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Hecke algebras and shellings of Bruhat intervals

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Introduction

This paper is concerned with connections between the Iwahori-Hecke algebra of a Coxeter group W and the combinatorial structure of Bruhat intervals. The main result is that the Kazhdan-Lusztig polynomials and inverse Kazhdan-Lusztig polynomials associated to a pair of elements of W can be defined using a natural labelling (by the reflections T of W) of the edges of a graph associated to the corresponding Bruhat interval. The definition involves certain orderings of the edges of the graph, which can be regarded as describing a refinement in this context of the combinatorial notion of shellability of the interval. In fact, CL-shellability of Bruhat intervals has been proved by Björner and Wachs [3, 4], and a byproduct of the results here is a proof of EL-shellability of Bruhat intervals, which is a stronger result conjectured by Björn er.

The construction of the Kazhdan-Lusztig polynomial $P_{x,y}$ here yields a family of polynomials which "interpolate" between $q^{-(l(y)-l(x))/2}P_{x,y}(q)$ and $q^{(l(y)-l(x))/2}P_{x,y}(q^{-1})$. These polynomials are parametrized by certain subsets of T which we call initial sections of reflection orders; the polynomials corresponding to the finite subsets are, up to some renormalization, precisely the coefficients arising when one expresses products $T_vC'_w$ as linear combinations of the basis elements T_u (where for v, $w \in W$, T_v and C'_w respectively denote the corresponding standard and Kazhdan-Lusztig basis elements of the Hecke algebra). It is known that for finite Weyl groups and also for "universal" Coxeter groups (see [5, 7]) that these coefficients are Laurent polynomials in $q^{1/2}$ with non-negative integral coefficients. We conjecture that the polynomials interpolating between $q^{-(l(y)-l(x))/2}P_{x,y}(q)$ and $q^{(l(y)-l(x))/2}P_{x,y}(q^{-1})$, (and also similar polynomials defined here for the inverse Kazhdan-Lusztig polynomial

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$Q_{x,y}(q)$) also have non-negative integral coefficients for arbitrary Coxeter systems, extending the conjecture by Kazhdan and Lusztig on non-negativity of the coefficients of Kazhdan-Lusztig polynomials. Non-negativity of $P_{x,y}$ and $Q_{x,y}$ is known for crystallographic Coxeter groups (by interpreting them in terms of intersection cohomology of Schubert varieties for associated Kac-Moody groups) but non-negativity of the interpolating polynomials is not known even in that case.

The interpolating polynomials defined here are closely analogous to certain polynomials associated to shellings of face lattices of convex polytopes by Richard Stanley (see [15, section 6] where the relevant polynomials are denoted $f(P_i, I_i, x)$). Stanley's polynomials are also conjectured to have non-negative coefficients in general (this has been proved, in the special case of rational convex polytopes, for a subset of these polynomials by interpreting them in terms of intersection cohomology of associated toric varieties).

In subsequent papers, we will define Bruhat-like orders $\leqslant_A$ on W parametrized by initial sections A of the reflection orders of T, and extend the definitions of Kazhdan-Lusztig polynomials and the corresponding interpolating polynomials to certain intervals in these orders. It will be shown that these polynomials and Stanley's polynomials may be given a common construction involving labellings (by elements of a vector space) of the edges of the Hasse diagram of the corresponding poset. We also begin an attempt to construct quadratic algebras from these labelled posets, the representation theory of which would provide a common explanation for many conjectural properties of these families of polynomials, and which should be related to classical Lie representation theory in the case of crystallographic Coxeter groups (especially finite and affine Weyl groups).

In the case of face lattices, Stanley's polynomials may be defined as sums (with certain simple polynomial coefficients) of a special subset of these polynomials which are analogous to Kazhdan-Lusztig polynomials. There is a similar expression for interpolating polynomials for Kazhdan-Lusztig polynomials as sums of Kazhdan-Lustig polynomials, but this is not suitable as a definition since in this context the relevant coefficients are more complex (they include the polynomials $R_{x,y}$ of [11], for instance) and are not a priori well-defined. To overcome this difficulty, we utilize a representation of the Hecke algebra as an algebra of functions under a twisted convolution product.

We explain this representation first in the simplest case of a finite Coxeter system (W, S). Let $\mathscr{R} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ be the ring of Laurent polynomials in an indeterminate $q^{1/2}$ over $\mathbb{Z}$. Let $a \mapsto \bar{a}$ be the ring involution of $\mathscr{R}$ determined by $q^{1/2} \mapsto q^{-1/2}$. The set of functions $W \times W \to \mathscr{R}$ becomes an $\mathscr{R}$-algebra $\mathscr{H}$ under the twisted convolution product

$$(fg)(v, w) = \sum_{y \in W} f(vy^{-1}, yw)g(y, w)$$

for $f, g: W \times W \to \mathscr{R}$ and $v, w \in W$. Moreover, the ring $\mathscr{H}$ has an involution $f \to \bar{f}$ defined by $\bar{f}(v, w) = \overline{f(v, ww_0)}$ where w_0 is the longest element of W.

One can show that the generic Hecke algebra $\mathscr{H}(W)$ of (W, S) over $\mathscr{R}$ can be embedded as a $\mathscr{R}$-subalgebra of $\mathscr{H}$; for $r \in S$, the standard generator T_r of $\mathscr{H}(W)$ corresponds to the function $W \times W \to \mathscr{R}$ with $(r, v) \mapsto q^{1/2}$ for all $v \in W$, $(1, v) \mapsto q - 1$ for $v \in W$ with $rv < v$ in Bruhat order, and $(w, v) \mapsto 0$ for other $(w, v) \in W \times W$. Moreover, $f \mapsto \bar{f}$ induces the Kazhdan-Lusztig involution $\sum_{w \in W} a_w T_w \mapsto \sum_{w \in W} \bar{a}_w T_{w^{-1}}^{-1}$ on $\mathscr{H}(W)$. Let $c_w: W \times W \to \mathscr{R}$ correspond to the Kazhdan-Lusztig basis element C'_w of $\mathscr{H}(W)$. For $x \leqslant w$, one has that $c_w(x, 1) = q^{-(l(w)-l(x))/2} P_{x,w}(q)$ and $c_w(x, w_0) = q^{(l(w)-l(x))/2} P_{x,w}(q^{-1})$. The interpolating polynomials for $P_{x,w}$ are precisely the Laurent polynomials $c_w(x, v)$ for varying $v \in W$. For finite W, a reduced expression $w_0 = r_1 \cdots r_n$ for the longest element of W induces a total order $\preceq$ on T by the rule $r_1 \preceq r_1 r_2 r_1 \preceq \cdots \preceq r_1 \cdots r_{n-1} r_n r_{n-1} \cdots r_1$ (recall each element of T occurs precisely once in the above listing). One can use these orderings of T and the above description of the Kazhdan-Lusztig involution to give a characterization of the function c_w completely in terms of its values.

For infinite W, the absence of a longest element precludes the possibility of such a simple description of c_w, and the symmetry of the values of c_w is lost (in general, for instance, one cannot obtain $q^{(l(w)-l(x))/2} P_{x,w}(q^{-1})$ as a value of c_w). One therefore considers instead the Hecke algebra $\mathscr{H}(W)$ as a subalgebra of an algebra of functions $W \times \mathscr{A} \to \mathscr{R}$ under essentially the same twisted convolution product as before, where $\mathscr{A}$ is a certain subset of the power set of T endowed with a left W-action. Roughly, an element of $\mathscr{A}$ may be thought of as the set of reflections in the positive roots (of the reflection representation of W) lying on one side of some (linear) hyperplane. The W-action is given by $(w, A) \mapsto N(w) + wAw^{-1}$ where $+$ denotes symmetric difference and $N(w) = \{t \in T \mid l(tw) < l(w)\}$. The map $w \mapsto N(w)$ identifies W (with left W-action by multiplication) with the elements of $\mathscr{A}$ with finite cardinality, and complementation in T of elements of $\mathscr{A}$ corresponds to right multiplication by w_0 in finite Coxeter systems. The interpolating polynomials are then the values assumed by the functions $c_w: W \times \mathscr{A} \to \mathscr{R}$ corresponding to the Kazhydan-Lusztig basis elements C'_w.

To characterize the c_w for infinite Coxeter groups, and to describe EL-shellings of Bruhat intervals, we use certain total orderings $\preceq$ of T with the property that all their initial sections are elements of $\mathscr{A}$. For finite W, these "reflection orders" are precisely the orders on T obtained from a reduced expression for the longest element as described above. To describe the orders in general, recall that positive roots are naturally in bijection with T. The orders on the positive roots which correspond to reflection orders on T may be characterized as follows; the restriction of the order to the positive roots lying on the plane spanned by any two positive roots is one of the two possible orders in which a ray from the origin, undergoing a full rotation in the plane beginning at a negative root, would sweep through the positive roots on that plane. It is most convenient for our purposes to define reflection orders by an algebraic condition equivalent to this, and to define $\mathscr{A}$ as the set of all initial sections of all reflection orders on T.

The arrangement of this paper is as follows. Section 1 provides a framework for our later constructions by describing a procedure which associates a family of Laurent polynomials to certain directed graphs with edges labelled by elements of a totally ordered set. Section 2 is concerned with the definition and properties of reflection orders and the set $\mathscr{A}$ of their initial sections; these facts will be used extensively in future papers. Section (3.1)–(3.3) describes in detail the representation of the Hecke algebra as an algebra of functions under twisted convolution product, leading to the interpolating polynomials for the Kazhdan-Lusztig polynomials. In (3.4), this result is applied to give a natural characterization of the polynomial $R_{x,y}$ $(x, y \in W)$ [11] as a generating function for a set of paths in the corresponding Bruhat interval. Sections (3.4)–(3.7) describe the dual construction to that in (3.4)–(3.7), giving interpolating polynomials for the $Q_{x,y}$. Sections (3.8)–(3.9) list some conjectures generalizing those of [7], [8, 7.16] and describe some evidence for them. Finally, in Section 4, we apply (3.4) to show that reflection orders give rise to EL-shellings of Bruhat intervals.

A number of results in this paper appear in [8], notably the definition and basic properties of reflection orders, and the result (3.4) (which was proved there by a different argument that does not depend on the Hecke algebra).

