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A Fourier summation formula for the symmetric space GL(*n*)/GL(*n* – 1)

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A. Introduction

The question of describing the decomposition of the restriction of an irreducible complex representation π of a group G to a subgroup H of G is fundamental in representation theory. The Frobenius reciprocity law: $\mathrm{Hom}_H(\pi, \rho) = \mathrm{Hom}_G(\pi, \mathrm{Ind}(\rho; G, H))$ (see, e.g. [BZ1], Theorem 2.28) asserts that the restriction $\pi \mid H$ of π to H has the irreducible H-module ρ as a quotient precisely when the G-module π embeds in the G-module $\mathrm{Ind}(\rho; G, H)$ induced to G from ρ on H. Since $\mathrm{Hom}_H(\pi, \rho) = \mathrm{Hom}_H(\pi \otimes \check{\rho}, \mathbb{C})$, where $\check{\rho}$ is the H-module contragredient to ρ, the question of the multiplicity of ρ in π can be stated in terms of linear forms on $\pi \otimes_H \check{\rho}$. The study of such forms for real groups, especially when H is the group of fixed points of an involution on a real group G, has led to the rapidly expanding subject of harmonic analysis on such symmetric spaces G/H (if ρ is trivial; $(G \times H)/H$ in general); see, e.g., Flensted-Jensen [FJ], Oshima-Matsuki [OM], Bien [Bi].

Various facts are known also when G is a p-adic reductive group. As an example we recall a result of Gelfand-Kazhdan [GK] and Bernstein-Zelevinski [BZ2], which asserts that the restriction of an irreducible admissible *generic* (= having a Whittaker model) representation π of $G = \mathrm{GL}(n, F)$, where F is a non-archimedean field, to its subgroup $H = \mathrm{GL}(n-1, F)$ $\left(H \hookrightarrow G \text{ via } h \mapsto \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right)$, contains each irreducible admissible generic representation ρ of H with multiplicity one. Equivalently, there exists a unique up-to-a-scalar non-zero H-invariant linear form on $\pi \otimes \check{\rho}$. Recently J. Bernstein showed this (unpublished) for all irreducible admissible π and ρ, not necessarily generic, namely that $(\mathrm{GL}(n, F), \mathrm{GL}(n-1, F))$ – and more generally $(\mathrm{GL}(n, F) \times \mathrm{GL}(n-1, F), \mathrm{GL}(n-1, F))$ and $(O(n, F) \times O(n-1, F), O(n-1, F))$ – is a "Gelfand pair" (see [DP] when F is $\mathbb{R}$ and π is unitary, for the pair $(\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n-1, \mathbb{R}))$).

When F is a global field with a ring $\mathbb{A}$ of adeles, $\pi = \otimes \pi_v$ an irreducible cuspidal (hence generic) representation of $\mathbb{G} = \mathbf{G}(\mathbb{A})$, $\mathbf{G} = \mathrm{GL}(n)$, and $\rho = \otimes \rho_v$ an irreducible cuspidal representation of $\mathbb{H} = \mathbf{H}(\mathbb{A})$, $\mathbf{H} = \mathrm{GL}(n-1)$, the local result implies that there exists at most one (up-to-a-scalar) non-zero form on $\pi \otimes \check{\rho}$. Such a form actually exists, since the local forms have the property (a proof is given in a remark at the end of this Introduction) that for almost all v they are non-zero at $\eta_n \otimes \check{\eta}_{n-1}$; here η_n is a non-zero K_v-fixed vector in π_v, $K_v = \mathbf{G}(R_v)$, $R_v =$ ring of integers in the completion F_v of F at the non-archimedean place v, and $\check{\eta}_{n-1}$ is a non-zero K_v^H-fixed vector in $\check{\rho}_v$, where $K_v^H = \mathbf{H}(R_v)$.

But there is a purely global, automorphic, statement, of number theoretic interest, concerning a specific shape of this linear form on $\pi \otimes \check{\rho}$. *The question is whether the global form is a multiple of the automorphically defined bilinear form $B = B_{1/2}$ on $\pi \otimes \check{\rho}$, where*

$$B_s(\phi_n, \bar{\phi}_{n-1}) = \int_{H\backslash\mathbb{H}} \phi_n\left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right) \bar{\phi}_{n-1}(h)|\det h|^{s-1/2}\,\mathrm{d}h;$$

ϕ_n ranges over $\pi \subset L_0^2(G\backslash\mathbb{G})$ and ϕ_{n-1} over $\rho \subset L_{0,\omega}^2(H\backslash\mathbb{H})$. We again take the algebraic group $\mathbf{G}$ to be $GL(n)$, and assume that the central character of π is unitary and fixed, and that, ω, of ρ, is unitary. Then $\check{\rho}$ consists of the complex conjugates $\bar{\phi}_{n-1}$ of the ϕ_{n-1} in ρ. The cuspidal representations π, ρ are realized in the spaces $L_0^2(G\backslash\mathbb{G})$, $L_{0,\omega}^2(H\backslash\mathbb{H})$ of cusp forms (which transform under the center via the fixed character in the case of $\mathbf{G}$ and via ω in the case of $\mathbf{H}$). The integral defining B_s is clearly convergent since the cusp form ϕ_n is rapidly decreasing (and so is ϕ_{n-1}).

To answer this question, consider the Fourier expansion of the cusp form ϕ_n

$$\phi_n(g) = \sum_{p \in N_H\backslash H} W_n(pg)$$

with respect to the character $\psi(x) = \psi(\Sigma_{1 \leqslant i<n} x_{i,i+1})$ of $N\backslash\mathbb{N}$, where $\mathbf{N}$ is the upper triangular unipotent subgroup of $\mathbf{G}$, and ψ is a non-trivial complex character of $\mathbb{A}$ mod F. Here $\mathbf{N}_H = \mathbf{N} \cap \mathbf{H}$, and

$$W_n(g) = \int_{N\backslash\mathbb{N}} \phi_n(xg)\bar{\psi}(x)\,\mathrm{d}x \text{ satisfies } W_n(xg) = \psi(x)W_n(g).$$

Then

$$B_s(\phi_n, \bar{\phi}_n) = \int_{N_H\backslash\mathbb{H}} W_n\left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right) \bar{\phi}_{n-1}(h)|\det h|^{s-1/2}\,\mathrm{d}h$$

$$= \int_{\mathbb{N}_H\backslash\mathbb{H}} W_n\left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right) \int_{N_H\backslash\mathbb{N}_H} \bar{\phi}_{n-1}(xh)\psi(x)\,\mathrm{d}x\,|\det h|^{s-1/2}\,\mathrm{d}h$$

$$= \int_{\mathbb{N}_H\backslash\mathbb{H}} W_n\left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right) \bar{W}_{n-1}(h)|\det h|^{s-1/2}\,\mathrm{d}h.$$

This last integral is "Eulerian", that is, can be expressed as a product of local integrals, when W_n and W_{n-1} factorize as local products:

$$W_n((g_v)) = \prod_v W_{n,v}(g_v), \quad W_{n-1}((h_v)) = \prod_v W_{n-1,v}(h_v).$$

In general, W_n and W_{n-1} are finite linear combinations of such local products. At almost all places the local component is the normalized (value $\mathrm{vol}(K_v)^{-1}$ or $\mathrm{vol}(K_v^H)^{-1}$ at the identity) right K_v- (or K_v^H)-invariant Whittaker function $W_{n,v}^0$ or $W_{n-1,v}^0$.

Using Shintani's explicit form [Sh] of these invariant Whittaker functions, and the theory of Schur functions [M], a computation – relegated to the remark at the end of this Introduction – shows that the local integral

$$\int_{N_{H,v}\backslash H_v} W_{n,v}^0\left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right) \bar{W}_{n-1,v}^0(h)|\det h|_v^{s-1/2}\,\mathrm{d}h$$

is equal to the local L-function $L(s, \pi_v \otimes \check{\sigma}_v)$ associated to the unramified components π_v and $\check{\rho}_v$ of π and $\check{\rho}$ at v. At the remaining finite set of places of F where π, ρ or ψ are ramified, or $(W_{n,v}, W_{n-1,v})$ are not $(W_{n,v}^0, W_{n-1,v}^0)$, the analysis of [JPS], Theorem 2.7, shows that the local integrals are convergent for $\mathrm{Re}(s)$ large, and relate them to a local factor $L(s, \pi_v \otimes \check{\rho}_v)$, which is now defined to be the normalized generator of the fractional principal ideal generated by these local integrals (see [JS2], Theorem 5.1, for the archimedean case). The product $L(s, \pi \otimes \check{\rho}) = \Pi_v L(s, \pi_v \otimes \check{\rho}_v)$ has analytic continuation to the entire complex plane as a holomorphic function in s which satisfies a functional equation relating its value at s and $1-s$, and the automorphic criterion alluded to above is as follows.

The bilinear form on $\pi \otimes \check{\rho}$ is a multiple of B, namely B is not identically zero on $\pi \otimes \check{\rho}$, precisely when $L(s, \pi \otimes \check{\rho})$ does not vanish at $s = 1/2$.

It is clear from the argument above that when ρ is not generic, but π is still cuspidal, then B_s, which is still defined by a convergent integral, is zero.

In the analogous situation of the pair $\mathbf{G} = \mathrm{SO}(n)$ and $\mathbf{H} = \mathrm{SO}(n-1)$, B. Gross and D. Prasad [GP] conjectured in particular that (1) $\dim_{\mathbb{C}} \mathrm{Hom}_{H_v}(\pi_v \otimes \check{\rho}_v, \mathbb{C}) \leqslant 1$ for every irreducible admissible G_v-module π_v and H_v-module ρ_v, and that (2) for cuspidal representations $\pi = \otimes\pi_v$ of $\mathbb{G}$ and $\rho = \otimes\rho_v$ of $\mathbb{H}$ with $\mathrm{Hom}_{H_v}(\pi_v \otimes \check{\rho}_v, \mathbb{C}) = \mathbb{C}$ for all v, the form B on $\pi \otimes \check{\rho}$ is non zero precisely when $L(\frac{1}{2}, \pi \otimes \check{\rho}) \neq 0$, where $L(s, \pi \otimes \check{\rho})$ is the standard L-function associated to $\pi \otimes \check{\rho}$. When $n = 3$ the pair with $\mathbf{G} = \mathrm{SO}(3) = \mathrm{PGL}(2)$ had been studied by Waldspurger [W] who in fact took $\mathbf{H}$ to be an elliptic torus of $\mathbf{G}$ which splits over a quadratic extension E of F, and showed that $B \neq 0$ precisely when (in addition to the local condition $\mathrm{Hom}_{H_v}(\pi_v, \rho_v) = \mathbb{C}$ for all v) $L(1/2, \Pi \otimes \rho) \neq 0$, where Π is the base-change of the cuspidal π to $\mathrm{PGL}(2, \mathbb{A}_E)$ and ρ is a character of $\mathbb{A}_E^\times/E^\times = \mathbf{H}(\mathbb{A})/\mathbf{H}(F)$. When $n = 4$ the groups SO(4) and SO(3) are related to $\mathrm{GL}(2) \times \mathrm{GL}(2)$ and PGL(2), the local question was treated by Prasad's thesis [P], and the global (for some F, π and ρ) by Harris and Kudla [HK] using techniques of Garrett [G], Piatetski-Shapiro and Rallis [PR]. The multiplicity of ρ in π is naturally related in these cases to that of ρ' in π', where ρ', π' are the corresponding representations of the inner forms of $\mathbf{G}$ and $\mathbf{H}$ (when these exist).

Conversations with D. Prasad on the conjecture of [GP] were a source of inspiration to the present work. While visiting Prasad, in email correspondence concerning the archimedean case of the conjecture made in [F2] and studied in [F3] for the pair $G = \mathrm{GL}(n, E)$ and $H = \mathrm{GL}(n, F)$ (more precisely $\mathbf{G} = \mathrm{Res}_{E/F}(\mathrm{GL}(n)/F)$, $\mathbf{H} = \mathrm{GL}(n)/F$, E/F = quadratic extension of local or global fields of characteristic $\neq 2$), F. Bien alluded to work which was identified for us by J. G. M. Mars as that of van Dijk and his collaborators; see [DP] and references there. In [DP] the $H = \mathrm{GL}(n-1, \mathbb{R})$-invariant distributions on unitary $G = \mathrm{SL}(n, \mathbb{R})$-modules π were studied.

Theorem 5.1 of [DP] essentially says that the unitary irreducible non-trivial G-modules π which are H-spherical, namely admit a non-zero H-invariant linear form, are of the form $I(\mathbb{1} \times \tau; G, P)$, normalizedly induced from the representation $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \tau(c)$ of the standard parabolic subgroup $P = P_{n-2,2}$ of G of type $(n-2, 2)$ (thus $a \in \mathrm{GL}(n-2, \mathbb{R})$, $c \in \mathrm{GL}(2, \mathbb{R})$, $\det a \det c = 1$), where τ is a unitary infinite dimensional representation of $\mathrm{PGL}(2, \mathbb{R})$ (or $\mathrm{GL}(2, \mathbb{R})$, with a trivial central character). This work was another source of inspiration for our work. We were especially intrigued by the occurrence in a new context for us of "small" representations of the type which attracted the attention of Kazhdan, Savin, and others (see, e.g. [FKS]).

Since packets are singletons, and by virtue of multiplicity one and rigidity theorems in the global case, it is more natural to work with the group $\mathbf{G} = \mathrm{GL}(n)$, than with $\mathrm{SL}(n)$. An analogue over a non-archimedean field F of the theorem [DP] of van Dijk and Poel is proven in Proposition 0 in the Appendix below. It would do no harm to extend our perspective a little and consider a character $\xi(h) = \xi(\det h)$ of $H = \mathrm{GL}(n-1, F)$, where ξ is a character of $F^\times$. It asserts that *the irreducible admissible unitarizable* $G = \mathrm{GL}(n, F)$*-modules* π *which admit a non-zero linear form which transforms under* H *via* ξ *must be* $\xi(\det)$, *or of the shape* $I(\xi \times \tau; G, P)$, *normalizedly induced from the parabolic of type* $(n-2, 2)$ *where* ξ *is viewed here as a character of* $\mathrm{GL}(n-2, F)$, *and* τ *is an irreducible* *unitarizable infinite dimensional representation of the* 2×2 *factor of the Levi subgroup.* The proof of Proposition 0 is based on the Gelfand-Kazhdan [GK] and Bernstein-Zelvinski theory [BZ2] concerning the restriction of a representation of $\mathrm{GL}(n, F)$ to the subgroup P_n of [BZ2], Section 3. We show in Proposition 0.1 in the Appendix that these $I(\xi\times\tau; G, P)$ do have a form which transforms under H via ξ. Consequently if an irreducible unitary automorphic infinite dimensional representation π of $\mathbb{G} = \mathbf{G}(\mathbb{A})$ admits a non-zero form which transforms under $\mathbb{H} = \mathbf{H}(\mathbb{A})$ according to $\xi(h) = \boldsymbol{\xi}(\det h)$, where now $\boldsymbol{\xi}$ is a character of $\mathbb{A}^\times/F^\times$, then π is of the form $I(\xi\times\tau; \mathbb{G}, \mathbb{P})$, normalizedly induced from the parabolic of type $(n-2, 2)$, where ξ is the associated character of $\mathrm{GL}(n-2, \mathbb{A})$ and τ is an automorphic unitary representation of $\mathrm{GL}(2, \mathbb{A})$ with no one dimensional components.

The restriction of an irreducible representation of $\mathrm{GL}(n)$ over a finite field $\mathbb{F}_q$ to the subgroup $\mathrm{GL}(n-1, q)$ was considered by Thoma [Th], and by Zelevinsky [Z2], Corollary 13.8, p. 148. Their results ("branching rule") in the finite field case are analogous to those of Proposition 0, in the p-adic case. The case of the compact pair $U(n, \mathbb{R})$, $U(n-1, \mathbb{R})$, and that of the analytic finite-dimensional representations of $\mathrm{GL}(n, \mathbb{C})$ (and $\mathrm{GL}(n-1, \mathbb{C})$), is also reviewed in the Appendix, following the proof of Proposition 0.1, using the "Gelfand-Cetlin" basis technique of [Zh].

Our main interest in this paper is in the purely global, or automorphic, notion of $\mathbb{G}$-modules with a form transforming under $\mathbb{H}$ via ξ, or more precisely in the bilinear form B on $\pi\otimes\xi^{-1}$. This B would be the linear form on π of the shape

$$B(\phi) = \int_{H\backslash\mathbb{H}} \phi\left(\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix}\right) \xi(h)^{-1}\,\mathrm{d}h.$$

It was noted above that this form is identically zero if π is cuspidal. If π is not cuspidal then it can be realized in the space of automorphic forms by means of Eisenstein series $\phi(g) = E(g, \Phi, \rho, \lambda)$, when $\pi \simeq I(\rho, \lambda)$, where ρ is a discrete series representation of a (standard, not necessarily proper) parabolic subgroup $\mathbb{P}$ of $\mathbb{G}$ (ρ is trivial on the unipotent radical $\mathbb{N}$ of $\mathbb{P}$), $\lambda\in i\mathfrak{A}_P^*$ where $\mathfrak{A}_P^*$ is some real space, and Φ lies in the $\mathbb{G}$-module $I(\rho, \lambda)$ normalizedly induced from the data $\rho\otimes e^{\langle\lambda, H\rangle}$ on $\mathbb{P}$.

