begin{aligned} I(f; p, q) = \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^{2z+1})[&-\Psi_f(2z+1+(0, Q-1, P+Q+1)) \\ &- \Psi_f(2z+1+(-1, Q+1, P+Q)) \\ &+ \Psi_f(2z+1+(-1, Q, P+Q+1)) \\ &\times (1 - q_F + q_F|_{\mathrm{if} Q=-1} + q_F|_{\mathrm{if} P=-1}) \\ &+ q_F\Psi_f(2z+1+(-2, Q+1, P+Q+1)) \\ &+ q_F\Psi_f(2z+1+(-1, Q-1, P+Q+2))]. \end{aligned}$$

Changing index from z to $z - (Q+1)/2$ and applying Theorem 2, we get

$$\begin{aligned} I(f; p, q) = -\chi^{-1}(q) \sum_{z \in \mathbf{Z}} \chi(\varpi_F^{2z}) \\ \times \Bigg[&(1+q_F)\Phi'_{f'}\left(z + \left(-\frac{Q+1}{2}, 0, \frac{P+1}{2}\right)\right) \\ &- (1+q_F)q_F^2\Phi'_{f'}\left(z + \left(-\frac{Q+1}{2}, -1, \frac{P+3}{2}\right)\right)\Bigg|_{\mathrm{if}\ Q>0} \\ &- (1+q_F)q_F^2\Phi'_{f'}\left(z + \left(-\frac{Q+3}{2}, 1, \frac{P+1}{2}\right)\right)\Bigg|_{\mathrm{if}\ P>0} \\ &+ (1+q_F)q_F^4\Phi'_{f'}\left(z + \left(-\frac{Q+3}{2}, 0, \frac{P+3}{2}\right)\right)\Bigg|_{\mathrm{if}\ P>0,\ Q>0} \\ &-(1+q_F)q_F^4\Phi'_{f'}\left(z + \left(-\frac{Q+3}{2}, 0, \frac{P+3}{2}\right)\right)\Bigg|_{\mathrm{if}\ P=Q=-1}\Bigg]. \end{aligned}$$

From this expression we can easily see that the last non-zero part of the theorem is valid when $P > 0$, $Q > 0$. When $P = Q = -1$, we note that $(-Q-3, 0, P+3) = (-2, 0, 2)$ and $q_F^4\Phi_f(2z + (-Q-3, 0, P+3)) = q_F^2\Phi_f(2z + (-Q-1, -2, P+3)) = q_F^2\Phi_f(2z + (-Q-3, 2, P+1))$, because they are essentially Satake coefficients which are independent of the order of their entries. Thus the theorem is valid for $P = Q = -1$. When $P = -1$, $Q > 0$ we point out that the two missing terms cancel each other: $-(1+q_F)q_F^2\Phi_f(2z+(-Q-3,2,P+1)) + (1+q_F)q_F^4\Psi_f(2z+(-Q-3,0,P+3)) = 0$, and hence the theorem holds in this case. Similarly from $-(1+q_F)q_F^2\Phi_f(2z+(-Q-1,-2,P+3)) + (1+q_F)q_F^4\Phi_f(2z+(-Q-3,0,\mathrm{P}+3)) = 0$ when $P>0$, $Q=-1$, we get the theorem for $P>0$, $Q=-1$.

9. The orbital integral $J(f')$

Recall from Section 2 that $J(f';p,q) = \int \Omega({}^t\bar{n}zan)\chi(z)\theta_E(n)\,\mathrm{d}n\,\mathrm{d}^{\times}z$, where $n \in N_E$, $z \in Z_F$ and $a = \mathrm{diag}(1,q,pq)$. Since $\Omega(x) = 0$ unless $\det(x) \in N(E^{\times})$, the integral with respect to z is actually taken over those z with $z\det(a) \in N(E^{\times})$. By $\det a = pq^2$ we may change variables and get $J(f';p,q) = \chi^{-1}(pq^2)\int \Omega({}^t\bar{n}z\bar{z}bn)\chi\circ N(2z)\theta_E(n)\,\mathrm{d}n\,\mathrm{d}^{\times}z$, where $n \in N_E$, $z \in Z_E$ and $b = \mathrm{diag}(1/(pq^2),\ 1/(pq),\ 1/q)$. According to the definition of the function Ω we can write the orbital integral in terms of f':

$$J(f';p,q) = \chi^{-1}(pq^2)\int_{H_F}\int_{N_E}\int_{Z_E} \times f(zhvn)\chi\circ N(z)\theta_E(n)\,\mathrm{d}h\,\mathrm{d}n\,\mathrm{d}^{\times}z$$

where v is a matrix such that ${}^t\bar{v}\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}v = b$.

The choice of v depends on p and q. When $Q = \mathrm{ord}(q)$ is even, we may choose

$$v = \frac{1}{\alpha_1}\begin{pmatrix} 1/\alpha & 1 & \\ -1/(2\alpha p) & 1/(2p) & \\ & & 1 \end{pmatrix}$$

where α, $\alpha_1 \in E^{\times}$ with $\alpha\bar{\alpha} = -q$, $\alpha_1\bar{\alpha}_1 = q$. Hence $J(f';p,q) = \chi^{-1}(pq)J^{[1]}(f';p,q)$ if $Q \equiv 0 \pmod 2$, where

$$J^{[1]}(f';p,q) = \int_{\alpha\bar{\alpha}=-q}\mathrm{d}\alpha\int_{H_F}\int_{N_E}\int_{Z_E} \times f'\left(zh\begin{pmatrix} 1/\alpha & 1 & \\ -1/(2\alpha p) & 1/(2p) & \\ & & 1 \end{pmatrix}n\right)\chi\circ N(z)\theta_E(n)\,\mathrm{d}h\,\mathrm{d}n\,\mathrm{d}^{\times}z.$$

Here the measure $\mathrm{d}\alpha$ is normalized so that $\int_{\alpha\bar{\alpha}=-q}\mathrm{d}\alpha = 1$.

When $P = \text{ord}(p)$ is even, we may set

$$v = \frac{1}{\beta_1 q}\begin{pmatrix} \beta & 1 & \\ -\beta q/2 & q/2 & \\ & & 1 \end{pmatrix}\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

where β, $\beta_1 \in E^\times$ with $\beta\bar{\beta} = -p$ and $\beta_1\bar{\beta}_1 = p$. Thus $J(f';p,q) = J^{[2]}(f';p,q)$ if $P \equiv 0 (\text{mod } 2)$, where

$$J^{[2]}(f';p,q) = \int_{\beta\bar{\beta}=-p} \mathrm{d}\beta \int_{H_F}\int_{N_E}\int_{Z_E}$$

$$\times f'\left(zh\begin{pmatrix} \beta & 1 & \\ -\beta q/2 & q/2 & \\ & & 1 \end{pmatrix}\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} n \right)\chi \circ N(z)\theta_E(n)\,\mathrm{d}h\,\mathrm{d}n\,\mathrm{d}^\times z$$

with $\int_{\beta\bar{\beta}=-p} \mathrm{d}\beta = 1$.

Finally when $P \equiv Q (\text{mod } 2)$ we use

$$v = \frac{1}{\gamma_1}\begin{pmatrix} 1/\gamma & 1 & \\ -p/(2\gamma) & p/2 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

where $\gamma, \gamma_1 \in E^\times$ such that $\gamma\bar{\gamma} = -pq$, $\gamma_1\bar{\gamma}_1 = pq$. Consequently we have $J(f';p,q) = \chi^{-1}(q)J^{[3]}(f';p,q)$ if $P \equiv Q (\text{mod } 2)$, where

$$J^{[3]}(f';p,q) = \int_{\gamma\bar{\gamma}=-pq} \mathrm{d}\gamma \int_{H_F}\int_{N_E}\int_{Z_E}$$

$$\times f'\left(zh\begin{pmatrix} 1/\gamma & 0 & 1 \\ -p/(2\gamma) & 0 & p/2 \\ 0 & 1 & 0 \end{pmatrix} n \right)\chi \circ N(z)\theta_E(n)\,\mathrm{d}h\,\mathrm{d}n\,\mathrm{d}^\times z$$

with $\int_{\gamma\bar{\gamma}=-pq} \mathrm{d}\gamma = 1$.

Therefore we need to show that $\zeta(q)\chi^{-1}(pq)J^{[1]}(f';p,q) = I(f;p,q)$ when $Q \equiv 0 (\text{mod } 2)$; $\zeta(q)J^{[2]}(f';p,q) = I(f;p,q)$ when $P \equiv 0 (\text{mod } 2)$, $Q \equiv 1 (\text{mod } 2)$; and $\zeta(q)\chi^{-1}(q)J^{[3]}(f';p,q) = I(f;p,q)$ when $P \equiv Q \equiv 1$ (mod 2). The second case, however, can be deduced from the first, by exactly the same argument as we used in Section 6. Consequently what we need to do is to calculate $J^{[1]}(f')$ and $J^{[3]}(f')$.

The computations of $J^{[1]}(f')$ and $J^{[3]}(f')$ are quite similar but the integral $J^{[1]}(f')$ is much more complicated than $J^{[3]}(f')$. By this reason we will write a theorem on $J^{[3]}(f')$ below without proof and give the detailed computation of $J^{[1]}(f')$ in Sections 10 and 12.