1. Some constructions in incidence algebras

(1.1) Let $\mathscr{R} = \mathbb{Z}[u, u^{-1}]$ be the ring of Laurent polynomials in an indeterminate u, and let $a \mapsto \bar{a}$ and $a \mapsto \hat{a} (a \in \mathscr{R})$ denote the two ring involutions of $\mathscr{R}$ determined by $\bar{u} = u^{-1}$ and $\hat{u} = -u$ respectively. Set $\alpha = u - u^{-1}$ and note that $\bar{\alpha} = \hat{\alpha} = -\alpha$. Let $\mathscr{R}^+ = \{\Sigma c_n u^n \in \mathscr{R} \mid \text{all } c_n \geqslant 0\}$.

Fix a locally finite poset $(X, \leqslant)$; thus, X is a set, $\leqslant$ is a partial order on X and each interval $[x,z] = \{y \in X \mid x \leqslant y \leqslant z\} (x, z \in X)$ is a finite set. Let $\mathscr{M} = \mathscr{M}_{\mathscr{R}}(X)$ denote the incidence algebra of X over $\mathscr{R}$. Recall that the set underlying $\mathscr{M}$ is the set of functions $f: X \times X \to \mathscr{R}$ such that $f(x, y) = 0$ unless $x \leqslant y$, and that for $f, g \in \mathscr{M}$

$$(fg)(x, z) = \sum_{y \in [x,z]} f(x, y)g(y, z) \quad (x, z \in X).$$

Define ring involutions $f \mapsto \bar{f}$ and $f \mapsto \hat{f}$ of $\mathscr{M}$ by the formulae $\bar{f}(x, y) = \overline{f(x, y)}$, $\hat{f}(x, y) = \widehat{f(x, y)} (f \in \mathscr{M}, x, y \in X)$.

Let r denote any element of $\mathscr{M}$ satisfying conditions (i), (ii) below:

(i) $r(x, x) = 1 \quad (x \in X)$
(ii) $r^{-1} = \bar{r}$.

For example, take $r = a^{-1}\bar{a}$ where $a \in \mathscr{M}$ satisfies $a(x, x) = 1 (x \in X)$. Elements r satisfying (i), (ii) arise naturally in [11], [16] and [6].

(1.2) PROPOSITION. *Let* $r \in \mathscr{M}$ *satisfy* (i), (ii) *above.*

(i) *There exists a unique* $p \in \mathscr{M}$ *satisfying* (a)–(c) *below:*
  (a) $p(x, x) = 1 \quad (x \in \mathscr{M})$
  (b) $p(x, y) \in u^{-1}\mathbb{Z}[u^{-1}] \quad (x, y \in X, x \neq y)$
  (c) $p = r\bar{p}$
(ii) *There exists a unique* $q \in \mathscr{M}$ *satisfying* (a)′–(c)′ *below:*
  (a)′ $q(x, x) = 1 \quad (x \in \mathscr{M})$
  (b)′ $q(x, y) \in u^{-1}\mathbb{Z}[u^{-1}] \quad (x, y \in X; x \neq y)$
  (c)′ $q = \bar{q}r$
(iii) *If* $\bar{r} = \hat{r}$ *(i.e.* $r(x, y) \in \mathbb{Z}[\alpha]$ *for all* $x, y \in X$*) then* $p^{-1} = \hat{q}$.

Proof. The proof of (i) is by an argument in [14] which we briefly recall. Fix $x < y$ $(x, y \in X)$. Suppose $p(z, y)$ is defined for all z with $x < z \leqslant y$ and satisfies

$$p(z, y) = \sum_{w \in [z,y]} r(z, w)\bar{p}(w, y) \quad \text{for all such } z. \tag{1.3}$$

Let $\beta(x, y) = \sum_{z \in (x,y]} r(x, z)\bar{p}(z, y)$. One must show that there exists a unique element $p(x, y) \in u^{-1}\mathbb{Z}[u^{-1}]$ satisfying $p(x, y) - \overline{p(x, y)} = \beta(x, y)$. This will be the case provided $\overline{\beta(x, y)} = -\beta(x, y)$.

But

$$\begin{aligned}
\overline{\beta(x, y)} &= \sum_{z \in (x,y]} \bar{r}(x, z)p(z, y) \\
&= \sum_{z \in (x,y]} \bar{r}(x, z) \sum_{w \in [z,y]} r(z, w)\overline{p(w, y)} \\
&= \sum_{w \in (x,y]} \left(\sum_{z \in (x,w]} \bar{r}(x, z)r(z, w) \right) \overline{p(w, y)} \\
&= \sum_{w \in (x,y]} -r(x, w)\bar{p}(w, y) \\
&= -\beta(x, y) \text{ as required.}
\end{aligned}$$

A similar argument proves (ii).

(iii) Suppose $\bar{r} = \hat{r}$. Then $\hat{q}p = \hat{q}r\bar{p} = \widehat{q^{r}}\,\bar{p} = \bar{\hat{q}}\bar{p} = \overline{\hat{q}p}$.

If $x, y \in X$ and $x \neq y$, we have $\hat{q}p(x, y) \in u^{-1}\mathbb{Z}[u^{-1}]$ and $\hat{q}p(x, y) = \overline{\hat{q}p(x, y)}$, hence $\hat{q}p(x, y) = 0$. Since $\hat{q}p(x, x) = 1$ $(x \in X)$, it follows that p is invertible and $p^{-1} = \hat{q}$.

(1.4) We now describe a simple construction which produces pairs of mutually inverse elements in $\mathscr{M}$; in many natural situations these elements satisfy (i), (ii) of (1.1).

Fix any subset E of $\{(x, y) \in X \times X \mid x < y\}$; regard E as the edge set of a directed graph $\Omega = (X, E)$. For $n \in \mathbb{N}$, let C_n denote the set of paths of length n in Ω; that is,

$$C_n = \{(x_0, \ldots, x_n) \in X^{n+1} \mid (x_{i-1}, x_i) \in E \text{ for } i = 1, \ldots, n\}.$$

For $x, y \in X$ let

$$C_n(x, y) = \{(x_0, \ldots, x_n) \in C_n \mid x_0 = x,\ x_n = y\}.$$

If $\tau = (x_0, \ldots, x_n) \in C_n$, define $\tau_i \in C_2$ $(i = 1, \ldots, n-1)$ by $\tau_i = (x_{i-1}, x_i, x_{i+1})$. For any subset I of C_2, let

$$A_n^I(x, y) = \{\tau \in C_n(x, y) \mid \tau_i \in I (i = 1, \ldots, n-1)\}$$

and define $r_I \in \mathscr{M}$ by

$$r_I(x, y) = \sum_{n \in \mathbb{Z}} \#(A_n^I(x, y))\alpha^n \quad (x, y \in X).$$

(1.5) PROPOSITION. *For any subset I of C_2, $r_I^{-1} = \bar{r}_J$ where $J = C_2 \backslash I$.*

Proof. It is sufficient to show that if $x < z$ then

$$\sum_{y \in [x,z]} r_I(x, y)\bar{r}_J(y, z) = 0.$$

The left-hand side is equal to

$$\sum_{n > 0} \sum_{\tau \in C_n(x,z)} \sum_{k \in D(\tau)} \alpha^k \bar{\alpha}^{n-k}$$

where for $\tau = (x_0, \ldots, x_n) \in C_n(x, z)$,

$$D(\tau) = \{k \mid 0 \leqslant k \leqslant n,\ \tau_i \in I (i = 1, \ldots, k-1),\ \tau_i \in J (i = k+1, \ldots, n-1)\}$$

But for $n > 0$ and $\tau \in C_n(x, z)$, either $D(\tau) = 0$ or $D(\tau) = \{j-1, j\}$ for some $j (1 \leqslant j \leqslant n)$; in either case,

$$\sum_{k \in D(\tau)} \alpha^k \bar{\alpha}^{n-k} = 0.$$

(1.6) A special instance of the construction in (1.4) arises when the edges of E are labelled by elements of a totally ordered set, and I is the set of paths of length 2 with decreasing label. For our applications, we need to examine this situation more closely.

For $x \in X$, let $E_x = \{(y, z) \in E \mid y = x \text{ or } z = x.\}$ Assume that $\preceq$ is a total order on a set Λ and $\lambda: E \to \Lambda$ is a function which restricts to an injection $\lambda|_{E_x}: E_x \to \Lambda$ for each $x \in X$. Let $\mathscr{A}'$ be the set of all initial sections of Λ i.e. $\mathscr{A}' = \{A \subseteq \Lambda | a \prec b \text{ for all } a \in A, b \in \Lambda \backslash A\}$.

Suppose given a function $h_\varnothing: X \to \mathscr{R}$ such that for all $x \in X$, there are only finitely many $y \in X$ such that $y \geqslant x$ and $h_\varnothing(y) \neq 0$. It is easily seen that there is a unique function $h: X \times \mathscr{A}' \to \mathscr{R}$ with the following properties:

(i) $h(x, \varnothing) = h_\varnothing(x) \quad (x \in X)$
(ii) $h(x, A) = 0$ unless there exists $y \geqslant x$ with $h(y, \varnothing) \neq 0$.
(iii) for each $x \in X$, there exists a finite subset Λ' of Λ such that $f(x, A) = f(x, B)$ if $A \cap \Lambda' = B \cap \Lambda' (A, B \in \mathscr{A})$
(iv) if $t \in \Lambda$ and $A = \{t' \in \Lambda | t' \prec t\}$ then

$$h(x, A \cup \{t\}) = \begin{cases} h(x, A) + \alpha h(x', A) & \text{if } x' \in X, (x, x') \in E \text{ and } \lambda(x, x') = t \\ h(x, A) & \text{if no such } x' \in X \text{ exists.} \end{cases}$$

The following fact is also easily verified:

(1.7) PROPOSITION. *With notation as in* (1.6), *define* $I \subseteq C_2$ *by*

$$I = \{(x, y, z) \in C_2 | \lambda(x, y) > \lambda(y, z)\}.$$

Then

$$h(x, \Lambda) = \sum_{y \geqslant x} r_I(x, y) h(y, \varnothing).$$

(1.8) Suppose now that in (1.6), (1.7) Λ and λ are such that $r_I = r_{C_2 \backslash I}$. It follows from (1.2)(i) and (1.7) that for any $y \in X$ there is a unique function $h: X \times \mathscr{A}' \to \mathscr{R}$ satisfying conditions (iii), (iv) of (1.6) and the following conditions

$$h(x, A) = 0 \text{ unless } x \leqslant y$$

$$h(y, \varnothing) = 1 \quad \text{and} \quad h(x, \varnothing) \in u^{-1}\mathbb{Z}[u^{-1}] \quad (x \neq y)$$

$$h(x, \varnothing) = \overline{h(x, \Lambda)} \quad (x \in X).$$

(1.9) The construction described in (1.6) may also be applied to the opposite poset of X. Specifically, $E' = \{(x, y) \in X \times X \mid (y, x) \in E\}$ is the edge set of a directed graph, and E' is naturally labelled by $\lambda': E' \to \Lambda$ where $\lambda'(x, y) = \lambda(y, x)((x, y) \in E')$.

