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Asymptotics and characteristic cycles for representations of complex groups

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Introduction

Let $G_{\mathbb{R}}$ be a connected semisimple linear Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$. Write $\mathfrak{g}$ for the complexification of $\mathfrak{g}_{\mathbb{R}}$. Suppose M is an irreducible admissible representation of $G_{\mathbb{R}}$, and Θ is its global distribution character on $G_{\mathbb{R}}$. Via the exponential map, Θ lifts to an invariant eigendistribution θ on a neighborhood of the origin in $\mathfrak{g}_{\mathbb{R}}$. According to Barbasch and Vogan [B-V], θ admits an asymptotic expansion near the origin and the Fourier transform of its leading term is a linear combination of invariant Liouville measures on "nilpotent" coadjoint $G_{\mathbb{R}}$-orbits in $\mathfrak{g}_{\mathbb{R}}^*$, the linear dual to $\mathfrak{g}_{\mathbb{R}}$. The resulting nilpotent $G_{\mathbb{R}}$-orbits with multiplicity will be referred to as the asymptotic cycle of M. On the other hand, let $K_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathbb{R}}$ and K its complexification. The $K_{\mathbb{R}}$-finite part of M is naturally an algebraic $(U(\mathfrak{g}), K)$-module, where $U(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$. The characteristic cycle $\mathscr{V}(M)$ of M is defined algebraically with respect to the natural filtration (by the degree) of $U(\mathfrak{g})$, and is a finite sum of "nilpotent" K-orbits in $(\mathfrak{g}/\mathfrak{k})^*$ where $\mathfrak{k}$ is the complexification of $\mathfrak{k}_{\mathbb{R}}$, the Lie algebra of $K_{\mathbb{R}}$. In [V2], D. Vogan conjectured that the two invariants – asymptotic and characteristic cycles – are in fact the same under the Sekiguchi correspondence [S] which sets up a natural bijection between the relevant nilpotent $G_{\mathbb{R}}$-orbits and K-orbits. In this paper, we give a geometric interpretation of the "multiplicities" appearing in the characteristic cycle $\mathscr{V}(M)$, and when $G_{\mathbb{R}}$ is itself a complex group, we deduce the conjecture from results of Rossmann [R] and Joseph [J2].

Our starting point is to consider both invariants for the coherent family $\{M(\mu)\}$ of M; the parameter μ ranges over weights of finite-dimensional representations of $G_{\mathbb{R}}$. First of all, the multiplicity (of the invariant measure on an arbitrary $G_{\mathbb{R}}$-orbit) appearing in the asymptotic cycle of $M(\mu)$ is in fact a harmonic homogeneous polynomial function in the infinitesimal character of $M(\mu)$ [B-V]. To describe our method and results on the characteristic cycles, let $\mathscr{M}$ be the localized irreducible $\mathscr{D}_\lambda$-module for M on the flag variety X of $\mathfrak{g}$ according to Beilinson and Bernstein [B-B1]. The characteristic cycle $\mathscr{C}h(\mathscr{M})$ of

$\mathcal{M}$ is an algebraic cycle on T^*X, the cotangent space of X, defined as the support with multiplicity of gr $\mathcal{M}$ with respect to a good filtration on $\mathcal{M}$. The passage from the characteristic cycle of $\mathcal{M}$ to that of M is made through the moment map from T^*X to $\mathfrak{g}^*$; an explicit formula relating them in certain cases (e.g. λ is sufficiently dominant) is given by Borho and Brylinski [Bo-Br]. We formulate an extension of their result in full generality (see 2.5.2). In the context of a coherent family, we show that the multiplicity (of an arbitrary K-orbit) appearing in $\mathscr{V}(M(\mu))$ is given by the dimension of the Euler characteristic of the cohomology groups on a certain fiber of the moment map with coefficients in gr $\mathcal{M}$ twisted by an invertible sheaf; moreover its leading part depends only on $\mathscr{C}h(\mathcal{M})$ (see 2.5 and 2.6). As a byproduct of this, the multiplicity is a polynomial function in the infinitesimal character of $M(\mu)$ with the "right" degree. It is pointed out to us by D. Vogan that this multiplicity function has no terms of degree lower than the "right" one; this follows from considerations on the Weyl group representations (see 1.6). Thus the characteristic cycle of M can be recovered completely from that of $\mathcal{M}$ via the moment map.

When $G_{\mathbb{R}}$ is itself a complex group, various aspects of these invariants have been studied extensively (see references in say [Bo-Br]). In the area of the distribution characters of Harish-Chandra modules, Rossmann [R] has obtained character formulae in terms of the homology classes of the flag variety X; moreover the asymptotics of θ is expressed in terms of the characteristic cycle of the $\mathscr{D}_\lambda$-module $\mathcal{M}$ when the infinitesimal character λ is integral (notations being those in the first paragraph). On the other hand, based on an observation of J. Bernstein, Joseph [J2] interpreted the asymptotics formula of Rossmann's as the asymptotics of a certain dimension function q_Z attached to some fixed-point variety Z in X. These dimension functions are exactly the Euler characteristic appearing in the setup of the moment map. Putting these results together, we deduce the conjecture in this case. To a large extent, this is just an organization of results of Rossmann's in the context of Vogan's conjecture. The chain of argument presented here is in fact entirely general. In other words, a geometric theory for distribution characters in the more general case similar to that of Rossmann's would imply the conjecture using the same argument.

As for the organization of the paper, Vogan's conjecture is stated in 1.5.1 after a brief statement on the behavior of the characteristic cycles of Harish-Chandra modules under the coherent continuation (1.4.2). The counterpart for the $\mathscr{D}$-modules is treated in Section 2. In the last section, we collect results from [R] and deduce the conjecture in the case of a complex group. In the Appendix, we give a brief account on the setup of Rossmann's integral formula, which is used in Section 3. The original version of this paper only deals with the following weak form of Vogan's conjecture: replacing the multiplicity functions appearing in $\mathscr{V}(M)$ by their leading parts in the identification of the two invariants (cf. 1.5.2). We would like to thank D. Vogan and K. Vilonen for helpful comments and discussions. Vogan kindly explained to us the equivalence of his conjecture

to the above weak version. The formulation in 2.5.2 was inspired by their comments.

1. Asymptotic and characteristic cycles

1.1. As in the introduction, let $G_{\mathbb{R}}$ be a connected linear semisimple group and $K_{\mathbb{R}}$ a maximal compact subgroup; deleting the subscript $\mathbb{R}$ amounts to taking the complexification. Following Section 5 of [V3], write $\mathcal{N}^*$ for the nilpotent cone in $\mathfrak{g}^*$, also set $\mathcal{N}^*_{\mathbb{R}} = \mathcal{N}^* \cap \mathfrak{g}^*_{\mathbb{R}}$ and $\mathcal{N}^*_{\mathfrak{k}} = \mathcal{N}^* \cap (\mathfrak{g}/\mathfrak{k})^*$. Note that $G_{\mathbb{R}}$ and K act on $\mathcal{N}^*_{\mathbb{R}}$ and $\mathcal{N}^*_{\mathfrak{k}}$ respectively with finitely many orbits [K-R]. In the following, "dim" always means complex dimension unless specified otherwise.

1.1.1. THEOREM (Sekiguchi). *There is a natural bijection between $G_{\mathbb{R}}$-orbits in $\mathcal{N}^*_{\mathbb{R}}$ and K-orbits in $\mathcal{N}^*_{\mathfrak{k}}$. Suppose that O_r and O are a pair of corresponding $G_{\mathbb{R}}$-orbit and K-orbit respectively, then $\dim_{\mathbb{R}} O_r = 2 \dim O$.*

For a proof and a more extended statement see [S] and [V3]. For convenience, we use the notations O_r and O for corresponding orbits as above whenever no confusion should occur.

1.2. We now recall the notion of a coherent continuation. First of all, since there is only one G-conjugacy class of Cartan subalgebras in $\mathfrak{g}$, it is convenient to talk about an "abstract" Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and refer things there. Let us also fix a positive system $\Phi^+(\mathfrak{g}, \mathfrak{h})$. Recall that $\lambda \in \mathfrak{h}^*$ is called dominant if $\langle \lambda, \check{\alpha} \rangle$ is not a negative integer for any $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})$. Write Λ for the lattice of weights of all finite-dimensional representations of $G_{\mathbb{R}}$. Also a Harish-Chandra module means a $(U(\mathfrak{g}), K)$-module of finite length. An element in the Grothendieck group of the category of Harish-Chandra modules will be called a virtual character (or a virtual representation).