THEOREM 5. *Suppose* $f' \in \mathcal{H}_E$ *satisfies* $\Phi'_{f'}(z) = q_F^4 \Phi'_{f'}(z + (-1, 0, 1))$ *for any* $z \in \mathbf{Z}$. *Assume* $P \equiv Q \equiv 1 (\bmod 2)$. *Then*

$$\begin{aligned} J^{[3]}(f'; p, q) = \sum_{z \in \mathbf{Z}} \chi(\varpi_F^{2z}) \Big[& \Phi'_{f'}\Big(z + \Big(-\frac{Q+1}{2}, 0, \frac{P+1}{2}\Big)\Big) \\ & - q_F^2 \Phi'_{f'}\Big(z + \Big(-\frac{Q+1}{2} - 1, 1, \frac{P+1}{2}\Big)\Big) \\ & - q_F^2 \Phi'_{f'}\Big(z + \Big(-\frac{Q+1}{2}, -1, \frac{P+1}{2} + 1\Big)\Big) \\ & + q_F^4 \Phi'_{f'}\Big(z + \Big(-\frac{Q+3}{2}, 0, \frac{P+3}{2}\Big)\Big)\Big](1+q) \\ & \text{if } P \geqslant -1, Q \geqslant -1; \\ = 0 \quad & \text{otherwise.} \end{aligned}$$

Comparing Theorem 5 with the last non-zero part of Theorem 4 in Section 8, we conclude that $\zeta(p)\chi^{-1}(q) J^{[3]}(f'; p, q) = I(f; p, q)$ when $P \equiv Q \equiv 1 (\bmod 2)$.

10. The orbital integral $J^{[1]}(f')$

In this section we always assume $Q \equiv 0 (\bmod 2)$. Using the Iwasawa decomposition of H_F we may specify a Haar measure on H_F and write

$$J^{[1]}(f'; p, q) = \int_{\alpha\bar{\alpha} = -q} \mathrm{d}\alpha \int_{F^\times} \int_E \int_F |b|^4 \, \mathrm{d}^\times b \, \mathrm{d}x \, \mathrm{d}\lambda$$

$$\times \int_{N_E} \int_{Z_E} f'\left(z \begin{pmatrix} b & & \\ & 1/b & \\ & & \cdot\cdot \end{pmatrix} \begin{pmatrix} 1 & -2x\bar{x} + 2\lambda\sqrt{\tau} & 2x \\ & 1 & 0 \\ & -2\bar{x} & 1 \end{pmatrix} \right.$$

$$\times \begin{pmatrix} 1/\alpha & 1 & \\ -1/(2\alpha p) & 1/(2p) & \\ & & 1 \end{pmatrix} n \Bigg) \chi \circ N(z)\theta_E(n)\, \mathrm{d}n\, \mathrm{d}^\times z.$$

Let $B = \operatorname{ord}(b)$, $X = \operatorname{ord}(x)$ and $\Lambda = \operatorname{ord}(\lambda)$. Then there are four cases:

$$f'\left(z\begin{pmatrix} b & & \\ & 1/b & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & -2x\bar{x}+2\lambda\sqrt{\tau} & 2x \\ & 1 & 0 \\ & -2\bar{x} & 1 \end{pmatrix}\begin{pmatrix} 1/\alpha & 1 & \\ -1/(2\alpha p) & 1/(2p) & \\ & & 1 \end{pmatrix} n\right)$$

$$= f'\left(z\begin{pmatrix} 1/(2\alpha pb) & 0 & 0 \\ & 2b & 2bx \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & -\alpha & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} n\right)$$

if $-B \leqslant X$, $-B \leqslant B + \min(\operatorname{ord}(p + x\bar{x}), \Lambda)$;

$$= f'\left(z\begin{pmatrix} \bar{x}/(\alpha p) & 0 & 1 \\ & 2b & b(-p+x\bar{x}+\lambda\sqrt{\tau})/\bar{x} \\ & & 1/(b\bar{x}) \end{pmatrix}\begin{pmatrix} 1 & -\alpha & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} n\right)$$

if $X < -B$, $X \leqslant B + \min(\operatorname{ord}(p + x\bar{x}), \Lambda)$;

$$= f'\left(z\begin{pmatrix} b(p+x\bar{x}-\lambda\sqrt{\tau})/(\alpha p) & 2b & 2bx \\ & 1/[b(p+x\bar{x}-\lambda\sqrt{\tau})] & x/[b(p+x\bar{x}-\lambda\sqrt{\tau})] \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & -\alpha & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} n\right)$$

if $B + \min(\operatorname{ord}(p + x\bar{x}), \Lambda) < \min(-B, X)$, $B + X \geqslant 0$;

$$= f'\left(z\begin{pmatrix} b(p+x\bar{x}-\lambda\sqrt{\tau})/(\alpha p) & 2b & 2bx \\ & -2\bar{x}/(p+x\bar{x}-\lambda\sqrt{\tau}) & (p-x\bar{x}-\lambda\sqrt{\tau})/(p+x\bar{x}-\lambda\sqrt{\tau}) \\ & & 1/(b\bar{x}) \end{pmatrix}\right.$$

$$\left.\times \begin{pmatrix} 1 & -\alpha & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} n\right)$$

if $B + \min(\operatorname{ord}(p + x\bar{x}), \Lambda) < \min(-B, X)$, $B + X < 0$.

We will denote by $J^{[1](1)}(f'), \ldots, J^{[1](4)}(f')$ the integrals corresponding to these cases. Thus $J^{[1]}(f') = J^{[1](1)}(f') + \cdots + J^{[1](4)}(f')$.

THEOREM 6. *Assume $f' \in \mathscr{H}_E$ satisfies $\Phi'_{f'}(z) = q^4\Phi'_{f'}(z + (-1, 0, 1))$ for all $z \in \mathbf{Z}$. Then*

$$J^{[1](1)}(f'; p, q) = \sum_{\substack{z\in\mathbf{Z}, B\geqslant -P/2, \\ -(2P+Q)/4\leqslant B\leqslant 0}} \chi(\varpi_F^{2z})$$

$$\times \Psi'_{f'}(z + (-P - (Q/2) - B, B, 0)) \int_{\alpha\bar{\alpha}=-q} \psi_E(\alpha)\, \mathrm{d}\alpha$$

$$\text{if } P \geqslant 0,\ P + (Q/2) \geqslant 0;$$

$$= 0 \text{ otherwise};$$

$$J^{[1](2)}(f'; p, q)$$

$$= \sum_{\substack{z \in \mathbf{Z}, \\ -(P+Q)/2 \leqslant B \leqslant -(P/4)}} \chi(\varpi_F^{2z})$$

$$\times \Psi'_{f'}(z + (-(P+Q)/2, B, -(P/2) - B)) q_F^{-2B-(P/2)}$$

$$\times (-1)^{P/2} \int_{\varpi_F^{P/2} R_F^\times} \psi\left(x - \frac{p}{x}\right) \mathrm{d}x$$

$$\text{if } P < 0,\ P \equiv 0 (\mathrm{mod}\ 2),\ (P/2) + Q \geqslant 0;$$

$$= \sum_{\substack{z \in \mathbf{Z}, \\ -(P+Q)/2 \leqslant B < -P/2}} \chi(\varpi_F^{2z})$$

$$\times \Psi'_{f'}(z + (-(P+Q)/2, B, -(P/2) - B)) q_F^{-2B-P}(1 + q_F^{-1})$$

$$\text{if } P \geqslant 0,\ P \equiv 0 (\mathrm{mod}\ 2),\ Q > 0;$$

$$= 0 \text{ otherwise};$$

$$J^{[1](3)}(f'; p, q)$$

$$= \sum_{\substack{z \in \mathbf{Z}, \\ 0 \leqslant L \leqslant (P-1)/2}} \chi(\varpi_F^{2z}) \Psi'_{f'}\left(z + \left(-P - \frac{Q}{2} + L, -L, 0\right)\right)$$

$$\times ((1 + q_F^{-1}) q_F^{-2L+P} - 1) \qquad \text{if } P > 0,\ Q \geqslant 0;$$

$$= \sum_{\substack{z \in \mathbf{Z}, \\ 0 \leqslant L \leqslant (2P+Q)/4}} \chi(\varpi_F^{2z}) \Psi'_{f'}\left(z + \left(-P - \frac{Q}{2} + L, -L, 0\right)\right)$$

$$\times ((1 + q_F^{-1}) q_F^{-2L+P} - 1) \int_{\alpha\bar{\alpha} = -q} \psi_E(\alpha)\, \mathrm{d}\alpha \qquad \text{if } Q < 0,\ P + \frac{Q}{2} \geqslant 0;$$

$$= 0 \text{ otherwise};$$

$$J^{[1](4)}(f'; p, q)$$

$$= \sum_{\substack{z \in \mathbf{Z}, k < 0, l - k \leqslant (P-1)/2, \\ 0 \leqslant l - k \leqslant (2P+Q)/4}} \chi(\varpi_F^{2z}) \Psi'_{f'}\left(z + \left(-P - \frac{Q}{2} + l, k - l, -k\right)\right)(1 - q_F^{-2})$$

$$\times\, (q_F^{-2l-2k+P}(1+q_F^{-1}) - q_F^{-4k}) \int_{\alpha\bar{\alpha}=-q} \psi_E(\alpha)\, \mathrm{d}\alpha \tag{8}$$

if $P > 0, 2P + Q \geqslant 0$;

$$+ \sum_{z\in\mathbf{Z},k<0} \chi(\varpi_F^{2z})\Psi'_{f'}\left(z + \left(-P - \frac{Q}{2} - 1 + k, 1, -k\right)\right)$$