The problem raised by this realization is that the Eisenstein series is slowly increasing (in a Siegel domain) and is no longer rapidly decreasing. Consequently the integral which should have defined $B(\phi)$ does not converge. To overcome this problem it is natural to apply B to the truncation $\Lambda^T E$ of the Eisenstein series, where the truncation operator Λ^T, for T in $\mathfrak{A}_0^+$, is the one introduced by Arthur [A2] to develop the trace formula. Since E is slowly increasing, $\Lambda^T E$ (for a sufficiently regular T) is rapidly decreasing, and the integral which defines $B(\Lambda^T E)$ converges (absolutely).

We computed $B(\Lambda^T E)$ in two important cases. The first is when $\mathbf{P} = \mathbf{P}_{(n-2,2)}$ and $\rho = \rho_1 \times \rho_2$, where ρ_1 is a character of $\mathbb{A}^\times / F^\times$, or of $\mathrm{GL}(n-2, \mathbb{A})/\mathrm{GL}(n-2, F)$ via the determinant map, and ρ_2 is a cuspidal $\mathrm{GL}(2, \mathbb{A})$-module. Then λ lies in the one dimensional (over $\mathbb{R}$) space $i\mathfrak{A}_P^* \simeq i\mathbb{R}$, and the result of the computation is (a linear combination of) the product of a slowly increasing function in λ, and t^λ/λ, where t is the projection of T to a line in the positive chamber. The multiple is zero unless $\rho_1 = \xi$, and then it is the value at $(n-1)/2$ of an L-function of $\rho_2 \otimes \xi^{-1}$, depending on Φ. In any case the result is supported on the line $I(\rho_1 v^{\lambda/(n-2)} \times \rho_2 v^{-\lambda/2})$ of representations (here $v(x) = |x|$, $x \in \mathbb{A}^\times$), and not only at $\lambda = 0$ as Proposition 0, Appendix, which is the non-archimedean analogue of [DP], would suggest.

The second case is when $\rho = \rho_1 \times \rho_2 \times \rho_3$ is a character of the minimal parabolic subgroup $\mathbf{P} = \mathbf{B}$ of $\mathbf{G} = \mathrm{PGL}(3)$. The result of a lengthy computation shows that $B(\Lambda^T E)$ is a linear combination of terms of the form: Product of a nice function in λ, depending on Φ, and a factor of the form $t^{l(\lambda)}/l(\lambda)$ or $t_1^{l_2(\lambda)} t_2^{l_2(\lambda)}/l_1(\lambda) l_2(\lambda)$, where t, t_i are components of T and the l, l_i are linear forms in the components of λ. Here λ lies in the two dimensional (over $\mathbb{R}$) space $i\mathfrak{A}_B^* (\simeq i\mathbb{R}^2)$, and not in a one-dimensional subspace as could have been predicted by Proposition 0, Appendix, and [DP]. Some of the forms l_i are not homogeneous. But the kernels of the homogeneous forms l_i, l do define the representations $\pi \simeq I(\rho, \lambda)$ which are permitted by Proposition 0, Appendix, and [DP], to have $\mathbb{H} = \mathrm{GL}(2, \mathbb{A})$-invariant forms.

To explain this phenomenon note that the representation $I(\rho, \lambda)$ occurs in a series of representations. As λ varies over the space $i\mathfrak{A}_P^*$, and ρ through a set of representatives for the set of orbits $\rho \otimes e^{\langle \lambda, H \rangle}$ of discrete series representations of the various parabolic subgroups (more precisely, their Levi components), all automorphic representations are obtained. In particular, for any test function $f \in C_c^\infty(\mathbf{G}(\mathbb{A}))$, the convolution operator $r(f)$ on the space of automorphic forms is an integral operator:

$$(r(f)\Phi)(g) = \int_{G \backslash \mathbb{G}} K_f(g, h) \Phi(h) \,\mathrm{d}h,$$

whose kernel has the spectral decomposition

$$K_f(g, h) = \sum_{\mathbf{P}} n(P)^{-1} \sum_{\rho} \int_{i\mathfrak{A}_P^*} \sum_{\Phi_1, \Phi_2} (I(f, \rho, \lambda)\Phi_1, \Phi_2) E(g, \Phi_2, \rho, \lambda) \bar{E}(h, \Phi_1, \rho, \lambda) \,\mathrm{d}\lambda;$$

see Arthur [A1]; the orthonormal bases Φ_i of $I(\rho, \lambda)$ have standard finiteness properties. The matrix coefficient $(I(f, \rho, \lambda)\Phi_1, \Phi_2)$ is rapidly decreasing in $\lambda \in i\mathfrak{A}_P^*$ as $|\lambda| \to \infty$, being the Mellin transform of a Schwartz function. Hence the integrals and sums here are absolutely convergent.

Our strategy is then to apply the truncation operator Λ^T to the second variable, h, in $K_f(g, h)$, multiply by $\xi(h)$ and integrate over $H\backslash\mathbb{H}$. Changing the order of integration over h and λ we obtain an integral over $\lambda \in i\mathfrak{A}_P^*$ of an integrand which has the factor $\bar{B}(\Lambda^T E(h, \Phi_1, \rho, \lambda))$. Also we multiply this kernel by a character $\psi(g)$ of the compact group $N\backslash\mathbb{N}$, where **N** is the upper triangular subgroup of **G**, and integrate over $g \in N\backslash\mathbb{N}$. Another factor in the integrand is then the Fourier coefficient $E_\psi(\Phi_2, \rho, \lambda)$ of $E(g, \Phi_2, \rho, \lambda)$. By virtue of the computation of $B(\Lambda^T E)$, the rapid decay of the matrix coefficient, and the elementary Lemma 10, asserting that $\lim_{t\to\infty} \int_{i\mathbb{R}} f(\lambda)(t^\lambda/\lambda)\,\mathrm{d}\lambda = f(0)$ if f is a Schwartz function on $i\mathbb{R}$, the limit of

$$\iint \Lambda^T K_f(x, h)\psi(x)\xi(h)\,\mathrm{d}x\,\mathrm{d}h \text{ as } T \to \infty$$

(T sufficiently regular) can be taken, and the $I(\rho, \lambda)$ which contribute to this limit are precisely those which are permitted by Proposition 0, Appendix, and [DP], to have a non-zero $\mathbb{H} = \mathbf{H}(\mathbb{A})$ invariant form.

On the other hand the integral (*) $\int_{N\backslash\mathbb{N}} \int_{H\backslash\mathbb{H}} K_f(x, h)\xi(h)\psi(x)\,\mathrm{d}h\,\mathrm{d}x$ converges absolutely, and so is equal to $\lim_{T\to\infty} \iint \Lambda^T K_f(x, h)\psi(x)\xi(h)\,\mathrm{d}h\,\mathrm{d}x$. Indeed, the kernel $K_f(g, h)$ of $r(f)$ has the simpler "geometric" expansion $\Sigma_{\gamma\in G} f(g^{-1}\gamma h)$, and an elementary computation shows that (*) is integrable, equal to zero unless ψ has index at most two, and can be expressed as a sum of a certain new type of orbital integrals, the orbit being $\mathbf{U}(\mathbb{A})g\mathbf{H}(\mathbb{A})$ for some subgroup **U** of **N**, when ψ has index two. Note that in general, given ψ and a non-trivial character ψ of $\mathbb{A} \bmod F$, there is $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n-1}) \in F^n$ such that $\psi = \psi_{\boldsymbol{\alpha}}$, where $\psi_{\boldsymbol{\alpha}}(x) = \psi(\Sigma_{1\leqslant i<n} \alpha_i x_{i,i+1})$ on $x \in N\backslash\mathbb{N}$. The *index* of $\psi = \psi_{\boldsymbol{\alpha}}$ is the number of non-zero entries in $\boldsymbol{\alpha}$. In dealing with this "geometric" side, it is more convenient to work with another embedding of $\mathrm{GL}(n-1)$ as **H** in $\mathbf{G} = \mathrm{GL}(n)$; see the Statement of Result, or Geometric Side, below.

Our Fourier summation formula is the resulting identity of a sum of orbital integrals on one hand, and a sum of distributions supported on the variety of representations of the form $I(\rho_1 \times \rho_2; \mathbb{G}, \mathbb{P})$, *where* **P** *is the parabolic of type* $(n-2, 2)$, ρ_2 *is an automorphic generic representation of* $\mathrm{GL}(2, \mathbb{A})$, *and* ρ_1 *is a character of* $\mathbb{A}^\times/F^\times$ *and so also of* $\mathrm{GL}(n-2, \mathbb{A})/\mathrm{GL}(n-2, F)$, *via the determinant.*

It is called "Fourier" since it involves the Fourier coefficient $E_\psi(\Phi_2, \rho, \lambda)$, and the character ψ occurs also in the orbital integral. It would be misleading to call our formula a "trace formula", as we did in an analogous context in [F2], since no traces feature in the formula. It is a summation formula, comparing a sum of integrals with a sum (possibly continuous) of distributions parametrized by representations. *Our original question concerns the identification of the representations which occur in this parametrizing set.*

The proof of the summation formula is complete only in the case of $n=3$. Indeed, the computation of $\int_{H\backslash\mathbb{H}}\xi^{-1}(h)\Lambda^T E(h,\Phi,\rho,\lambda)\,dh$ is carried out for all parabolic subgroups $\mathbf{P}$ only in the case of $\mathbf{G}=\mathrm{GL}(3)$. For $n\geqslant 4$ it is merely shown that if $\int\xi^{-1}\Lambda^T E$ has the expected form, then comparison with the geometric side implies that only $\pi=I(\rho,\lambda)$ with index two (in the obvious sense) occur, and these are of the form $I(\rho_1\times\rho_2;\mathbb{G},\mathbb{P}_{(n-2,2)})$ as above, or induced from a character of a parabolic of type (n_1,n_2,n_3). It would be natural to conjecture that at least two of the n_1, n_2, n_3 are equal to 1 if π is in the support of the summation formula, but we did not go beyond computing $\int\xi^{-1}\Lambda^T E$ when $n_i=1$ $(i=1,2,3)$, that is, $n=3$. To obtain the formula in the $n\geqslant 4$ case we used a consequence of the theory of the Bernstein center (see [BD] or [B]) which permits choosing a component f_v of f such that $\pi_v(f_v)$ is zero unless π_v is a constituent of an induced $I_v(\mu_1\times\cdots\times\mu_{n-2}\times\rho_2)$, where ρ_2 is supercuspidal on $\mathrm{GL}(2,F_v)$ and μ_i are characters of $F_v^\times$.

The case of $n=2$ is also studied in full, mainly as an example to shed light on the general case. This is similar to a case treated by Jacquet [J2] – although his truncation seems to be slightly different than the one we use (see the computations of [J1], p. 211, on which [J2], p. 127, is based) – to reprove Waldspurger's beautiful theorem [W] about a cuspidal $\mathrm{PGL}(2,\mathbb{A})$-module π, that there is a character η of $\mathbb{A}^\times/F^\times$ with $\eta^2=1$ and $L(\frac{1}{2},\pi\otimes\eta)\neq 0$, if and only if π has square integrable components or $\varepsilon(\frac{1}{2},\pi)=1$ if not.

In the case of $n=2$, a similar summation formula is compared in [J2] with an analogous formula which is obtained on integrating the kernel $k_{\tilde f}(x,y)$ of a convolution operator $\tilde r(\tilde f)$ on $L^2(\tilde G\backslash\tilde{\mathbb{G}})$, against a character $\psi(x^{-1}y)$, on x, $y\in N\backslash\mathbb{N}$, where now $\mathbf{N}=\left\{\begin{pmatrix}1&*\\0&1\end{pmatrix}\right\}$. In [J2] the group $\tilde{\mathbf{G}}$ is taken to be the two-fold covering group of SL(2). For $n\geqslant 3$ the group $\tilde{\mathbf{G}}$ with which our summation formula should be compared is GL(2). In the case of $n=3$ the required identities of Fourier orbital integrals are proven in [F4] for general and spherical functions (see Propositions 7 and 16 there). This is the case of a place which splits in the quadratic extension of [F4]. These identities permit a comparison of our formula with the summation formula of [F2] on $\mathrm{GL}(2,\mathbb{A})$ obtained there on integrating the kernel $K_f(x,y)$ multiplied by $\psi(x^{-1}y)$, on x, $y\in\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})$. Once executed, such a comparison would show that the support of our Fourier summation formula for GL(3) consists of all $I(\xi\times\tau)$, where τ is a cuspidal representation of $\mathrm{GL}(2,\mathbb{A})$ or one induced from a unitary character of the upper triangular subgroup of $\mathrm{GL}(2,\mathbb{A})$. It will be interesting to carry out the transfer of orbital integrals for such a comparison also for $n>3$, but we have not done this. As the present paper is already sufficiently long, and the comparison of our formula with that for $\mathrm{GL}(2,\mathbb{A})$ is similar to the comparisons of [F2] and [F4], this will not be done here.

It is interesting to note the occurrence of the factor of the form

$$L(\rho_2 \otimes \xi^{-1}, (n-1)/2)$$

in the term in the summation formula which is parametrized by

$$\pi = I(\rho_1 \otimes \rho_2; \mathbb{G}, \mathbb{P}_{(n-2,2)})$$

where ρ_2 is a cuspidal GL(2, $\mathbb{A}$)-module and ρ_1 a character (necessarily ξ), and $n \geqslant 2$ ($n = 2$ included). Trying to approximate between the case of a form B on $\pi \otimes \xi^{-1}$, ξ a character of $H\backslash\mathbb{H}$, which underlies our summation formula, and that of B on $\pi \otimes \check{\rho}$, where π and ρ are cuspidal on $\mathbb{G}$ and $\mathbb{H}$ as mentioned at the beginning of this Introduction, one may wish to deal with the question in the general context of $\pi \otimes \check{\rho}$, where π is an automorphic $\mathbb{G}$-module, and ρ is a discrete-series (irreducible) representation of $\mathbb{H}$. Moeglin and Waldspurger [MW1] have shown that each such ρ is the unique subrepresentation of the $\mathbb{G}$-module $I(\rho_m \nu^{(k-1)/2} \times \rho_m \nu^{(k-3)/2} \times \cdots \times \rho_m \nu^{-(k-1)/2})$ which is normalizedly induced from the indicated representation of the (Levi factor of the) parabolic subgroup of type $(m, \ldots, m)$, where $mk = n-1$, ρ_m is a cuspidal GL(m, $\mathbb{A}$)-module, and $\nu(x) = |x|$ ($x \in \mathbb{A}^\times$).

It is tempting to ask whether it is true that if $\pi \otimes \check{\rho}$ admits a non-zero form which is automorphic (such as B, or in the sense of occurring in the support of a suitable global summation formula as here), then (at least the least degenerate, or unitarizable) π is of the form $I(\pi_1 \times \pi_{m+1})$, induced from the parabolic of type $(n-m-1, m+1)$, where π_1 is a character and π_{m+1} is a generic automorphic GL($m+1$, $\mathbb{A}$)-module, *and* the standard L-function $L(s, \pi_{m+1} \otimes \check{\rho}_m)$ does not vanish at $k/2$. The extreme cases where $m = n-1$, $k = 1$, and $m = 1$, $k = n-1$, are those elaborated on in this Introduction. The second condition is non-trivial only when $k = 1$, since by Jacquet-Shalika [JS1], [JS2], and Shahidi [Sh1], the L-function $L(s, \pi_{m+1} \otimes \check{\rho}_m)$ does not vanish on $\mathrm{Re}(s) \geqslant 1$. We have no further evidence to answer the question affirmatively or otherwise. But it is important to understand that the occurrence of the factor $L(\rho_2 \otimes \xi^{-1}, (n-1)/2)$ in our formula suggests that the condition that $L(s, \pi \times \rho)$ does not vanish at $s = 1/2$ occurs only when ρ is cuspidal, as in the example discussed above for $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$, in Waldspurger [W] for $\mathrm{SO}(3) \times \mathrm{SO}(2)$, and in Harris-Kudla [HK] for $\mathrm{SO}(4) \times \mathrm{SO}(3)$. In the case $U(3) \times U(2)$ of [F4] this L-function condition does not appear since ρ is taken there to be a character, namely a non-cuspidal discrete series representation of $U(2, \mathbb{A})$.

Our techniques are likely to be applicable with other pairs, such as SO(n), SO($n-1$), and $U(n)$, $U(n-1)$, but only when ρ is a character. This is indeed done in [F4] in the case of $U(3)$, $U(2)$, where global and local applications concerning representations of $U(3)$ with a $U(2)$-invariant linear form, are deduced. It would be interesting to apply these techniques in the other situations too.