1.2.1. COHERENT CONTINUATION. *Suppose M is an irreducible Harish-Chandra module with an infinitesimal character given by (the Weyl group orbit of) a dominant weight $\lambda \in \mathfrak{h}^*$. Then there exists a family of virtual characters $\{M(\mu): \mu \in \Lambda\}$ satisfying*

(1) *For any finite-dimensional representation F of $G_{\mathbb{R}}$ with weights (counted with multiplicity) $\Delta(F) \subset \mathfrak{h}^*$, then as virtual characters,*

$$M(\mu) \otimes F = \sum_{\nu \in \Delta(F)} M(\mu + \nu).$$

(2) *$M(\mu)$ has an infinitesimal character given by $\lambda + \mu \in \mathfrak{h}^*$; moreover it's 0 or an irreducible Harish-Chandra module whenever $\lambda + \mu$ is dominant, and the latter is the case whenever $\lambda + \mu$ is nonsingular.*

(3) *$M(0) = M$.*

This is a special case of the coherent continuation given in [H-S] and [V1]. Given an M as above, we will exhibit such a family in 2.5.1.

1.3. According to Theorem 3.1 of [B-V], any invariant eigendistribution θ on an invariant neighborhood of 0 in $\mathfrak{g}_{\mathbb{R}}$ admits an asymptotic expansion near 0 with coefficients given by tempered distributions on $\mathfrak{g}_{\mathbb{R}}$. Moreover when θ is the "local" character of an irreducible Harish-Chandra module say M, then the Fourier transform of the leading term in the expansion is a linear combination of invariant measures on nilpotent $G_{\mathbb{R}}$-orbits in $\mathcal{N}_{\mathbb{R}}^*$ i.e., $\sum m(O_r, M)\cdot\beta(O_r)$ where $\beta(O_r)$ is the Liouville measure on the coadjoint orbit O_r. We define the asymptotic cycle of M as the following finite sum

$$\mathscr{A}_S(M) = \sum m(O_r, M)\cdot O_r. \tag{1.3.1}$$

1.3.2. PROPOSITION 4.7 of [B-V]. *Let M and $\{M(\mu)\}$ be as in* 1.2.1, *and O_r a $G_{\mathbb{R}}$-orbit in $\mathcal{N}_{\mathbb{R}}^*$. If $m(O_r, M) \neq 0$, then $\dim_{\mathbb{R}} O_r$ is the Gelfand-Kirillov dimension of M and the function sending $\lambda + \mu$ to $m(O_r, M(\mu))$ is a harmonic homogeneous polynomial function of degree $\frac{1}{2}(-\dim_{\mathbb{R}} O_r + \dim(\mathfrak{g}/\mathfrak{h}))$.*

To emphasize the role of the coherent continuation, we call the above polynomial $m_{\mathrm{asy}}(O_r, M)$ as an element of $S(\mathfrak{h})$; it's set to be zero if $\beta(O_r)$ does not contribute to the leading term.

1.4. Suppose that M is a Harish-Chandra module and $\{M_j\}$ a K-stable good filtration (for details see Section 2 of [V3]). The graded object $\operatorname{gr} M$ is then naturally a finitely generated module over $S(\mathfrak{g}) \simeq \mathbb{C}(\mathfrak{g}^*)$. The support with multiplicity on $\mathfrak{g}^*$ of $\operatorname{gr} M$, denoted by $\mathscr{V}(M)$, is the characteristic cycle of M; the support alone is the characteristic variety and will be denoted by $V(M)$. Due to the actions of K and $Z(\mathfrak{g})$ (the center of $U(\mathfrak{g})$), the support $V(M)$ is a union of closures of K-orbits in $\mathcal{N}_{\mathfrak{k}}^*$. We write

$$\mathscr{V}(M) = \sum m(O, M)\cdot \bar{O}. \tag{1.4.1}$$

where $m(O, M)$ are nonnegative integers and the sum ranges over irreducible components of $V(M)$. Note that the characteristic cycle is independent of the choice of K-stable good filtrations and is additive in the associated Grothendieck groups.

1.4.2. PROPOSITION. *Let M and $\{M(\mu)\}$ be as in* 1.2.1, *then*

(1) *The characteristic variety of every constituent of $M(\mu)$ is contained in $V(M)$.*
(2) *$V(M) = V(M(\mu))$ for dominant μ.*
(3) *Suppose $\bar{O}$ is an irreducible component of $V(M)$, then the function sending $\lambda + \mu$ to $m(O, M(\mu))$ is a polynomial function of degree $\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{h}) - \dim O$.*

Due to (1), we can make sense of the multiplicity in (3) above for general μ: simply summing up the multiplicities of $\bar{O}$ over the constituents. A more extended version of this proposition, which interprets the multiplicity geometrically, will be proved in 2.3 and 2.5. The fact that the multiplicity $m(O, M(\mu))$ is given by a polynomial function in $\lambda + \mu$ is also a consequence of the arguments in [V4] (cf. Lemmas 4.1 and 4.3); in fact the argument also implies that this function is harmonic. As in 1.3.2, we denote this polynomial (as a function in $\lambda + \mu$) by $m_{\text{alg}}(O, M)$ as an element in $S(\mathfrak{h})$.

1.5. We now recall the conjecture given in [V2].

1.5.1. CONJECTURE (Vogan). *Suppose M is an irreducible Harish-Chandra module and $\{O_r, O\}$ is a Sekiguchi pair* (cf. 1.1.1), *then $m_{\text{asy}}(O_r, M) = m_{\text{alg}}(O, M)$ as polynomials in $S(\mathfrak{h})$ (equivalently $m(O_r, M) = m(O, M)$).*

Given a polynomial q in $S(\mathfrak{h})$, we denote by $\operatorname{gr} q$ the leading homogeneous part of q. We consider the following weak form of 1.5.1.

1.5.2. CONJECTURE. *Let M, O_r, and O be as in* 1.5.1, *then $m_{\text{asy}}(O_r, M) = \operatorname{gr} m_{\text{alg}}(O, M)$.*

As explained in the introduction, this weak version is in fact equivalent to 1.5.1 thanks to the homogeneity of $m_{\text{alg}}(O, M)$ (cf. 1.6 below). Since most of our considerations will be centered around $\operatorname{gr} m_{\text{alg}}(O, M)$, we keep this weak version for a convenient reference.

1.6. This subsection deals with the homogeneity of the multiplicity $m_{alg}(O, M)$; all the materials presented here were communicated to us by D. Vogan.

1.6.1. PROPOSITION (Vogan). *In the setting of* 1.4.2, *the polynomial $m_{\text{alg}}(O, M)$ has no terms of lower degree than $\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{h}) - \dim O$.*

In particular this shows that $m_{\text{alg}}(O, M)$ is homogeneous and that 1.5.2 is equivalent to 1.5.1. We sketch a proof of this result.

First of all it follows from the definition of the coherent continuation (given in 2.5.1) that the family originated from M is the same as the one originated from $M(\mu)$ whenever $\lambda + \mu$ is regular dominant; therefore we may as well assume that λ is regular dominant. Let W_λ be the Weyl group generated by the λ-integral roots (i.e., those with $\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$). Then W_λ acts on the lattice V of virtual representations (for $G_{\mathbb{R}}$) having the same infinitesimal character as λ by the coherent continuation. To describe the action, first note that V has a basis (over $\mathbb{Z}$) represented by the irreducible Harish-Chandra modules. For $w \in W$ and M an irreducible Harish-Chandra module, we set $w \cdot M$ to be the virtual representation $M(w^{-1} \cdot \lambda - \lambda)$ given in 1.2.1. Now fix a nilpotent G-orbit $\mathbb{O}$ in $\mathcal{N}^*$ and consider the sublattice $V_{\mathbb{O}}$ of virtual representations in V whose irreducible constituents have primitive ideals with associated variety contained in the closure of $\mathbb{O}$. Then $V_{\mathbb{O}}$ is a W_λ-invariant sublattice of V (see [V4]). Let $S^m(\mathfrak{h})$ be the subspace of $S(\mathfrak{h})$ of homogeneous polynomials of degree m.