$$\times\, q_F^{-4k}(q_F^{-2} - q_F^{P}(1+q_F^{-1})) \int_{\alpha\bar{\alpha}=-q} \psi_E(\alpha)\, \mathrm{d}\alpha \tag{9}$$

if $P \geqslant -1, P + \frac{Q}{2} \geqslant -2$;

$$+ \sum_{z\in\mathbf{Z},k<0} \chi(\varpi_F^{2z})\Psi'_{f'}\left(z + \left(-\frac{2P+Q-2}{4} + k, -\frac{2P+Q+2}{4}, -k\right)\right)$$

$$\times\, q_F^{-4k}(1+q_F^{-1})(q_F^{-1} - q_F^{-(Q/2)-2}(1+q_F^{-1})) \int_{\alpha\bar{\alpha}=-q} \psi_E(\alpha)\, \mathrm{d}\alpha \tag{10}$$

if $Q \leqslant -4, P + \frac{Q}{2} \geqslant -1, P + \frac{Q}{2} \equiv 1 (\mathrm{mod}\, 2)$;

$$+ \sum_{\substack{z\in\mathbf{Z},l<P/2,\\ P/2<l-k\leqslant(P+Q)/2}} \chi(\varpi_F^{2z})\Psi'_{f'}\left(z + \left(-P - \frac{Q}{2} + l, k-l, -k\right)\right) q_F^{-2k-2l+P}$$

$$\times\, (1-q_F^{-2})(1+q_F^{-1}) \qquad \text{if } P \geqslant 0, P \equiv 0 (\mathrm{mod}\, 2), Q \geqslant 0; \tag{11}$$

$$- \sum_{z\in\mathbf{Z},l<P/2} \chi(\varpi_F^{2z})\Psi'_{f'}\left(z + \left(-P - \frac{Q}{2} + l, -\frac{P+Q}{2} - 1, \frac{P+Q}{2} + 1 - l\right)\right)$$

$$\times\, q_F^{-4l+2P+Q}(1+q_F^{-1}) \qquad \text{if } P \geqslant 0, Q \geqslant 0, P \equiv 0 (\mathrm{mod}\, 2); \tag{12}$$

$$+ \sum_{\substack{z\in\mathbf{Z},l<P/2,\\ (P-2)/4\leqslant l-k\leqslant(P+Q)/2}} \chi(\varpi_F^{2z})\Psi'_{f'}\left(z + \left(-P - \frac{Q}{2} + l, k-l, -k\right)\right)$$

$$\times\, q_F^{-2l-2k+P}(1-q_F^{-2})(-1)^{P/2} q_F^{P/2} \int_{\varpi_F^{P/2}R_F^\times} \psi\left(x - \frac{p}{x}\right) \mathrm{d}x \tag{13}$$

if $P < 0, P \equiv 0 (\mathrm{mod}\, 2), \frac{P}{2} + Q \geqslant 0$;

$$- \sum_{\substack{z\in\mathbf{Z},\\ k\leqslant(P-2)/4}} \chi(\varpi_F^{2z})\Psi'_{f'}\left(z + \left(-P - \frac{Q}{2} + \frac{P-2}{4} + k, -\frac{P-2}{4}, -k\right)\right)$$

$$\times\, q_F^{-4k+P}(1+q_F^{-1})(-1)^{P/2}\int_{\varpi_F^{P/2}R_F^{\times}}\psi\!\left(x-\frac{p}{x}\right)\mathrm{d}x \tag{14}$$

if $P<0,\ P\equiv 2(\mathrm{mod}\,4),\ \frac{P}{2}+Q\geqslant -1$;

$$-\sum_{\substack{z\in\mathbf{Z}\\ k\leqslant -(Q/2)-2}}\chi(\varpi_F^{2z})\Psi'_{f'}\left(z+\left(-\frac{P}{2}+1+k,\,-\frac{P+Q}{2}-1,\,-k\right)\right)$$

$$\times\, q_F^{-4k-Q-4}(-1)^{P/2}q_F^{P/2}\int_{\varpi_F^{P/2}R_F^{\times}}\psi\!\left(x-\frac{p}{x}\right)\mathrm{d}x \tag{15}$$

if $P<0,\ P\equiv(\mathrm{mod}\,2),\ \frac{P}{2}+Q\geqslant -2$;

$$+\sum_{\substack{z\in\mathbf{Z},k<0,\\ 0\leqslant l-k\leqslant(2P+Q)/4}}\chi(\varpi_F^{2z})\Psi'_{f'}\left(z+\left(-P-\frac{Q}{2}+l,\,k-l,\,-k\right)\right)$$

$$\times\, q_F^{-4k}(1-q_F^{-2})\int_{\alpha\bar\alpha=-q}\psi_E(\alpha)\,\mathrm{d}\alpha \qquad \text{if } P>0,\ 2P+Q\geqslant 0; \tag{16}$$

$$-\sum_{z\in\mathbf{Z},k<0}\chi(\varpi_F^{2z})\Psi'_{f'}\left(z+\left(-P-\frac{Q}{2}-1+k,\,1,\,-k\right)\right)q_F^{-4k-2}$$

$$\times\int_{\alpha\bar\alpha=-q}\psi_E(\alpha)\,\mathrm{d}\alpha \qquad \text{if } P\geqslant -1,\ 2P+Q\geqslant -4; \tag{17}$$

$$-\sum_{z\in\mathbf{Z},k<0}\chi(\varpi_F^{2z})\Psi'_{f'}\left(z+\left(-\frac{2P+Q-2}{4}+k,\,-\frac{2P+Q+2}{4},\,-k\right)\right)$$

$$\times\, q_F^{-4k-1}(1+q_F^{-1})\int_{\alpha\bar\alpha=-q}\psi_E(\alpha)\,\mathrm{d}\alpha \tag{18}$$

if $P>0,\ 2P+Q\geqslant -2,\ 2P+Q\equiv 2(\mathrm{mod}\,4),\ Q\leqslant -4$;

$$+\sum_{z\in\mathbf{Z},k<0}\chi(\varpi_F^{2z})\Psi'_{f'}\left(z+\left(-\frac{P+Q}{2}+k,\,-\frac{P}{2},\,-k\right)\right)$$

$$\times\, q_F^{-4k}(1-2q_F^{-2}-q_F^{-3}) \qquad \text{if } P\geqslant 0,\ Q\geqslant 0,\ P\equiv 0(\mathrm{mod}\,2); \tag{19}$$

$$-\sum_{z\in\mathbf{Z},k<0}\chi(\varpi_F^{2z})\Psi'_{f'}\left(z+\left(-\frac{P+Q}{2}+k,\,-\frac{P}{2},\,-k\right)\right)q_F^{-4k-1}(1+q_F^{-1})^2$$

$$\times \int_{\alpha\bar{\alpha}=-q} \psi_E(\alpha)\,\mathrm{d}\alpha \qquad \text{if } P \geqslant 0, P \equiv 0 (\mathrm{mod}\, 2), Q = -2; \tag{20}$$

$$- \sum_{z\in\mathbf{Z}, k<0} \chi(\varpi_F^{2z})\Psi'_{f'}\left(z + \left(-\frac{P+Q}{2} + k, -\frac{P}{2}, -k\right)\right) q_F^{-4k-1} q_F^{-2})$$

$$\times \int_{\varpi_F^{-1}R_F^\times} \psi\left(x - \frac{p}{x}\right)\mathrm{d}x \qquad \text{if } P = -2, Q \geqslant 0. \tag{21}$$

This theorem will be proved in Section 12.

We remark that the integral $\int_{\alpha\bar{\alpha}=-q} \psi_E(\alpha)\,\mathrm{d}\alpha$ in Theorem 6 was discussed in [Z], p. 24, and [Y], p. 92:

LEMMA 8. *When* $Q < 0$, $Q \equiv 0(\mathrm{mod}\ 2)$, *we have*

$$(1 + q_F^{-2}) \int_{\alpha\bar{\alpha}=-q} \psi_E(\alpha)\,\mathrm{d}\alpha$$

$$= (-1)^{Q/2} q_F^{Q/2} \int_{\varpi_F^{Q/2}R_F^\times} \psi\left(x - \frac{q}{x}\right)\mathrm{d}x.$$

We then observe that the expressions of $J^{[1](1)}(f'), \ldots, J^{[1](4)}(f')$ are all given in terms of infinite series of the forms $\Sigma_{k\leqslant 0} \Psi'_{f'}(\lambda + (k, 0, -k)) q_F^{-4k}$ and $\Sigma_{l\leqslant 0} \Psi'_{f'}(\lambda + (l, -l, 0)) q_F^{-2l}$. By Lemma 4 in Section 4 we can rewrite the first kind of infinite series as finite sums of $\Phi'_{f'}$. The second kind of infinite series is given by the following lemma.

LEMMA 9. *For* $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *with* $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ *we have*

$$\sum_{\lambda_2-\lambda_3\leqslant l\leqslant 0} q^{-2l}\Psi'_{f'}(\lambda + (l, -l, 0))$$

$$= \Phi'_{f'}(\lambda) - q_F^2 \Phi'_{f'}(\lambda + (0, -1, 1))$$

$$- q_F^4 \Phi'_{f'}(\lambda + (-1, 0, 1)) + q_F^6 \Phi'_{f'}(\lambda + (-1, -1, 2))$$

$$+ q_F^{2\lambda_3-2\lambda_2+6} \Phi'_{f'}(\lambda_1 + \lambda_2 - \lambda_3 - 2, \lambda_3 + 1, \lambda_3 + 1)$$

$$- q_F^{2\lambda_3-2\lambda_2+8} \Phi'_{f'}(\lambda_1 + \lambda_2 - \lambda_3 - 2, \lambda_3, \lambda_3 + 2).$$

Proof. By (5) in Section 3.