Suppose that $h': X \times \mathscr{A} \to \mathscr{R}$ is obtained by applying the construction in (1.6) to the reverse poset of X, using E' as edge set and λ' as edge-labelling, and that $h: X \times \mathscr{A} \to \mathscr{R}$ is as in (1.6). Moreover, suppose that the following holds:

$$\{x \in X \mid h(x, A)h'(x, A) \neq 0 \text{ for some } A \in \mathscr{A}\}$$

is finite, Then it is easily checked that

$$\sum_{x \in X} h(x, A)\overline{h'(x, A)} = \sum_{x \in X} h(x, \varnothing)\overline{h'(x, \varnothing)} \quad \text{for all } A \in \mathscr{A}.$$

Similarly, if $l: X \to \mathbb{Z}$ is a function such that $l(x) - l(y)$ is an odd integer for all $(x, y) \in E$, then

$$\sum_{x \in X} (-1)^{l(x)} h(x, A)h'(x, A) \text{ is independent of } A \in \mathscr{A}.$$

(1.10) We mention, without proof, the following facts.

(i) Let $r \in \mathscr{M}$ satisfy $r(x, x) = 1 (x \in X)$, $r(x, y) \in \alpha\mathbb{Z}[\alpha]$ $(x, y \in X, x \neq y)$ and set $s = \bar{r}^{-1}$. Define

$$R_{j,p} = \sum_{k=1}^{j+1} (-1)^{j+1-k} \binom{p-k}{j+1-k} R_p^k \in \mathscr{M} \quad (0 \leqslant j \leqslant p-1)$$

where $R_p^k(x, y)$ is the coefficient of α^p in $(r-1)^k(x, y)$, and define $S_{j,p}$ similarly using s instead of r.

Then it may be shown that $R_{j,p} = S_{p-1-j,p} (0 \leqslant j \leqslant p - 1)$. These equations are of interest when, for example,

(a) X is an Eulerian poset [16] and $r(x, y) = \alpha^{l(x,y)} (x \leqslant y)$ where $l(x, y)$ denotes the length of the interval $[x, y]$, or

(b) X is a Coxeter group with Bruhat order, and $r = R$ as defined in (3.1).

(ii) Suppose that in (i), $r = r_I$ for some $I \subseteq C_2$. Then it may be shown that

$$R_{j,p}(x, y) = \#\{\tau \in C_p(x, y) \mid \#\{i \mid 1 \leqslant i \leqslant p-1, \tau_i \notin I\} = j\} (0 \leqslant j \leqslant p-1).$$

2. Orderings of reflections of Coxeter groups

Let (W, S) be a Coxeter system and $l: W \to \mathbb{N}$ denote the corresponding length function. Without loss of generality, assume that (W, S) is realized geometrically as a group of isometries of a real vector space V as in [10, 3], and adopt the notation there. In particular, Π denotes the set of simple roots, Φ^+ the set of positive roots and for non-isotropic $\alpha \in V$, $r_\alpha: V \to V$ denotes the reflection in α.

Let $T = \bigcup_{w \in W} wSw^{-1}$ be the set of reflections of (W, S) and regard the power set $\mathscr{P}(T)$ as an abelian group under symmetric difference:

$$A + B = (A \cup B) \backslash (A \cap B) \quad (A, B \subseteq T).$$

Define $N: W \to \mathscr{P}(T)$ by $N(w) = \{t \in T | l(tw) < l(w)\}$; then N is the unique function $W \to \mathscr{P}(T)$ satisfying $N(r) = \{r\} (r \in S)$ and $N(xy) = N(x) + xN(y)x^{-1} (x, y \in W)$.

Recall from [9] that if W' is any reflection subgroup of W (i.e. $W' = \langle W' \cap T \rangle$) then $\chi(W') = \{t \in T | N(t) \cap W' = \{t\}\}$ is a set of Coxeter generators for W'. A reflection subgroup W' of W is said to be dihedral if $\#(\chi(W')) = 2$.

(2.1) DEFINITION. A total order $\preceq$ on T is called a reflection order if for any dihedral reflection subgroup W' of W either $r \prec rsr \prec \cdots \prec srs \prec s$ or $s \prec srs \prec \cdots \prec rsr \prec r$ where $\{r, s\} = \chi(W')$.

Here, for example, $r \prec rsr \prec \cdots \prec srs \prec s$ means that

$$\begin{cases} (rsr\cdots)_{2m+1} \preceq (rsr\cdots)_{2n+1} & (1 \leqslant 2m + 1 \leqslant 2n + 1 \leqslant \mathrm{ord}(rs)) \\ (srs\cdots)_{2m+1} \preceq (srs\cdots)_{2n+1} & (1 \leqslant 2n + 1 \leqslant 2m + 1 \leqslant \mathrm{ord}(rs))\,\mathrm{I} \\ (rsr\cdots)_{2m+1} \preceq (srs\cdots)_{2n+1} & (1 \leqslant 2m + 1,\ 2n + 1 \leqslant \mathrm{ord}(rs)). \end{cases}$$

(2.2) Before proving the existence of reflection orders, it is convenient to note the following more geometric formulation of their definition.

Any total order $\preceq$ on T determines a total order $\preceq'$ on Φ^+ by the condition $\alpha \preceq' \beta$ iff $r_\alpha \preceq r_\beta$. The order $\preceq$ is a reflection order iff the order $\preceq'$ has the following property: if $\alpha, \beta, \gamma \in \Phi^+$, $\alpha \preceq' \gamma$ and $\beta = c\alpha + d\gamma$ where $c \geqslant 0$, $d \geqslant 0$ then $\alpha \preceq' \beta \preceq' \gamma$.

To see this, one checks first by direct calculation that the result holds if $\Pi = \{\alpha, \beta\}$ where

$$(\alpha\,|\,\beta)\in\left\{-\cos\frac{\pi}{n}\,\middle|\, n\in\mathbb{N},\, n\geqslant 2\right\}\cup(-\infty,\ -1].$$

The result for general (W, S) reduces to this case since

(i) If W' is a dihedral reflection subgroup of (W, S) and $\chi(W')=\{r_\alpha, r_\beta\}$ $(\alpha,\beta\in\Phi^+)$ then $(\alpha\,|\,\beta)\in\{-\cos\pi/n|n\in\mathbb{N},\, n\geqslant 2\}\cup(-\infty,\ -1]$ [9, (4.4)]
(ii) if α, β, $\gamma\in\Phi^+(\alpha\neq\gamma)$ then $\langle r_\alpha,\ r_\beta,\ r_\gamma\rangle$ is dihedral iff $\beta\in\mathbb{R}\alpha+\mathbb{R}\gamma$ (see [9], (3.2)).

(2.3) PROPOSITION. *Let I, J be disjoint subsets of S and let $W_I=\langle I\rangle$, $W_J=\langle J\rangle$ be the corresponding parabolic subgroups of W. Then there is a reflection order $\preceq$ on T such that*

(i) $t\prec t'$ *if* $t\in W_I\cap T$ *and* $t'\in T\backslash W_I$
(ii) $t\prec t'$ *if* $t\in T\backslash W_J$ *and* $t'\in W_J\cap T$

Proof. Let U denote the affine hyperplane

$$U=\left\{\sum_{\alpha\in\Pi} c_\alpha\alpha\,\middle|\,\sum_{\alpha\in\Pi} c_\alpha=1\right\}$$

of V spanned by Π and define

$$\Psi=\left\{\alpha\in U\,|\,(\alpha\,|\,\alpha)>0,\ \frac{\alpha}{(\alpha\,|\,\alpha)^{1/2}}\ c_\alpha\in\Phi^+\right\}.$$

Note that the map $\Psi\to\Phi^+$ defined by $\alpha\mapsto\alpha/(\alpha\,|\,\alpha)^{1/2}$ is a bijection.

Let $f\colon V\to\mathbb{R}$ denote an arbitrary $\mathbb{R}$-linear map such that for $\alpha\in\Pi$,

$$f(\alpha)\begin{cases}=0 & \text{if } r_\alpha\in I\\ =1 & \text{if } r_\alpha\in J\\ \in(0,\ 1) & \text{if } r_\alpha\in S\backslash(I\cup J)\end{cases}.$$

Set $P_0=\{v\in V\,|\,f(v)>0\}$. Note that $P=P_0$ satisfies conditions (i)–(iii) below: (i) $0\notin P$ (ii) if x, $y\in P$ then $x+y\in P$ (iii) if $x\in P$ and $c\in\mathbb{R}$, $c>0$ then $cx\in P$. Choose a maximal subset $P\supseteq P_0$ of V satisfying (i)–(iii) above (if V is finite-dimensional, this can be done by a simple direct argument). Define a total order $\preceq'$ on Ψ by setting $\alpha\preceq'\beta$ iff $\beta-\alpha\in P\cup\{0\}$ $(\alpha,\beta\in\Psi)$. Note that if α, β, $\gamma\in\Psi$, $\alpha\preceq\gamma$ and $\beta=c\alpha+(1-c)\gamma$ where $0\leqslant c\leqslant 1$, then $\alpha\preceq'\beta\preceq'\gamma$. Using (2.2), one sees that the total order $\preceq$ on T defined by $r_\alpha\preceq r_\beta$ iff $\alpha\preceq'\beta$ $(\alpha,\beta\in\Psi)$ is a reflection order.