1.6.2. THEOREM (Joseph-Casian). *Assume that λ is regular and dominant as above.*

(1) *If $m < \frac{1}{2}(\dim(\mathfrak{g}/\mathfrak{h}) - \dim \mathbb{O})$, then $\mathrm{Hom}_{W_\lambda}(V_{\mathbb{O}}, S^m(\mathfrak{h})) = 0$.*
(2) *If $m = \frac{1}{2}(\dim(\mathfrak{g}/\mathfrak{h}) - \dim \mathbb{O})$, then $\mathrm{Hom}_{W_\lambda}(V_{\mathbb{O}}, S^m(\mathfrak{h}))$ is caused only by certain irreducible representations $\rho_1, \ldots, \rho_r$ of W_λ, each occurring exactly once in $S^m(\mathfrak{h})$; moreover the representation of W generated by any one of the ρ_i is irreducible.*

Assuming the theorem, we now prove 1.6.1. It's well known that the associated variety of the primitive ideal of M is the closure of a single G-orbit in $\mathcal{N}^*$ (this is due to Borho-Brylinski and Joseph, for a simple proof of this see Corollary 4.7 in [V3]); call this orbit $\mathbb{O}$. If O is a K-orbit such that $\bar{O}$ is an irreducible component of $V(M)$, then $O \subset \mathbb{O}$ and $\dim O = \frac{1}{2} \dim \mathbb{O}$ (see Corollary 5.20 in [V3]). Since the map sending $M(\mu)$ to $m_{alg}(O, M(\mu))$ is W_λ-equivariant, Proposition 1.6.1 follows from 1.6.2(1).

As for the theorem, let V' be the lattice of virtual $\mathfrak{g}$-modules with irreducible constituents in the category $\mathcal{O}$ with fixed infinitesimal character (say λ as above); one defines the sublattice $V'_{\mathbb{O}}$ in V' similarly as above. Then W_λ acts on V' and on $V'_{\mathbb{O}}$ in the same fashion as it acts on V and $V_{\mathbb{O}}$ given above; moreover this representation of W_λ is just the Goldie rank representation (see [J1]). The theorem then follows from Joseph's result (see Theorems 5.4 and 5.5 of [J1]). The case of Harish-Chandra modules is reduced to the case of the category $\mathcal{O}$ by Theorems 2.11 and 3.4 in [Ca].

To conclude this section, we remark that the representation of W in 1.6.2(2) is the Springer representation attached to $\mathbb{O}$ (and a trivial representation of a component group); this is due to Hotta and Ginsburg (and many others see the reference in [J2]). In particular, the representation of W_λ on $m_{\mathrm{alg}}(O, M)$ is the same as the Goldie rank representation and generates the Springer representation attached to $\mathbb{O}$.

2. Characteristic cycles of $\mathcal{D}$-modules

2.1. Let X be the flag variety of $\mathfrak{g}$. Each point in X represents a Borel subalgebra say $\mathfrak{b}$ in $\mathfrak{g}$ which determines a positive root system in $(\mathfrak{g}, \mathfrak{h})$; we follow the convention that the nilradical $\mathfrak{n}^-$ of $\mathfrak{b}$ is the span of eigenvectors with negative roots. For each $\lambda \in \mathfrak{h}^*$, a twisted sheaf of differential operators (t.d.o.) $\mathcal{D}_\lambda$ is constructed in [B-B1]. Being a t.d.o., $\mathcal{D}_\lambda$ is naturally filtered by the order of differential operators (locally); we denote this filtration by $\{(\mathscr{D}_\lambda)_j\}$. With respect to this filtration, for any $\lambda \in \mathfrak{h}^*$,

$$\operatorname{gr} \mathscr{D}_\lambda = \operatorname{gr} \mathscr{D}_X = p_*(\mathcal{O}_{T^*X}) \tag{2.1.1}$$

where p is the projection from T^*X onto X.

Let $\mathscr{M}$ be a coherent $(\mathscr{D}_\lambda, K)$-module on X: a coherent $\mathscr{D}_\lambda$-module with an algebraic action of K (in the sense of Chapter 1, Section 3 in [M]), such that the action of $\mathscr{D}_\lambda$ is K-equivariant and compatible with the action of K (i.e., the action of $\mathfrak{k}$ as a subalgebra of $\mathfrak{g} \subset \Gamma(X, \mathscr{D}_\lambda)$ coincides with the differential of the K-action). Recall that [Bo-Br] an increasing filtration $\{\mathscr{M}_j\}_{j \in \mathbb{Z}}$ of $\mathscr{M}$ consisting of coherent $\mathcal{O}_X$-modules is called a good filtration if $\mathscr{M}_j = 0$ for some small j, $\bigcup_j \mathscr{M}_j = \mathscr{M}$, $(\mathscr{D}_\lambda)_i \mathscr{M}_j \subseteq \mathscr{M}_{i+j}$ for all i, j, and the equality holds for all i when j is sufficiently large. We call it K-stable if moreover $\mathscr{M}_j$ is K-equivariant for all j. A K-stable good filtration always exists on a coherent $(\mathscr{D}_\lambda, K)$-module. The support with multiplicity of $\operatorname{gr} \mathscr{M}$ on T^*X (via (2.1.1)) is the characteristic cycle $\mathscr{C}h(\mathscr{M})$ of $\mathscr{M}$, the support alone will be called the characteristic variety and denoted by $Ch(\mathscr{M})$. Note that the definition is independent of the choice of K-stable good filtration and it is additive in the Grothendieck groups (see [B]).

2.2. For each $\mu \in \Lambda$, let $\mathcal{O}(\mu)$ be the induced G-equivariant invertible sheaf on X. For a $\mathscr{D}_\lambda$-module $\mathscr{M}$, the twisted sheaf $\mathscr{M}(\mu) := \mathcal{O}(\mu) \otimes \mathscr{M}$ is a $\mathscr{D}_{\lambda+\mu}$-module.

2.2.1. LEMMA. *If $\{\mathscr{M}_j\}$ is a K-stable good filtration on $\mathscr{M}$, then so is $\{\mathscr{M}_j(\mu)\}$ on $\mathscr{M}(\mu)$. With respect to these filtrations, as $\mathcal{O}_{T^*X}$-modules,*

$$\operatorname{gr}(\mathscr{M}(\mu)) = \operatorname{gr} \mathscr{M} \otimes_{\mathcal{O}_{T^*X}} p^* \mathcal{O}(\mu);$$

in particular $\mathscr{C}h(\mathscr{M}) = \mathscr{C}h(\mathscr{M}(\mu))$.

This follows from the exactness of tensoring with $\mathcal{O}(\mu)$, 2.1.1, and (for the last part) that $p^* \mathcal{O}(\mu)$ is invertible on T^*X.

2.3. We recall results from [J2]. For each coherent $\mathcal{O}_X$-module $\mathscr{F}$, consider the dimension function, for $\mu \in \Lambda$,

$$q_{\mathscr{F}}(\mu) := \sum_i (-1)^i \dim \mathrm{H}^i(X, \mathscr{F}(\mu)). \tag{2.3.1}$$

Choosing a free resolution of $\mathscr{F}$ by $\{\mathcal{O}(\mu) \colon \mu \in \Lambda\}$ (by Borel-Serre, see [H]), it follows from the Weyl dimension formula that $q_{\mathscr{F}}$ is a polynomial function; hence an element in $S(\mathfrak{h})$. For a closed algebraic subset Z in X, write q_Z for $q_{i_*(\mathcal{O}_Z)}$

where i is the embedding. Also let c_Z be the polynomial in $S(\mathfrak{h})$ determined by the following function from Λ

$$\mu \mapsto c_Z(\mu) = \int_Z \exp \sigma_\mu,$$

where σ_μ is the first Chern class of $\mathcal{O}(\mu)$.

2.3.2. PROPOSITION (Bernstein-Joseph, Corollary 6.7 in [J]). *If Z is a closed subvariety of X, then* $\operatorname{gr} q_Z = c_Z$ *as elements in* $S(\mathfrak{h})$.