Now we can apply Lemmas 4 and 9 to each sum in Theorem 6 and rewrite $J^{[1]}(f')$ in terms of $\Phi'_{f'}$. Since there are so many cases, the computation is rather lengthy, but it is very similar to the proof of Theorem 4 and hence we will not give the detail here. If we collect the results, we see that

$\zeta(q)\chi^{-1}(pq)J^{[1]}(f';p,q)$ has exactly the same expressions as the integral $I(f;p,q)$ in Theorem 4 under the condition $Q \equiv 0 (\mathrm{mod}\ 2)$. Therefore the identity $I(f;p,q) = \zeta(q)J(f';p,q)$ is proved for $Q \equiv 0 (\mathrm{mod}\ 2)$ save the proof of Theorems 3 and 6.

11. The proof of Theorem 3

Recall from Section 2 that

$$I(f;p,q) = \int_{(F)^3}\int_{N_F}\int_{Z_F} f\left(z\begin{pmatrix}1 & x_1 & x_3\\ & 1 & x_2\\ & & 1\end{pmatrix}\begin{pmatrix} & & pq\\ & q & \\ 1 & & \end{pmatrix}y\right)$$

$$\times\, \psi(x_1+x_2)\theta(y)\zeta\chi(z)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}y\,\mathrm{d}^{\times}z.$$

We denote $P = \mathrm{ord}(p)$, $Q = \mathrm{ord}(q)$ and $X_i = \mathrm{ord}(x_i)$ for $i = 1, 2, 3$. Since the function f is bi-invariant under K_F, there are six cases:

$$f\left(z\begin{pmatrix}1 & x_1 & x_3\\ & 1 & x_2\\ & & 1\end{pmatrix}\begin{pmatrix} & & pq\\ & q & \\ 1 & & \end{pmatrix}y\right)$$

$$= f\left(z\begin{pmatrix}1 & & \\ & q & \\ & & pq\end{pmatrix}y\right) \qquad \text{if } X_1, X_2, X_3 \geqslant 0;$$

$$= f\left(z\begin{pmatrix}1 & 0 & 0\\ & x_1 q & pq\\ & & pq/x_1\end{pmatrix}y\right) \qquad \text{if } X_1 < 0, X_2, X_3 \geqslant 0;$$

$$= f\left(z\begin{pmatrix}x_2 & q & 0\\ & q/x_2 & 0\\ & & pq\end{pmatrix}y\right) \qquad \text{if } X_2 < 0, X_2 \leqslant X_3, \mathrm{ord}(x_1x_2 - x_3) \geqslant 0;$$

$$= f\left(z\begin{pmatrix}x_2 & q & 0\\ & q(x_1x_2 - x_3)/x_2 & pq\\ & & pq/(x_1x_2 - x_3)\end{pmatrix}y\right)$$

if $X_2 < 0, X_2 \leqslant X_3, \mathrm{ord}(x_1x_2 - x_3) < 0;$

$$= f\left(z\begin{pmatrix} x_3 & x_1q & pq \\ & x_1q/x_3 & pq/x_3 \\ & & pq/x_1 \end{pmatrix} y\right)$$

if $X_3 < 0, X_3 < X_2, X_1 \leqslant \operatorname{ord}(x_1x_2 - x_3)$;

$$= f\left(z\begin{pmatrix} x_3 & x_1q & pq \\ & q(x_1x_2 - x_3)/x_3 & x_2pq/x_3 \\ & & pq/(x_1x_2 - x_3) \end{pmatrix} y\right)$$

if $X_3 < 0, X_3 < X_2, X_1 > \operatorname{ord}(x_1x_2 - x_3)$.

We will denote the integral corresponding to the case (i) by $I^{(i)}(f)$ for $i = 1, \ldots, 6$. Since the last case is the most complicated one, we will give the computation of $I^{(6)}(f)$ in full detail and only list the final expressions of $I^{(1)}(f), \ldots, I^{(5)}(f)$:

$$I^{(1)}(f; p, q) = \sum_{x \in \mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z + (0, Q, P + Q));$$

$$I^{(2)}(f; p, q) = -\sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z + (0, Q - 1, P + Q + 1))$$

if $Q > 0, P \geqslant -1$;

$$= \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f\left(z + \left(0, Q + \frac{P}{2}, Q + \frac{P}{2}\right)\right) \int_{\varpi_F^{P/2} R_F^\times} \psi\left(x - \frac{p}{x}\right) \mathrm{d}x$$

if $P < -1, P \equiv 0 \pmod 2, \frac{P}{2} + Q \geqslant 0$;

$= 0$ otherwise;

$$I^{(3)}(f; p, q) = -\sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z + (-1, Q + 1, P + Q))$$

if $Q \geqslant -1, P > 0$;

$$= \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f\left(z + \left(\frac{Q}{2}, \frac{Q}{2}, P + Q\right)\right) \int_{\varpi_F^{Q/2} R_F^\times} \psi\left(x - \frac{q}{x}\right) \mathrm{d}x$$

if $Q < -1, Q \equiv 0 \pmod 2, P + \frac{Q}{2} \geqslant 0$;

$= 0$ otherwise;

$$
\begin{aligned}
I^{(4)}(f;p,q) &= q_F \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z+(-1,Q-1,P+Q+2))\\
&\quad \text{if } Q \geqslant -1, P \geqslant -1;\\
&\quad + (1-q_F)\sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z+(-1,Q,P+Q+1))\\
&\quad \text{if } Q \geqslant -1, P \geqslant 0;\\
&\quad + \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z+(-1,Q,P+Q+1))\\
&\quad \text{if } Q \geqslant -1, P = -1;\\
&= -q_F \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f\left(z+\left(-1,\frac{P}{2}+Q,\frac{P}{2}+Q+1\right)\right)\\
&\quad \times \int_{\varpi_F^{P/2}R_F^\times} \psi\left(x-\frac{p}{x}\right)\mathrm{d}x\\
&\quad \text{if } Q \geqslant -1, P < -1, P \equiv 0 (\operatorname{mod} 2), Q+\frac{P}{2} \geqslant -1;\\
&= 0 \text{ otherwise};
\end{aligned}
$$

$$
\begin{aligned}
I^{(5)}(f;p,q) &= \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)[(1-q)\Psi_f(z+(-1,Q,P+Q+1))\\
&\quad + q_F\Psi_f(z+(-2,Q+1,P+Q+1))]\\
&\quad \text{if } P \geqslant -1, Q \geqslant 0;\\
&= \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)[\Psi_f(z+(-1,Q,P+Q+1))\\
&\quad + q_F\Psi_f(z+(-2,Q+1,P+Q+1))]\\
&\quad \text{if } P \geqslant -1, Q = -1;\\
&= -q_F \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f\left(z+\left(\frac{Q}{2}-1,\frac{Q}{2},P+Q+1\right)\right)\\
&\quad \times \int_{\varpi_F^{Q/2}R_F^\times} \psi\left(x-\frac{q}{x}\right)\mathrm{d}x\\
&\quad \text{if } P \geqslant -1, Q < -1, Q \equiv 0 (\operatorname{mod} 2);\\
&= 0 \text{ otherwise};
\end{aligned}
$$

Now let us study the integral $I^{(6)}(f)$. In this last case

$$I^{(6)}(f;p,q) = \sum_{\substack{z\in\mathbf{Z},X_3<0,\\ 2X_3-L\leqslant Q,2L-X_3\leqslant P}} \zeta\chi(\varpi_F^z)$$

$$\times\ \Psi_f(z + (X_3, Q + L - X_3, P + Q - L))$$

$$\times \int_{\substack{x_3\in\varpi_F^{X_3}R_F^\times, x_2\in\varpi_F^{X_3+1}R_F,\\ x_1\in\varpi_F^{L+1}R_F, x_1x_2-x_3\in\varpi_F^L R_F^\times}}$$

$$\times\ \psi\left(x_1 + x_2 - \frac{x_1 q}{x_3} - \frac{x_2 p}{x_1x_2 - x_3}\right) \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3.$$

To compute the integral $T = \int \psi(x_1 + x_2 - (x_1q/x_3) - x_2p/(x_1x_2 - x_3))\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3$ we consider two cases: (i) $X_2 < 0$ and (ii) $X_2 \geqslant 0$. We will denote by $I^{(6.1)}(f)$ and $I^{(6.2)}(f)$ the corresponding expressions we get from these cases. We need a trivial lemma which will be quoted repeatedly.