Now suppose α, $\beta\in\Psi$ and $r_\alpha\in W_I$ but $r_\beta\notin W_I$. Then one may write

$\alpha = \sum_{\gamma\in\Pi} c_\gamma \gamma$ where $\sum_{\gamma\in\Pi} c_\gamma = 1$ and $c_\gamma = 0$ unless $r_\gamma \in I$, and similarly, $\beta = \sum_{\gamma\in\Pi} d_\gamma \gamma$ where $\sum_{\gamma\in\Pi} d_\gamma = 1$ and $d_\gamma \neq 0$ for some $\gamma \in \Pi$ with $r_\gamma \in S\backslash I$. It follows that $f(\alpha) = 0$ and $f(\beta) > 0$ whence $\beta - \alpha \in P_0 \subseteq P$, which gives $r_\alpha \preceq r_\beta$. Similarly, $t \prec t'$ if $t \in T\backslash W_J$ and $t' \in W_J \cap T$.

(2.4) REMARKS

(i) The reverse of a reflection order is a reflection order.

(ii) Let (W', S') be a reflection subsystem of (W, S) (i.e. W' is a reflection subgroup of W and $S' = \chi(W')$). The restriction of a reflection order on T to an order on $W' \cap T$ is a reflection order on the reflections of (W', S')

(iii) If $\preceq$ is a reflection order, $r \in S$, $t \in T$ and $r \preceq t$ then $r \preceq t'$ for all $t' \in T \cap \langle r, t\rangle$ (since $r \in \chi(\langle r, t\rangle)$). In particular, $r \preceq t$ iff $r \preceq rtr$.

(2.5) PROPOSITION. *Let $\preceq$ be a reflection order on T, and $r \in S$. Then the partial order $\preceq'$ on T defined by*

$$t \preceq' t' \text{ iff } \begin{cases} t = r \\ \text{or } (t \neq r,\ r \prec t' \text{ and } t \preceq t') \\ \text{or } (t \neq r,\ t' \prec r \text{ and } rtr \prec rt'r \end{cases}$$

is a reflection order on T.

Proof. To show that $\preceq'$ is transitive, it will suffice to show that if $t_1, t_2, t_3 \in T$ and $t_1 \prec' t_2$, $t_2 \prec' t_3$ then $t_1 \prec' t_3$. Note that $r \notin \{t_2, t_3\}$ so $r \prec' t_3$. Hence we may assume $t_1 \neq r$. Consider first the case $t_3 \prec r$. Then $rt_3r \prec r$ so $rt_2r \prec rt_3r \prec r$ (since $t_2 \prec' t_3$) and thus $t_2 \prec r$. From $t_1 \prec' t_2$ it follows that $rt_1r \prec rt_2r$. Hence $rt_1r \prec rt_3r$, proving $t_1 \prec' t_3$ as required. Now suppose $r \prec t_2$. Since $t_2 \prec' t_3$, it follows that $r \prec t_3$; hence $t_1 \prec t_2 \prec t_3$ which gives $t_1 \prec' t_3$. The remaining case is $t_2 \prec r \prec t_3$. Here, $t_1 \prec' t_2$ and $rt_2r \prec r$ so $rt_1r \prec rt_2r \prec r$. Therefore $t_1 \prec r \prec t_3$, proving $t_1 \prec' t_3$. The proof that $\preceq'$ is reflexive and antisymmetric is similar (and simpler).

To show that $\preceq'$ is a reflection order, fix a dihedral reflection subgroup W' of W and write $\chi(W') = \{t, t'\}$ where $t \prec t'$. One must check that either

$$t \prec' tt't \prec' \cdots \prec' t'tt' \prec' t' \quad \text{or} \quad t' \prec' t'tt' \prec' \cdots \prec' tt't \prec' t.$$

Consider the case $t \prec r \prec t'$. Then $t \prec tt't \prec \cdots \prec t'tt' \prec t'$. Now $\chi(rW'r) = \{rtr, rt'r\}$ (by [9, (3.2) (i)]) and $rtr \prec r \prec rt'r$, so $rtr \prec rtt'tr \prec \cdots \prec rt'tt'r \prec rt'r$. Note that if $t_1 \prec r \prec t_2$ then $t_1 \prec' t_2$. It follows that $t \prec' tt't \prec' \cdots \prec' t'tt' \prec' t'$. The remaining cases ($t \prec t' \prec r$, $r \prec t \prec t'$, $t = r$, $t' = r$) are treated similarly.

(2.6) Since $N(xy) = N(x) + xN(y)x^{-1}$ $(x, y \in W)$, there is a W-action on the set $\mathscr{P}(T)$ defined by $w \cdot A = N(w) + wAw^{-1}$ $(w \in W, A \subseteq T)$. Note that $w \cdot (A + T) = (w \cdot A) + T$ $(w \in W, A \subseteq T)$.

Let $\mathscr{A}$ denote the set of all initial sections of reflection orders on T. Thus, if $A \subseteq T$, then $A \in \mathscr{A}$ iff there is a reflection order $\preceq$ on T such that $a \prec b$ for all $a \in A$ and $b \in T \backslash A$.

(2.7) LEMMA. *If $A \in \mathscr{A}$ then $A + T \in A$ and $w \cdot A \in \mathscr{A}$ for all $w \in W$.*

Proof. Let $A \in \mathscr{A}$. Then $A + T \in \mathscr{A}$ by (2.4)(i). To prove $w \cdot A \in \mathscr{A} (w \in W)$ it is sufficient to show that $r \cdot A \in \mathscr{A} (r \in S)$. Suppose first that $r \notin A$; if A is an initial section of the reflection order $\preceq$, it is easily checked that $r \cdot A$ is an initial section of the reflection order $\preceq'$ defined in (2.5). On the other hand, if $r \in A$ then $rA = r \cdot (A + T) + T \in \mathscr{A}$ by what has already been shown.

(2.8) COROLLARY. *If A, $B \in \mathscr{A}$ are initial sections of a reflection order $\preceq$, $y \in W$ and $N(y^{-1}) \cap A = N(y^{-1}) \cap B$ then there is a reflection order $\preceq'$ of which $y \cdot A$ and $y \cdot B$ are both initial sections.*

Proof. The proof of this easily reduces to the case $y \in S$, when it is clear from the proof of (2.7).

(2.9) LEMMA. *Let $A \in \mathscr{A}$ and $t \in T \backslash A$. Then $t \cdot A = A + \{t\}$ iff there exists a reflection order $\preccurlyeq$ on T such that $A = \{t' \in T | t' \prec t\}$.*

Proof. Suppose $A \in \mathscr{A}$ and $t \cdot A = A + \{t\}$. Let $\preccurlyeq$, $\preccurlyeq'$ be reflection orders of which A, $t \cdot A$ are initial sections, and set $B = T \backslash (A + \{t\})$. There is a unique partial order $\preccurlyeq''$ on T such that

(i) the restrictions of $\preccurlyeq$ and $\preccurlyeq''$ to partial orders on A coincide
(ii) the restrictions of $\preccurlyeq'$ and $\preccurlyeq''$ to partial orders on B coincide
(iii) $t' \preccurlyeq'' t''$ for all $t' \in A \cup \{t\}$, $t'' \in B \cup \{t\}$.

It is easily checked that $\preccurlyeq''$ is a reflection order and $A = \{t' \in T \mid t' \prec t\}$.

Conversely, suppose that $\preccurlyeq$ is a reflection order and $A = \{t' \in T \mid t' \prec t\}$. To show that $t \cdot A = A + \{t\}$, it is sufficient to show that for any dihedral reflection subgroup W' of (W, S) with $t \in W'$ we have $(t \cdot A) \cap W' = (A + \{t\}) \cap W'$ i.e. $[N(t) \cap W'] + t(A \cap W')t^{-1} = (A \cap W') + \{t\}$. By (2.4)(i) and [9, (3.3)(ii)], this is just the assertion of the lemma for the dihedral Coxeter system $(W', \chi(W'))$. Therefore it is sufficient to check the claim when (W, S) is dihedral, which is easily done.

(2.10) REMARK. Suppose that $\preccurlyeq$, $\preccurlyeq'$ are reflection orders on T and that A is an initial section of both $\preccurlyeq$ and $\preccurlyeq'$; set $B = T \backslash A$. Then there is a unique order $\preccurlyeq''$ on T satisfying conditions (i), (ii) of the proof of (2.8) and such that $a \prec'' b$ for all $a \in A$, $b \in B$. It is easily checked that $\preccurlyeq''$ is a reflection order.

In the remainder of this section, we give a number of additional facts concerning reflection orders and their initial sections. These won't be needed in subsequent sections, but are of some independent interest.

(2.11) LEMMA. *Let* Γ *be a finite subset of* Φ^+ *and* $A = \{r_\alpha \mid \alpha \in \Gamma\} \subseteq T$. *Then the following are equivalent:*

(i) $A \in \mathscr{A}$
(ii) $A = N(w)$ *for some* $w \in W$
(iii) *there exists a non-zero linear function* $\phi: V \to \mathbb{R}$ *such that* $\Gamma = \Phi^+ \cap \phi^{-1}((0, \infty))$

Proof. We prove that (i) ⇒ (ii) by induction on $\#(A)$. If $\#(A) = 0$, (ii) holds. Otherwise, let $\preccurlyeq$ be a reflection order of which A is an initial section and let t denote the maximum element of A in the total order induced by $\preccurlyeq$. Then $A = \{t' \in T \mid t' \preccurlyeq t\}$. By (2.9) and (2.7), $A \backslash \{t\} = t \cdot A \in \mathscr{A}$. By induction, $t \cdot A = N(w)$ for some $w \in W$, so $A = t \cdot N(w) = N(tw)$.

To prove (ii) ⇒ (iii), define $\phi: V \to \mathbb{R}$ by $\phi(\sum_{\alpha \in \Pi} c_\alpha \alpha) = \sum_{\alpha \in \Pi} c_\alpha$. Then $\Phi^+ = \phi^{-1}((0, \infty)) \cap \Phi$. If $A = N(w)(w \in W)$ then $\Gamma = \Phi^+ \cap w(-\Phi^+) = [(-\phi \circ w^{-1})^{-1}(0, \infty)] \cap \Phi^+$.