Local L functions

The Whittaker function computation alluded to above is a minor variation on that given in [F1], p. 305. In the notations of [F1] we consider the integral

$$\Psi(s, W_n, W_r) = \int_{N\backslash G} W_n\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right) W_r(g)|\det g|^s \, dg$$

where $W_r = W_r^0$ is the normalized unramified $\bar{\psi}$-Whittaker function of the unramified $G = \mathrm{GL}(r, F)$-module ρ with Hecke parameters $y_1, \dots, y_r$, and $W_n = W_n^0$ is the normalized unramified ψ-Whittaker function of the unramified $\mathrm{GL}(n, F)$-module π with Hecke parameters $x_1, \dots, x_n$; $n > r$. We take ψ which is trivial on the ring R of integers in the non-archimedean field F, but not on $\boldsymbol{\pi}^{-1}R$, where $\boldsymbol{\pi}$ is a uniformizer.

The normalized unramified Whittaker function has been computed by Shintani [Sh]. His result is recorded in the Lemma of [F1], p. 305. Using this lemma, in the notations of [F1], our integral takes the form

$$\sum_{\lambda} W_n(\boldsymbol{\pi}^{(\lambda,0)}) W_r(\boldsymbol{\pi}^{\lambda}) |\boldsymbol{\pi}^{\lambda}|^s \delta_r^{-1}(\boldsymbol{\pi}^{\lambda}),$$

where the sum ranges over $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}^r$, $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0$, we put $(\lambda, 0)$ for $(\lambda_1, \dots, \lambda_r, 0, \dots, 0) \in \mathbb{Z}^n$, and emphasize the dependence of the modular function δ of [F1], p. 305, on $\mathrm{GL}(m)$, by the index $m (= r$ or $n)$. Again by the Shintani lemma this sum is

$$\sum_{\lambda} s_{(\lambda,0)}(x) \delta_n^{1/2}(\boldsymbol{\pi}^{(\lambda,0)}) s_{\lambda}(y) \delta_r^{1/2}(\boldsymbol{\pi}^{\lambda}) |\boldsymbol{\pi}^{\lambda}|^s \delta_r^{-1}(\boldsymbol{\pi}^{\lambda}).$$

But

$$\delta_n(\boldsymbol{\pi}^{(\lambda,0)}) = \delta_r(\boldsymbol{\pi}^{\lambda}) |\boldsymbol{\pi}^{\lambda}|^{n-r}.$$

Hence the sum is

$$\sum_{\lambda} s_{(\lambda,0)}(x, 0) s_{(\lambda,0)}(q^{-s-(n-r)/2}(y, 0)) = \prod_{i,j} (1 - x_i y_j q^{-s-(n-r)/2})^{-1}$$

by virtue of homogeneity properties of the Schur function s_{λ} ((3.1), p. 24 of Macdonald [M]), of the homomorphism $\rho_{m,n}$ of [M], p. 24, between (3.2) and (3.3), and the identity (4.3) of [M], p. 33, which was used already in [F1], p. 305.

The last product is equal to the local L-function

$$L\left(s+\frac{n-r}{2}, \pi\otimes\rho\right)$$

attached to $\pi\otimes\rho$, at $s+(n-r)/2$. This is the required result as mentioned above when $r=n-1$ and s is replaced by $s-1/2$.

B. Statement of result

To simplify the notations we work with $\mathbf{G}=\mathrm{PGL}(n)$. The summation formula is an equality of two sums of distributions on $\mathbf{G}(\mathbb{A})$, $\mathbb{A}$ = ring of adeles of a global field F of characteristic $\neq 2$, namely complex valued linear functions in $f\in C_c^\infty(\mathbf{G}(\mathbb{A}))$. These distributions depend on a (unitary, complex valued) character ξ (to simplify the notations we take ξ of order dividing n) of the idele class group $\mathbb{A}^\times/F^\times$, and on an additive character $\psi\neq 1$ of $\mathbb{A}$ mod F into $\mathbb{C}^\times$. The "geometric" side of the summation formula – see Proposition 1 – is

$$\sum_{b\in F^\times}\Psi(g_b;\, f;\, \xi;\, \psi)+\delta_{3,n}\Psi(g_0;\, f;\, \xi;\, \psi)+\delta_{2,n}[\Psi(g_0^+;\, f;\, \xi;\, \psi)+\Psi(g_0^-;\, f;\, \xi;\, \psi)].$$

Here $\delta_{i,n}$ is 1 if $i=n$, and 0 if $i\neq n$. For $b\in F^\times$ we put $g_b=\mathrm{diag}(1,\dots,1,b)\in\mathbf{G}(F)$. Also

$$g_0=\begin{pmatrix}1&0&-1\\1&0&1\\0&1&0\end{pmatrix},\quad g_0^+=\begin{pmatrix}1&-1\\1&1\end{pmatrix},\quad g_0^-=\begin{pmatrix}1&1\\1&-1\end{pmatrix}.$$

To introduce Ψ, note that the centralizer

$$\mathbf{H}=\left\{h=\begin{pmatrix}a&p&b\\ {}^tq&z&-{}^tq\\ b&-p&a\end{pmatrix}\in\mathbf{G};\quad hx_0h^{-1}=x_0=\begin{pmatrix}1&0&1\\0&0&0\\1&0&1\end{pmatrix}\right\}$$

of the $n\times n$ matrix x_0 (which has four non-zero entries, at the corners; also, p, q are row vectors of length $n-2$, and z is an $(n-2)\times(n-2)$ matrix) in $\mathbf{G}$, is isomorphic to $\mathrm{GL}(n-1)$ when $n\geqslant 2$. Denote by $\iota:\mathbf{H}\to\mathrm{GL}(n-1)$, this isomorphism. In the case of $n=2$ it is given by

$$\iota\left(\begin{pmatrix}a&b\\b&a\end{pmatrix}\right)=\mathrm{diag}((a+b)/(a-b),\ 1).$$

Put $\xi(h) = \xi(\det \iota(h))(= \xi(\det h)$ since $\xi^n = 1)$; it is a character of $\mathbf{H}(\mathbb{A})/\mathbf{H}(F)$ in $\mathbb{C}^\times$. Also denote by $\mathbf{U}$ the group of $n \times n$ matrices of the form

$$u = \begin{pmatrix} 1 & p & b \\ 0 & I & {}^t q \\ 0 & 0 & 1 \end{pmatrix},$$

where I is the identity $(n-2)\times(n-2)$ matrix, and put $\psi(u) = \psi(p_1 + q_{n-2})$ where $p = (p_1, \ldots, p_{n-2})$, $q = (q_1, \ldots, q_{n-2})$. Then ψ is a non-trivial character of $\mathbf{U}(\mathbb{A})/\mathbf{U}(F)$ in $\mathbb{C}^\times$. Denoting by $\mathrm{d}u$ and $\mathrm{d}h$ Haar measures on $\mathbf{U}(\mathbb{A})$ and $\mathbf{H}(\mathbb{A})$, the "geometric" distributions are

$$\Psi(g;\, f;\, \xi;\, \psi) = \int_{\mathbf{U}(\mathbb{A}) \cap g\mathbf{H}(\mathbb{A})g^{-1} \backslash \mathbf{U}(\mathbb{A})} \int_{\mathbf{H}(\mathbb{A})} f(u^{-1}gh)\xi(h)\psi(u)\, \mathrm{d}h\, \mathrm{d}u.$$

The spectral side of the summation formula is more difficult to express, and to obtain. In any case we now write $\mathbf{H}$ for the subgroup

$$\left\{ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{G};\ h \in \mathrm{GL}(n-1) \right\} \quad \text{of } \mathbf{G} = \mathrm{PGL}(n),$$

and write the spectral side in three different cases, when $n = 2$, when $n \geqslant 3$ for a special f, and for $n = 3$ with a (more) general f.

In the case of $\mathbf{G} = \mathrm{PGL}(2)$ the spectral side is the sum of the following terms. The main term – see (2)1 below – is

$$\sum_\pi \sum_\Phi W_\psi(\pi(f)\Phi)\bar{L}_\Phi(1/2,\, \pi \otimes \xi^{-1}).$$

The first sum ranges over all cuspidal irreducible $\mathbf{G}(\mathbb{A})$-modules π, and the second over an orthonormal basis $\{\Phi\}$ of smooth functions in the automorphic realization of $\pi \subset L_0^2(\mathbf{G}(F)\backslash \mathbf{G}(\mathbb{A}))$. Here

$$W_\psi(\Phi) = \int_{\mathbf{N}(F)\backslash \mathbf{N}(\mathbb{A})} \Phi(u)\psi(x)\, \mathrm{d}u, \quad \mathbf{N} = \left\{ u = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\},$$

is the ψ-Whittaker functional on the space of automorphic forms, and

$$\begin{aligned} L_\Phi(t,\, \pi \otimes \xi^{-1}) &= \int_{F^\times \backslash \mathbb{A}^\times} \Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) |a|^{t-1/2} \xi(a)^{-1} d^\times a \\ &= \int_{\mathbb{A}^\times} W_\psi\left(\pi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \Phi \right) |a|^{t-1/2} \xi(a)^{-1} d^\times a \end{aligned}$$

is the L-function of $\pi \otimes \xi^{-1}$ which is associated with Φ, at t.

The other terms are

$$\frac{1}{2} \sum_{\mu} \sum_{\Phi} \int_{i\mathbb{R}} E_{\psi}(I(f, \mu, \lambda)\Phi, \mu, \lambda) L_{\bar{\Phi}}(\tfrac{1}{2} - \lambda, \xi/\mu) L_{\bar{\Phi}}(\tfrac{1}{2} - \lambda, (\mu\xi)^{-1})$$
$$\cdot L_{\bar{\Phi}}(1 - 2\lambda, \mu^{-2})^{-1} \, d\lambda$$

and

$$\frac{\pi}{2} \sum_{\mu} \sum_{\Phi} \{E_{\psi}(I(f, \mu, \tfrac{1}{2})\Phi, \mu, \tfrac{1}{2})[\delta(\mu\xi)\bar{\Phi}(1) + \delta(\mu\xi)\bar{\Phi}(w)]$$
$$- E_{\psi}(I(f, \mu, -1/2)\Phi, \mu, -1/2)[\delta(\mu\xi)(M(w, \mu^{-1}, -1/2)\bar{\Phi})(1)$$
$$+ \delta(\mu/\xi)(M(w, \mu^{-1}, -1/2)\bar{\Phi})(w)]\}.$$

The sums over μ are taken over a set of representatives of unitary characters μ of $\mathbb{A}^{\times}/F^{\times}$, up to multiplication by $v^{i\lambda}$, $\lambda \in \mathbb{R}$, $v(x) = |x|$, $x \in \mathbb{A}^{\times}/F^{\times}$. Then Φ ranges over an orthonormal basis – consisting of smooth functions – in the space of the normalizedly induced PGL(2, $\mathbb{A}$)-module $I(\mu, \lambda)$ thus Φ satisfies

$$\Phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} g\right) = \mu(a/c)|a/c|^{\lambda + 1/2}\Phi(g).$$

We put

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and } M(w, \mu, \lambda): I(\mu, \lambda) \to I(\mu^{-1}, -\lambda)$$

is the standard intertwining operator. Also $\delta(\mu/\xi^i)$ $(i = 1, -1)$ is 0 if $\mu \neq \xi^i$ and 1 if $\mu = \xi^i$, on $\mathbb{A}^0 = \{x \in \mathbb{A}^{\times}; |x| = 1\}$. If $\mu = \xi^i$ on $\mathbb{A}^0$ we may – and do – choose the representative μ to satisfy $\mu = \xi^i$ on $\mathbb{A}^{\times}$.

In applications, the continuous sum over $I(\mu, \lambda)$, $\lambda \in i\mathbb{R}$, is of little or no importance, and so are the contributions associated with $I(\mu, \pm 1/2)$ (since no cuspidal representation has a component of the form $I(\mu_v, \pm 1/2)$).

Next we describe the spectral side in the general $n \geqslant 3$ case, for a test function $f \in C_c^{\infty}(\mathbf{G}(\mathbb{A}))$ of the form $f = f^u f_u$, such that the component f_u at some non-archimedean place u of F has the following property. Fix a supercuspidal PGL(2, F_u)-module ρ_{2u}, and write $I(\rho_{2u}, \lambda)$, $\lambda = (\lambda_1, \ldots, \lambda_{n-2}, \lambda_n)$ for the G_n-module normalizedly induced from the representation

$$v_u^{\lambda_1} \times \cdots \times v_u^{\lambda_{n-2}} \times \rho_{2u} \otimes v_u^{\lambda_n}$$

of the (Levi subgroup of the) parabolic subgroup of type $(1,1,\ldots,1,2)$. Here $\nu_u(x)=|x|_u$ and $\lambda_1+\cdots+\lambda_{n-2}+2\lambda_n=0$. Then: *$f_u$ has the property that $\pi_u(f_u)$ is 0 unless π_u is a constituent of $I(\rho_{2u},\lambda)$ for some λ.*

The theory of the Bernstein center has the corollary, recorded as Proposition 12 below, that there exist plenty of non-zero functions $f_u\in C_c^\infty(G_u)$ with this property. We need such f_u to dispose of continuous sums of representations which contribute to the summation formula, whose computation is beyond the scope of this paper. Also we emphasize that our computation in the $n>3$ case is only sketched, and as such it is incomplete.

Then the spectral side is

$$\frac{1}{2}\sum_{\rho_2}\sum_{\Phi} E_\psi(I(f,\rho,0)\Phi,\rho,0)$$

$$[L_{(M(s_2,\check\rho_2,0)\bar\Phi)_{\mathbb{K},\xi^{-1}}}(\rho_2\otimes\xi^{-1},(n-1)/2)+L_{\bar\Phi_{\mathbb{K},\xi^{-1}}}(\check\rho_2\otimes\xi,(n-1)/2)].$$

Here ρ_2 ranges over all cuspidal representations of $\mathrm{GL}(2,\mathbb{A})$ (with the supercuspidal component ρ_{2u} at u) whose central character is ξ^{2-n}; ρ is the representation $\xi\times\rho_2$ of $\mathrm{GL}(n-2,\mathbb{A})\times\mathrm{GL}(2,\mathbb{A})$, extended trivially to $\mathbf{P}(\mathbb{A})$, $\mathbf{P}$ being the parabolic of type $(n-2,2)$; Φ ranges over an orthonormal smooth basis for the $\mathbf{G}(\mathbb{A})$-module $I(\rho,0)$ normalizedly induced from ρ on $\mathbf{P}(\mathbb{A})$. The L-functions are associated to the indicated functions – for whose definition see Propositions 9 and 11 – in the spaces of the cuspidal $\mathrm{GL}(2,\mathbb{A})$-modules $\rho_2\otimes\xi^{-1}$ and $\check\rho_2\otimes\xi$. They are evaluated at $(n-1)/2$, in the domain of absolute convergence when $n\geqslant 4$, and on the edge of the critical strip when $n=3$.

The upshot of this is that (up to the minor local assumption at u) the support of the summation formula consists of the $\mathbf{G}(\mathbb{A})$-modules $I(\rho)$ normalizedly induced from the standard parabolic with Levi factor $\mathrm{GL}(n-2,\mathbb{A})\times\mathrm{GL}(2,\mathbb{A})$, and the representation $\rho=\xi\otimes\rho_2$ on it, where ρ_2 is an automorphic $\mathrm{GL}(2,\mathbb{A})$-module with central character ξ^{2-n}.

When $n=3$, thus $\mathbf{G}=\mathrm{PGL}(3)$, our computation of the spectral side in the summation formula is complete, for a function $f=f^uf_u$ where f_u is no longer required to have the property with respect to the supercuspidal ρ_{2u}. The function f_u is nevertheless restricted to be spherical and have the following property. Denote by $I_u(\lambda_1,\lambda_2,\lambda_3)$ the G_u-module normalizedly induced from the character $(b_{ij})\mapsto\prod_{1\leqslant i\leqslant 3}|b_i|_u^{\lambda_i}$ of the upper triangular subgroup. Here

$$\lambda_i\in\mathbb{C}\quad\text{with } |x|_u^{\lambda_1+\lambda_2+\lambda_3}=1\quad\text{for all } x\in F_u^\times.$$

Then f_u is taken as follows. *f_u satisfies* $\mathrm{tr}\,\pi_u(f_u)=0$ *if* $\pi_u=I_u(\lambda_1,\lambda_2,\lambda_3)$ *and* (1) $\lambda_i-\lambda_j=\pm 1$ *for some* $i\neq j$, *or* (2) $\lambda_1=\lambda_2=\lambda_3$.