LEMMA 10. *Assume the order of the character ψ to be zero. Then the integral $\int_{\varpi_F^X R_F^\times} \psi(x - (b/x))\, \mathrm{d}x$ vanishes unless either $B \geqslant -1$, $-1 \leqslant X \leqslant B + 1$, or $B < -1$, $B \equiv 0 (\mathrm{mod}\, 2)$, $X = B/2$, where $B = \mathrm{ord}(b)$.*

11.1. *The computation of $I^{(6.1)}(f)$.* Setting $x_1 = (x_3/x_2) + x$ with $x \in \varpi_F^{L-X_2} R_F^\times$ we get $T = \int \psi((x_3/x_2) + x + x_2 - (q/x_2) - (xq/x_3) - (p/x))\, \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x$ where $x_2 \in \varpi_F^{X_2} R_F^\times$, $x_3 \in \varpi_F^{X_3} R_F^\times$ and $x \in \varpi_F^{L-X_2} R_F^\times$. By Lemma 10 the integral with respect to x_3 vanishes unless either $\mathrm{ord}(xq/x_2) \geqslant -1$, $X_3 - X_2 = -1$, or $\mathrm{ord}(xq/x_2) < -1$, $\mathrm{ord}(xq/x_2) \equiv 0 (\mathrm{mod}\, 2)$, $X_3 - X_2 = \frac{1}{2}\mathrm{ord}(xq/x_2)$. Denote by $I^{(6.1.1)}(f)$ and $I^{(6.1.2)}(f)$ the corresponding expressions.

11.1.1. *The integral $I^{(6.1.1)}(f)$.* In this case the conditions are $X_3 < -1, L < -1$, $2X_3 - L < Q$, $2L - X_3 \leqslant P$, $X_2 = X_3 + 1$, and $X = L - X_3 - 1$, and the integral becomes $T = -q_F^{-X_3-1} \int \psi(x - (p/x))\, \mathrm{d}x \int \psi(x_2 - (q/x_2))\, \mathrm{d}x_2$, where $x \in \varpi_F^{L-X_3-1} R_F^\times$ and $x_2 \in \varpi_F^{X_3+1} R_F^\times$. Applying Lemma 10 to these two integrals with respect to x and x_2 we get

$$I^{(6.1.1)}(f;p,q) = -\, q_F \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z + (-2, Q, P + Q + 2))$$

if $P \geqslant -1, Q \geqslant -1$;

$$= q_F \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f\left(z + \left(-2, Q + \frac{P}{2} + 1, Q + \frac{P}{2} + 1\right)\right)$$

$$\times \int_{\varpi_F^{P/2} R_F^\times} \psi\left(x - \frac{p}{x}\right) \mathrm{d}x$$

if $Q \geqslant -1, P < -1, P \equiv 0 (\mathrm{mod}\, 2)$;

$= 0$ otherwise.

11.1.2. *The integral* $I^{(6.1.2)}(f)$. Since $\mathrm{ord}(xq/x_2) < -1$, $\mathrm{ord}(xq/x_2) \equiv 0 (\mathrm{mod}\, 2)$, $X_3 - X_2 = \frac{1}{2}\mathrm{ord}(xq/x_2)$, we have

$$I^{(6.1.2)}(f; p, q) = \sum_{\substack{z\in\mathbf{Z}, X_3<-1, X_3\leqslant(Q-2)/2,\\ X_3\leqslant(P+2Q)/3,\, X_3<X_2<0,\, X_3+X_2<Q}} \times \zeta\chi(\varpi_F^z)\Psi_f(z + (X_3, X_3, P + 2Q - 2X_3)) \times \int_{\substack{x_2\in\varpi_F^{X_2}R_F^\times, x_3\in\varpi_F^{X_3}R_F^\times,\\ x\in\varpi_F^{2X_3-X_2-Q}R_F^\times}} \times \psi\left(\frac{x_3}{x_2} + x + x_2 - \frac{q}{x_2} - \frac{xq}{x_3} - \frac{p}{x}\right) dx_2\, dx_3\, dx.$$

We will consider three cases: (i) $X_3 < Q$, (ii) $X_3 = Q$ and (iii) $X_3 > Q$, and calculate $I^{(6.1.2.1)}(f)$, $I^{(6.1.2.2)}(f)$, and $I^{(6.1.2.3)}(f)$.

11.1.2.1. *The case of* $I^{(6.1.2.1)}(f)$. Since $x_3 < Q$ and $\mathrm{ord}(x_3 - q) = X_3$, we may apply Lemma 10 to the integral with respect to x_2 in $I^{(6.1.2)}(f)$ and conclude that $\int \psi(x_2 + (x_3 - q)/x_2)\, dx_2$ vanishes unless $X_3 < -1$, $X_3 \equiv 0 (\mathrm{mod}\, 2)$, $X_2 = X_3/2$. Assume $X_3 < -1$, $X_3 \equiv 0 (\mathrm{mod}\, 2)$ and $X_2 = X_3/2$. We may also apply Lemma 10 to the integral with respect to x. Then $\int \psi(x \dot{-} (xq/x_3) - (p/x))\, dx$ vanishes unless either $P \geqslant -1$, $2X_3 - X_2 - Q = 1$, or $P < -1$, $P \equiv 0 (\mathrm{mod}\, 2)$, $2X_3 - X_2 - Q = P/2$. Therefore the integral with respect to x_2 and x vanishes unless either $Q \equiv 1 (\mathrm{mod}\, 3)$, $2(Q - 1)/3 < Q$, $2(Q - 1)/3 \leqslant (Q/2) - 1$, $X_3 = 2(Q - 1)/3 < -1$, or $P < -1$, $P \equiv 0 (\mathrm{mod}\, 2)$, $P + 2Q < -3$, $P + 2Q \equiv 0 (\mathrm{mod}\, 3)$, $X_3 = (P + 2Q)/3$. The formal case, however, is impossible, because it would imply $Q \leqslant 1$, $Q \geqslant 1$, $Q = 1$, $2(Q - 1)/3 \leqslant (Q/2) - 1$. The latter case implies $\Psi_f(z + (X_3, X_3, P + 2Q - 2X_3)) = \Psi_f(z + ((P + 2Q)/3,\ (P + 2Q)/3,\ (P + 2Q)/3)) = 0$ by our assumption in Section 7. Consequently $I^{(6.1.2.1)}(f; p, q) = 0$.

11.1.2.2. *The case of* $I^{(6.1.2.2)}(f)$. When $X_3 = Q$ we have

$$I^{(6.1.2.2)}(f; p, q) = \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z + (Q, Q, P)) \sum_{Q<X_2<0} \int_{\substack{x_3\in\varpi_F^{Q}R_F^\times,\, x_2\in\varpi_F^{X_2}R_F^\times,\\ x\in\varpi_F^{Q-X_2}R_F^\times}}$$

$$\times\ \psi\left(x_2 - \frac{q-x_3}{x_2} - \frac{x}{x_3}(q-x_3) - \frac{p}{x}\right) \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x$$

for $Q < -1$, $Q \leqslant P$.

If $P = Q$, then $\Psi_f(z + (Q, Q, P)) = 0$ by the assumption in Section 7. Now we assume $P > Q$. By Lemma 10 the integral $\int \psi(x_2 - (q - x_3)/x_2)\, \mathrm{d}x_2$ vanishes unless either $\mathrm{ord}(q - x_3) \geqslant -1$, $X_2 = -1$, or $\mathrm{ord}(q - x_3) < -1$, $\mathrm{ord}(q - x_3) \equiv 0 (\mathrm{mod}\, 2)$, $X_2 = \frac{1}{2}\mathrm{ord}(q - x_3)$.

We first consider the case of $\mathrm{ord}(q - x_3) \geqslant -1$, $X_2 = -1$. Since $X_3 = Q < -1$, we have $x_3 \in q + \varpi_F^{-1}R_F$, $x(q - x_3)/x_3 \in R_F$, $p/x \in R_F$, and

$$\int_{\substack{X_2=-1,\\ \mathrm{ord}(q-x_3)\geqslant -1}} \psi\left(x_2 - \frac{q-x_3}{x_2} - \frac{x}{x_3}(q-x_3) - \frac{p}{x}\right) \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x = -(1 - q_F^{-1})q_F^{-Q}.$$

Next we consider the case of $\mathrm{ord}(q - x_3) < -1$, $\mathrm{ord}(q - x_3) \equiv 0(\mathrm{mod}\, 2)$, $X_2 = \frac{1}{2}\mathrm{ord}(q - x_3)$. Since $\mathrm{ord}(x(q - x_3)/x_3) = X_2 < 0$, $\mathrm{ord}(p/x) = P - Q + X_2 > X_2$, applying Lemma 10 to the integral with respect to x, we know that it vanishes unless $X_2 = -1$, $\mathrm{ord}(q - x_3) = -2$, $X_3 = Q \leqslant -2$, $X = Q + 1$, $Q < P$. Thus

$$\int_{\substack{X_2=-1,\\ \mathrm{ord}(q-x_3)= -2}} \psi\left(x_2 - \frac{q-x_3}{x_2} - \frac{x}{x_3}(q-x_3) - \frac{p}{x}\right) \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x$$

$$= -\, q_F^{-Q-1} \qquad \text{if } Q < -2;$$

$$= -\, q_F^{-Q-1} + q_F^{Q-1} \int_{\varpi_F^{-1}R_F^\times} \psi\left(x_2 - \frac{q}{x_2}\right) \mathrm{d}x_2 \qquad \text{if } Q = -2.$$

Therefore

$$I^{(6.1.2.2)}(f; p, q) = -\, q_F^{-Q} \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z + (Q, Q, P))$$

$$\text{if } Q < -2,\ Q < P;$$

$$= -\, q_F^{-Q} \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)\Psi_f(z + (Q, Q, P))$$

$$\times \left(1 - q_F^{-1} \int_{\varpi_F^{-1}R_F^\times} \psi\left(x - \frac{q}{x}\right) \mathrm{d}x\right)$$

$$\text{if } Q = -2 < P;$$

$$= 0 \text{ otherwise.}$$

11.1.2.3. *The case of* $I^{(6.1.2.3)}(f)$. When $X_3 > Q$, we have

$$I^{(6.1.2.3)}(f; p, q) = \sum_{\substack{z\in\mathbf{Z}, X_3 \leqslant (P+2Q)/3,\\ Q<X_3\leqslant (Q/2)-1}} \times \zeta\chi(\varpi_F^z)\Psi_f(z + (X_3, X_3, P + 2Q - 2X_3)) \times \sum_{X_3<X_2<Q-X_3} \int_{\substack{x\in\varpi_F^{2X_3-X_2-Q}R_F^\times,\\ x_3\in\varpi_F^{X_3}R_F^\times, x_2\in\varpi_F^{X_2}R_F^\times}} \times \psi\left(x - \frac{p}{x} + x_2 - \frac{q}{x_2} + \frac{x_3}{x_2} - \frac{xq}{x_3}\right) \mathrm{d}x_3\, \mathrm{d}x_2\, \mathrm{d}x$$

if $Q \leqslant -4$, $Q + 3 \leqslant P$.