In particular, the degree of q_Z is $\dim Z$. The following lemma says that $\operatorname{gr} q_{\mathscr{F}}$ depends only on the support with multiplicity of $\mathscr{F}$. The lemma is standard; we include a proof for the lack of a direct reference.

2.3.3. LEMMA. *Let $\mathscr{F}$ be as above and assume that its support is of dimension m. Write $\{Z_i\}$ for the set of irreducible components of dimension m of* $\operatorname{Supp}(\mathscr{F})$, *and n_i the multiplicity of $\mathscr{F}$ along Z_i. Then* $\operatorname{gr} q_{\mathscr{F}} = \sum n_i \operatorname{gr} q_{Z_i} = \sum n_i c_{Z_i}$.

Proof. Consider first the case that $Z = \operatorname{Supp}(\mathscr{F})$ is irreducible. For μ sufficiently dominant, $\mathscr{F}(\mu)$ is generated by its global sections (Serre's theorem). Since $\operatorname{gr}(\mathscr{F}(\mu)) = \operatorname{gr} \mathscr{F}$, we may assume that $\mathscr{F}$ is generated by its global sections. Then $\mathscr{F} = \sum \mathcal{O}_Z(\lambda_i)$ in the Grothendieck group of coherent O_X-modules (see the proof of Theorem 5.19 in [H]). The multiplicity is the number of the summand. This gives the lemma in this case, for $\operatorname{gr} q_{\mathcal{O}_Z(\lambda_i)} = \operatorname{gr} q_Z$.

In general, let Z be an irreducible component of dimension m of the support of $\mathscr{F}$; write i for the embedding of Z into X. Then the natural morphism $\mathscr{F} \to i_* i^* \mathscr{F}$ has its cokernel supported in a closed set of dimension less than m; the support of its kernel has dimension m and the number of its m-dimensional irreducible components is strictly less than that of $\mathscr{F}$. Since $\operatorname{Supp}(i_* i^* \mathscr{F}) = Z$ is irreducible and q is additive, the lemma follows from 2.3.2 by an induction on the number of top-dimensional irreducible components. □

2.4. Suppose now that M is an irreducible Harish-Chandra module with an infinitesimal character given by a dominant $\lambda \in \mathfrak{h}^*$. Then there exists a unique irreducible $(\mathscr{D}_\lambda, K)$-module $\mathscr{M}$ on X with $M = \Gamma(X, \mathscr{M})$ [B-B1]. For each $\mu \in \Lambda$, consider the virtual character

$$M(\mu) = \sum_i (-1)^i \mathrm{H}^i(X, \mathscr{M}(\mu)). \tag{2.4.1}$$

2.4.2. PROPOSITION. *With assumptions as above, the family $\{M(\mu)\}$ is a coherent family of M in the sense of* 1.2.1.

Condition (3) in 1.2.1 is trivial, (1) follows from the proof of the key lemma and (2) follows from the main theorem of [B-B1].

2.5. We now give a proof of Proposition 1.4.2. Following the setup in 2.4, write Δ_λ for the localization functor. The argument in Section 1.8 of [Bo-Br] shows that a good filtration on M induces one on the localized $\mathscr{D}_\lambda$-module $\Delta_\lambda M$ which, λ being dominant, has the irreducible module $\mathscr{M}$ as a quotient; thus the good filtration passes onto $\mathscr{M}$. It follows from Theorem 1.9(c) of [Bo-Br] that $V(M) = \gamma(Ch(\mathscr{M}))$. Moreover, for μ dominant, $M(\mu) = \Gamma(X, \mathscr{M}(\mu))$ and by the same reasoning,

$$V(M(\mu)) = \gamma(Ch(\mathscr{M}(\mu))). \tag{2.5.1}$$

Recall that γ is the moment map from T^*X to $\mathfrak{g}^*$; this gives 1.4.2(2).

2.5.2. PROPOSITION. *Under the above setup, for arbitrary $\mu \in \Lambda$ and integer k,*

(1) *the characteristic variety of the Harish-Chandra module* $\mathrm{H}^k(X, \mathscr{M}(\mu))$ *satisfies:*

$$V(\mathrm{H}^k(X, \mathscr{M}(\mu))) \subseteq \mathrm{Supp}(R^k\gamma_*(\mathrm{gr}\,\mathscr{M}(\mu))) \subseteq V(M);$$

(2) *there is a good filtration on each* $\mathrm{H}^k(X, \mathscr{M}(\mu))$ *and with respect to these filtrations, in the Grothendieck group of coherent* $\mathcal{O}_{\mathfrak{g}^*}$*-modules supported on* $V(M)$, *we have*

$$\sum_k (-1)^k \,\mathrm{gr}\,\mathrm{H}^k(X, \mathscr{M}(\mu)) = \sum_i (-1)^i R^i\gamma_*(\mathrm{gr}\,\mathscr{M}(\mu)).$$

Note that, γ being proper, the higher direct images $R^k\gamma_*(\mathrm{gr}\,\mathscr{M}(\mu))$ are coherent sheaves of $\mathcal{O}_{\mathfrak{g}^*}$-modules. In (2) we identify finitely generated $S(\mathfrak{g})$-modules with coherent sheaves of $\mathcal{O}_{\mathfrak{g}^*}$-modules. Proposition 2.5.2 is a slight extension of results of Borbo-Brylinski [Bo-Br]; a proof of this will be given in 2.7. Assuming this we proceed to the proof of 1.4.2. In view of the definition (2.4.1), immediately 1.4.2(1) follows from 2.5.2(2).

To prove 1.4.2(3), suppose that O is a K-orbit in $\mathscr{N}_{\mathfrak{k}}^*$ such that $\bar{O}$ is an irreducible component of $V(M)$. As in the discussion after 1.4.2, the multiplicity of $\bar{O}$ in $M(\mu)$ (for arbitrary $\mu \in \Lambda$) is given by

$$\begin{aligned} m(O, M(\mu)) &:= \sum_k (-1)^k m(O, \mathrm{H}^k(X, \mathscr{M}(\mu)) \\ &= \sum_i (-1)^i m(O, R^i\gamma_*(\mathrm{gr}\,\mathscr{M}(\mu))). \end{aligned} \tag{2.5.3}$$

The first equality is the definition and the second equality follows from 2.5.2(2).

Now $Ch(\mathscr{M}(\mu))$ is a union of conormal varieties $\overline{T_Z^* X}$ for certain K-orbits Z in X. In light of (2.5.1), if $\bar{O} \subset \gamma(\overline{T_Z^* X})$ then $\gamma^{-1}(O)$ is dense in $\overline{T_Z^* X}$. In particular, for $v \in O$, the fiber $F_v := \gamma^{-1}(v) \cap Ch(\mathscr{M}(\mu))$ is equidimensional closed subset of dimension $\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{h}) - \dim O$ in $\overline{T_Z^* X}$; write j for this embedding. Through the projection p, the fiber F_v is isomorphic to a closed subset in X.

Since the moment map γ is proper and K-equivariant, $R^k\gamma_* \operatorname{gr}(\mathscr{M}(\mu))$ is locally free of finite rank on O for each k by Grauert's theorem (see Corollary 12.9 in [H]). Moreover its fiber at v is given by

$$\mathrm{H}^k(F_v, j^* \operatorname{gr}(\mathscr{M}(\mu))) \simeq \mathrm{H}^k(X, p_* j^* \operatorname{gr}(\mathscr{M}(\mu))) \simeq \mathrm{H}^k(X, (p_* j^* \operatorname{gr} \mathscr{M})(\mu)). \quad (2.5.4)$$

The second isomorphism follows from 2.2.1 and the projection formula. Therefore by (2.5.3) for all μ, noting that $p_* j^* \operatorname{gr} \mathscr{M}$ is coherent and recalling the definition of q in 2.3,

$$m(O, M(\mu)) = \sum (-1)^i \dim \mathrm{H}^i(F_v, j^* \operatorname{gr}(\mathscr{M}(\mu))) = q_{p_* j^* \operatorname{gr} \mathscr{M}}(\mu). \quad (2.5.5)$$

On account of the dimension of the fiber F_v, this gives 1.4.2(3).

More systematically, for each K-orbit Z in X, the image of $\overline{T_Z^* X}$ under γ is the closure of a K-orbit say O in $\mathscr{N}_\mathfrak{k}^*$, we write F_Z for the typical fiber in $\overline{T_Z^* X}$ over O.