The implication (iii) ⇒ (i) does not depend on finiteness of Γ. Suppose $\Gamma = \Phi^+ \cap \phi^{-1}((0, \infty))$. It is easily seen that there is a total order $\leqslant$ on V (compatible with the vector space operations) such that $\Gamma = \{\alpha \in \Phi^+ \mid \alpha < 0\}$. As in the proof of (2.3), $\leqslant$ gives rise to a reflection order of T, and $A = \{r_\gamma \mid \gamma \in \Gamma\}$ is an initial section of this order.

(2.12) REMARK. Let $\Gamma \subseteq \Phi^+$. Say that Γ is closed if the conditions $\alpha, \beta \in \Gamma$, $\gamma \in \Phi^+$, $\gamma = c\alpha + d\beta$ where $c, d \geqslant 0$ imply that $\gamma \in \Gamma$.

Note that if $\{r_\alpha \mid \alpha \in \Gamma\} \in \mathscr{A}$ then both Γ and $\Phi^+ \backslash \Gamma$ are closed. The converse is open.

(2.13) PROPOSITION. *Let* (W, S) *be a finite Coxeter system with longest element* w_0, *and* $t_1, \ldots, t_n (n = l(w_0))$ *be the elements of* T. *Then the total order* $\preccurlyeq$ *on* T *such that* $t_1 \prec t_2 \prec \cdots \prec t_n$ *is a reflection order iff there is a reduced expression* $w_0 = r_1 \cdots r_n (r_i \in S)$ *such that* $t_i = r_1 \cdots r_{i-1} r_i r_{i-1} \cdots r_1$ $(i = 1, \ldots, n)$.

Proof. Suppose $t_1 \prec \cdots \prec t_n$ defines a reflection order. Using (2.11), define $v_i \in W$ $(i = 0, \ldots, n)$ by $\{t_1, \ldots, t_i\} = N(v_i)$. Write $v_i = v_{i-1}\, r_i (r_i \in W;\ i = 1, \ldots, n)$. Then $N(v_i) = N(v_{i-1}) + v_{i-1}\ N(r_i) v_{i-1}^{-1}$ from which $N(r_i) = \{v_{i-1}^{-1} t_i v_{i-1}\}$. Hence $r_i \in S$ $(i = 1, \ldots, n)$. We now have $N(r_1 \cdots r_i) = \{t_1, \ldots, t_i\}$ $(i = 0, \ldots, n)$ so $t_i = r_1 \cdots r_i \cdots r_1$. Also, $l(r_1 \cdots r_n) = \#\{t_1, \ldots, t_n\} = n = l(w_0)$ so $r_1 \cdots r_n = w_0$.

The converse follows from (2.11) and the following fact, which is valid for arbitrary Coxeter systems: if a total order $\prec$ on T is such that all its initial sections are elements of $\mathscr{A}$, then $\prec$ is a reflection order. The proof of this fact easily reduces to the case when (W, S) is dihedral, and is left to the reader.

(2.14) PROPOSITION. *Let* $\preccurlyeq$ *denote a fixed reflection order on* T. *Fix* $w \in W$ *and write* $N(w) = \{t_1, \ldots, t_n\}$ *where* $t_1 \prec \cdots \prec t_n$. *Then for* $1 \leqslant i \leqslant n$,

$$\#\{t \in N(t_i w) \mid t_i \prec t\} = \frac{l(w) + l(t_i w) + 1}{2} - i.$$

Proof. The proof is by induction on $l(w)$. The result holds if $l(w) = 0$. Suppose $l(w) \geqslant 1$ and choose $r \in S \cap N(w)$. Write $r = t_{i_0} (1 \leqslant i_0 \leqslant n)$. We show first that

$$\#\{t \in N(rw) \mid r \prec t\} = l(w) - i_0.$$

Let W' be any maximal dihedral reflection subgroup of W with $r \in W'$. Now $\#(N(w) \cap W') = \#(N(rw) \cap W') + 1$ and either $t \preccurlyeq r$ for all $t \in W' \cap T$ or $t \preccurlyeq r$ for all $t \in W' \cap T$. Hence

$$\#\{t \in N(rw) \cap W' \mid t \succ r\} = \#\{t \in N(w) \cap W' \mid t \succ r\}.$$

Summing over all the distinct maximal dihedral reflection subgroups W' with $r \in W'$, one finds

$$\#\{t \in N(rw) \mid t \succ r\} = \#\{t \in N(w) \mid t \succ r\} = l(w) - i_0.$$

Now consider the case $1 \leqslant i \leqslant n$, $i \neq i_0$. Suppose first $i < i_0$. Then $t_i \prec r$, $rt_i r \prec r$. Let $\preccurlyeq'$ be the reflection order on T defined in (2.5).

Suppose firstly that $r \in N(t_i w)$. The map $t \mapsto rtr$ induces a bijection

$$\{t \in N(t_i w) \mid t \prec t_i\} \to \{t \in N(rt_i w) \mid t \prec' rt_i r\}.$$

The result now follows by applying the inductive hypothesis to

$$\#\{t \in N((rt_i r)rw) \mid rt_i r \prec' t\}$$

The case $r \notin N(t_i w)$ is similar; one notes that $t \mapsto rtr$ induces a bijection

$$\{t \in N(t_i w) \mid t \prec t_i\} \to \{t \in N(rt_i w) \backslash \{r\} \mid t \prec' rt_i r\}.$$

This proves the assertion of the proposition for $i \leqslant i_0$. The result for $i \geqslant i_0$ follows by applying the result for $i \leqslant i_0$ to the reverse order of $\preccurlyeq$.

3. Reflection orders and the Hecke algebra

(3.1) As in (1.1), let $\mathscr{R} = \mathbb{Z}[u, u^{-1}]$ where u is an indeterminate. Hence forward, we write $q^{1/2}$ in place of u (thus, $u^n = q^{n/2}$, $n \in \mathbb{Z}$) in accordance with standard notation for the Hecke algebra.

Recall that the generic Hecke algebra $\mathscr{H}(W)$ of (W, S) over $\mathscr{R}$ is the unital associative $\mathscr{R}$-algebra which has a free $\mathscr{R}$-basis $\{\tilde{T}_w\}_{w\in W}$ and multiplication determined by

$$\tilde{T}_r\tilde{T}_w = \begin{cases} \tilde{T}_{rw} & \text{if } rw > w \\ \tilde{T}_{rw} + \alpha\tilde{T}_w & \text{if } rw < w \quad (r\in S,\ w\in W) \end{cases}$$

where $\alpha = q^{1/2} - q^{-1/2}$ [11].

There is a ring involution $h \mapsto \bar{h}$ of $\mathscr{R}$ defined by $\sum_{w\in W} a_w\tilde{T}_w \mapsto \sum_{w\in W} \bar{a}_w\tilde{T}_{w^{-1}}^{-1}$ where $a \mapsto \bar{a}(a\in\mathscr{R})$ is as in (1.1). For $w\in W$, define $R(x, w)\in\mathscr{R}$ by

$$\tilde{T}_{w^{-1}}^{-1} = \sum_{x\in W} R(x, w)\tilde{T}_x$$

Regard R as an element of the incidence algebra $\mathscr{M}$ of W (1.1) where W is equipped with Bruhat order; then one has $R^{-1} = \bar{R}$. Let $p\in\mathscr{M}$ be the element determined by (1.2)(i) (with $r = R$). For fixed $w\in W$, $\sum_{x\in W} p(x, w)\tilde{T}_x = C'_w$ where C'_w is the unique element of $\tilde{T}_w + \sum_{x<w} q^{-1/2}\mathbb{Z}[q^{-1/2}]\tilde{T}_x$ such that $\bar{C}'_w = C'_w$ [11, (1.1c)]. Note that $R = p\bar{p}^{-1}$.

(3.2) Let $\mathscr{A}$ be as defined as in (2.6). For $A\in\mathscr{A}$ and $y\in W$, we will sometimes write yA for $y\cdot A$. Let $\mathscr{H}'$ denote the set of functions $f\colon W\times\mathscr{A}\to\mathscr{R}$ such that $\{w\in W \mid f(w, A)\neq 0 \text{ for some } A\in\mathscr{A}\}$ is finite. Regard $\mathscr{H}'$ as an $\mathscr{R}$-module in the natural way.

Note that $\mathscr{H}'$ becomes an associative $\mathscr{R}$-algebra under the product defined by

$$(fg)(w, A) = \sum_{\substack{x,y\in W\\ xy=w}} f(x, yA)g(y, A) \quad (f, g\in\mathscr{H}', (w, A)\in W\times\mathscr{A}).$$

The identity element is $\delta\colon W\times\mathscr{A}\to\mathscr{R}$ where for $(w, A)\in W\times\mathscr{A}$, $\delta(w, A) = 1$ *if* $w = 1$ and $\delta(w, A) = 0$ otherwise. There is an $\mathscr{R}$-antilinear ring involution of $\mathscr{H}'$, denoted as usual by $h\mapsto\bar{h}$ $(h\in\mathscr{H}')$, such that for $h\in\mathscr{H}'$, $\bar{h}(w, A) = \overline{h(w, A+T)}$.

Let $\mathscr{H}$ denote the set of those $f\in\mathscr{H}'$ such that (3.2.1), (3.2.2) below hold:

(3.2.1) For any $w\in W$, there is a finite subset T' of T such that if $A, B\in\mathscr{A}$ are initial sections of the reflection order $\preccurlyeq$ and $A\cap T' = B\cap T'$ then $f(w, A) = f(w, B)$.

(3.2.2) If $A\in\mathscr{A}$, $t\in T\backslash A$ and $t\cdot A = A + \{t\}$ then

$$f(w, tA) = \begin{cases} f(w, A) & \text{if } wt < w \\ f(w, A) + \alpha f(wt, A) & \text{if } wt > w. \end{cases}$$

(3.3) PROPOSITION

(i) *The set* $\mathscr{H}$ *defined above is a subalgebra of* $\mathscr{H}'$.
(ii) *The map* $\theta: \mathscr{H} \to \mathscr{H}(W)$ *defined by* $\theta(f) = \sum_{w \in W} f(w, \varnothing)\tilde{T}_w$ *is an isomorphism of* $\mathscr{R}$*-algebras.*
(iii) *If* $f \in \mathscr{H}$ *then* $\bar{f} \in \mathscr{H}$ *and* $\overline{\theta(f)} = \theta(\bar{f})$.