Since $\operatorname{ord}(xq/x_3) < X$, $\operatorname{ord}(x - (xq/x_3)) = X_3 - X_2 < 0$, the integral $\int \psi(x - (xq/x_3) - (p/x))\,\mathrm{d}x$ vanishes unless either $P + Q - X_3 \geqslant -1$, $X_3 - X_2 = -1$, or $P + Q - X_3 < -1$, $P + Q - X_3 \equiv 0 (\mathrm{mod}\ 2)$, $X_3 - X_2 = (P + Q - X_3)/2$, by Lemma 10.

When $P + Q - X_3 < -1$, $P + Q - X_3 \equiv 0 (\mathrm{mod}\, 2)$, $X_3 - X_2 = (P + Q - X_3)/2$, we note that $\operatorname{ord}(q - x_3) = Q$ and the integral $\int \psi(x_2 - (q - x_3)/x_2)\,\mathrm{d}x_2$ vanishes unless $Q \equiv 0 (\mathrm{mod}\, 2)$, $X_2 = Q/2$. Thus $P + 2Q \equiv 0 (\mathrm{mod}\, 3)$, $X_3 = (P + 2Q)/3$, and $\Psi_f(z + (X_3, X_3, P + 2Q - 2X_3)) = \Psi_f(z + ((P + 2Q)/3, (P + 2Q)/3, (P + 2Q)/3)) = 0$ by the assumption in Section 7.

When $P + Q - X_3 \geqslant -1$, $X_3 - X_2 = -1$, the integral becomes

$$\int \psi\left(x - \frac{p}{x} + x_2 - \frac{q}{x_2} + \frac{x_3}{x_2} - \frac{xq}{x_3}\right) \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x$$

$$= q_F^{Q-2X_3-1} \int_{\varpi_F^{X_2}R_F^\times} \psi\left(x_2 - \frac{q}{x_2}\right) \mathrm{d}x_2,$$

because $\int \psi(x_3/x_2)\,\mathrm{d}x_3 = -q_F^{-X_3-1}$. Since $X_2 = X_3 + 1 \leqslant Q/2 \leqslant -2$, the integral $\int \psi(x_2 - (q/x_2))\,\mathrm{d}x_2$ vanishes unless $Q \equiv 0 (\mathrm{mod}\, 2)$, $X_2 = Q/2$. Thus we get

$$I^{(6.1.2.3)}(f; p, q) = q_F \sum_{z\in\mathbf{Z}} \zeta\chi(\varpi_F^z)$$

$$\times \Psi_f\left(z + \left(\frac{Q}{2} - 1, \frac{Q}{2} - 1, P + Q + 2\right)\right) \int_{\varpi_F^{Q/2} R_F^\times} \psi\left(x - \frac{q}{x}\right) \mathrm{d}x$$

$$\text{if } Q \leqslant -4, Q \equiv 0 (\mathrm{mod}\, 2), P + (Q/2) \geqslant -2;$$

$$= 0 \text{ otherwise.}$$

11.2. *The computation of* $I^{(6.2)}(f)$. Now we assume $X_2 \geqslant 0$. Then $X_1 + X_2 > L$, $L = \mathrm{ord}(x_1 x_2 - x_3) = X_3$, and

$$I^{(6.2)}(f; p, q) = \sum_{\substack{z \in \mathbf{Z}, \\ L \leqslant \min(-1, P, Q)}} \times \zeta\chi(\varpi_F^z) \Psi_f(z + (L, Q, P + Q - L)) \times \int_{\substack{x_1 \in \varpi_F^{L+1} R_F, \\ x_3 \in \varpi_F^L R_F^\times}} \psi\left(x_1 - \frac{x_1 q}{x_3}\right) \mathrm{d}x_1 \, \mathrm{d}x_3 .$$

Observe that the integral with respect to x_1 vanishes unless $\mathrm{ord}(x_3 - q) \geqslant -1$. When $\mathrm{ord}(x_3 - q) \geqslant -1$, the integral $\int \psi(x_1 - (x_1 q / x_3)) \, \mathrm{d}x_1 \, \mathrm{d}x_3$ equals $q_F^{-X_3 - 1} \int \mathrm{d}x_3$. Therefore

$$I^{(6.2)}(f; p, q) = (q_F - 1) \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z) \Psi_f(z + (-1, Q, P + Q + 1))$$

$$\text{if } Q \geqslant -1, P \geqslant -1;$$

$$= q_F^{-Q} \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z) \Psi_f(z + (Q, Q, P))$$

$$\text{if } Q < -1, Q \leqslant P;$$

$$= 0 \text{ otherwise.}$$

Collecting the results in these subsections, we have

$$I^{(6)}(f; p, q) = -\, q_F \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z) \Psi_f(z + (-2, Q, P + Q + 2))$$

$$+ (q_F - 1) \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z) \Psi_f(z + (-1, Q, P + Q + 1))$$

$$\text{if } Q \geqslant -1, P \geqslant -1;$$

$$= q_F \sum_{z \in \mathbf{Z}} \zeta\chi(\varpi_F^z) \Psi_f\left(z + \left(-2, Q + \frac{P}{2} + 1, Q + \frac{P}{2} + 1\right)\right)$$

$$\times \int_{\varpi_F^{P/2} R_F^\times} \psi\left(x - \frac{p}{x}\right) \mathrm{d}x$$

$$\text{if } Q \geqslant -1, P < -1, P \equiv 0 (\mathrm{mod}\, 2);$$

$$= q_F \sum_{z \in \mathbf{Z}} \zeta \chi(\varpi_F^z) \Psi_f\left(z + \left(\frac{Q}{2} - 1, \frac{Q}{2} - 1, P + Q + 2\right)\right)$$

$$\times \int_{\varpi_F^{-1} R_F^\times} \psi\left(x - \frac{q}{x}\right) \mathrm{d}x$$

$$\text{if } Q \leqslant -2, Q \equiv 0 (\mathrm{mod}\, 2), P + \frac{Q}{2} \geqslant -2;$$

$$= 0 \text{ otherwise.}$$

Adding the results for $I^{(1)}(f), \ldots, I^{(6)}(f)$, we complete the proof of Theorem 3.

12. The proof of Theorem 6

We will only consider the integral $J^{[1](4)}(f')$because the calculations of others are similar and less complicated. By Section 10 the integral is

$$J^{[1](4)}(f'; p, q) = \sum_{\substack{z \in \mathbf{Z}, k<0, \\ 2k \leqslant l, 2l \leqslant k + P + (Q/2)}} \chi(\varpi_F^{2z})$$

$$\times \Psi'_{f'}\left(z + \left(-P - \frac{Q}{2} + l, k - l, -k\right)\right) q_F^{-4k}$$

$$\times \sum_{X > l} q_F^{4X} \int_{\substack{\alpha\bar{\alpha} = -q, \\ \min(\mathrm{ord}(p + x\bar{x}), \Lambda) = l - k + X}} \omega(p, \alpha, x, \lambda) \, \mathrm{d}\alpha \, \mathrm{d}x \, \mathrm{d}\lambda$$

where $\omega(p, \alpha, x, \lambda) = \psi_E(\alpha - 2\alpha p/(p + x\bar{x} - \lambda\sqrt{\tau}) + (p + x\bar{x} - \lambda\sqrt{\tau})/2\bar{x} - x)$ and $k = B + X$. We will consider three cases: (1) $X > l - k$, (2) $X < l - k$ and (3) $X = l - k$, and denote by $J^{[1](4.i)}(f')$, $i = 1, 2, 3$, the corresponding expressions.

12.1. *The computation of* $J^{[1](4.1)}(f')$. Since $X > l - k$, the condition $\min(\mathrm{ord}(p + x\bar{x}), \Lambda) = l - k + X$ can be written as $\min(P, \Lambda) = l - k + X$. Hence the conditions in the expression of $J^{[1](4.1)}(f')$ are $k < 0$, $2k \leqslant l$,

$2l \leqslant k + P + (Q/2)$, $2l < k + P$, $l - k < X \leqslant P - l + k$, $\min(P, \Lambda) = l - k + X$.