2.5.6. COROLLARY. *Retaining the same assumption on M and $\mathscr{M}$, and suppose $\bar{O}$ is an irreducible component of $V(M)$. Write*

$$\mathscr{C}h(\mathscr{M}) = \sum_Z m(\overline{T_Z^* X}, \mathscr{M}) \cdot \overline{T_Z^* X},$$

then

$$m_{\mathrm{alg}}(O, M) = \operatorname{gr}(m_{\mathrm{alg}}(O, M)) = \sum_{\gamma(\overline{T_Z^* X}) = \bar{O}} m(\overline{T_Z^* X}, \mathscr{M}) \cdot c_{F_Z}.$$

For notations see 1.4.2. This follows from (2.5.5), 2.3.3 and 1.6.1. In particular, the polynomial $m_{\mathrm{alg}}(O, M)$ depends only on the characteristic cycle of $\mathscr{M}$. We conclude this subsection with the following corollary.

2.5.7. COROLLARY (Borho-Brylinski). *Let $\mathscr{M}$ be a coherent $\mathscr{D}_\lambda$-module on X, with respect to a good filtration, we have*

$$\mathscr{V}(\Gamma(X, \mathscr{M}(\mu))) = \mathscr{S}\mathrm{upp}(\gamma_* \operatorname{gr}(\mathscr{M}(\mu)))$$

for μ sufficiently dominant.

When μ is sufficient dominant, the higher cohomology groups vanish in (2.5.4) by Serre's Theorem (see p. 121 in [H]). The corollary then follows from (2.5.3).

2.6. Let Z be a K-orbit in X and write i for the embedding of Z in X. Denote by

$\mathscr{D}^i_\lambda$ the sheaf of differential endomorphisms of the $\mathcal{O}_Z$-module $i^*(\mathscr{D}_\lambda) = \mathcal{O}_Z \otimes_{i^{-1}\mathcal{O}_X} i^{-1}(\mathscr{D}_\lambda)$ which are also (right) $i^{-1}(\mathscr{D}_\lambda)$-module endomorphisms; $\mathscr{D}^i_\lambda$ is a t.d.o on Z (for details see [B-B2]). Recall ([B-B1], [C1]) that a set of standard data consists of (Z, τ, λ); where Z is a K-orbit in X, and τ is an invertible K-equivariant $\mathscr{D}^i_\lambda$-module on Z. The associated standard module is

$$\mathscr{I}(Z, \tau, \lambda) = i_+(\tau) = i_*(\mathscr{D}_{\lambda, X \leftarrow Z} \otimes_{\mathscr{D}^i_\lambda} \tau). \tag{2.6.1}$$

Without defining precisely $\mathscr{D}_{\lambda, X \leftarrow Z}$, we note that it is a $i^{-1}(\mathscr{D}_\lambda) - \mathscr{D}^i_\lambda$ bimodule. Also note that i is affine (this is due to [B-B1]; a proof can be found in Section 4 of [H-M-S-W]), i_+ can be defined without going into the derived category formalism. For the mere emphasis on the orbit Z, we simply call these standard objects based on Z.

2.6.2. PROPOSITION. *Suppose $\mathscr{I}_1$ and $\mathscr{I}_2$ are two standard modules based on the same K-orbit Z (with perhaps different infinitesimal characters), then $\mathscr{C}h(\mathscr{I}_1) = \mathscr{C}h(\mathscr{I}_2)$.*

Proof. Let $\mathscr{I}_{1j}$ and $\mathscr{I}_{2j}$ be K-stable good filtrations on $\mathscr{I}_1$ and $\mathscr{I}_2$ respectively. It suffices to show that they have the same support with multiplicity on T^*U for an affine open cover $\{U\}$ for X. Since $\mathscr{D}_\lambda$ is a t.d.i, it is isomorphic to $\mathscr{D}_X$ locally (leaving $\mathcal{O}_X$ unchanged). Since $Z \cap U$ is affine, τ is isomorphic to $\mathcal{O}_{Z \cap U}$; in view of (2.6.1), we may consider both standard modules as $i_+(\mathcal{O}_{Z \cap U})$. The two filtrations remain good under the restriction to U. By the remark at the end of 2.1, the proposition follows. □

When λ is dominant, the global section space $I(Z, \tau, \lambda) = \Gamma(X, \mathscr{I}(Z, \tau, \lambda))$ is a standard Harish-Chandra module. Therefore by Theorem 1.9(c) in [Bo-Br] (as in (2.5.2)), we have

2.6.3. COROLLARY. *For the nontrivial standard modules, $V(I(Z, \tau, \lambda))$ depends only on the orbit Z.*

2.7. We now give a proof of Proposition 2.5.2. The proof is homological algebra in nature and we will follow [C-E] closely. We begin with the general setup: let $\mathscr{M}$ be a coherent $(\mathscr{D}_\lambda, K)$-module with a good filtration $\{\mathscr{M}_j\}$ where λ here is arbitrary. In accordance with notations used in [C-E], we set $F^p\mathscr{M} = \mathscr{M}_{-p}$. Fix a finite open affine cover for X and write $\mathscr{A}$ for the Cěch complex of $\mathscr{M}$. By Section 4 of Chapter XV in [C-E], there is a spectral sequence $E^{p,q}_r(\mathscr{A})$.

Note that $\mathrm{H}^*(\mathscr{A}) = \mathrm{H}^*(X, \mathscr{M})$ and a natural filtration on it is given by

$$F^p\mathrm{H}^k(\mathscr{A}) = \mathrm{Im}(\mathrm{H}^k(F^p\mathscr{A}) \to \mathrm{H}^k(\mathscr{A})). \tag{2.7.1}$$

2.7.2. LEMMA.

(1) $E^{p,q}_1(\mathscr{A}) = \mathrm{H}^{p+q}(F^p\mathscr{A}/F^{p+1}\mathscr{A}) \simeq \mathrm{H}^{p+q}(X, F^p\mathscr{M}/F^{p+1}\mathscr{M})$.

(2) *For each k*, $\{F^p\mathrm{H}^k(\mathscr{A})\}$ *is a good filtration on* $\mathrm{H}^k(X, \mathscr{M})$.

Proof. For (1), note that $F^p\mathscr{M}$ is a quasi-coherent $\mathscr{O}_X$-module, therefore the Cěch complex of $F^p\mathscr{M}/F^{p+1}\mathscr{M}$ is given by $F^p\mathscr{A}/F^{p+1}\mathscr{A}$. As for (2), write $\mathscr{B}$ for the Cěch complex for $\mathscr{D}_\lambda$ with respect to the chosen covering on X. Then the filtered complex $\mathscr{B}$ acts on $\mathscr{A}$ and the action passes onto their spectral sequences (see exercises 1, 2 on pp. 336–337 of [C-E]). Now $F^pU(\mathfrak{g})$ maps to $F^p\mathrm{H}^0(\mathscr{B})$, the filtration $\{F^p\mathrm{H}^k(\mathscr{A})\}$ is compatible with that of $U(\mathfrak{g})$. Now for p suffuciently large $F^p\mathscr{A} = 0$ by the assumption, thus $F^p\mathrm{H}^k(\mathscr{A}) = 0$; moreover the filtration on $\mathscr{A}$ is regular and the spectral sequence converges to E_∞. It remains to show that

$$\mathrm{gr}\,\mathrm{H}^k(\mathscr{A}) = \oplus_p F^p\mathrm{H}^k(\mathscr{A})/F^{p+1}\mathrm{H}^k(\mathscr{A}) = \oplus_{p+q=k} E_\infty^{p,q}(\mathscr{A})$$

is finitely generated as a $\mathrm{gr}\,U(\mathfrak{g}) = S(\mathfrak{g})$-module. Since E_∞^* is a subquotient of E_1^* (for the spectral sequence converges), it suffices to show that $\oplus_{p+q=k} E_1^{p,q} = \mathrm{H}^k(X, \mathrm{gr}\,\mathscr{M})$ (by (1)) is finitely generated. By (2.1.1), this is also given by $\mathrm{H}^0(\mathfrak{g}^*, R^k\gamma_*\,\mathrm{gr}\,\mathscr{M})$ by the Leray spectral sequence. Since γ is proper and $\mathrm{gr}\,\mathscr{M}$ is coherent as $\mathscr{O}_{T^*X}$-module, $R^k\gamma_*\,\mathrm{gr}\,\mathscr{M}$ is a coherent $\mathscr{O}_{\mathfrak{g}^*}$-module (by Grauert's Theorem). This gives the lemma. □

For the proof of Proposition 2.5.2, replacing $\mathscr{M}$ above by $\mathscr{M}(\mu)$ and using the filtration given in (2.7.1), we have

$$V(\mathrm{H}^k(X, \mathscr{M}(\mu))) = \mathrm{Supp}(\mathrm{gr}\,\mathrm{H}^k(X, \mathscr{M}(\mu))).$$

Now $\mathrm{gr}\,\mathrm{H}^k(X, \mathscr{M}(\mu))$ is a subquotient of $\mathrm{H}^k(X, \mathrm{gr}\,\mathscr{M}(\mu))$ (this is the E_1 term), and the latter—when considered as a sheaf on $\mathfrak{g}^*$—is just $R^k\gamma_*(\mathrm{gr}\,\mathscr{M}(\mu))$. This gives the first inclusion of 2.5.2(1); the second inclusion then follows from 2.2.1 and (2.5.1).