The requirement (1) will not affect any possible applicability of the summation formula, since no representation of $\mathbf{G}(\mathbb{A})$ of the form $I(\xi\times\rho_2)$, where ρ_2 is a cuspidal GL(2, $\mathbb{A}$)-module with the central character ξ^{-1}, has a component which is the unramified constituent of the induced representation of the form $I_u(\lambda_1, \lambda_2, \lambda_3)$, with $\lambda_i - \lambda_j = \pm 1$ for some $i \neq j$. The requirement (2) will not affect applicability either, since if $I(\xi \times \rho_2)$ has the component $I_v(\mu_v, \mu_v, \mu_v)$ for almost all places v of F, where μ_v is a character of $F_v^\times$ of order 3 (or 1), then $\xi^3 = 1$, and $\rho_{2v} \simeq I_v(1) \otimes \xi_v$ for almost all v. But there is no cuspidal representation $\rho_2 \otimes \xi^2$ of PGL(2, $\mathbb{A}$) whose component is the same as that of the principal series representation $I(\mathbb{1})$ at almost all places $\left(I(\mathbb{1})\right.$ is the PGL(2, $\mathbb{A}$)-module normalizedly induced from the trivial representation of $\left.\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}\right)$. Thus the requirement here on the component f_u is put to simplify the computations, and is not important. An analogous requirement in the case of PGL(2) would annihilate the terms associated with $I(\mu, \pm 1/2)$, which – as noted in the discussion of the case of PGL(2) above – are not important.

For $f = f^u f_u$ with such a component f_u, the spectral side is the sum of the terms parametrized by $I(\xi \times \rho_2)$, cuspidal ρ_2 on GL(2, $\mathbb{A}$), as described above for $n \geqslant 3$, and terms parametrized by a line of representations, of the form $I(\rho_1 v^{i\lambda} \times \rho_2 v^{-i\lambda} \times \rho_3)$, $i\lambda \in i\mathbb{R}$. As explained at the end of this paper, these new terms are integrals over $i\mathbb{R}$, with integrand containing $E_\psi(I(f, \rho, \lambda)\Phi, \rho, \lambda)$, and the expressions labeled ((3)i,j); $i = 4, 5$; $j = 1, 2, 3, 4$; and ((3)6.j), $1 \leqslant j \leqslant 5$.

The term corresponding to ((3)4.1) take the form

$$\frac{1}{4} \sum_\rho \delta(\xi/\rho_3) \int_{i\mathbb{R}} \sum_\Phi E_\psi(I(f, \rho, \lambda)\Phi, \rho, \lambda)\varepsilon(\lambda_1/2, \rho_1/\rho_3)$$
$$\cdot L_{M\Phi}(1 - \lambda_1/2, \rho_3/\rho_1) L_{M\Phi}(1 + \lambda_1/2, \rho_3/\rho_2) L_{M\Phi}(1 + \lambda_1, \rho_1/\rho_2)^{-1}\, \mathrm{d}\lambda_1.$$

Here $\lambda = (\lambda_1/2, -\lambda_1/2, 0)$, and $\rho = \rho_1 \times \rho_2 \times \rho_3$ is a character of $(\mathbb{A}^\times/F^\times)^3$ with $\rho_1\rho_2\rho_3 = 1$, namely $I(\rho, \lambda) = I(\rho_1 v^{\lambda_1/2} \times \rho_2 v^{-\lambda_2/2} \times \rho_3)$, Φ ranges over an orthonormal smooth basis for (the trivialized vector bundle) $I(\rho, \lambda)$, M is some intertwining operator and $L_{M\Phi}(\rho_i/\rho_j)$ is an L-function, attached to the character ρ_i/ρ_j.

The other (twelve) terms have a similar shape. It will be too long to write out all these terms, although this can be easily derived from our computations. This description, and convergence properties of the integrals and sums, lend themselves to separation arguments used to derive applications from such summation formulae (see, e.g. [FK], Theorem 2).

The main conclusion from our computations is however the following:

THEOREM. *The support of the summation formula is concentrated only on those automorphic* (*unitary*) *PGL*(3, $\mathbb{A}$)*-modules of the form* $I(\xi \times \rho_2)$, *normalizedly induced from a maximal parabolic subgroup, where* ρ_2 *is an automorphic* (*unitary*) *generic GL*(2, $\mathbb{A}$)*-module* (*with central character* ξ^{-1}).

Appendix. Representations with one dimensional quotient

As suggested in the Introduction, we shall consider now irreducible representations π of $G_n = \mathrm{GL}(n, F)$, where F is a local field, which admit a linear form which transforms under G_{n-1} according to a character ξ. Here G_{n-1} is embedded in G_n via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. The following propositions are not used in this paper, but they shed light on our global theory, being local analogues. In Proposition 0, whose proof was suggested to us by J. Bernstein, we consider only unitarizable representations, since every component of an automorphic representation is such. The description is completed in Proposition 0.1, following communication from D. Prasad [P3], who worked out the case of G_3 in general.

Put $\nu(x) = |x|$, $x \in F^\times$. Denote by $I_{(n-2,2)}(\xi \times \rho_2)$ the representation of G_n normalizedly induced from the representation $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \xi(a)\rho_2(c)$ of the standard parabolic subgroup of type $(n-2, 2)$. Here ρ_2 is a representation of G_2, and $\xi(a) = \xi(\det a)$, for a in G_{n-2}. Similarly we introduce $I_{(n-1,1)}(\xi_1 \times \xi_2)$, where ξ_i are characters of $F^\times$. Let P_n denote the group of matrices in G_n whose bottom row is $(0, \dots, 0, 1)$.

0. PROPOSITION. *Let π be a unitarizable irreducible admissible representation of G_n, such that $\mathrm{Hom}_{G_{n-1}}(\pi, \xi) \neq \{0\}$, where ξ is a unitary character of $F^\times$. Then $\pi = \xi$ or there is an irreducible admissible unitarizable representation ρ_2 of G_2 such that $\pi = I_{(n-2,2)}(\xi \times \rho_2)$.*

Proof. The proof is based on the analysis – developed in Bernstein-Zelevinsky [BZ2], Section 3 – of the restriction τ of a representation of G_n to its subgroup $P = P_n$. According to [BZ2], (3.5), p. 452, there exists a natural filtration $\tau = \tau_1 \supset \tau_2 \supset \cdots \supset \tau_n \supset 0$, such that $\tau_k/\tau_{k+1} = \Phi^{k-1}\Psi(\tau^{(k)})$, where $\tau^{(k)}$ is the kth derivative of τ – see [BZ2], (3.5). This is a representation of G_{n-k}. The functors $\Psi = \Psi^+ = i_{V,1}$: Alg $G_{n-1} \to$ Alg P_n and $\Phi = \Phi^+ = i_{V,\theta}$: Alg $P_{n-1} \to P_n$ are defined in [BZ2], (3.2), where V is the unipotent radical of the parabolic of type $(n-1, 1)$, and θ indicates a non-trivial (additive) character of F, and also of V, via $\theta((v_{ij})) = \theta(v_{n-1,n})$. The induction i is normalized – on [BZ2], p. 444 – by the character $\mathrm{mod}_V^{1/2}(m) = |\det m|^{1/2}$ of $m \in P$.

It is also useful to recall – from [BZ2], (4.4) and (4.5), pp. 454–5 – that if π is a supercuspidal G_n-module then its kth derivative $\pi^{(k)}$ is 0 $(0 < k < n)$, and $\pi^{(n)} = 1$ when π is also irreducible. Further, the composition series of $I(\rho_1 \times \rho_2)^{(k)}$ consists of $I(\rho_1^{(i)} \times \rho_2^{(k-i)})$, and if ξ is a one dimensional representation of G_n then $\xi^{(k)}$ is 0 unless $k = 1$, where the character $\nu^{-1/2}\xi$ of G_{n-1} is obtained.

Now if $\mathrm{Hom}_{G_{n-1}}(\tau, \xi) \neq 0$, then $\mathrm{Hom}_{G_{n-1}}(\tau_k/\tau_{k+1}, \xi) \neq 0$, for some $k \geqslant 1$.

If $k=1$ then $\tau_1/\tau_3 = \Psi(\tau^{(1)})$, and the restriction $\Psi(\tau^{(1)})|G_{n-1}$ is $\tau^{(1)}\nu^{1/2}$. Hence

$$\mathrm{Hom}_{G_{n-1}}(\Psi(\tau^{(1)}), \xi) = \mathrm{Hom}_{G_{n-1}}(\tau^{(1)}\nu^{1/2}, \xi) \neq 0$$

precisely when $\nu^{1/2}\tau^{(1)}$ has ξ as a quotient. If $\pi | P = \tau$, and π is a quotient of $I(\rho_1 \times \cdots \times \rho_r)$, where ρ_i are irreducible supercuspidal not necessarily unitarizable representations of G_{j_i}, then the composition series of $I(\rho_1 \times \cdots \times \rho_r)^{(1)}$ consists of

$$I(\rho_1 \times \cdots \times \hat{\rho}_i \times \cdots \times \rho_r), \quad \text{where } \mathrm{G}_{j_i} = \mathrm{GL}(1, F).$$

The hat over ρ_i indicates that ρ_i is omitted. Such a composition factor may have the quotient $\xi\nu^{-1/2}$ only if it is of the form

$$\xi\nu^{-1/2} \otimes I(\nu^{(n-2)/2} \times \nu^{(n-4)/2} \times \cdots \times \nu^{-(n-2)/2})$$

(by Zelevinsky [Z]), namely π is a quotient of

$$I = \xi \otimes I(\nu^{(n-3)/2} \times \nu^{(n-5)/2} \times \cdots \times \mu \times \cdots \times \nu^{-(n-1)/2}),$$

where μ is a character. However, by [Z] and Tadic [T], the only unitarizable quotient of I is ξ, obtained when $\mu = \nu^{n-1/2}$.

REMARK. If $\mu \neq \nu^{[n-(2k+1)]/2}$ $(1 \leqslant k \leqslant n)$, then (by [Z]) I has the unique irreducible quotient

$$\pi = \xi \otimes I_{(n-1,1)}(\nu^{-1/2} \times \mu),$$

and the composition series of

$$\pi^{(1)} = (\xi \otimes I_{(n-1,1)}(\nu^{-1/2} \times \mu))^{(1)}$$

consists of

$$I_{(n-2,1)}(\xi\nu^{-1} \times \xi\mu) \quad \text{and} \quad \xi\nu^{-1/2}.$$

We have $\mathrm{Hom}_{G_{n-1}}(\pi, \xi) \neq \{0\}$ precisely when $\xi\nu^{-1/2}$ is a quotient, not a sub, in this composition series. But we have not investigated $\mathrm{Hom}_{G_{n-1}}(\pi, \xi)$ for these π, nor for the quotients π of I when $\mu = \nu^{[n-(2k+1)]/2}$ $(1 \leqslant k \leqslant n)$.

If $k \geqslant 2$ the quotient τ_k/τ_{k+1} is of the form $\Phi(\beta_{n-2})$, where β_{n-2} is a representation of P_{n-1}. Note that

$$\Phi(\beta_{n-2})|G_{n-1} = \mathrm{ind}_{P_{n-1} \times V_n}^{P_n}(\nu^{1/2}\beta_{n-2} \times \theta)|G_{n-1} = \mathrm{ind}_{P_{n-1}}^{G_{n-1}}(\nu^{1/2}\beta_{n-2}),$$

since $P_n/P_{n-1}V_n = G_{n-1}/P_{n-1}$. Here ind indicates unnormalized induction. By Frobenius reciprocity ([BZ1], (2.29)) we then have

$$\mathrm{Hom}_{G_{n-1}}(\Phi(\beta_{n-2}), \xi) = \mathrm{Hom}_{G_{n-1}}(\mathrm{ind}_{P_{n-1}}^{G_{n-1}}(\nu^{1/2}\beta_{n-2}), \xi)$$
$$= \mathrm{Hom}_{P_{n-1}}(\nu^{-1}\nu^{1/2}\beta_{n-2}, \xi).$$

This is 0 unless β_{n-2} is 1 on V_n, and if β_{n-2} is 1 on V_n we obtain $\mathrm{Hom}_{G_{n-2}}(\nu^{-1/2}\beta_{n-2}|G_{n-2}, \xi)$. In particular $\mathrm{Hom}_{G_{n-1}}(\tau_k/\tau_{k+1}, \xi) = 0$ if $k \geqslant 3$.

If $k = 2$ then $\beta_{n-2}|G_{n-2} = \Psi(\tau^{(2)})|G_{n-2} = \tau^{(2)}\nu^{1/2}$, and $\mathrm{Hom}_{G_{n-1}}(\tau_2/\tau_3, \xi) \neq 0$ if and only if $\mathrm{Hom}_{G_{n-2}}(\tau^{(2)}, \xi) \neq 0$, namely ξ is a quotient of $\tau^{(2)}$. As in the case of $k = 1$ we represent π as a quotient of an induced $I = I(\rho_1 \times \cdots \times \rho_r)$ from supercuspidals ρ_i of G_{j_i}. The second derivative $I^{(2)}$ is glued – in the terminology of [BZ2] – from $I(\rho_1 \times \cdots \times \hat{\rho}_i \times \cdots \times \rho_r)$, where ρ_i is a representation of G_2, and from $I(\rho_1 \times \cdots \times \hat{\rho}_l \times \cdots \times \hat{\rho}_j \times \cdots \times \rho_r)$, where ρ_l and ρ_j are characters (of G_1). By [Z] these can have ξ as a quotient only when the non-deleted $\rho_1 \times \rho_2 \times \cdots$ are of the form

$$\xi\nu^{(n-3)/2} \times \xi\nu^{(n-5)/2} \times \cdots \times \xi\nu^{-(n-3)/2}.$$

In the first case, where $\rho_i \in \mathrm{Alg}\, G_2$ it is not linked – in the terminology of [BZ2] and [Z] – to the other characters, our π must be equivalent – by the irreducibility criterion of [BZ2] – to the irreducible $I_{(n-2,2)}(\xi \times \rho_2)$, and then $\pi^{(2)}$ is ξ. In the second case, where ρ_l, ρ_j are characters, if they are not linked to the $\xi\nu^*$, the same conclusion is obtained (again by [BZ2]). If they are it is easy to conclude from [T] that the only unitarizable quotient of

$$\xi \otimes I(\nu^{(n-3)/2} \times \cdots \times \rho_l \times \cdots \times \rho_j \times \cdots \times \nu^{-(n-3)/2}),$$

$n > 3$ can be ξ, or $\xi \otimes I_{(n-2,2)}(\mathbb{1} \times \mathbb{1}_2)$, or $\xi \otimes I_{(n-2,1,1)}(\mathbb{1} \times 1 \times 1)$.

Let π be irreducible on G_3 with $\mathrm{Hom}_{G_2}(\pi, \xi) \neq \{0\}$. Then $\mathrm{Hom}_{G_2}(\nu^{1/2}\pi^{(1)}, \xi) \neq \{0\}$ or $\mathrm{Hom}_{G_1}(\pi^{(2)}, \xi) \neq \{0\}$. Noting that the Steinberg representations sp of G_2 and st of G_3 satisfy $sp^{(1)} = \nu^{1/2}$, $st^{(1)} = \nu^{1/2}sp$, and $st^{(2)} = \nu$, computing the derivatives we conclude that π is ξ, or $I_{(1,2)}(\xi \times \rho)$, or $I_{(1,2)}(\mu \times \xi\nu^{\pm 1/2})$ (but neither $I_{(2,1)}(\xi\nu^{-1/2} \times \xi\nu)/\xi$, nor its contragredient $I_{(1,2)}(\xi\nu^{-1} \times \xi\nu^{1/2})/\xi$; neither $\pi = I_{(1,2)}(\mu \times \xi\nu^{-1/2}sp)(\mu \neq \xi)$ nor $\pi = \xi\nu^{-1}st$ have $\mathrm{Hom}_{G_2}(\pi, \xi) \neq \{0\}$, since both have $\mathrm{Hom}_{G_2}(\check{\pi}, \xi^{-1}) = \{0\}$, and $\mathrm{Hom}_{G_2}(\pi, \xi) \simeq \mathrm{Hom}_{G_2}(\check{\pi}, \xi^{-1})$ by [GK]: $\check{\pi}(g) = \pi({}^tg^{-1})$), or $\pi = I_{(1,2)}(\xi \times \xi\nu^{-3/2}sp)/\xi\nu^{-1}st$ or its contragredient $I_{(2,1)}(\xi\nu^{3/2}sp \times \xi)/\xi\nu st$. (That each of these, except $I_{(1,2)}(\xi \times \rho)$, with $\dim \rho = 1$, $\rho \neq \xi\nu^{\pm 1/2}$, has $\mathrm{Hom}_{G_2}(\pi, \xi) \neq \{0\}$, is shown next.) The proposition follows.

0.1 PROPOSITION. *An irreducible G_n-module π of the form $I_{(n-1,1)}(\xi\nu^{\pm 1/2} \times \mu)$, or $I_{(n-2,2)}(\xi \times \xi\rho)$, where μ is a character and ρ is infinite dimensional or the character $\nu^{\pm(n-2)/2}$, has $\mathrm{Hom}_{G_{n-1}}(\pi, \xi) \neq \{0\}$. If $\dim \rho = 1$ but $\rho \neq \nu^{\pm(n-2)/2}$, then $\mathrm{Hom}_{G_{n-1}}(\pi, \xi) = \{0\}$, where $\pi = I_{(n-2,2)}(\xi \times \xi\rho)$. The irreducible G_3-modules*

π with $\mathrm{Hom}_{G_2}(\pi, \xi) \neq \{0\}$ *are the irreducible* $\pi = \xi$, $I_{(1,2)}(\xi \times \rho)$ $(\dim \rho > 1)$, $I_{(1,2)}(\mu \times \xi\nu^{\pm 1/2})$, $I_{(1,2)}(\xi \times \xi\nu^{-3/2}sp)/\xi\nu^{-1}st$ *and its contragredient* $I_{(2,1)}(\xi\nu^{3/2}sp \times \xi)/\xi\nu st$.