LEMMA 11. *When* $X > l - k$, *the integral* $\int \omega(p, \alpha, x, \lambda)\, dx\, d\lambda$, *where* $\min(P, \Lambda) = l - k + X$ *and* $x \in \varpi_E^X R_E^\times$, *vanishes unless either (i)* $P + (Q/2) \geqslant -5$, $-1 \leqslant l - k \leqslant (2P + Q + 2)/4$; *or (ii)* $P + (Q/2) \leqslant -6$, $P + (Q/2) \equiv 0 (\mathrm{mod}\ 3)$, $l - k = (2P + Q)/6$.

Proof. For $x \in \varpi_E^X R_E^\times$ we set $x = x_1(1 + y)$ with $x_1 \in \varpi_E^X(R_E^\times/1 + \varpi_E^M R_E)$ and $y \in \varpi_E^M R_E$ for some $M > 0$. At the same time for $\lambda \in \varpi_F^{l-k+X} R_F^\times$ or $\lambda \in \varpi_F^{l-k+X} R_F$ with $\min(P, \Lambda) = l - k + X$ we set $\lambda = \lambda_1 + \eta$ with $\lambda_1 \in \varpi_F^{l-k+X}(R_F^\times/1 + \varpi_F^{X+k-l+M} R_F)$ or $\lambda_1 \in \varpi_F^{l-k+X}(R_F/\varpi_F^{X+k-l+M} R_F)$ and $\eta \in \varpi_F^{2X+M} R_F$. We will show that it is possible to choose M so that the integral with respect to y and η vanishes, except for the above cases.

If $M \geqslant -[(2P + Q)/4)] + l - k$, then

$$\psi_E\left(-\frac{2\alpha p}{P + x\bar{x} - \lambda\sqrt{\tau}}\right) = \psi_E\left(-\frac{2\alpha p}{p + x_1\bar{x}_1 - \lambda_1\sqrt{\tau}} + \frac{2\alpha p}{(p + x_1\bar{x}_1 - \lambda_1\sqrt{\tau})^2}(2x_1\bar{x}_1 y - \eta_1\sqrt{\tau})\right) \tag{22}$$

where $\eta_1 = \eta + x_1\bar{x}_1(y - \bar{y})/\sqrt{\tau} \in \varpi_F^{2X+M} R_F$. Similarly when $M \geqslant (k - l)/2$, we have

$$\psi_E\left(\frac{p + x\bar{x} - \lambda\sqrt{\tau}}{2\bar{x}} - x\right) = \psi_E\left(\frac{p + x_1\bar{x}_1 - \lambda_1\sqrt{\tau}}{2\bar{x}_1} - x_1 - \frac{p + x_1\bar{x}_1 - \lambda_1\sqrt{\tau}}{2\bar{x}_1} y - \frac{\eta_1\sqrt{\tau}}{2\bar{x}_1}\right). \tag{23}$$

When $l - k \geqslant [(2P + Q)/4] + 1$ we can choose $M > 0$ satisfying $-[(2P + Q)/4] + l - k \leqslant M < -P - (Q/2) + 2l - 2k$ so that (22) is a non-trivial character of y and hence $\int \psi_E(-2\alpha p/(p + x\bar{x} - \lambda\sqrt{\tau}))\, dy\, d\eta = 0$. If $l - k < [(2P + Q)/4] + 1$, then (22) is indeed independent of y and η_1. On the other hand when $l - k \leqslant -2$, we can choose $M > 0$ with $(k - l)/2 \leqslant M < k - l$ and hence $\int \psi_E((p + x\bar{x} - \lambda\sqrt{\tau})/2\bar{x} - x)\, dy\, d\eta = 0$. If $l - k \geqslant -1$, then (23) is independent of y and η.

When $P + (Q/2) \geqslant -5$ and $l - k \leqslant -2$, it is possible to choose $M > 0$ such that $(k - l)/2 \leqslant M < k - l$, $M \geqslant -[(2P + Q)/4] + l - k$. Thus (22) is independent of y and η, while (23) is a non-trivial character of y, and hence $\omega(p, \alpha, x, \lambda)$ is also a non-trivial character of y. Therefore when

$P + (Q/2) \geqslant -5$, the integral in the lemma vanishes if $l - k \leqslant -2$. Likewise the integral equals zero when $P + (Q/2) \geqslant -5$ and $l - k \geqslant [(2P + Q)/4] + 1$.

When $P + (Q/2) \leqslant -6$, we may use the same techniques to show that the integral of $\omega(p, \alpha, x, \lambda)$ is zero. But it is possible that both (22) and (23) are non-trivial characters of y. If this is the case, $\omega(p, \alpha, x, \lambda)$ remains a non-trivial character of y when $\operatorname{ord}(4\alpha p x_1\bar{x}_1/(p + x_1\bar{x}_1 - \lambda_1\sqrt{\tau})^2) \neq \operatorname{ord}((p + x_1\bar{x}_1 - \lambda_1\sqrt{\tau})/2\bar{x}_1)$, i.e., $l - k \neq (2P + Q)/6$. Therefore the integral of $\omega(p, \alpha, x, \lambda)$ vanishes when $P + (Q/2) \leqslant -6$ unless $l - k = (2P + Q)/6$.

□

By this lemma we can write $J^{[1](4.1)}(f')$ as

$$\begin{aligned}
J^{[1](4.1)}(f'; p, q) = & \sum_{\substack{z\in\mathbf{Z}, k<0, 2k\leqslant l, 2l<k+P,\\ -1\leqslant l-k\leqslant(2P+Q+2)/4, l-k\leqslant(P-1)/2}} \chi(\varpi_F^{2z}) \\
& \times \Psi'_{f'}\left(z + \left(-P - \frac{Q}{2} + l, k - l, -k\right)\right) q_F^{-4k} \\
& \times \sum_{l-k<X\leqslant k-l+P} q_F^{4X} \int_{\substack{\alpha\bar{\alpha}=-q, x\in\varpi_E^X R_E^\times,\\ \min(P,\Lambda)=l-k+X}} \\
& \times \omega(p, \alpha, x, \lambda)\, \mathrm{d}\alpha\, \mathrm{d}x\, \mathrm{d}\lambda \qquad (24) \\
& \text{if } P \geqslant -1, P + \frac{Q}{2} \geqslant -3; \\
= & \sum_{z\in\mathbf{Z}, k\leqslant(2P+Q)/6} \chi(\varpi_F^{2z}) \\
& \times \Psi'_{f'}\left(z + \left(-\frac{2(2P+Q)}{6} + k, -\frac{2P+Q}{6}, -k\right)\right) q_F^{-4k} \\
& \times \sum_{(2P+Q)/6<X\leqslant P-(2P+Q)/6} \int \omega(p, \alpha, x, \lambda)\, \mathrm{d}\alpha\, \mathrm{d}x\, \mathrm{d}\lambda \qquad (25) \\
& \text{if } P + (Q/2) \leqslant -6, P + (Q/2 \equiv 0 (\mathrm{mod}\ 3); \\
= & \ 0 \text{ otherwise.}
\end{aligned}$$

By the assumption at the end of Section 7 the sum with respect to k in (25) vanishes. Thus (25) equals zero. To calculate (24) we consider four cases:

(1) $0 \leqslant l - k < (2P + Q + 2)/4$;
(2) $l - k = -1$, $(2P + Q + 2)/4 > -1$;
(3) $P + (Q/2) \equiv 1 (\mathrm{mod}\ 2)$, $l - k = (2P + Q + 2)/4 \geqslant 0$; and

(4) $P+(Q/2)=-3$, $l-k=-1$,
and denote the corresponding expressions by $J^{[1](4.1.i)}(f')$, $i=1,\ldots,4$.

To calculate $J^{[1](4.1.1)}(f')$, we note that $\omega(p,\alpha,x,\lambda)=\psi_E(\alpha)$ when $0\leqslant l-k<(2P+Q+2)/4$. Consequently $J^{[1](4.1.1)}(f')$ equals (8) in Theorem 6.

In the second case we note that $\omega(p,\alpha,x,\lambda)=\psi_E(-\alpha[(p+\lambda\sqrt{\tau})/(p-\lambda\sqrt{\tau})]+[(p-\lambda\sqrt{\tau})/(2\bar{x})])$. Integrating this expression we get (9) from $J^{[1](4.1.2)}(f')$.

For $J^{[1](4.1.3)}(f')$ we have $\omega(p,\alpha,x,\lambda)=\psi_E(\alpha_1(1-2py\bar{y}))$ where $\alpha_1=\alpha[(p+\lambda\sqrt{\tau})/(p-\lambda\sqrt{\tau})]$ and $y=x/(p-\lambda\sqrt{\tau})$.

LEMMA 12. *If* $Q\leqslant -4$, $Q\equiv 0(\mathrm{mod}\,2)$, $P+(Q/2)\geqslant -1$ *and* $P+(Q/2)\equiv 1(\mathrm{mod}\,2)$, *then*

$$\int_{\substack{\alpha_1\bar{\alpha}_1=-q,\\ y\in\varpi_E^{-(2P+Q+2)/4}R_E^{\times}}}\psi_E(\alpha_1(1-2py\bar{y}))\,\mathrm{d}\alpha_1\,\mathrm{d}y$$
$$=-(1+q_F^{-1})q_F^{P+(Q/2)}\int_{\alpha\bar{\alpha}=-q}\psi_E(\alpha)\,\mathrm{d}\alpha.$$

Proof. Similar to the proof of Lemma 5.

By this lemma we get (10) from the third case.