Now the convergence of the spectral sequence above implies that

$$\sum(-1)^{p+q}E_\infty^{p,q} = \sum(-1)^{p+q}E_1^{p,q}.$$

Therefore, in light of the above discussion on supports and 2.7.2, we have in the category of coherent $\mathscr{O}_{\mathfrak{g}^*}$-modules supported on $V(M)$,

$$\sum(-1)^k\,\mathrm{gr}\,\mathrm{H}^k(X, \mathscr{M}(\mu)) = \sum(-1)^k\mathrm{H}^k(X, \mathrm{gr}\,\mathscr{M}(\mu)) = \sum(-1)^i R^i\gamma_*(\mathrm{gr}\,\mathscr{M}(\mu)).$$

This completes the proof of Proposition 2.5.2(2). □

3. Complex groups case

3.1. Throughout this section $G_{\mathbb{R}}$ will be a connected semisimple complex linear group. We follow the setup and collect results from [R]. We use the compact real form of $G_{\mathbb{R}}$ for the maximal compact subgroup $K_{\mathbb{R}}$. Write $^{-}$ for the conjugation on the complex group $G_{\mathbb{R}}$ with respect to the real form $K_{\mathbb{R}}$; this is also the Cartan involution on $G_{\mathbb{R}}$. Recall that G is the complexification of $G_{\mathbb{R}}$. We make the following identification:

$$G \simeq G_{\mathbb{R}} \times G_{\mathbb{R}} \quad \text{as complex groups.}$$

$$K \simeq \Delta(G_{\mathbb{R}}) = \{(g, g): g \in G_{\mathbb{R}}\} \subset G.$$

$$G_{\mathbb{R}} \simeq \{(g, \bar{g}): g \in G_{\mathbb{R}}\} \subset G.$$

$$X \simeq X_0 \times X_0 \quad \text{as complex varieties.}$$

where X_0 is the flag variety of $\mathfrak{g}_{\mathbb{R}}$ as a complex Lie algebra. Let $\mathscr{X}$ (respectively $\mathscr{S}$) be the union of conormal varieties of the K-orbits (respectively $G_{\mathbb{R}}$-orbits) in X. Then the automorphism $\imath: (x, y) \mapsto (x, \bar{y})$ of $\mathfrak{g}$ (as a real Lie algebra) exchanges the $G_{\mathbb{R}}$-objects and the K-objects; for example $\imath(\mathscr{S}) = \mathscr{X}, \ldots$ etc. This way all the $G_{\mathbb{R}}$-objects inherit a complex structure from the corresponding K-objects. Note that this is not the Matsuki correspondence. Recall that γ is the moment map.

3.1.1. LEMMA. *Let Z be a K-orbit in X, then the dense orbits in $\gamma(\overline{T_Z^* X})$ and $\gamma(\imath(\overline{T_Z^* X}))$ correspond to each other under the Sekiguchi correspondence in* 1.1.1.

Proof. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ via the nondegenerate Killing form, we work in $\mathfrak{g}$. Write $\mathscr{N}_{\mathbb{R}}$ for the nilpotent cone in $\mathfrak{g}_{\mathbb{R}}$, then $\mathscr{N} \simeq \mathscr{N}_{\mathbb{R}} \oplus \mathscr{N}_{\mathbb{R}}$. Choose a strictly normal triple $\{h, e, f\}$ in $\mathfrak{g}$ (as in [S]) such that $e = (v, -v)$ lies in the dense orbit of $\gamma(\overline{T_Z^* X})$. Then $\imath(e)$ lies in the dense $G_{\mathbb{R}}$-orbit in $\gamma(\imath(\overline{T_Z^* X}))$. Note that the $\mathbb{C}$-span of the first components of the above triple gives an $\mathfrak{sl}(2, \mathbb{C})$ in $\mathfrak{g}_{\mathbb{R}}$ (as complex algebras) and whose complexification in $\mathfrak{g}$ contains the above triple. This reduces the lemma to the case when $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}(2, \mathbb{C})$. In that case, the orbits in question are the dense orbits and the lemma follows from the dimension consideration (cf. 1.1.1). □

3.2. For details see [RII]. For subsets $\mathscr{A}$ in T^*X and B in $\mathscr{N}^*$, write $\mathscr{A}(B)$ for the intersection $\mathscr{A} \cap \gamma^{-1}(B)$ where γ is the moment map. Denote by $\mathrm{H}_*(\mathscr{X})$ for the Borel-Moore homology with coefficient in $\mathbb{Z}$. The decomposition of $\mathscr{N}_{\mathfrak{k}}^*$ into K-orbits O leads to a filtration on $\mathrm{H}_{2n}(\mathscr{X})$ according to the closure relations among the orbits (here $n = \dim X$):

$$O \subset \bar{O}' \Rightarrow \mathrm{H}_{2n}(\mathscr{X}(\bar{O}')) \supset \mathrm{H}_{2n}(\mathscr{X}(\bar{O})). \tag{3.2.1}$$

The subquotients of this filtration are

$$\mathrm{H}_{2n}(\mathscr{X}(O)) = \mathrm{H}_{2n}(\mathscr{X}(\bar{O})) \Big/ \sum_{O' \subset \partial \bar{O}} \mathrm{H}_{2n}(\mathscr{X}(\bar{O}')).$$

Together with Borel's picture of the cohomology of the flag variety X, we have the following maps

$$\mathrm{H}_{2n}(\mathscr{Z}) \to \mathrm{gr}\,\mathrm{H}_{2n}(\mathscr{Z}) = \oplus H_{2n}(\mathscr{Z}(O)) \to \mathrm{H}_*(X) \simeq \mathscr{H}(\mathfrak{h}). \tag{3.2.2}$$

The second arrow is through the fibration $\mathscr{Z}(O) \to O$ for each K-orbit O, and the second isomorphism is given by the map $Z \mapsto c_Z$ in 2.3; $\mathscr{H}(\mathfrak{h})$ is the space of harmonic polynomials in $S(\mathfrak{h})$.

Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$; it then acts on the homologies in (3.2.2). According to Section 2 of [RII], there is a "proper" homotopy action of W on T^*X which gives rise to a W-action on $\mathrm{H}_*(\mathscr{Z})$; moreover the action induces one on gr $\mathrm{H}_{2n}(\mathscr{Z})$. The W-action on $\mathrm{H}_*(X)$ is induced from the following action of W on X: for $w \in W$ and $\mathfrak{b}_x$ the Borel subalgebra of $\mathfrak{g}$ representing a point $x \in X$, set $w \cdot x$ to be the point in X represented by the Borel subalgebra $w^{-1} \cdot \mathfrak{b}$—we identify the abstract Cartan subalgebra (cf. 1.2) to a Cartan subalgebra in $\mathfrak{b}$. The natural W-action on $S(\mathfrak{h})$ induces one on $\mathscr{H}(\mathfrak{h})$. An important aspect of these actions lies in

3.2.3. W-EQUIVARIANCE. *The maps in* (3.2.2) *are W-equivariant.*

This is in essence the Specialization Theorem of Hotta-Springer (see Lemma 3.1 and Corollary 3.2 in [RII]). For each $\Gamma \in \mathrm{H}_{2n}(\mathscr{Z})$, we write C_Γ for the image in the composite map of (3.2.2). Note that via ι we can replace $\mathscr{Z}$ by $\mathscr{S}$ everywhere.