Proof. Since $\mathrm{Hom}_{G_{n-1}}(\pi, \xi) = \mathrm{Hom}_{G_{n-1}}(\pi\xi^{-1}, \mathbb{1})$, we assume that $\xi = 1$. Using the map $\varphi \mapsto \varphi | G_{n-1}$ (=restriction to G_{n-1}), "Mackey's theory" (see [BZ1], (1.8)) implies that $I_{(n-1,1)}(\nu^{-1/2} \times \mu) | G_{n-1}$ has trivial quotient. By [GK] π has a G_{n-1}-invariant form iff its contragredient $\check{\pi}$ does. Hence

$$\mathrm{Hom}_{G_{n-1}}(I_{(n-1,1)}(\nu^{\pm 1/2} \times \mu), \mathbb{1}) \neq \{0\}.$$

The same map and reference imply that $I_{(n-2,2)}(\mathbb{1} \times \rho) | G_{n-1}$ has the quotient $I_{(n-2,1)}(\nu^{1/2} \times \rho')$, where $\rho'(a) = \rho(\mathrm{diag}(a, 1))$. Now $I_{(n-2,1)}(\nu^{1/2} \times \mu)$ has a trivial quotient precisely when $\mu = \nu^{-(n-2)/2}$ (by [Z]). Hence $\mathrm{Hom}_{G_{n-1}}(\pi, \mathbb{1}) \neq \{0\}$ for the irreducible $\pi = I_{(n-2,2)}(\mathbb{1} \times \nu^{-(n-2)/2})$, and its contragredient $I_{(n-2,2)}(\mathbb{1} \times \nu^{(n-2)/2})$. Moreover, for any infinite dimensional G_2-module ρ, and any character μ, we have $\mathrm{Hom}_{G_1}(\rho, \mu) \neq \{0\}$, by [W1], Propositions 9, 10, or [W3], Lemmas 8, 9, pp. 219–220. Hence $\mathrm{Hom}_{G_1}(\rho', \nu^{-(n-2)/2}) \neq \{0\}$, and $\mathrm{Hom}_{G_{n-1}}(I_{(n-2,2)}(\mathbb{1} \times \rho), \mathbb{1}) \neq \{0\}$ for infinite dimensional ρ. The first assertion follows.

If $\dim \rho = 1$, consider also the kernel of the map $\varphi \mapsto \varphi | G_{n-1}$, from $\pi = I_{(n-2,2)}(\mu \times \rho)$ to $I_{(n-2,1)}(\mu\nu^{1/2} \times \rho)$. By Corollary 5.1 below, G_n is the disjoint union of PG_{n-1} $(P = P_{(n-2,2)})$, $Pr(n-2, n)G_{n-1}$, and $P\kappa G_{n-1}$. Hence Mackey's theory ([BZ1], (1.8)) implies that there are two constituents in this kernel, as follows. The set of $\{g \mapsto \varphi(r(n-2, n)g); \varphi \in \pi\}$ is easily seen to be the space of $I_{(n-3,2)}(\mu \times \rho\nu^{-1/2})$; this G_{n-1}-module has the trivial quotient only when $\mu = \nu$ and $\rho = \nu^{-(n-4)/2}$. The set of $\{g \mapsto \varphi(\kappa g); \varphi \in \pi\}$ is the space of the unnormalizedly compactly induced G_{n-1}-module $\tau = \mathrm{ind}((\mu\nu \times 1 \times \rho\nu^{-(n-2)/2}); G_{n-1}, Q)$, where Q is the group of matrices in the standard parabolic subgroup of G_{n-1} of type $(n-3, 1, 1)$, whose $(n-2, n-2)$-entry is 1, and the indicated representation of Q is trivial on the unipotent radical. Now Frobenius reciprocity ([BZ1], (2.29)) implies that

$$\mathrm{Hom}_{G_{n-1}}(\tau, \mathbb{1}_{n-1}) = \mathrm{Hom}_Q((\mu\nu^{-1} \times 1 \times \rho\nu^{(n-2)/2}), \mathbb{1}_Q);$$

this is non-zero only when $\mu = \nu$ and $\rho = \nu^{-(n-2)/2}$. This proves the second assertion.

By Proposition 13 below, G_3 is the disjoint union of $P = P_3$, $Pr(23)G_2$, and $Pr(23)uG_2$. Here $r(23)$ is the matrix with entry 1 at (1, 1), (2, 3), (3, 2), and 0 elsewhere; u has 1 along the diagonal, -1 at (2, 3), and 0 elsewhere. It follows from the proof of Proposition 0 that π is not supercuspidal; hence π is a constituent of $I_{(2,1)}(\rho \times \mu)$. The map $\varphi \mapsto \varphi | G_2$ takes $I_{(2,1)}(\rho \times \mu)$ onto $\rho\nu^{1/2}$. The map $\varphi \mapsto f$, where $f(g) = \varphi(r(23)g)$, takes $I_{(2,1)}(\rho \times \mu)$ to $I_{(1,1)}(\rho' \times \mu\nu^{-1/2})$. The set of $\{g \mapsto \varphi(r(23)ug);\ \varphi \in I_{(2,1)}(\rho \times \mu)\}$, is the space of $\mathrm{ind}(v^{1/2}\rho|P_2; G_2, P_2)$ (unnormalized compact induction). The cosets P and $Pr(23)G_2$ are closed and disjoint, while $Pr(23)uG_2$ is open. Hence [BZ1], (1.8), asserts that we have the exact sequence (*):

$$0 \to \mathrm{ind}(v^{1/2}\rho|P_2;\ G_2,\ P_2) \to I_{(2,1)}(\rho \times \mu)|G_2 \to \rho v^{1/2} \oplus I_{(1,1)}(\rho' \times \mu v^{-1/2}) \to 0.$$

The quotient in this sequence has a trivial quotient when $\rho = v^{-1/2}$, or $\mu = 1$ and $\rho = v^{1/2}$, or $\mu = 1$ and $\dim \rho > 1$ (since then $\mathrm{Hom}_{G_1}(\rho, v^{1/2}) \neq \{0\}$). Hence $\mathrm{Hom}_{G_2}(\pi, \mathbb{1}) \neq \{0\}$ for irreducible π of the form $I_{(2,1)}(\rho \times 1)$, $\dim \rho > 1$, $I_{(2,1)}(v^{\pm 1/2} \times \mu)$, and $I_{(2,1)}(v^{3/2}sp \times 1)/vst$ (since $\mathrm{Hom}_{G_2}(I_{(2,1)}(v^{3/2}sp \times 1), \mathbb{1}) \neq \{0\}$ as we have just seen, and $\mathrm{Hom}_{G_2}(vst, \mathbb{1}) = \{0\}$ as noted at the end of the proof of Proposition 0).

Finally, if $I_{(2,1)}(\rho \times \mu)$ admits a G_2-invariant form which does not factorize through the quotient in the exact sequence (*), then

$$0 \neq \mathrm{Hom}_{G_2}(\mathrm{ind}(v^{1/2}\rho|P_2;\ G_2,\ P_2), \mathbb{1}) = \mathrm{Hom}_{P_2}(v^{-1/2}\rho, \mathbb{1});$$

the last equality is Frobenius reciprocity of [BZ1], (2.29). Since $\mathrm{Hom}_{P_2}(v^{-1/2}\rho, \mathbb{1}) = \mathrm{Hom}_{G_1}(\rho_N, \mathbb{1})$, where ρ_N is the normalized module of coinvariants of ρ ([BZ2], p. 444), we conclude that ρ must be $v^{1/2}$, or $v^{-1/2}sp$ (as $1_N = v^{-1/2}$ and $sp_N = v^{1/2}$), or induced $I_{(1,1)}(\mu_1 \times \mu_2)$, with $\mu_1 = 1$ or $\mu_2 = 1$. But it has already been shown above that $\mathrm{Hom}_{G_2}(I_{(2,1)}(\rho \times \mu), \mathbb{1}) \neq \{0\}$ for these ρ, except for $\rho = v^{-1/2}sp$, where $\mathrm{Hom}_{G_2}(I_{(2,1)}(\rho \times \mu), \mathbb{1}) = \{0\}$ by the end of the proof of Proposition 0. The proposition follows.

Finite groups

We shall very briefly note now that the analogue of Proposition 0 – in the case of the finite group $G_n = \mathrm{GL}(n, q)$ over the field with q elements – follows from the "branching rule", i.e. Corollary 13.8 of [Z2], p. 148, whose proof shares much with the proof of Proposition 0 above. We shall use the notations of [Z2]. The irreducible representations of G_n are parametrized by partition valued functions $\varphi \in S_n(C, P)$ ([Z2], (4.19), pp. 68–9) on the set C of irreducible cuspidal representations ρ in $R(q) = \bigoplus_{n \geq 0} R(G_n)$, with $\deg(\varphi) = \Sigma_{\rho \in C} \deg(\rho)|\varphi(\rho)|$ equals n. Corresponding to such $\varphi\colon C \to P$, $\deg(\varphi) = n$, we have then a representation $\{\varphi\}$ in the set $\Omega(G_n) = \mathrm{Irr}(G_n)$. Corollary 13.8 of [Z2], p. 148, asserts, for $\varphi_n \in S_n(C, P)$ and $\varphi_{n-1} \in S_{n-1}(C, P)$, that the multiplicity of $\{\varphi_{n-1}\}$ in the restriction of $\{\varphi_n\}$ to G_{n-1} is equal to the number of $\varphi'' \in S(C, P)$ with $\varphi''(\rho) \dashv \varphi_n(\rho)$ and $\varphi''(\rho) \dashv \varphi_{n-1}(\rho)$ for all $\rho \in C$. The notation $\mu \dashv \lambda$ means (see [Z2], (4.3), p. 50), that μ is obtained by removing at most one point from each row of the Young diagram of λ.

The case studied in this paper is that of φ_{n-1} parametrizing the one dimensional representation ξ_{n-1} of G_{n-1}. We denote by ξ the associated irreducible necessarily cuspidal representation of G_1. Then $\varphi_{n-1}(\xi) = (n-1) \in P_{n-1}$, and $\varphi_{n-1}(\rho) = (0)$ for all $\rho \in C$, $\rho \neq \xi$. If $\varphi''(\rho) \dashv \varphi_{n-1}(\rho)$ (for all $\rho \in C$) then $\varphi''(\xi) = (n-2)$ or $(n-1)$, and $\varphi''(\rho) = (0)$ for all $\rho \neq \xi$ in C. If also $\varphi''(\rho) \dashv \varphi_n(\rho)$ (for all $\rho \in C$), then there are 6 possibilities.

(1) $\varphi_n(\xi) = (n-2)$, $\varphi_n(\rho_2) = (1)$ for some $\rho_2 \in C(G_2)$, in which case $\{\varphi_n\} = I_{(n-2,2)}(\xi_{n-2} \times \rho_2)$, where ρ_2 is an irreducible cuspidal representation of G_2.
(2) $\varphi_n(\xi) = (n-2)$, $\varphi_n(\eta_i) = (1)$ for two distinct characters $\eta_i \neq \xi$, where $\{\varphi_n\} = I_{(n-2,1,1)}(\xi_{n-2} \times \eta_1 \times \eta_2)$.
(3) $\varphi_n(\xi) = (n-2, 1)$, $\varphi_n(\eta) = (1)$ for some $\eta \neq \xi$, where $\{\varphi_n\}$ is the unique irreducible constituent of $I_{(n-2,1,1)}(\xi_{n-2} \times \xi \times \eta)$ (whose length is 2 by [Z2], (c), p. 46) specified by [Z2], (4.1), (9.4–5), as lying also in

$$I_{(2,1,\ldots,1,1)}(s(\xi)_2 \times \xi \times \cdots \times \xi \times \eta),$$

where $s(\xi)_2$ indicates the generic (=non-degenerate) constituent of $I_{(1,1)}$ $(\xi \times \xi) \in R(G_2)$.
(4) $\varphi_n(\xi) = (n-1)$, $\varphi_n(\eta) = (1)$ for some $\eta \neq \xi$, and then $\{\varphi_n\} = I_{(n-1,1)}(\xi_{n-1} \times \eta)$.
(5) $\varphi_n(\xi) = (n-1, 1)$, where $\{\varphi_n\}$ is the irreducible constituent of $I_{(n-1,1)}(\xi_{n-1} \times \xi)$ specified by [Z2] as lying also in $I_{(2,1,\ldots,1)}(s(\xi)_2 \times \xi \times \cdots \times \xi)$.
(6) $\varphi_n(\xi) = (n)$, where $\{\varphi_n\}$ is the one dimensional representation $\xi_n = \xi(\det)$ of G_n.

The analogy with the p-adic case is apparent.

Real compact groups

As noted in the Introduction, the irreducible unitary representations of $\mathrm{SL}(n, \mathbb{R})$ with a non-zero $\mathrm{GL}(n-1, \mathbb{R})$-invariant linear form are determined in [DP]. We shall observe here that the answer to the analogous question in the context of the compact groups $U(n, \mathbb{R})$ and $U(n-1, \mathbb{R})$ is classical.

We first note that there is a natural bijection between the set of irreducible representations of the unitary group $U(n, \mathbb{R})$, and the set of irreducible analytic representations of the group $\mathrm{GL}(n, \mathbb{C})$, given by analytic continuation, or Weyl's "Unitarian trick" (see [Zh], Section 41, Theorem 1, p. 111, Section 42, Theorem 2, p. 113, and Section 44, p. 118).

Second, one knows that every such representation is uniquely determined by a sequence $(m_1, \ldots, m_n)$ of n integers with $m_i \geqslant m_{i+1}$ $(1 \leqslant i < n)$ ("highest weight"), and is denoted here by $R_n(m_1, \ldots, m_n)$; see [Zh], Section 48, Theorem 3, p. 132, and Section 49, Theorem 4, p. 133.

Third, the restriction of $R_n(m_1, \ldots, m_n)$ to $U(n-1, \mathbb{R})$ is completely reducible, and it contains $R_{n-1}(k_1, \ldots, k_{n-1})$, necessarily with multiplicity one, precisely when $m_i \geqslant k_i \geqslant m_{i+1}$ for all $i (1 \leqslant i < n)$; see [Zh], Section 66, Theorem 2, p. 186.

Now the character $\xi(z) = z^k$ defines the character $\xi = R_{n-1}(k, \ldots, k)$ of $U(n-1, \mathbb{R})$, and it is unitary when $k \in \mathbb{Z}$ is 0. Clearly the restriction of $\pi = R_n(m_1, \ldots, m_n)$ to $U(n-1, \mathbb{R})$ contains a copy of ξ precisely when $m_2 = \cdots = m_{n-1} = k$, and then $\pi = \xi \otimes R_n(m_1 - k, 0, \ldots, 0, m_n - k)$. The representations $R_n(l_1, 0, \ldots, 0, l_n)$, $l_1 \geqslant 0 \geqslant l_n$, are in bijection with the representations $\rho_2 = R_2(l_1, l_n)$ of $U(2, \mathbb{R})$. When ξ is taken to be unitary ($\xi = 1, k = 0$) and π is taken to have a unitary (namely trivial) central character, then ρ_2 ranges through all of the unitary irreducible representations of the complexification of $U(2, \mathbb{R})$. Of course π can be viewed as an analytic irreducible unitary representation of $\mathrm{GL}(n, \mathbb{C})$, and ρ_2 as an analytic irreducible unitary representation of $\mathrm{GL}(2, \mathbb{C})$.

This case then agrees with our p-adic result, that the irreducible admissible unitarizable non-trivial π on $\mathrm{GL}(n, F)$ with a non-zero linear form which transforms under $\mathrm{GL}(n-1, F)$ via the unitary character ξ, is of the form $\pi = \xi \otimes I_{(n-2,2)}(\mathbb{1} \times \rho_2)$, where ρ_2 is an irreducible admissible unitarizable representation of $\mathrm{GL}(2, F)$.