Proof. Let $\theta: \mathscr{H} \to \mathscr{H}(W)$ be the $\mathscr{R}$-linear map $f \mapsto \sum_{w \in W} f(w, \varnothing)\tilde{T}_w$. We first show that θ is injective. Let Ω denote the Bruhat graph of (W, S) [10, (1.1)]; recall that Ω is defined to be the directed graph with vertex set W and edge set $E = \{(wt, w) \mid w \in W, t \in N(w^{-1})\}$. Define a function $\lambda: E \to T$ by $\lambda(x, y) = x^{-1}y((x, y) \in E)$. Applying the discussion of (1.6) (for a fixed reflection order $\preccurlyeq$ on T) and making use of (2.9), one sees that θ is injective.

Next we show that the set of those $f \in \mathscr{H}'$ satisfying (3.2.1) is a subalgebra of $\mathscr{H}'$. Suppose $f, g \in \mathscr{H}'$ satisfy (3.2.1) and fix $w \in W$. Let $X = \{x \in W \mid f(x, A) \neq 0 \text{ for some } A \in \mathscr{A}\}$ and $Y = \{x^{-1}w \mid x \in X\}$; note X, Y are finite by definition of $\mathscr{H}'$. For $x \in X$ (resp. $y \in Y$) choose a finite subset S_x (resp. S'_y) of T such that $f(x, A) = f(x, B)$ (resp. $g(y, A) = g(y, B)$) if A, B are initial sections of a reflection order $\preccurlyeq$ and $A \cap S_x = B \cap S_x$ (resp. $A \cap S'_y = B \cap S'_y$). Let $T' = \bigcup_{y \in Y}(N(y^{-1}) \cup S'_y \cup y^{-1}(\bigcup_{x \in X} S_x)y)$. Suppose A, B are initial sections of a reflection order $\preccurlyeq$ with $A \cap T' = B \cap T'$. By (2.8), for any $y \in Y$ there is a reflection order of which yA and yB are both initial sections. Moreover, for $x \in X$ and $y \in Y$, $f(x, yA) = f(x, yB)$ since $yA \cap S_x = yB \cap S_x$; also $g(y, A) = g(y, B)(y \in Y)$. Hence

$$\begin{aligned}(fg)(w, A) &= \sum_{\substack{(x,y) \in X \times Y \\ xy = w}} f(x, yA)g(y, A) \\ &= \sum_{\substack{(x,y) \in X \times Y \\ xy = w}} f(x, yB)g(y, B) = (fg)(w, B),\end{aligned}$$

proving fg satisfies (3.2.1).

Now we show that the set of those $f \in \mathscr{H}'$ satisfying (3.2.2) is a subalgebra of $\mathscr{H}'$. Suppose $f, g \in \mathscr{H}'$ satisfy (3.2.2). Suppose $A \in \mathscr{A}$ and $t \in T \backslash A$ satisfy $tA = A + \{t\}$. Note that $yty^{-1}(yA) = yA + \{yty^{-1}\}(y \in W)$. Let $x, y \in W$. If $yt > y$, it follows that $yty^{-1} \notin yA$, that $xyty^{-1} > x$ iff $xyt > xy$ and that

$$f(x, ytA) = \begin{cases} f(x, yA) & xyt < xy \\ f(x, yA) + \alpha f(xyty^{-1}, yA) & xyt > xy. \end{cases}$$

On the other hand, if $yt < y$ then $yty^{-1} \in yA$, and $xyty^{-1} > x$ iff $xyt < xy$.

For any $x, y \in W$ write

$$I(x, y) = \begin{cases} 1 & x \leqslant y \\ 0 & \text{otherwise} \end{cases}$$

If $w \in W$, we have

$$(fg)(w, tA) = \sum_{xy=w} f(x, yty^{-1}(yA))g(y, tA)$$

$$= \sum_{\substack{xy=w \\ yt>y}} [f(x, yA) + \alpha I(x, xyty^{-1})f(xyty^{-1}, yA)][g(y, A) + \alpha g(yt, A)]$$

$$+ \sum_{\substack{xy=w \\ yt<y}} [f(x, yA) - \alpha I(x, xyty^{-1})f(xyty^{-1}, yA)]g(y, A)$$

$$= \sum_{xy=w} f(x, yA)g(y, A) + \alpha I(wt, w)h(w, A) + \alpha I(w, wt)k(w, A)$$

where

$$h(w, A) = \sum_{\substack{xy=w \\ yt>y}} f(x, yA)g(yt, A) - \sum_{\substack{xy=w \\ yt<y}} f(xyty^{-1}, yA)g(y, A)$$

and

$$k(w, A) = \sum_{\substack{xy=w \\ yt>y}} [(f(x, yA) + \alpha f(xyty^{-1}, yA))g(yt, A) + f(xyty^{-1}, yA)g(y, A)].$$

Now if $wt < w$,

$$h(w, A) = \sum_{\substack{xy=w \\ yt>y}} f(x, ytA)g(yt, A) - \sum_{\substack{xy=w \\ yt<y}} f(xyty^{-1}, yA)g(y, A) = 0$$

and if $wt > w$,

$$k(w, A) = \sum_{\substack{xy=w \\ yt>y}} [f(x, ytA)g(yt, A) + f(xyty^{-1}, yA)g(y, A)]$$

$$= (fg)(wt, A).$$

Hence $(fg)(w, tA) = (fg)(w, A) + \alpha I(w, wt)(fg)(wt, A)$ as required to complete the proof of (i).

Define $e_r \in \mathscr{H}$ $(r \in S)$ by

$$e_r(w, A) = \begin{cases} 1 & w = r \\ \alpha & w = 1,\ r \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then $\theta(e_r) = \tilde{T}_r$ and $\theta(e_r h) = \tilde{T}_r \theta(h)$ $(h \in \mathscr{H})$. This implies that θ is bijective (since $\mathscr{H}(W)$ is generated as an $\mathscr{R}$-algebra by $\{\tilde{T}_r\}_{r \in S}$); hence $\mathscr{H}$ is generated as an $\mathscr{R}$-algebra by $\{e_r\}_{r \in S}$ and θ is an $\mathscr{R}$-algebra isomorphism, proving (ii). To prove (iii), it is sufficient to note that $\bar{e}_r \in \mathscr{H}$ and $\theta(\bar{e}_r) = \tilde{T}_r^{-1}$ $(r \in S)$.

(3.4) COROLLARY. *Let $\preccurlyeq$ denote a fixed reflection order on T and for x, $w \in W$ let $R(x, w)$ be as defined in* (3.1). *Then*

$$R(x, w) = \sum_{n \in \mathbb{N}} \sum_{(t_1, \dots, t_n)} \bar{\alpha}^n$$

where for fixed $n \in \mathbb{N}$, the inner sum is taken over those $(t_1, \dots, t_n) \in T^n$ such that $x < xt_1 < xt_1t_2 < \cdots < xt_1 \cdots t_n = w$ and $t_n \prec t_{n-1} \prec \cdots \prec t_1$.

Proof. Let Ω, E and λ be as in the proof of (3.3). As in (1.6), define $I \subseteq C_2$ by

$$I = \{(x, y, z) \in C_2 \mid \lambda(x, y) > \lambda(y, z)\}.$$

Recall the definition of $r_I \in \mathscr{M}$ from (1.3), and note that (3.4) simply asserts that $R = \bar{r}_I$.

Fix $w \in W$ and let $f = \theta^{-1}(C'_w)$. Note that $f(x, \varnothing) = p(x, w)$ where p is as in (3.1). Let $\mathscr{A}'$ denote the set of initial sections of $\preccurlyeq$ and let $h: X \times \mathscr{A}' \to \mathscr{R}$ denote the restriction of the function f to $X \times \mathscr{A}'$. Then h satisfies (1.6) (ii)–(iv), and $h(x, A) = 0$ unless $x \leqslant w$. It follows from (1.7) and (3.3) (iii) that

$$\overline{h(x, \phi)} = h(x, T) = \sum_{y \in W} r_I(x, y) h(y, \phi)$$

i.e. $\bar{p}(x, w) = \sum_{y \in W} r_I(x, y) p(y, w)$. Since x, w are arbitrary, $\bar{p} = r_I p$.

Hence $R = p\bar{p}^{-1} = \bar{r}_I$ as required.

(3.5) One may also apply the construction in (1.6) to the reverse Bruhat order to obtain a certain module for $\mathscr{H}$. The following sections sketch the details, paralleling (3.1)–(3.3).

Let $\mathscr{K}(W)$ denote the set of possibly infinite formal $\mathscr{R}$-linear combinations $\sum_{w \in W} a_w \tilde{t}_w (a_w \in \mathscr{R})$, regarded as an $\mathscr{R}$-module in the natural way. It is easily checked that there is a left $\mathscr{H}(W)$-module structure on $\mathscr{K}(W)$ such that

$$\tilde{T}_r\left(\sum_{w\in W} a_w\tilde{t}_w\right)=\sum_{w\in W} b_w\tilde{t}_w$$

where

$$b_w=\begin{cases}a_{rw} & \text{if } rw<w\\ a_{rw}+\alpha a_w & \text{if } rw>w.\end{cases}$$

There is an $\mathscr{R}$-antilinear map $\mathscr{K}(W)\to\mathscr{K}(W)$ defined by

$$\sum_{w\in W} a_w\tilde{t}_w\mapsto\sum_{w\in W}\left(\sum_{z\leqslant w}\bar{a}_z R(z,w)\right)\tilde{t}_w;$$

we denote this map by $k\mapsto\bar{k}(k\in\mathscr{K}(W))$. Then $k\mapsto\bar{k}$ is an involution (i.e. $\bar{\bar{k}}=k$, $k\in\mathscr{K}(W)$) and $\overline{hk}=\bar{h}\bar{k}$ $(h\in\mathscr{H}(W),\ k\in\mathscr{K}(W))$; the last claim may be proved using [11, (2.0.6)] but will be obvious after (3.7).

Let q be the element of the incidence algebra $\mathscr{M}$ defined by (1.2) (ii) (with $r=R$) and define $D'_w=\sum_{y\in W}q(w,y)\tilde{t}_y$ $(w\in W)$; note there is some conflict with the notation of [15]. Then D'_w is the unique element of $\tilde{t}_w+\sum_{y>w}q^{-1/2}\mathbb{Z}[q^{-1/2}]\tilde{t}_y$ such that $\bar{D}'_w=D'_w$.