3.3. Let M be an irreducible Harish-Chandra module as in 2.5 with $\mathscr{M}$ as its localization. Recall notations in 1.3.2, 3.2, and the orbit correspondence in 1.1.1.

3.3.1. THEOREM. *Let $M, \mathscr{M}$ be as above, O_r be a $G_{\mathbb{R}}$-orbit in $\mathscr{N}^*_{\mathbb{R}}$, and O in $\mathscr{N}^*_{\mathfrak{k}}$ its Sekiguchi pair. Suppose $\bar{O}$ is an irreducible component of $V(M)$, then as elements in $S(\mathfrak{h})$,*

$$m_{asy}(O_r, M) = C_{\mathscr{C}h(\mathscr{M})(\bar{O})}.$$

When $\bar{O}$ is not an irreducible component of $V(M)$, then the multiplicity is 0. The essence of the theorem lies in the case when $\mathscr{D}_\lambda = \mathscr{D}_X$, and is due to Rossmann (and results of Kashiwara and Tanisaki, see [RII]).

In terms of notations in 2.5.6, noting that the map C in (3.2.2) is $\mathbb{Z}$-linear, we see that

$$C_{\mathscr{C}h(\mathscr{M})(\bar{O})} = \sum_{\gamma(\overline{T_Z^* X}) = \bar{O}} m(\overline{T_Z^* X}, \mathscr{M}) \cdot c_{F_z}.$$

In light of 2.3.2 (i.e., gr $q = c$), this gives 1.5.2, therefore

3.3.2. COROLLARY. *The conjecture* 1.5.1 *holds in case* $G_{\mathbb{R}}$ *is a connected complex semisimple linear group.*

3.4. Let us first fix a parameterization for K-orbits in X. Recall the setup in 3.1. Fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ such that $\overline{\mathfrak{h}_{\mathbb{R}}} = \mathfrak{h}_{\mathbb{R}}$. For the following, we identify the abstract Cartan subalgebra $\mathfrak{h}$ to $\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}$ in $\mathfrak{g}$. Let $W_{\mathbb{R}}$ be the Weyl group for $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}})$, then $W = W_{\mathbb{R}} \times W_{\mathbb{R}}$ and $W_{\mathbb{R}}$ corresponds to the diagonal subgroup of W in the identification of 3.1. Fix a Borel subalgebra $\mathfrak{b}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ containing $\mathfrak{h}_{\mathbb{R}}$ such that its opposite Borel subalgebra is given by $\overline{\mathfrak{b}_{\mathbb{R}}}$. Let z_1 in X be the point representing the Borel subalgebra $\mathfrak{b}_{\mathbb{R}} \times \mathfrak{b}_{\mathbb{R}}$ in $\mathfrak{g}$; and (by abuse of notation) for $y \in W$, $z_y = y^{-1} \cdot (\mathfrak{b}_{\mathbb{R}} \times \mathfrak{b}_{\mathbb{R}})$. Similarly write s_1 for $\imath(z_1) = \mathfrak{b}_{\mathbb{R}} \times \overline{\mathfrak{b}_{\mathbb{R}}}$ and set $s_y = y^{-1} \cdot s_1$. Then the K-orbits are given by the $Z_y = K \cdot z_y$ with y ranging over a set of representatives for the coset space $W/W_{\mathbb{R}}$ (note: Z_1 is the open orbit). We denote the standard $(\mathscr{D}_\lambda, K)$-module based on Z_y by $\mathscr{I}_y(\lambda)$; and $I_y(\lambda)$ for the corresponding Harish-Chandra module (cf. 2.6).

Retaining the notations as above, we proceed to the proof of 3.3.1. By the remark following 1.6.1 on the coherent continuation, we may as well assume that λ is regular dominant. Now the coherent family $\{M(\mu)\}$ is parameterized by $\lambda + \Lambda$ and both sides of 3.3.1 are polynomial functions, it suffices to show this for all $\lambda + \mu$ with μ dominant. On the level of Grothendieck groups, we have (λ is assumed to be regular dominant) for any dominant μ,

$$\mathscr{M}(\mu) = \sum m_y \cdot \mathscr{I}_y(\lambda + \mu) \quad \text{and} \quad M(\mu) = \sum m_y \cdot I_y(\lambda + \mu). \tag{3.4.1}$$

Here for each $y \in W/W_{\mathbb{R}}$, $m_y \in \mathbb{Z}$ depends only on M. Note that the infinitesimal character is given by the parameter $\lambda + \mu$; we denote this by χ in the following. Let w_0 be the longest element in $W_{\mathbb{R}}$ and set $w \in W$ to be the involution given by (id, w_0); also set $\rho = (\rho_{\mathbb{R}}, \rho_{\mathbb{R}})$ be the half sum of positive roots for $\mathfrak{g}$. The following lemma relates the characteristic cycle and the distribution character of a standard Harish-Chandra module.

3.4.2. LEMMA. *For* $\chi = \lambda + \mu$ *with* μ *dominant,*

$$\theta_{I_y(\chi)} = \frac{1}{(2\pi i)^n} \int_{p_{w\chi}(\imath w \mathscr{C}h(\mathscr{I}_y(\rho)))} e^{x_{w\chi} - \bar{\sigma}_{w\chi}}.$$

We refer to the Appendix for the definition of this integral (as a distribution). For notations, the twisted moment map $p_{w\chi}$ is defined with respect to the base point $s_1 = \mathfrak{b}_{\mathbb{R}} \times \overline{\mathfrak{b}_{\mathbb{R}}}$ (together with the Cartan subalgebra $\mathfrak{h}$, cf. (A.1)), and in the lower bound of the integral, w applies to the cycle $\mathscr{C}h(\mathscr{I}_y(\rho)) \in \mathrm{H}_{2n}(\mathscr{X})$ according to the proper homotopy action indicated before 3.2.3. Note that $\imath w \mathscr{C}h(\mathscr{I}_y(\rho))$ is a cycle in $\mathscr{S}$.

Proof. First note that, according to the duality theorem of [H-M-S-W], the standard modules $I_y(\chi)$ are principal series Harish-Chandra module for $G_{\mathbb{R}}$; their character formulae are well-known (see Section 6 of [RII] or Section 3 of [H-S]). In fact, $\theta_{I_y(\chi)}$ is given (and determined completely) by the following analytic function on the regular part of $\mathfrak{h}_{\mathbb{R}}$:

$$\frac{1}{\prod_{\alpha\in\Phi^+}\alpha}\sum_{x\in W} e^{(yx)^{-1}\chi}.$$

On the other hand, the integral appearing in the lemma is a $G_{\mathbb{R}}$-invariant eigendistribution determined by the cycle $\iota w\mathscr{C}h(\mathscr{I}_y(\rho))$ (see Appendix). The integral makes sense if χ is replaced by any regular element in $\mathfrak{h}^*$; in fact it extends analytically on the entire $\mathfrak{h}^*$, and is given, on the regular part of $\mathfrak{h}_{\mathbb{R}}$, by

$$\frac{1}{\prod_{\alpha\in\Phi^+}\alpha}\sum_{x\in W} n_x e^{(yx)^{-1}\chi}, \tag{3.4.3}$$

for some integers $\{n_x: x\in W\}$ depending only on the cycle $\iota w\mathscr{C}h(\mathscr{I}_y(\rho))$ (see (1) in Section 5 of [RII]).