C. Geometric side

Put $\mathbf{G} = \mathrm{PGL}(n)$, $n \geqslant 2$, and consider $L = L^2(\mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}))$, where F is a global field (char $F \neq 2$) and $\mathbb{A}$ denotes its ring of adeles. Then $\mathbf{G}(\mathbb{A})$ acts on L by $(r(g)\Phi)(h) = \Phi(hg)$, $g, h \in \mathbf{G}(\mathbb{A})$, $\Phi \in L$. For any f in the space $C_c^\infty(\mathbf{G}(\mathbb{A}))$ of smooth compactly supported complex valued functions on $\mathbf{G}(\mathbb{A})$, the convolution operator $r(f)$ is defined by

$$(r(f)\Phi)(g) = \int_{\mathbf{G}(\mathbb{A})} f(h)\Phi(gh)\,\mathrm{d}h,$$

where $\mathrm{d}h$ is a fixed Haar measure on $\mathbf{G}(\mathbb{A})$. Clearly

$$(r(f)\Phi)(g) = \int_{\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})} K_f(g, h)\Phi(h)\,\mathrm{d}h, \quad \text{where } K_f(g, h) = \sum_{\gamma \in \mathbf{G}(F)} f(g^{-1}\gamma h).$$

Define

$$x_0 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

to be an $n \times n$ matrix whose entries are 0 except at $(1,1)$, $(1,n)$, $(n,1)$, (n,n) where the entry is 1. The centralizer $\mathbf{H} = \{g \in \mathbf{G}; gx_0g^{-1} = x_0\}$ of x_0 in $\mathbf{G}$ consists of matrices of the form

$$\begin{pmatrix} a & p & b \\ {}^tq & z & -{}^tq \\ b & -p & a \end{pmatrix},$$

where a, b are scalars; p, q are row vectors of length $n-2$, ${}^tq$ indicates the transpose of q; and z an $(n-2)\times(n-2)$ matrix. This $\mathbf{H}$ is isomorphic to $\mathrm{GL}(n-1)$, since $w = I - x_0$ is conjugate in $\mathbf{G}$ to $\mathrm{diag}(1, \ldots, 1, 1, -1) \in \mathbf{G}$. Note that when $n=2$ this w is conjugate to $\mathrm{diag}(-1,1)$ by $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and the isomorphism is $\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mapsto (a+b)/(a-b)$.

Denote by $\mathbf{U}$ the group of $n \times n$ matrices of the form

$$u = \begin{pmatrix} 1 & p & z \\ 0 & I & {}^tq \\ 0 & 0 & 1 \end{pmatrix}$$

where here I is the identity $(n-2)\times(n-2)$ matrix. A complex valued character $\psi \neq 1$ of $\mathbb{A}/F$ defines a character $\psi \neq 1$ of $\mathbf{U}(\mathbb{A})/\mathbf{U}(F)$ by $\psi(u) = \psi(p_1 + q_{n-2})$, where $p = (p_1, \ldots, p_{n-2})$ and $q = (q_1, \ldots, q_{n-2})$. Denote by ξ a unitary character of the idele class group $\mathbb{A}^\times/F^\times$ and put $\xi(h) = \xi(\det \iota(h))$ for $h \in \mathbf{H}(\mathbb{A})$; det means "determinant", and ι the isomorphism from $\mathbf{H}$ to $\mathrm{GL}(n-1)$. Note that $\xi(\det \iota(h)) = \xi(\det h)$ since $\xi^n = 1$. We shall integrate the product of $K_f(u, h)$, $\xi(h)$ and $\psi(u)$ over $u \in \mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})$ and $h \in \mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$, and obtain

1. PROPOSITION. *We have*

$$\int_{\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})} \int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})} K_f(u,\, h)\xi(h)\psi(u)\,\mathrm{d}u\,\mathrm{d}h$$
$$= \sum_{b \in F^\times} \Psi(g_b;\, f;\, \xi;\, \psi) + \delta_{3,n}\Psi(g_0;\, f;\, \xi;\, \psi)$$
$$+ \delta_{2,n}[\Psi(g_0^+;\, f;\, \xi;\, \psi) + \Psi(g_0^-;\, f;\, \xi;\, \psi)].$$

Here

$$\Psi(g;\, f;\, \xi;\, \psi) = \int_{\mathbf{U}(\mathbb{A})/\mathbf{U}(\mathbb{A}) \cap g\mathbf{H}(\mathbb{A})g^{-1}} \int_{\mathbf{H}(\mathbb{A})} f(ugh)\xi(h)\psi(u)^{-1}\,\mathrm{d}u\,\mathrm{d}h,$$

and $g_b = \mathrm{diag}(1, \ldots, 1, b) \in \mathbf{G}(F)$, *if* $b \neq 0$, *and*

$$g_0 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{if } n=3,\ g_0^+ = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad g_0^- = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{if } n=2.$$

Note that $\mathbf{U} \cap g\mathbf{H}g^{-1} = \{I\}$ *for* $g = g_b (b \neq 0)$ *or* $g_0^{\pm}$, *but it is*

$$\left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{if } g = g_0 (n=3).$$

This is the geometric half of our summation formula. It is to be compared below with the integral over $\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A}) \times \mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ of the product by $\psi(u)$ and $\xi(h)$ of the spectral expression for the kernel $K_f(u, h)$ of the convolution operator $r(f)$ on L.

To prove the proposition it suffices to show that if $\Psi(g; f; \xi, \psi) \neq 0$ for g in $\mathbf{G}(F)$ then g lies in $\mathbf{U}(F)g_b\mathbf{H}(F)$ for some b in F, and this follows from the local analogue, asserting that if $\Psi(g; f_v; \xi; \psi_v) \neq 0$ for g in G_v then g lies in $U_v g_b H_v$ for some b in F_v. Here v denotes any place of F and F_v is the associated completion of F; we put $G_v = \mathbf{G}(F_v)$, $H_v = \mathbf{H}(F_v)$, $U_v = \mathbf{U}(F_v)$; $\psi_v(u) = \boldsymbol{\psi}_v(p_1 + q_{n-2})$ is a character of U_v defined using a character $\boldsymbol{\psi}_v \neq 1$ of F_v; and f_v lies in the space $C_c^\infty(G_v)$ of smooth compactly supported complex valued functions on G_v. The local integral is defined in analogy with the global integral:

$$\Psi(g, f_v, \xi_v, \psi_v) = \int_{U_v/U_v \cap gH_v g^{-1}} \int_{H_v} f_v(ugh)\xi_v(h)\psi_v(u)^{-1}\, du\, dh.$$

Note that

$$\Psi(g; f, \xi, \psi) = \prod_v \Psi(g_v; f_v, \xi, \psi_v) \quad \text{if } g = (g_v),\ \xi = \otimes \xi_v,\ \psi = \otimes \psi_v,\ f = \otimes f_v,$$

$$dh = \otimes dh_v \quad \text{and} \quad du = \otimes du_v.$$

2. LEMMA. *If* $\Psi(g; f_v; \xi_v; \psi_v) \neq 0$ *then* $g \in U_v g_b H_v$ *for some* $b \in F_v$.

Proof. To simplify the notations the index v is omitted in the course of the proof, and so is the reference to ψ, ξ. The integral $\Psi(g; f)$ satisfies $\Psi(gh; f) = \xi^{-1}(h)\Psi(g; f)(h \in H)$, hence its support depends only on the image of g in G/H. The homogeneous space G/H is isomorphic to the space X of $n \times n$ matrices (over F) of rank 1 and trace 2 via the map $g \mapsto gx_0g^{-1}$. Note that $x_0 = {}^t\varepsilon\varepsilon$, where $\varepsilon = (1, 0, \ldots, 0, 1)$. The integral $\Psi(g; f)$ is then viewed as a function $\Xi(x)$ on X which satisfies $\Xi(uxu^{-1}) = \psi(u)\Xi(x)$. The image of the double coset Ug_bH, $b \neq 0$, in X is the set of the matrices

$$(ug_b{}^t\varepsilon)(\varepsilon g_b^{-1}u^{-1}) = {}^t(1+zb,\ bq,\ b)(1,\ -p,\ p{}^tq - z + b^{-1}) \quad \text{if } u = \begin{pmatrix} 1 & p & z \\ 0 & I & {}^tq \\ 0 & 0 & 1 \end{pmatrix},$$

namely the matrices in X whose $(n, 1)$ entry is $b \neq 0$. To prove the lemma it suffices to show that for any matrix x in X whose $(n, 1)$ entry $x_{n,1}$ is 0 there exists $u \in U$ with $uxu^{-1} = x$ and $\psi(u) \neq 1$. Indeed, if $g \in G$ has the image x, namely $gx_0g^{-1} = x$, there would exist $h \in H$ (with $\det h = 1$) such that $ugh = g$. Then $\Psi(ugh; f) = \psi(u)\Psi(g; f)$ is necessarily zero.

A matrix x in X whose last non-zero row is the lth, and its first non-zero column is the fth, is of the form $x = {}^tvw$, where $v = (v_1, \ldots, v_l, 0, \ldots, 0)$, $v_l \neq 0$, and $w = (0, \ldots, 0, w_f, \ldots, w_n)$, $w_f \neq 0$. If $l \geqslant 3$ and $f > 1$ then $uxu^{-1} = x$, where u has $q = 0$, and top row $(1, yv_l, 0, \ldots, 0, -yv_2, 0, \ldots)$, with the entry $-yv_2$ at the lth place. If $f \leqslant n-2$ and $l < n$ then $uxu^{-1} = x$, where u has $p = 0$ and its last column is ${}^t(0, \ldots, 0, -yw_{n-1}, 0, \ldots, 0, yw_f, 1)$, with the entry $-yw_{n-1}$ at the fth place. If $l \leqslant 2$ and $f \geqslant n-1$, then $n = 2$ or $n = 3$, since $\operatorname{tr} x = 2$. If $n = 3$ then $l = 2 = f$, and U acts by conjugation transitively on the orbit of

$$g_0x_0g_0^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with stabilizer $U \cap g_0Hg_0^{-1}$ as stated in the proposition. If $n = 2$ then U acts simply transitively on the orbit of

$$g_0^+x_0(g_0^+)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

and on the orbit of

$$g_0^-x_0(g_0^-)^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

This completes the proof of the lemma and Proposition 1.

REMARK. Choosing the character ψ to be of the form $\psi(u) = \psi(p_1 + q_1)$, a similar lemma is obtained but with a term indexed by a suitable g_0 is present for all $n > 2$.

Denote by $\mathbf{U}'$ the group of unipotent upper triangular matrices in $\mathbf{G}$ whose top row is $(1,0,\ldots,0)$ and last column is ${}^t(0,\ldots,0,1)$. Then $\mathbf{U}'$ consists of the identity matrix only, unless $n \geqslant 4$ as we now assume. For any $y \in G_v$, denote by ${}^yf_v$ the function ${}^yf_v(g) = f_v(yg)$. Clearly $\Psi(g; {}^yf_v; \xi_v; \psi_v)$ is independent of y if $y \in U'_v$, since this integral is non-zero only if $g \in U_v g_b H_v$ for some $b \in F_v^\times$. Any character of $\mathbf{U}'(F)\backslash\mathbf{U}'(\mathbb{A})$ is of the form

$$\psi'_\alpha(u') = \psi\left(\sum_{2 \leqslant i \leqslant n-2} \alpha_i u_{i,i+1}\right),$$

for some $\alpha = (\alpha_2, \ldots, \alpha_{n-2}) \in F^{(n-3)}$. The unipotent upper triangular subgroup $\mathbf{N}_0$ of $\mathbf{G}$ is equal to $\mathbf{U}\mathbf{U}' = \mathbf{U}'\mathbf{U}$. Put $\psi_\alpha(uu') = \psi(u)\psi'_\alpha(u')$; it is a character of $\mathbf{N}_0(F)\backslash\mathbf{N}_0(\mathbb{A})$, and we have that

$$\iint_{\mathbf{U}'(F)\backslash\mathbf{N}_0(\mathbb{A}) \times \mathbf{H}(\mathbb{A})} f(ngh)\xi(h)\psi_\alpha(n)^{-1}\,\mathrm{d}n\,\mathrm{d}h$$

is 0 unless $\alpha = (0, \ldots, 0)$, in which case $\Psi(g; f; \xi; \psi)$ is obtained. Consequently

3. COROLLARY. *The integral*

$$\int_{\mathbf{N}_0(F)\backslash\mathbf{N}_0(\mathbb{A})} \int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})} K_f(n,h)\xi(h)\psi_\alpha(n)\,\mathrm{d}n\,\mathrm{d}h$$
$$= \int_{\mathbf{U}'(F)\backslash\mathbf{U}'(\mathbb{A})} \psi'_\alpha(u') \left[\int_{\mathbf{U}(F)\backslash\mathbf{U}(\mathbb{A})} \int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})} K_{({}^{u'}f)}(u,h)\xi(h)\psi(u)\,\mathrm{d}u\,\mathrm{d}h\right]\mathrm{d}u'$$

is 0 *unless* $\alpha = (0, \ldots, 0)$, *in which case it is equal to*

$$\sum_{b \in F^\times} \Psi(g_b; f; \xi, \psi) + \delta_{3,n}\Psi(g_0; f; \xi, \psi) + \delta_{2,n}[\Psi(g_0^+; f; \xi, \psi) + \Psi(g_0^-; f; \xi; \psi)].$$

The sum over $b \in F^\times$ *is finite.*

Only the last assertion remains to be proven. Thus consider $f(u\gamma h)$, with

$$u \in \mathbf{U}(\mathbb{A})/\mathbf{U}(F),\ h \in \mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A}) \quad \text{and} \quad \gamma \in \mathbf{G}(F).$$

If this $f(u\gamma h)$ makes a non-zero contribution to $K_f(u,h)$ then $\gamma x_0 \gamma^{-1}$ lies in the discrete subset $\mathbf{X}(F)$ of the set $\mathbf{X}(\mathbb{A})$, and also is in a compact which depends on the support of f and on the compact $\mathbf{U}(\mathbb{A})/\mathbf{U}(F)$. Hence the image of $\gamma \in \mathbf{G}(F)$ in $\mathbf{G}(F)/\mathbf{H}(F)$ lies in a finite set $\{\gamma_i \mathbf{H}(F)\}$ of cosets (and $h \in \mathbf{H}(\mathbb{A})$ ranges over the compact

$$\left(\bigcup_i (\gamma_i^{-1} \cdot \mathbf{U}(\mathbb{A})/\mathbf{U}(F) \cdot \operatorname{supp} f)) \cap \mathbf{H}(\mathbb{A}) \text{ in } \mathbf{H}(\mathbb{A})\right),$$

as required.

D. The case of PGL(2)

Let us consider separately, and briefly, the well-known spectral expression for the kernel $K_f(g,h)$ in the case where $\mathbf{G} = \mathrm{PGL}(2)$. This is recorded here to motivate the discussion for $n \geqslant 3$ below. We shall truncate this spectral expression with respect to the second variable h, integrate over

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{in } \mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A}),\ \mathbf{N} = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\},$$

after multiplying by $\psi(g) = \psi(x)$, where $\psi \neq 1$ is a character of $\mathbb{A}$ mod F, and integrate over

$$h = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \quad \text{in } \mathbf{A}(F)\backslash\mathbf{A}(\mathbb{A}),$$

$\mathbf{A}$ being the diagonal group in PGL(2), after multiplying by $\xi(a)$, ξ being a character of order 1 or 2 of $\mathbb{A}^\times/F^\times$ in $\mathbb{C}^\times$. The Eisenstein series $E(g, \Phi, \mu, \lambda)$, the truncation operator Λ^T, and the spectral expression for the kernel are defined below in the case of a general n. Hence the standard definitions will not be recalled here separately in the case of $n = 2$. We obtain (the first figure 2 in ((2)1) below refers to $n = 2$, as we now deal with PGL(2))

$$\sum_{\pi} \sum_{\Phi \in \pi} W_\psi(\pi(f)\Phi)\bar{L}_\Phi(\tfrac{1}{2}, \pi \otimes \xi^{-1}) \tag{(2)1}$$

$$+\frac{1}{2} \sum_{\mu} \int_{i\mathbb{R}} \sum_{\Phi} E_\psi(I(f, \mu, \lambda)\Phi, \mu, \lambda) \int_{F^\times\backslash\mathbb{A}^\times} \Lambda^T \bar{E}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \Phi, \mu, \lambda\right) \xi(a)\, \mathrm{d}^\times a\, \mathrm{d}\lambda. \tag{(2)2}$$

Here the first sum ranges over all cuspidal irreducible representations π of PGL(2, $\mathbb{A}$), and Φ ranges over an orthonormal basis – consisting of smooth functions – for the space of $\pi \subset L_0^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))$. The Whittaker functional is defined by

$$W_\psi(\pi(f)\Phi) = \int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} (\pi(f)\Phi)(u)\psi(u)\,\mathrm{d}u,$$

and

$$L_\Phi(t, \pi \otimes \xi^{-1}) = \int_{F^\times\backslash\mathbb{A}^\times} \Phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) |a|^{t-1/2}\xi(a)^{-1}\,\mathrm{d}^\times a$$

is the L-function of $\pi \otimes \xi^{-1}$ at t which is associated with Φ. Since Φ is a cusp form, its Fourier expansion with respect to

$$\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A}) \quad \text{is } \Phi(g) = \sum_{\alpha\in F^\times} W_\psi\left(\pi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\Phi\right),$$

and so

$$L_\Phi(t, \pi\otimes\xi^{-1}) = \int_{\mathbb{A}^\times} W_\psi\left(\pi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)\Phi\right)|a|^{t-1/2}\xi(a)^{-1}\,\mathrm{d}^\times a.$$

At each non-archimedean place v where ξ_v is unramified and $W_\psi(\pi(g)\Phi)$ is right $K_v = \mathrm{PGL}(2, R_v)$ invariant, the local factor at v of this global integral is easily computed (as in the Remark – which is based on [F1], p. 305 – following the Introduction above) to be the local L-factor $L_v(t, \pi_v\otimes\xi_v^{-1})$ attached to $\pi_v\otimes\xi_{v_1}^{-1}$. The infinite product converges for a sufficiently large t, and it has analytic continuation to the entire complex plane. The local factors have no zeroes, and no poles on the half plane $\mathrm{Re}(t)\geqslant\frac{1}{2}$.