(3.6) Let $\mathscr{K}'$ denote the set of functions $f\colon W\times\mathscr{A}\to\mathscr{R}$, regarded as an $\mathscr{R}$-module in the natural way. For $h\in\mathscr{H}'$ and $k\in\mathscr{K}'$, one may define $hk\in\mathscr{K}'$ by

$$(hk)(w,A)=\sum_{\substack{x,y\in W\\ xy=w}} h(x,yA+T)k(y,A).$$

It is easy to check that this makes $\mathscr{K}'$ into a left $\mathscr{H}'$-module. For $k\in\mathscr{K}'$, define $\bar{k}\in\mathscr{K}'$ by $\bar{k}(w,A)=\bar{k}(w,A+T)$. Then $\bar{\bar{k}}=k(k\in\mathscr{K}')$ and $\overline{hk}=\bar{h}\bar{k}(h\in\mathscr{H}',\ k\in\mathscr{K}')$.

Let $\mathscr{K}\subseteq\mathscr{K}'$ denote the set of those $f\in\mathscr{K}'$ which satisfy (3.2.1) above and (3.6.1) below:

(3.6.1) If $A\in\mathscr{A}$, $t\in T\backslash A$ and $tA=A+\{t\}$ then

$$f(w,tA)=\begin{cases}f(w,A) & \text{if } wt>w\\ f(w,A)+\alpha f(wt,A) & \text{if } wt<w\end{cases}$$

(3.7) PROPOSITION.

(i) *Regard $\mathscr{K}'$ as an $\mathscr{H}$-module via the imbedding $\mathscr{H}\hookrightarrow\mathscr{H}'$. Then $\mathscr{K}$ is an*

$\mathscr{H}$-submodule of $\mathscr{K}'$.

(ii) *Let $\eta: \mathscr{K} \to \mathscr{K}(W)$ be the $\mathscr{R}$-linear map defined by $\eta(f) = \sum_{w\in W} f(w, \varnothing)\tilde{t}_w$ $(f \in \mathscr{K})$. Then η is an isomorphism of $\mathscr{R}$-modules, and for $h \in \mathscr{H}$, $f \in \mathscr{K}$ one has $\eta(hf) = \theta(h)\eta(f)$ where θ is as in* (3.3)

(iii) *For $f \in \mathscr{K}$ one has $\bar{f} \in \mathscr{K}$ and $\eta(\bar{f}) = \overline{\eta(f)}$*

Proof. As in the proof of (3.3), one may show that η is injective, that $hf \in \mathscr{K}$ if $h \in \mathscr{H}$ and $f \in \mathscr{K}$ and that $\eta(e_r f) = \theta(e_r)\eta(f)(r \in S, f \in \mathscr{K})$. To prove (i) and (ii), it is therefore sufficient to show that n is surjective. Suppose $\sum_{w\in W} a_w t_w \in \mathscr{K}(W)$. For any reflection order $\preccurlyeq$, let $\mathscr{A}_{\preccurlyeq}$ denote the set of initial sections of $\preccurlyeq$. Using (1.6) one obtains a function $f_{\preccurlyeq}: W \times \mathscr{A}_{\preccurlyeq} \to \mathscr{R}$ such that $f_{\preccurlyeq}(w, \varnothing) = a_w$, (3.2.1) holds and (3.6.1) holds provided $A \in \mathscr{A}_{\preccurlyeq}$ and $tA \in \mathscr{A}_{\preccurlyeq}$; in fact, in (3.2.1) one may take $T' = N(w^{-1})$. To complete the proof of surjectivity of η, one must show that if $\preccurlyeq'$ is another reflection order on T with $A \in \mathscr{A}_{\preccurlyeq'}$ then $f_{\preccurlyeq}(w, A) = f_{\preccurlyeq'}(w, A)$. We prove this by induction on $l(w)$.

Let $\preccurlyeq''$ be the reflection order constructed in (2.10). Write $N(w^{-1}) \cap (T\backslash A) = \{t_1, \ldots, t_k\}$ where $t_1 \prec' \cdots \prec' t_k$ (note $t_1 \prec'' \cdots \prec'' t_k$) and define

$$A_i = \{t \in T \mid t \prec' t_i\} = \{t \in T \mid t \prec'' t_i\} \quad (i = 1, \ldots, k).$$

Note that $f_{\preccurlyeq'}(w, T) = f_{\preccurlyeq''}(w, T)$ by (1.7) and (3.4). Making repeated use of (3.2.2), one finds

$$\begin{aligned} f_{\preccurlyeq'}(w, A) &= f_{\preccurlyeq'}(w, T) - \alpha[f_{\preccurlyeq'}(wt_1, A_1) + \cdots + f_{\preccurlyeq'}(wt_k, A_k)] \\ &= f_{\preccurlyeq''}(w, T) - \alpha[f_{\preccurlyeq''}(wt_1, A_1) + \cdots + f_{\preccurlyeq''}(wt_k, A_k)] \end{aligned}$$

(by induction)

$$= f_{\preccurlyeq''}(w, A).$$

Similarly, by using (3.2.2) to relate $f_{\preccurlyeq}(w, A)$ and $f_{\preccurlyeq}(w, \varnothing)$, one has

$$f_{\preccurlyeq}(w, A) = f_{\preccurlyeq''}(w, A).$$

Hence

$$f_{\preccurlyeq}(w, A) = f_{\preccurlyeq'}(w, A),$$

completing the proof of (i) and (ii).

To prove (iii), one checks from the definition of $\mathscr{K}$ that $\bar{f} \in \mathscr{K}$ if $f \in \mathscr{K}$. Then $\eta(\bar{f}) = \overline{\eta(f)}$ follows from (1.7) and (3.4).

(3.8) For $w \in W$, let $c'_w = \theta^{-1}(C'_w) \in \mathscr{H}$ and $d'_w = \eta^{-1}(D'_w) \in \mathscr{K}$. We now state a number of conjectures concerning the c'_w and d'_w.

For $A \subseteq W$, let $\Omega_{(W,S)}(A)$ be the full subgraph of the Bruhat graph $\Omega_{(W,S)}$ on vertex set A [10]. Note that there is an obvious notion of isomorphism for pairs consisting of a directed graph together with a subset of its vertices.

CONJECTURE 1. For v, $w \in W$ and $A \in \mathscr{A}$, the Laurent polynomial $c'_w(v, A)$ (resp. $d'_v(w, A)$) is completely determined by the isomorphism type of the pair $(\Omega_{(W,S)}([v, w]), \{x \in [v, w] \mid v^{-1}x \in A\})$ (resp. of $(\Omega_{(W,S)}([v, w]), \{x \in [v, w] \mid w^{-1}x \in A\})$).

CONJECTURE 2. For v, $w \in W$ and $A \in \mathscr{A}$ we have $(a) c'_w(v, A) \in \mathscr{R}^+$ and $(b) d'_v(w, A) \in \mathscr{R}^+$.

CONJECTURE 3. For v, $w \in W$ we have (a) $c'_v c'_w \in \sum_{x \in W} \mathscr{R}^+ c'_x$ and (b) $c'_v d'_w \in \sum_{x \in W} \mathscr{R}^+ d'_x$. (One allows infinite sums in (b), though I know of no example in which the sum is not finite).

Conjectures 3(a), (b) may also be formulated directly in $\mathscr{H}(W)$ as follows: 3(a)$'$ $C'_x C'_y \in \sum_{x \in W} \mathscr{R}^+ C'_z (x, y \in W)$ 3$(b)'$ $C'_x C_y \in \sum_{z \in W} \mathscr{R}^+ C_z (x, y \in W)$.

(3.9) We now describe some special cases in which these conjectures are known to hold.

(i) For dihedral groups, Conjectures 1–3 may be checked by straightforward computation.

(ii) For finite Coxeter groups, Conjectures 3(a) and 3(b) are equivalent and the four conjectures 2(a), 2(b) above and 2$(a)'$, 2$(b)'$ below are all equivalent:

$$2(a)' \ \tilde{T}_x C'_y \in \sum_{z \in W} \mathscr{R}^+ \tilde{T}_z \quad (x, y \in W)$$

$$2(b)' \ \tilde{T}_{x^{-1}}^{-1} \tilde{T}_y \in \sum_{z \in W} \mathscr{R}^+ C_z \quad (x, y \in W)$$

(see [8] or [13]).

Now for finite Weyl groups, 3$(a)'$ is proved in [15] and 2$(b)'$ is implicit in [5], so Conjectures 2 and 3 both hold for finite Weyl groups (see [13], [12]).

For W of type H_3 and H_4, one has $P_{v,w} \in \mathscr{R}^+$ and $Q_{v,w} \in \mathscr{R}^+$ [1]; hence conjecture 2 holds in case $A = \emptyset$ and W is finite.

(iii) For crystallographic Coxeter systems, 3(a) is known [15] and conjecture 2 holds for $A = \emptyset$ (i.e. $P_{v,w}$, $Q_{v,w} \in \mathscr{R}^+$) ([15], [6]); the latter implies that in this case, the sums of the coefficients of $c'_w(v, A)$ and $d'_v(w, A)$ are nonnegative for any $A \in \mathscr{A}$.

(iv) For "universal" Coxeter systems (i.e. free products of cyclic groups of order 2), 3(a) and 3(b) are known by [8] and 2(a) may be proved by adapting the methods of [7]. Conjecture 2(b) for finite A is equivalent to 2(b)′ and is proved in [8].

(v) In general Coxeter systems, Conjecture 1 is known for $l(w) - l(v) \leqslant 4$. Conjecture 2 holds for $l(w) - l(v) \leqslant 3$. Moreover, if $l(w) - l(v) = 4$, the coefficient of $q^{n/2}$ in $c'_w(v, A)$ and $d'_v(w, A)$ is non-negative unless perhaps $n = 0$ (in particular, $P_{v,w} \in \mathscr{R}^+$ and $Q_{v,w} \in \mathscr{R}^+$; these last facts are proved in [8]).

(vi) Suppose that x, $y \in [v, w]$ and $x^{-1}y \in T$ imply $|l(x) - l(y)| = 1$; this is actually a condition on the isomorphism type of the poset $[v, w]$ (see [10]). Then $c'_w(v, A)$ and $d'_v(w, A)$ reduce to polynomials defined in [16] and Conjecture 1 holds trivially for $[v, w]$.