It then suffices to show that $n_x = 1$ for all x when $\chi = \rho$. This is given by Theorem 6.1 of [RII] (a brief explanation of this theorem is given after the proof); note that our convention for the localization theory in 2.1 is opposite to the one in Section 6 of [RII]. □

Due to its importance, we give a brief account on Theorem 6.1 of [RII]. For convenience of tracing its dependence on the variable χ, we denote the integral appearing in the lemma by $D(\chi)$. In light of the remark preceding (3.4.3), the Weyl group W acts on $\{D(v): v\in W\cdot\rho\}$ (by substituting v for $x^{-1}\cdot v$ with $x\in W$). The main result of [RI] gives a geometric interpretation of the integers $\{n_x\}$ in (3.4.3) in terms of the geometry of the cycle $\iota w\mathscr{C}h(\mathscr{I}_y(\rho))$. Using this information, the action of W on $\{D(v)\}$ given above is transferred to an action on $\mathrm{H}_{2n}(\mathscr{S})$ explicitly (Theorem 5.1 of [RII]). On the other hand, Kashiwara and Tanisaki have computed the characteristic cycles of $(\mathscr{D}_\rho, K)$-modules on X in terms of the Weyl group representation [K-T]. Theorem 6.1 of [RII] compares these two Weyl group representations explicitly.

We now continue the proof of 3.3.1. Since the characteristic variety of any nonzero coherent $(\mathscr{D}_\chi, K)$-module is of pure dimension $n = \dim X$, the map $\mathscr{C}h$ is $\mathbb{Z}$-linear. Therefore in light of 2.4.1, (3.4.1), and 2.6.1, we have

$$\mathscr{C}h(\mathscr{M}) = \mathscr{C}h(\mathscr{M}(\mu)) = \sum m_y\cdot\mathscr{C}h(\mathscr{I}_y(\chi)) = \sum m_y\cdot\mathscr{C}h(\mathscr{I}_y(\rho)).$$

Together with the above lemma, $\theta_{M(\mu)} = \sum m_y \cdot \theta_{I_y(\chi)}$ is given by (up to the constant $(2\pi i)^n$)

$$\int_{p_{w\chi}(\iota w(\sum m_y \cdot \mathscr{C}h(\mathscr{I}_y(\rho))))} e^{\chi_{w\chi} - \bar{\sigma}_{w\chi}} = \int_{p_{w\chi}(\iota w \mathscr{C}h(\mathscr{M}))} e^{\chi_{w\chi} - \bar{\sigma}_{w\chi}}. \tag{3.4.4}$$

Now suppose O_r and O be a Sekiguchi pair such that $\bar{O}$ is an irreducible component of $V(M)$. By Theorem 7.1 of [RII] (which computes the asymptotics of these integrals as χ tends to 0), recalling the notations in 3.2.3,

$$m(O_r, M(\mu)) = C_{(\iota w \mathscr{C}h(\mathscr{M}))(\overline{O_r})}(w\chi).$$

Transporting back to $\mathscr{L}$ via ι, in light of 3.1.1 and the W-equivariance of C (3.2.3), the above is just given by

$$C_{(w\mathscr{C}h(\mathscr{M}))(\bar{O})}(w\chi) = C_{\mathscr{C}h(\mathscr{M})(\bar{O})}(\chi).$$

This completes the proof of 3.3.1. □

Appendix

In this appendix, for the convenience of the reader, we collect the basic ingredients of Rossman's integral formula. For details see [RI]. We go back to the setting in Section 1; so $G_{\mathbb{R}}$ is a connected linear semisimple group. Write τ for the Cartan involution on $G_{\mathbb{R}}$ corresponding to the maximal compact subgroup $K_{\mathbb{R}}$. Let $U_{\mathbb{R}}$ be the compact real form for the complex group G. Then the flag variety X is a homogeneous space of $U_{\mathbb{R}}$.

Let $\mathfrak{h}$ be the (abstract) Cartan subalgebra of $\mathfrak{g}$ and fix a regular element λ in $\mathfrak{h}^*$ (cf. 1.2). To define a twisted moment map, choose a τ-stable Cartan subalgebra $(\mathfrak{h}_1)_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$, and a Borel subalgebra $\mathfrak{b}_1$ (in $\mathfrak{g}$) containing the Cartan subalgebra $\mathfrak{h}_1$ (the complexification of $(\mathfrak{h}_1)_{\mathbb{R}}$). Through the identification of $\mathfrak{h}$ to $\mathfrak{h}_1$, we consider λ to be a regular element in $\mathfrak{h}_1^*$. Let Ω_λ be the coadjoint G-orbit of λ in $\mathfrak{g}^*$ (via the invariant Killing form, $\mathfrak{h}_1^*$ is considered as a subset in $\mathfrak{g}^*$). We call the following map the twisted moment map with respect to the base point $\mathfrak{b}_1$ (together with $\mathfrak{h}_1$):

$$p_\lambda: T^*X \to \Omega_\lambda, \quad u \cdot (\mathfrak{b}_1, v) \mapsto u \cdot (\lambda + v) \tag{A.1}$$

for $u \in U_{\mathbb{R}}$ and $v \in (\mathfrak{g}/\mathfrak{b}_1)^*$, the fiber of T^*X over the point representing the Borel subalgebra $\mathfrak{b}_1$. This map is a $U_{\mathbb{R}}$-equivariant (but not G-equivariant) real-analytic bijection.

On Ω_λ there is a canonical two-form $\bar{\sigma}_\lambda$; which gives rise to the measure $\bar{\sigma}_\lambda^n$ on Ω_λ (where $n = \dim X$); the canonical Liouville measure is then $(1/(2\pi i)^n n!)\bar{\sigma}_\lambda^n$. For any $2n$-cycle Γ in T^*X with arbitrary support, consider the following "formal' operation: for $f \in C_0^\infty(\mathfrak{g}_{\mathbb{R}})$,

$$f \mapsto \int_{p_\lambda(\Gamma)} \left\{ \int_{\mathfrak{g}_{\mathbb{R}}} f(x) e^{\xi(x)} \, \mathrm{d}x \right\} \bar{\sigma}_\lambda^n(\mathrm{d}\xi). \tag{A.2}$$

Certainly this integral does not converge in general. However it is a simple observation (page 133 in [RI]) that it converges for those cycles $p_\lambda(\Gamma)$ with bounded real parts; the real part is the one with respect to the real form $\mathfrak{g}_{\mathbb{R}}$ on $\mathfrak{g}$. Moreover since $\bar{\sigma}_\lambda^n$ is a closed form on Ω_λ, the integral in (A.2) (when converges) depends only on the homology class of the $2n$-cycle $p_\lambda(\Gamma)$ on Ω (or Γ on T^*X); provided one defines a homology that respects the convergence of the integral (e.g. with bounded real parts).

As in 3.1, denote by $\mathscr{S} \subset T^*X$ the conormal varieties of $G_{\mathbb{R}}$-orbits (i.e., the unions of the conormal bundles of $G_{\mathbb{R}}$-orbits in X). Since $U_{\mathbb{R}}$ is compact, it follows that the real part of $p_\lambda(\mathscr{S})$ is uniformly bounded; hence the integral in (A.2) converges and defines a distribution on $\mathfrak{g}_{\mathbb{R}}$ for any $2n$-cycle Γ in $\mathscr{S}$, and the distribution depends only on the homology class of Γ. Moreover this distribution is a $G_{\mathbb{R}}$-invariant eigendistribution on $\mathfrak{g}_{\mathbb{G}}$: this is because that for any $g \in G_{\mathbb{R}}$, the cycle $g \cdot p_\lambda(\Gamma)$ is homotopic to $p_\lambda(\Gamma)$ ($G_{\mathbb{R}}$ is connected, and the homotopy respects the convergence of the integral), and for any $G_{\mathbb{R}}$-invariant polynomial p on $\mathfrak{g}^*$, we have $p(\xi) = \rho(\lambda)$ if $\xi \in \Omega_\lambda$.

For convenience, following [RI], we write the integral in (A.2) formally as the following integral (up to a factor $(-1)^n/(2\pi i)^n n!$)

$$\frac{1}{(2\pi i)^n} \int_{p_\lambda(\Gamma)} e^{x_\lambda - \bar{\sigma}_\lambda}. \tag{A.3}$$

Here x_λ is the function $x_\lambda(\xi) = \xi(x)$ on Ω_λ and the exponential is taken in the exterior algebra (the integral over a k-chain of an inhomogeneous differential form is the integral of its component of degree k). Thus for any Γ in $\mathrm{H}_{2n}(\mathscr{S})$ (see 3.2), integrating with $f \in C_0^\infty(\mathfrak{g}_{\mathbb{R}})$ in the sense of the operation (A.2), the integral (A.3) defines a distribution on $\mathfrak{g}_{\mathbb{R}}$. This is the distribution appearing in Lemma 3.4.2.

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