Note that the discrete spectrum of $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))$ consists in addition to the cuspidal π also of the one dimensional representations $\chi\colon g\mapsto\chi(\det g)$, where χ is a character of $\mathbb{A}^\times/F^\times$ of order two (or one). But $(\chi(f)\Phi)(u)$ is independent of $u\in\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})$ for $\Phi\in\chi(=\{\Phi\colon \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})\to\mathbb{C};\ \Phi(g)=\chi(g)\Phi(1)\})$, and so $W_\psi(\pi(f)\Phi)=0$ for such Φ. Hence such $\pi=\chi$ do not contribute to our summation formula.

The sum over μ ranges over a set of representatives for the set of connected components of unitary characters $x\mapsto\mu(x)$ of $\mathbb{A}^\times/F^\times$, a connected component consisting of $\mu\nu^{i\lambda}$, $\nu(x)=|x|$, $\lambda\in\mathbb{R}$. In the connected component of $\mu=\xi$ we take $\mu=\xi$ to be the representative. For $\Phi\in I(\mu,\lambda)$, thus

$$\Phi\left(\begin{pmatrix}a & *\\ 0 & b\end{pmatrix}g, \lambda\right) = \mu(a/b)|a/b|^{\lambda+1/2}\Phi(g,\lambda),$$

we have

$$E_\psi(\Phi, \mu, \lambda) = \int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} E(u, \Phi, \mu, \lambda)\psi(u)\,\mathrm{d}u.$$

The sum over Φ in ((2)2) ranges over an orthonormal basis for $I(\mu,\lambda)$ consisting of $\mathbb{K}$-finite functions Φ; this basis is independent of λ as Φ is determined by its restriction to $\mathbb{K}$.

The T is a sufficiently large positive number, and $\Lambda^T E(h, \Phi, \mu, \lambda)$ is described below for a general n in Proposition 14, for $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda) > 1/2$, to be:

$$\Lambda^T E(h, \Phi, \mu, \lambda) = \sum_{\gamma\in\mathbf{B}(F)\backslash\mathbf{G}(F)} \chi(H(\gamma h) < T)H(\gamma h)^{\lambda+1/2}\Phi(\gamma h)$$
$$-\sum_{\gamma\in\mathbf{B}(F)\backslash\mathbf{G}(F)} \chi(H(\gamma h) > T)H(\gamma h)^{1/2-\lambda}(M(w,\mu,\lambda)\Phi)(\gamma h),\quad w=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Note that $\Phi(g)=\Phi(g,-1/2)$. The characteristic function of the domain specified by the condition X is denoted by $\chi(X)$. For $g=(g_v)\in \mathrm{PGL}(2,\mathbb{A})$ we put $H(g)=\Sigma_v H_v(g_v)$, with $H_v(g_v)=|a_v|_v$ if $g_v\in \mathbf{N}(F_v)\begin{pmatrix} a_v & 0\\ 0 & 1\end{pmatrix}K_v$. In the higher rank case below an additive form of H will be used.

We shall integrate $\Lambda^T\bar{E}(h,\lambda)\xi(a)$ on $h=\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}$ over $a\in \mathbb{A}^\times/F^\times$. It is useful to note the simple

(2)3. LEMMA. *We have* $\mathbf{B}\backslash\mathbf{G}=I\cup w\cup w\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}\mathbf{A}$.

This follows at once from the Bruhat decomposition $\mathbf{G}=\mathbf{B}\cup\mathbf{B}w\mathbf{N}$.
Note that

$$H\left(w\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}\right)$$
$$=H\left(\begin{pmatrix} 0 & -1\\ a & 1\end{pmatrix}\right)=H\left(\begin{pmatrix} a/(1,a) & 0\\ 0 & (1,a)\end{pmatrix}\right)=|a|/\|(1,a)\|^2. \qquad ((2)4)$$

Further, $|a_v|_v/\|(1,a_v)\|^2$ is $|a_v|_v$ if $|a_v|_v\leqslant 1$ and $|a_v|_v^{-1}$ if $|a_v|_v\geqslant 1$ (in the non-archimedean case; in the archimedean case it is $|a_v|/(1+|a_v|^2)$), in any case it is $\leqslant 1$, and in particular $((2)4)<T$ if $T>1$, as we assume.

(2)5. LEMMA. *The integral*

$$\int_{F^\times/\mathbb{A}^\times}\xi^{-1}(a)\Lambda^T E\left(\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix},\lambda\right)\mathrm{d}^\times a$$

is the sum of 5 *terms* (*or* 6, *where the* 6*th is zero*):

$$\int_{|a|<T}(\mu/\xi)(a)|a|^{\lambda+1/2}\Phi(I)\,\mathrm{d}^\times a$$
$$=\delta(\mu/\xi)\Phi(I)T^{\lambda+1/2}/(\lambda+1/2),\quad a\in\mathbb{A}^\times/F^\times, \qquad ((2)5.1)$$

where, for a character χ *of* $\mathbb{A}^\times/F^\times$, $\delta(\chi)$ *is* 1 *if* χ *is* 1 *on* $\mathbb{A}^0=\{a\in\mathbb{A}^\times;|a|=1\}$, *and* $\delta(\chi)=0$ *otherwise,*

$$\int_{|a^{-1}|<T}(\mu\xi)(a)^{-1}|a^{-1}|^{\lambda+1/2}\Phi(w)\,\mathrm{d}^\times a$$
$$=\delta(\mu\xi)\Phi(w)T^{\lambda+1/2}/(\lambda+1/2),\quad a\in\mathbb{A}^\times/F^\times, \qquad ((2)5.2)$$

$$\int_{\mathbb{A}^\times} (|a|/\|(1, a)\|^2)^{\lambda+1/2}\Phi(k_a)\mu(a/(1, a)^2)\xi(a)^{-1}\,\mathrm{d}^\times a, \qquad ((2)5.3)$$

where $k_a \in \mathbb{K}$ depends on a and is easily computable from:

$$w\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} = n_a d_a k_a,$$

with n_a in $\mathbf{N}(\mathbb{A})$, $d_a = \mathrm{diag}(a/(1, a), (1, a))$,

$$-\int_{|a|>T} \mu(\xi)(a)^{-1}|a|^{1/2-\lambda}(M(w, \mu, \lambda)\Phi)(I)\,\mathrm{d}^\times a$$

$$= -\delta(\mu\xi)(M(w, \mu, \lambda)\Phi)(I)T^{1/2-\lambda}/(\tfrac{1}{2}-\lambda), \qquad ((2)5.4)$$

$$-\int_{|a^{-1}|>T} (\mu/\xi)(a)|a^{-1}|^{1/2-\lambda}(M(w, \mu, \lambda)\Phi)(w)\,\mathrm{d}^\times a$$

$$= -\delta(\mu/\xi)(M(w, \mu, \lambda)\Phi)(w)T^{1/2-\lambda}/(\tfrac{1}{2}-\lambda). \qquad ((2)5.5)$$

The third term ranges over the $a \in \mathbb{A}^\times$ with $|a|/\|(1, a)\|^2 < T$, namely over $\mathbb{A}^\times$, while the sixth ranges over the $a \in \mathbb{A}^\times$ with $|a|/\|(1, a)\|^2 > T$, namely over the empty set, hence it is 0 and we did not write it out. To compute the term ((2)5.3) note that for almost all v the function Φ is invariant under $K_v = \mathrm{PGL}(2, R_v)$, the characters μ_v and ξ_v are unramified, and the corresponding local factor is

$$\int_{|a_v|_v<1} (\mu_v/\xi_v)(a_v)|a_v|_v^{\lambda+1/2}\,\mathrm{d}^\times a_v + \int_{|a_v|_v>1} (\xi_v\mu_v)(a_v)^{-1}|a_v|_v^{-(\lambda+1/2)}\,\mathrm{d}^\times a_v + \int_{|a_v|_v=1} \mathrm{d}^\times a_v.$$

Denote by π_v a local uniformizer of F_v, and write ξ_v/μ_v for $(\xi_v/\mu_v)(\pi_v)$, and $\xi_v\mu_v$ for $(\xi_v\mu_v)(\pi_v)$ in the following computation. Recall that $|\pi_v| = q_v^{-1}$, where q_v is the cardinality of the residue field $R_v/(\pi_v)$ of R_v. Then the integral is equal to:

$$\sum_1^\infty (\mu_v/\xi_v)^n q_v^{-n(\lambda+1/2)} + \sum_1^\infty (\mu_v\xi_v)^{-n} q_v^{-n(\lambda+1/2)} + 1$$

$$= \frac{(\mu_v/\xi_v)q_v^{\lambda-1/2}}{1-(\mu_v/\xi_v)q_v^{-\lambda-1/2}} + \frac{(\mu_v\xi_v)q_v^{-\lambda-1/2}}{1-\mu_v\xi_v q_v^{-\lambda-1/2}} + 1$$

$$= (1-\mu_v^2 q_v^{-1-2\lambda})/(1-(\mu_v/\xi_v)q_v^{-\lambda-1/2})(1-\mu_v\xi_v q_v^{-\lambda-1/2})$$

$$= L_v(\lambda+1/2, \mu_v/\xi_v)L_v(\lambda+1/2, \mu_v\xi_v)/L_v(1+2\lambda, \mu_v^2).$$

Analogous computation can be carried out at the ramified places too, and a multiple – by a function holomorphic in $\lambda \in i\mathbb{R}$ of polynomial growth in $\lambda \in i\mathbb{R}$ as $|\lambda| \to \infty$ – of the same product of local L-factors, as defined e.g. in [JSP] in the non-archimedean, and in [JS2] in the archimedean cases, is obtained. We denote these local L-factors, which depend on Φ_v, by L_{Φ_v}. Note that $L_v = L_{\Phi_v}$ when Φ_v is the normalized (by $\Phi_v^0(1) = 1$) K_v-invariant function Φ_v^0 in $I(\mu_v, \lambda)$. The product over all v of the L_{Φ_v} is denoted by L_Φ. We then obtain

(2)6. LEMMA. *The term* ((2)5.3) *of the Lemma* (2)5 *is equal to*

$$L_\Phi(\lambda + 1/2, \mu/\xi)L_\Phi(\lambda + 1/2, \mu\xi)/L_\Phi(1 + 2\lambda, \mu^2).$$

This product of L-functions is holomorphic on $\lambda \in i\mathbb{R}$, since $L(1+2\lambda, \mu^2)$ has no zeroes on $\mathrm{Re}(\lambda) \geqslant 0$ (see, e.g. [JS1]), and is of polynomial growth in $\lambda \in i\mathbb{R}$ as $|\lambda| \to \infty$. Of course, $L_\Phi(\lambda + 1/2, \chi)$ is holomorphic on $\lambda \in i\mathbb{R}$ for any unitary character χ of $\mathbb{A}^\times/F^\times$.

Next we have to substitute the five terms of Lemma (2)5 in ((2)2), integrate over $\lambda \in i\mathbb{R}$, and take the limit as $T \to \infty$ (in this order!). For any choice of a test function f, the sums over μ and Φ in ((2)2) are finite. We fix then μ and Φ, and treat each of the 5 terms of Lemma (2)5 separately. Before we do that, note that for each Φ, $\Phi_1 \in I(\mu, \lambda)$, the matrix coefficient

$$c(\lambda) = c(f, \mu, \lambda; \Phi, \Phi_1) = (I(f, \mu, \lambda)\Phi, \Phi_1),$$

being the Mellin transform of a Schwartz function f, is rapidly decreasing (as $|\lambda| \to \infty$) in any vertical strip $a \leqslant \mathrm{Re}(\lambda) \leqslant b$, and so is the finite sum

$$E_\psi(I(f, \mu, \lambda)\Phi, \mu, \lambda) = \sum_{\Phi_1} (I(f, \mu, \lambda)\Phi, \Phi_1)E_\psi(\Phi_1, \mu, \lambda).$$

We shall use this observation with the vertical strip $-\frac{1}{2} - \varepsilon \leqslant \mathrm{Re}(\lambda) \leqslant \frac{1}{2} + \varepsilon$, for some small $\varepsilon > 0$.

Note also that it is not the integral of Lemma (2)5 which appears in (2)2, but rather its complex conjugate. For $\lambda \in i\mathbb{R}$, note that $\bar{\lambda}$ is $-\lambda$. We then replace $T^{\lambda + 1/2}/(\lambda + 1/2)$ by $T^{-\lambda + 1/2}/(-\lambda + 1/2)$ in ((2)5.1), ((2)5.2), and vice versa in ((2)5.4), ((2)5.5).

Substituting ((2)5.1) in place of $\int \xi^{-1}\Lambda^T E$ in ((2)2), we may change the line of integration from $\lambda \in i\mathbb{R}$ to $\lambda + \frac{1}{2} + \varepsilon$, $\lambda \in i\mathbb{R}$. By Cauchy's theorem the residue at $\lambda = \frac{1}{2}$ will be picked up. The corresponding contribution to ((2)5.1) is

$$\delta(\mu/\xi)\bar{\Phi}(1)\pi E_\psi(I(f, \mu, \tfrac{1}{2})\Phi, \mu, \tfrac{1}{2})$$
$$+ \tfrac{1}{2}\delta(\mu/\xi)\bar{\Phi}(1) \int_{i\mathbb{R}} E_\psi(I(f, \mu, \lambda + \tfrac{1}{2} + \varepsilon)\Phi, \mu, \lambda + \tfrac{1}{2} + \varepsilon)(T^{-\lambda - \varepsilon}/(-\lambda - \varepsilon))\, d\lambda. \qquad ((2)6.1)$$

As $T\to\infty$ the integral over $i\mathbb{R}$ here is absolutely convergent to zero.

The case of ((2)5.2), when placed in ((2)2), is treated in the same way, the limit as $T\to\infty$ is equal to

$$\delta(\mu\xi)\bar{\Phi}(w)\pi E_{\psi}(I(f,\mu,1/2)\Phi,\mu,1/2). \qquad ((2)6.2)$$

Next we substitute the expression obtained in Lemma (2)6 for ((2)5.3) instead of $\int\xi^{-1}\Lambda^T E$ in ((2)2). We obtain, noting that $\bar{\xi}=\xi^{-1}$, $\bar{\mu}=\mu^{-1}$, and $\bar{\lambda}=-\lambda$ for $\lambda\in i\mathbb{R}$,

$$\frac{1}{2}\int_{i\mathbb{R}} E_{\psi}(I(f,\mu,\lambda)\Phi,\mu,\lambda)L_{\bar{\Phi}}(-\lambda+1/2,\xi/\mu)$$
$$\cdot L_{\bar{\Phi}}(-\lambda+1/2,(\mu\xi)^{-1})L_{\bar{\Phi}}(1-2\lambda,\mu^{-2})^{-1}\,d\lambda. \qquad ((2)6.3)$$

The integrand is integrable on $i\mathbb{R}$ being (the product of a slowly increasing and) a rapidly decreasing function in λ, as $|\lambda|\to\infty$. It is independent of T.

The discussion of the terms ((2)5.4) and ((2)5.5) is similar to that of ((2)5.1) and ((2)5.2), except that the line of integration will be moved from $\lambda\in i\mathbb{R}$ to $\lambda-1/2-\varepsilon$, $\lambda\in i\mathbb{R}$. Before carrying this out we need to specify the dependence of $(M(w,\mu^{-1},-\lambda)\bar{\Phi})$ on λ. The operator $M(w,\mu^{-1},-\lambda)$ is not unitary in general, but it can be expressed (see [Sh2], p. 272) in the form

$$M(w,\mu^{-1},-\lambda)=m(\mu^{-1},-\lambda)\otimes_v R(\mu_v^{-1},-\lambda),$$

where $R(\mu_v^{-1},-\lambda): I(\mu_v^{-1},-\lambda)\to I(\mu_v,\lambda)$ is a unitary operator for all μ_v, λ, which maps $\Phi_v^0\in I(\mu_v^{-1},-\lambda)$ to $\Phi_v^0\in I(\mu_v,\lambda)$ whenever μ_v is unramified (and v nonarchimedean), and $(R(\mu_v^{-1},-\lambda)\Phi_v)(g)$ is holomorphic and slowly increasing in λ in any vertical band $a\leqslant\mathrm{Re}(\lambda)\leqslant b$, for any $\Phi_v\in I(\mu_v^{-1},-\lambda)$ and $g\in G_v$. Moreover, the scalar valued function

$$m(\mu^{-1},-\lambda)=L(-\lambda,\mu^{-2})/L(1-\lambda,\mu^{-2})\varepsilon(-\lambda,\mu^{-2})$$
$$=L(1+\lambda,\mu^2)/L(1-\lambda,\mu^{-2})$$

is holomorphic on $-1<\mathrm{Re}(\lambda)\leqslant 0$($L(1-\lambda,\mu^{-2})$ in the denominator has no zeroes in $\mathrm{Re}(\lambda)\leqslant 0$, see, e.g. [JS1]) except possibly for a simple pole on $\mathrm{Re}(\lambda)=0$ if μ^2 factorizes through $v(x)=|x|$. In this last case we may choose μ in its connected component to have $\mu^2=1$. Then $L(1+\lambda)$ in the numerator would have a pole at $\lambda=0$ in the number field case, and at any $\lambda\in i\mathbb{Z}/\log q$ in the function field case. But $L(1-\lambda)$ would also have a pole there, canceling the pole of the numerator, and $m(\mu,\lambda)$ would take the value -1 at $\lambda=0$. In conclusion, $M(\mu^{-1},-\lambda)$ is holomorphic in $-3/4\leqslant\mathrm{Re}\,\lambda\leqslant 0$ and of slow increase as $|\lambda|\to\infty$.