Finally we see that $J^{[1](4.1.4)}(f')$ equals zero because it can be written in terms of $\Sigma_{k<0}\Psi'_{f'}(z+(k+2,1,-k))q_F^{-4k}$, which vanishes according to our assumption in Section 7.

12.2. *The computation of* $J^{[1](4.2)}(f')$. When $X<l-k$, we deduce from $\min(\mathrm{ord}(p+x\bar{x}),\Lambda)=l-k+X$ that $P\equiv 0(\mathrm{mod}\ 2)$, $X=P/2$, $x\bar{x}\in -p+\varpi_F^{(P/2)+l-k}R_F$, and $l-k>P/2$.

LEMMA 13. *When* $P\equiv 0(\mathrm{mod}\,2)$, $l-k>P/2$, *the integral* $\int\omega(p,\alpha,x,\lambda)\,\mathrm{d}x\,\mathrm{d}\lambda$, *where* $x\in\varpi_E^{P/2}R_E^{\times}$, $x\bar{x}\in -p+\varpi_F^{(P/2)+l-k}R_F$ *and* min $(\mathrm{ord}(p+x\bar{x}),\Lambda)=(P/2)+l-k$, *vanishes unless in any one of the following cases*:

(1) $P\geqslant 0$, $Q\geqslant 0$, $P/2<l-k\leqslant[(P+Q)/2]+1$;
(2) $P<0$, $(P/2)+Q\geqslant -3$, $(P-2)/4\leqslant l-k\leqslant (P+Q)/2+1$;
(3) $P<0$, $(P/2)+Q<-3$, $P+(Q/2)\equiv 0(\mathrm{mod}\,3)$, $l-k=(2P+Q)/6$.

Proof. Similar to the proof of Lemma 11.

According to this lemma we consider six cases and denote by $J^{[1](4.2.i)}(f')$, $i=1,\ldots,6$, the corresponding expressions:

(1) $P\geqslant 0$, $Q>0$, $l<P/2$, $P/2<l-k\leqslant (P+Q)/2$;
(2) $P\geqslant 0$, $Q\geqslant 0$, $l<P/2$, $l-k=[(P+Q)/2]+1$;
(3) $P<0$, $(P/2)+Q\geqslant 0$, $l<P/2$, $(P-2)/4<l-k\leqslant (P+Q)/2$;

(4) $P<0$, $P\equiv 2 \pmod 4$, $(P/2)+Q\geqslant -1$, $k\leqslant (P-2)/4$, $l-k=(P-2)/4$;

(5) $P<0$, $(P/2)+Q\geqslant -3$, $k\leqslant -(Q/2)-2$, $l-k=[(P+Q)/2]+1$; and

(6) $P<0$, $(P/2)+Q<-3$, $P+(Q/2)\equiv 0 \pmod 3$, $k\leqslant (2P+Q)/6$, $l-k=(2P+Q)/6$.

We observe that $\omega(p,\alpha,x,\lambda)=1$ in case (1), $=\psi_E(-2\alpha p/(p+x\bar{x}-\lambda\sqrt{\tau}))$ in case (2), and $=\psi_E((p+x\bar{x}-\lambda\sqrt{\tau})/2\bar{x}-x)$ in cases (3) and (4). For case (5) we have a lemma:

LEMMA 14. *When* $P<0$, $P\equiv 0 \pmod 2$, $(P/2)+Q\geqslant -3$,

$$q_F^{-4k+2P}\int_{\substack{x\in\varpi_E^{P/2}R_E^\times,\, x\bar{x}\in -p+\varpi_F^{P+(Q/2)+1}R_F,\\ \alpha\bar{\alpha}=-q,\, \min(\mathrm{ord}(p+x\bar{x}),\Lambda)=P+(Q/2)+1}} \omega(p,\alpha,x,\lambda)\,\mathrm{d}\alpha\,\mathrm{d}x\,\mathrm{d}\lambda$$

$$=-q_F^{-4k-Q-4}(-1)^{P/2}q_F^{P/2}\int_{\varpi_F^{P/2}R_F^\times}\psi\left(x-\frac{p}{x}\right)\mathrm{d}x \qquad \text{if } \frac{P}{2}+Q\geqslant -2;$$

$$=0 \qquad \text{if } \frac{P}{2}+Q=-3.$$

Proof. Similar to the proof of Lemma 5.

By the above observation and Lemma 14 we get the expressions (11), ..., (15) in Theorem 6 from $J^{[1](4.2.1)}(f')$, ..., $J^{[1](4.2.5)}(f')$, respectively. Similar to (25), the expression $J^{[1](4.2.6)}(f')$ vanishes because of our assumption in Section 7.

12.3. *The computation of* $J^{[1](4.3)}(f')$. From $X=l-k$ and $\min(\mathrm{ord}(p+x\bar{x}),\Lambda)=2l-2k$ we observe that $l-k\leqslant P/2$. We will consider two cases: (1) $l-k<P/2$ and (2) $l-k=P/2$, and denote the corresponding expressions by $J^{[1](4.3.1)}(f')$ and $J^{[1](4.3.2)}(f')$.

12.3.1. *The integral* $J^{[1](4.3.1)}(f')$. Since $l-k<P/2$, the integral with respect to x and λ is taken over $x\in\varpi_E^{l-k}R_E^\times$ and $\lambda\in\varpi_F^{2l-2k}R_F$.

LEMMA 15. *When* $l-k<P/2$, *the integral* $\int\omega(p,\alpha,x,\lambda)\,\mathrm{d}x\,\mathrm{d}\lambda$, *where* $x\in\varpi_E^{l-k}R_E^\times$ *and* $\lambda\in\varpi_F^{2l-2k}R_F$, *vanishes unless either*

(1) $P+(Q/2)\geqslant -5$, $-1\leqslant l-k\leqslant (2P+Q+2)/4$, $l-k<P/2$; *or*

(2) $P+(Q/2)\leqslant -6$, $P+(Q/2)\equiv 0 \pmod 3$, $l-k=(2P+Q)/6$.

Proof. Similar to the proof of Lemma 11.

Lemma 15 suggests five cases:

(1) $P>0$, $P+(Q/2)\geqslant 0$, $k<0$, $l-k<P/2$, $0\leqslant l-k\leqslant (2P+Q)/4$;

(2) $P\geqslant -1$, $P+(Q/2)\geqslant -2$, $k<0$, $l-k=-1$;

(3) $P>0$, $P+(Q/2)\geqslant -1$, $k<0$, $l-k=(2P+Q+2)/4$;

(4) $P \geqslant -1$, $P + (Q/2) = -3$, $k < 0$, $l - k = -1$; and
(5) $P > Q$, $P + (Q/2) \leqslant -6$, $P + (Q/2) \equiv 0 (\mathrm{mod}\, 3)$, $k \leqslant (2P + Q)/6$, $l - k = (2P + Q)/6$.

Denote by $J^{[1](4.3.1.i)}(f')$ the corresponding expressions. By similar computation as in subsection 12.2 we get (16), (17) and (18) from the first three cases and prove that $J^{[1](4.3.1.4)}(f') = J^{[1](4.3.1.5)}(f') = 0$.

12.3.2. *The integral* $J^{[1](4.3.2)}(f')$. Now let $l - k = P/2$. Then

$$J^{[1](4.3.2)}(f'; p, q) = \sum_{\substack{z \in \mathbf{Z},\, k<0, \\ k \leqslant P/2,\, k \leqslant Q/2}} \chi(\varpi_F^{2z}) \times \Psi'_{f'}\left(z + \left(-\frac{P+Q}{2} + k, -\frac{P}{2}, -k\right)\right) q_F^{-4k+2P} \times \int_{\substack{\alpha\bar{\alpha} = -q, x \in \varpi_E^{P/2} R_E^{\times}, \\ \min(\mathrm{ord}(p + x\bar{x}), \Lambda) = P}} \omega(p, \alpha, x, \lambda)\, \mathrm{d}\alpha\, \mathrm{d}x\, \mathrm{d}\lambda$$

when $P \equiv 0 (\mathrm{mod}\, 2)$.

LEMMA 16. *Let* $P \equiv 0(\mathrm{mod}\ 2)$. *Then* $\int \omega(p, \alpha, x, \lambda)\, \mathrm{d}\alpha\, \mathrm{d}x\, \mathrm{d}\lambda$, *where* $\alpha\bar{\alpha} = -q$, $x \in \varpi_E^{P/2} R_E^{\times}$ *and* $\min(\mathrm{ord}(p + x\bar{x}), \Lambda) = P$, *vanishes unless either* $P \geqslant -2$, $Q \geqslant -2$; *or* $P = Q \leqslant -4$.

Proof. Similar to the proof of Lemma 11.

According to Lemma 16 we consider five cases: (1) $P \geqslant 0$, $Q \geqslant 0$; (2) $P \geqslant 0$, $Q = -2$; (3) $P = -2$, $Q \geqslant 0$; (4) $P = Q = -2$; and (5) $P = Q \leqslant -4$. In the first case, we have $\omega(p, a, x, \lambda) = 1$, and hence we get (19) for the theorem. In cases (2) and (3) we have $\omega(p, \alpha, x, \lambda) = \psi_E(\alpha - 2\alpha p/(p + x\bar{x} - \lambda\sqrt{\tau}))$ and $\psi_E((p + x\bar{x} - \lambda\sqrt{\tau})/2\bar{x} - x)$, respectively. Consequently we get (20) and (21). The last two cases yield nothing under our assumption in Section 7.

This completes the proof of Theorem 6.

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