(vii) For arbitrary (W, S), $d_1(w, A) = q^{-l(w)/2} q^{\#(A \cap N(w^{-1}))} (w \in W,\ A \in \mathscr{A})$; this follows from (2.14).

One expects a similar result to (vii) for $d'_v(w, A)$ when $Q_{v,w} = 1$ and for $c'_w(v, A)$ when $P_{v,w} = 1$.

(3.10) REMARK. Let $\mathscr{H}_1$ denote the subset of $\mathscr{H}$ consisting those $f \in \mathscr{H}$ such that for $w \in W$, $t \in T$ and $A \in \mathscr{A}$ with $wt > w$ and $tA = A + \{t\}$ one has

$$f(w, tA) \geqslant \begin{cases} q^{1/2} f(wt, A) & t \notin A \\ q^{-1/2} f(wt, A) & t \in A. \end{cases}$$

(i.e. the inequalities hold coefficient by coefficient). One may check that if f, $g \in \mathscr{H}_1$ then $fg \in \mathscr{H}_1$ and $\bar{f} \in \mathscr{H}_1$. It is natural to ask whether $c'_w \in \mathscr{H}_1 (w \in W)$; I have proved an analogous property in the situation of [16] for Eulerian lattices of rank $\leqslant 4$ with "nice" shellings (see (4.8)).

An affirmative answer to the question here would imply that $P_{x,w} \leqslant P_{y,w}$ if $y \leqslant x$; I have checked $P_{x,w} \leqslant P_{y,w}$ $(y \leqslant x)$ for W of type A_5, B_4, D_4 and H_3.

To conclude this section, we mention an identity involving the coefficients of the polynomials $R(x, w)$. The proof, which uses (3.4) and (2.14), will be omitted.

(3.11) PROPOSITION. *Write* $R(x, w) = \sum_{j \in \mathbb{N}} R_j(x, w) \bar{\alpha}^j$ $(R_j(x, w) \in \mathbb{N})$. *Then*

$$\binom{l(w) + 1}{k} = \sum_{x \leqslant w} \sum_{j=0}^{k} \binom{[l(w) - l(x) + 2 - j]/2}{k - j} R_j(x, w) \quad (0 \leqslant k \leqslant l(w) + 1)$$

where for $\alpha \in \mathbb{R}$ and $m \in \mathbb{N}$, $\displaystyle\binom{\alpha}{m} = \frac{\alpha(\alpha - 1)\cdots(\alpha - m + 1)}{m!}$.

The specialized result for $k = 1$ is just $l(w) = \sum_{x \leqslant w} R_1(x, w)$ i.e. $l(w) = \#\{x : x \leqslant w,\ x^{-1}w \in T\}$.

4. EL-labellings of Bruhat intervals

Let (W, S) be a Coxeter system; maintain the notation of section 2. For $I \subseteq K \subseteq R$, let $D_I^K = \{w \in W \mid I \subseteq N(w^{-1}) \cap S \subseteq K\}$ [2]. Note that if $v \in D_I^K$ then $N(v^{-1}) \cap W_I = W_I \cap T$ and $N(v^{-1}) \cap (S \backslash K) = \varnothing$.

If u, $w \in D_I^K$ and $u \leqslant w$, define $[u, w]_I^K = \{z \in D_I^K \mid u \leqslant z \leqslant w\}$. It is known that if u, $w \in D_I^K (u \leqslant w)$ then all maximal chains $u = u_0 < u_1 < \cdots < u_p = w$ in $[u, w]_I^K$ have the same length $p = l(w) - l(u)$ [4, 5.1 and 3,5].

Let $\preccurlyeq$ denote a fixed reflection order on T with the following properties:

(i) $t' \succ t$ if $t \in T \cap W_{S\backslash K}$ and $t' \in T \backslash W_{S \backslash K}$
(ii) $t' \succ t$ if $t' \in T \cap W_I$ and $t \in T \backslash W_I$.

(4.1) LEMMA. *Let u, $w \in W$ with $u \leqslant w$ and $l(w) - l(u) = 2$.*

(i) *There exist unique x, $y \in W$ such that $u < x < w$, $u < y < w$, $u^{-1}x \prec x^{-1}w$ and $u^{-1}y \succ y^{-1}w$. Moreover, $y^{-1}w \prec x^{-1}w$ and $u^{-1}x \prec u^{-1}y$.*
(ii) *If u, $w \in D_I^K$ and y is as in* (i), *then $y \in D_I^K$.*

Proof. The reflection subgroup of W generated by $\{z^{-1}v \mid z, v \in [u, w]\}$ is dihedral by [10, (3.1)]. Using (2.4)(ii), one sees that to prove (i) there is no loss of generality in assuming that (W, S) is dihedral (cf. the proof of [10, (2.1)]). We leave the verification that (i) holds if (W, S) is dihedral to the reader.

Now suppose that u, $w \in D_I^K$ but $y \notin D_I^K$. Then either $yr > y$ for some $r \in I$ or $yr < y$ for some $r \in S \backslash K$. Suppose first that $yr > y (r \in I)$. Since $wr < w$, it follows that $y = wr$, and $w^{-1}y = r \in W_I \cap T$. But $u^{-1}y \succ y^{-1}w$ so our assumption (ii) on $\prec$ implies $u^{-1}y \in W_I \cap T$. Hence $w^{-1}u \in W_I$. But

$$N(u^{-1}) \cap W_I = N(w^{-1}) \cap W_I = W_I \cap T$$

and

$$N(w^{-1}) = N(w^{-1}u) + w^{-1}uN(u^{-1})u^{-1}w$$

which gives a contradiction since $N(w^{-1}u) \cap W_I \neq \varnothing$. Hence $yr < y$ for all $r \in I$. Similarly, $yr > y$ for all $r \in S \backslash K$, so $y \in D_I^K$.

Note that the proof of (ii) could be adapted to directly prove that $[u, w]_I^K$ has the chain property for any u, $w \in D_I^K$ with $u \leqslant w$.

(4.2) Let u, $w \in W$ with $u \in W$ and $l(w) - l(u) = p$. For any maximal chain $m = (u_0, \ldots, u_p)$ $(w = u_0 > \cdots > u_p = u)$ in $[u, w]$, define a p-tuple $\lambda(m) = (u_0^{-1}u_1, \ldots, u_{p-1}^{-1}u_p) \in T^p$. Give T^p the lexicographic ordering induced by the ordering $\prec$ on T; thus, $(\lambda_1, \ldots, \lambda_p) < (\lambda'_1, \ldots, \lambda'_p) (\lambda_i, \lambda'_i \in T)$ if for some $i (1 \leqslant i \leqslant p)$, we have $\lambda_1 = \lambda'_1, \ldots, \lambda_{i-1} = \lambda'_{i-1}$ and $\lambda_i \prec \lambda'_i$.

(4.3) PROPOSITION. *Suppose that* u, $w \in D_I^K$ *in* (4.2). *Then*

(i) *there exists a unique maximal chain* m_0 *in* $[u, w]_I^K$ *such that* $\lambda(m_0) = (\lambda_1, \ldots, \lambda_p)$ *satisfies* $\lambda_1 \prec \cdots \prec \lambda_p$

(ii) *If* m *is any other maximal chain in* $[u, w]_I^K$ *then* $\lambda(m_0) < \lambda(m)$ *in the lexicographic order on* T^p.

Proof. Let m_0 be the maximal chain in $[u, w]_I^K$ whose associated p-tuple $\lambda(m_0) = (\lambda_1, \ldots, \lambda_p)$ is lexicographically first amongst those p-tuples arising from maximal chains of $[u, w]_I^K$. We show that $\lambda_1 \prec \cdots \prec \lambda_p$.

Write $m_0 = (u_0, \ldots, u_p)$ where $u_0 = w > u_1 > \cdots > u_p = u$. Suppose that $\lambda_i > \lambda_{i+1}$ for some $i(1 \leqslant i \leqslant p-1)$. Then $u_{i-1}^{-1}u_i \succ u_i^{-1}u_{i+1}$. The Bruhat interval $[u_{i-1}, u_{i+1}]$ has 2 atoms, of which u_i is one; let $u_i' \neq u_i$ be the other. By (4.1), we have $u_i' \in [u, w]_I^K$ and $u_{i-1}^{-1}u_i' \prec u_{i-1}^{-1}u_i$. It follows that $m_0' = (u_0, \ldots, u_{i-1}, u_i', u_{i+1}, \ldots, u_p)$ is a maximal chain in $[u, w]_I^K$ with $\lambda(m_o') < \lambda(m_0)$, contrary to choice of m_0. Hence $\lambda_i \prec \cdots \prec \lambda_p$.

To complete the proof, it is sufficient to show that there is at most one maximal chain m in $[u, w]$ such that

$$\lambda(m) = (\lambda_1', \ldots, \lambda_p')$$

satisfies

$$\lambda_i' \prec \cdots \prec \lambda_p'.$$

Now it is known (e.g. by [11], (2.0.b)–(2.0.c))] that $R_{u,w} = q^{1/2(l(w)-l(u))}\bar{R}(u, w)$ is a monic polynomial in q of degree $l(w) - l(u) = p$. Hence $R(u, w)$ is a monic poynomial in $\bar{\alpha} = q^{-1/2} - q^{1/2}$ of degree p. By (3.4), there is exactly one maximal chain m in $[u, w]$ such that $\lambda(m) = (\lambda_1', \ldots, \lambda_p')$ satisfies $\lambda_1' \prec \cdots \prec \lambda_p'$.

(4.4) In the terminology of [2], (4.3) asserts that the poset $[u, w]_I^K$ is EL-shellable. An argument similar to that above shows that if one equips T^p with the lexicographic ordering induced by the reverse of $\preccurlyeq$ on T, one obtains an EL-labelling of the opposite poset of $[u, w]_I^K$. It remains open whether the generalized quotients W/V of [4] are EL-shellable.

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Notes added in proof

(i) I can now show that the coefficients of q in $Q_{v,w}$ and $P_{v,w}$ are non-negative for general Coxeter systems. (ii) For a finite Weyl group, one can show all $c'_w \in \mathscr{H}_1$ in (3.10) using results in the preprint "Shuffled Verma modules and principal series modules over complex semisimple lie algebras" by R. Irving.