With this knowledge we replace $\int \xi^{-1}\Lambda^T E$ in ((2)2) by the right side of ((2)5.4), move the line of integration from $\lambda \in i\mathbb{R}$ to $\lambda - 1/2 - \varepsilon$, $\lambda \in i\mathbb{R}$, pick the residue at $\lambda = -1/2$, and obtain

$$-\delta(\mu\xi)\pi E_\psi(I(f, \mu, -1/2)\Phi, \mu, -1/2)(M(w, \mu^{-1}, -1/2)\bar{\Phi})(I)$$

$$-\tfrac{1}{2}\delta(\mu\xi)\int_{i\mathbb{R}} E_\psi(I(f, \mu, \lambda-\varepsilon-1/2)\Phi, \mu, \lambda-\varepsilon-1/2)$$

$$\cdot (M(w, \mu^{-1}, \lambda+\varepsilon+1/2)\bar{\Phi})(w)[T^{\lambda-\varepsilon}/(\lambda-\varepsilon)]\, d\lambda. \qquad ((2)6.4)$$

The integrand is holomorphic and rapidly decreasing in λ as $|\lambda| \to \infty$. The integral is absolutely convergent to zero as $T \to \infty$.

The case of ((2)5.5) is similarly treated, to yield, as $T \to \infty$, the limit

$$-\delta(\mu/\xi)\pi E_\psi(I(f, \mu, -1/2)\Phi, \mu, -1/2)(M(w, \mu^{-1}, -1/2)\bar{\Phi})(w). \qquad ((2)6.5)$$

The spectral side in our summation formula is then the sum of ((2)1) and the sum over μ and Φ of $((2)6.1) + \cdots + ((2)6.5)$.

E. On the general case

We now resume the discussion of the case of a general $n \geqslant 2$. Thus we note that there is another expression for the kernel $K_f(g, h)$, which we now recall from Arthur [A1], p. 935. It is based on Langlands' theory [L] of Eisenstein series (and Morris [M] in the function field case); see the recent clearer exposition of Moeglin-Waldspurger [MW2]. Thus let $\mathbf{P}$ denote a standard F-parabolic subgroup of $\mathbf{G}$, one which contains the upper triangular subgroup $\mathbf{P}_0$, let $\mathbf{N}$ be its unipotent radical, and $\mathbf{M}$ its Levi subgroup which contains the diagonal subgroup $\mathbf{A}$. Let $\Pi(\mathbf{M}(\mathbb{A}))$ be the set of equivalence classes of irreducible unitary discrete series representations of $\mathbf{M}(\mathbb{A})$. Put

$$X(\mathbf{M}) = \mathrm{Hom}(\mathbf{M}, \mathrm{GL}(1)), \quad \mathfrak{A}_{\mathbf{P}} = \mathrm{Hom}(X(\mathbf{M}), \mathbb{R})$$

for the Lie algebra of $\mathbf{M}$, and $\mathfrak{A}_P^* = X(\mathbf{M}) \otimes \mathbb{R}$ for its dual space. For $m = (m_v)$ in $\mathbf{M}(\mathbb{A})$ define the vector $H_M(m)$ in $\mathfrak{A}_P$ by

$$e^{\langle H_M(m), \chi\rangle} = |\chi(m)| = \prod_v |\chi_v(m_v)|_v, \quad \chi \in X(\mathbf{M}).$$

Extend H_M to a function on $\mathbf{G}(\mathbb{A}) = \mathbf{N}(\mathbb{A})\mathbf{M}(\mathbb{A})\mathbb{K}$ by $H_M(nmk) = H_M(m)$, where $\mathbb{K} = \Pi_v K_v$ and K_v is the standard maximal compact subgroup in G_v. If $\mathbf{M}(\mathbb{A})^1$ is the kernel of H_M on $\mathbf{M}(\mathbb{A})$, and $\mathbf{A}_M$ is the center of $\mathbf{M}$, then H_M is an isomorphism from

$$\mathbf{A}_M(\mathbb{A}) \cap \mathbf{M}(\mathbb{A})^1 \backslash \mathbf{A}_M(\mathbb{A}) \simeq \mathbf{M}(\mathbb{A})^1 \backslash \mathbf{M}(\mathbb{A})$$

to $\mathfrak{A}_P$. For any $\lambda \in \mathfrak{A}_{\mathbb{C}}^* = \mathfrak{A}_P^* \otimes_{\mathbb{R}} \mathbb{C}$ consider the character $g \mapsto e^{\langle \lambda, H_M(g) \rangle}$ on $\mathbf{G}(\mathbb{A})$, and denote its tensor product with $\rho \in \Pi(\mathbf{M}(\mathbb{A}))$ by ρ_λ. If $\lambda \in i\mathfrak{A}_P^*$ then ρ_λ is unitary, and the group $i\mathfrak{A}_P^*$ acts freely on $\Pi(\mathbf{M}(\mathbb{A}))$, making $\Pi(\mathbf{M}(\mathbb{A}))$ a differential manifold whose connected components are the orbits of $i\mathfrak{A}_P^*$.

For $\rho \in \Pi(\mathbf{M}(\mathbb{A}))$ denote by $H_P(\rho)$ the Hilbert space completion of the space $H_P^0(\rho)$ of smooth functions $\Phi: \mathbf{N}(\mathbb{A})\mathbf{M}(F) \backslash \mathbf{G}(\mathbb{A}) \to \mathbb{C}$ which are $\mathbb{K}$-finite, have the property that

$$\int_{\mathbb{K}} \int_{\mathbf{M}(F) \backslash \mathbf{M}(\mathbb{A})^1} |\Phi(mk)|^2 \, dm \, dk$$

is finite, and that for every $g \in \mathbf{G}(\mathbb{A})$ the function $m \mapsto \Phi(mg)$ on $\mathbf{M}(\mathbb{A})$ is a matrix coefficient of ρ. Let ρ_P be the vector in $\mathfrak{A}_P^*$ such that the modular function $\delta_P(p) = |\det(\mathrm{Ad}(p)|\tilde{N})|$ on $\mathbf{P}(\mathbb{A})$ is equal to $e^{2\langle \rho_P, H_M(p) \rangle}$; here $\tilde{N}$ is the Lie algebra of $\mathbf{N}$. For

$$\Phi \in H_{\mathbf{P}}(\rho) \quad \text{and} \quad \lambda \in \mathfrak{A}_{\mathbb{C}}^*$$

put

$$\Phi(g, \lambda) = \Phi(g) \, e^{\langle \rho_P + \lambda, H_M(g) \rangle} \quad (g \in \mathbf{G}(\mathbb{A})),$$

and denote by $I(\rho, \lambda)$ the right representation, $(I(h, \rho, \lambda)\Phi)(g, \lambda) = \Phi(gh, \lambda)$, of $(h \in)\mathbf{G}(\mathbb{A})$. The $\mathbf{G}(\mathbb{A})$-module $I(\rho, \lambda)$ is unitary for $\lambda \in i\mathfrak{A}_P^*$.

Denote by Δ_P the set of simple roots of $\mathbf{A}_M$ in $\mathbf{P}$. These are elements of $X(\mathbf{M}) \subset \mathfrak{A}_P^*$. The set $\Delta_0 = \Delta_{P_0}$ is a base for a root system, and there is a coroot $\alpha^\vee$ in $\mathfrak{A}_{P_0}$ for every root $\alpha \in \Delta_{\mathbf{P}}$. If $\mathbf{P}_1 \subset \mathbf{P}_2$ are parabolic subgroups, then the group $\mathbf{M}_{P_2} \cap \mathbf{P}_1$ is a parabolic subgroup of $\mathbf{M}_{P_2}$ with unipotent radical $\mathbf{N}_{P_1}^{P_2} = \mathbf{N}_{P_1} \cap \mathbf{M}_{P_2}$. The set $\Delta_{P_1}^{P_2}$ of simple roots of $\mathbf{A}_{M_1}$ in $\mathbf{M}_{P_2} \cap \mathbf{P}_1$ is a subset of Δ_{P_1} which spans a subspace $(\mathfrak{A}_{P_1}^{P_2})^*$ of $\mathfrak{A}_{P_1}^*$. We have $\mathfrak{A}_{P_1}^* = (\mathfrak{A}_{P_1}^{P_2})^* \oplus \mathfrak{A}_{P_2}^*$. Define

$$\mathfrak{A}_P^+ = \{H \in \mathfrak{A}_P; \langle \alpha, H \rangle > 0, \alpha \in \Delta_P\},$$

and

$$(\mathfrak{A}_P^*)^+ = \{\lambda \in \mathfrak{A}_P^*; \langle \lambda, \alpha^\vee \rangle > 0, \alpha \in \Delta_P\}.$$

Then $\rho_P \in (\mathfrak{A}_P^*)^+$.

Identify $\mathfrak{A}_{P_2}$ with the subspace $\{H \in \mathfrak{A}_{P_1}; \langle \alpha, H\rangle = 0, \alpha \in \Delta_{P_1}^{P_2}\}$, and denote by $\mathfrak{A}_{P_1}^{P_2}$ the subspace of $\mathfrak{A}_{P_1}$ which is annihilated by $\mathfrak{A}_{P_2}^*$. Then $\mathfrak{A}_{P_1} = \mathfrak{A}_{P_1}^{P_2} \oplus \mathfrak{A}_{P_2}$. Denote by $\hat{\Delta}_{P_1}^{P_2} = \{\tilde{\omega}_\alpha; \alpha \in \Delta_{P_1}^{P_2}\}$ the basis for $(\mathfrak{A}_{P_1}^{P_2})^*$ dual to the basis $\{\alpha^\vee; \alpha \in \Delta_{P_1}^{P_2}\}$ of $\mathfrak{A}_{P_1}^{P_2}$. Note that any root $\alpha \in \Delta_{P_1}^{P_2}$ is the restriction to $(\mathfrak{A}_{P_1}^{P_2})^*$ of a unique root $\beta \in \Delta_{P_0}^{P_2}$; $\alpha^\vee$ is defined to be the projection to $\mathfrak{A}_{P_1}^{P_2}$ of the vector $\beta^\vee$ in $\mathfrak{A}_{P_0}^{P_2}$. Let $\hat{\tau}_{P_1}^{P_2}$ be the characteristic function on $\mathfrak{A}_0$ of the $H \in \mathfrak{A}_0$ with $\langle \tilde{\omega}, H\rangle > 0$ for all $\tilde{\omega} \in \hat{\Delta}_{P_1}^{P_2}$. Put $\hat{\tau}_P = \hat{\tau}_P^G$. Note that $\hat{\tau}_G = 1$.

If $\mathbf{Q}$ is also a standard F-parabolic subgroup, denote by $W(\mathfrak{A}_P, \mathfrak{A}_Q)$ the set of elements s in the Weyl group W with $s\mathfrak{A}_P = \mathfrak{A}_Q$. Denote by w_s a representative in $\mathbf{G}(F)$ for the element s of W. For $\rho \in \Pi(\mathbf{M}(\mathbb{A}))$ and $\Phi \in H_P^0(\rho)$, and $\lambda \in \mathfrak{A}_{P,\mathbb{C}}^*$ with real part $\operatorname{Re}\lambda \in \rho_P + (\mathfrak{A}_P^*)^+$, define the Eisenstein series

$$E(g, \Phi, \rho, \lambda) = \sum_{\gamma \in \mathbf{P}(F)\backslash \mathbf{G}(F)} \Phi(\gamma g, \lambda)$$

and intertwining operator

$$(M(s, \rho, \lambda)\Phi)(g, s\lambda) = \int_{\mathbf{N}_Q(\mathbb{A}) \cap w_s \mathbf{N}_P(\mathbb{A}) w_s^{-1} \backslash \mathbf{N}_Q(\mathbb{A})} \Phi(w_s^{-1} ug, \lambda)\, \mathrm{d}u.$$

The functions $E(g, \Phi, \rho, \lambda)$ and $M(s, \rho, \lambda)\Phi$ can be continued as meromorphic functions in λ to $\mathfrak{A}_{\mathbb{C}}^*$. If $\lambda \in i\mathfrak{A}_P^*$ then $E(g, \Phi, \rho, \lambda)$ is smooth and slowly increasing in g, and $M(s, \rho, \lambda)$ is a unitary operator from $H_P(\rho_\lambda)$ to $H_Q(s\rho_{s\lambda})$. Denote by $n(P)$ the number of *chambers* of $\mathfrak{A}$, namely the connected components of the complement to the union of the hyperplanes orthogonal to the roots in Δ_P.

The representation theoretic expression for the kernel $K_f(g, h)$ is

$$\sum_{\mathbf{P}} n(P)^{-1} \sum_{\rho} \int_{i\mathfrak{A}_P^*} \sum_{\Phi} E(g, I(f, \rho, \lambda)\Phi, \rho, \lambda)\bar{E}(h, \Phi, \rho, \lambda)\, \mathrm{d}\lambda. \tag{3.1}$$

Here ρ ranges over a set of representatives for the connected components ($i\mathfrak{A}_P^*$-orbits) of $\Pi(\mathbf{M}(\mathbb{A}))$, and Φ over an orthonormal basis (chosen to have the finiteness properties of [A1], p. 926, l. − 12) for the space $H_P(\rho)$; $I(f, \rho, \lambda)$ is the convolution operator. By [A1], Lemma 4.4, p. 929, the sums over ρ and Φ and the integral over $i\mathfrak{A}_P^*$ are absolutely convergent. Note that $(I(f, \rho, \lambda)\Phi, \Phi')$ is a rapidly decreasing function in $|\lambda| \to \infty$, where $(\cdot, \cdot)$ indicates the inner product on $H_P(\rho)$.

Our summation formula is obtained on integrating $K_f(n, h)\xi(h)\psi_\alpha(n)$ over n in $\mathbf{N}_0(F)\backslash \mathbf{N}_0(\mathbb{A})$ and h in $\mathbf{H}(F)\backslash \mathbf{H}(\mathbb{A})$, using the spectral decomposition of the kernel, and comparing with the result of Corollary 3. Put

$$\phi_{\mathbf{N}}(g) = \int_{\mathbf{N}(F)\backslash \mathbf{N}(\mathbb{A})} \phi(ng)\, \mathrm{d}n$$

for a continuous function ϕ on $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$. Following [A2], p. 89, for a suitably regular point T in $\mathfrak{A}_0^+$ introduce

$$\Lambda^T\phi(g)=\sum_{\mathbf{P}}(-1)^{\dim(\mathbf{A})}\sum_{\gamma\in\mathbf{P}(F)\backslash\mathbf{G}(F)}\hat{\tau}_P(H(\gamma g)-T)\phi_{\mathbf{N}}(\gamma g);$$

here $\mathbf{P}$ ranges over the standard F-parabolic subgroups in $\mathbf{G}$.

Denote by $\Lambda^T K_f(g,h)$ the image of the function $h\mapsto K_f(g,h)$ under the operator Λ^T. Since $h\mapsto K_f(g,h)$ is slowly increasing, it follows from [A2], Lemma 1.4, that $\Lambda^T K_f(g,h)$ is rapidly decreasing as a function of $h\in\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$. Since $K_f(g,h)$ is integrable over $h\in\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ and $\Lambda^T K_f(g,h)\to K_f(g,h)$ as $T\to\infty$, and $\mathbf{N}_0(F)\backslash\mathbf{N}_0(\mathbb{A})$ is compact, we conclude that

$$\lim_{T\to\infty}\int_{\mathbf{N}_0(F)\backslash\mathbf{N}_0(\mathbb{A})}\int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})}\Lambda^T K_f(u,\ h)\xi(h)\psi(u)\,\mathrm{d}u\,\mathrm{d}h$$

$$=\int_{\mathbf{N}_0(F)\backslash\mathbf{N}_0(\mathbb{A})}\int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})}K_f(u,\ h)\xi(h)\psi(u)\,\mathrm{d}u\,\mathrm{d}h. \tag{3.2}$$

The function $E(g,\Phi,\rho,\lambda)$ is slowly increasing in $g\in\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$, hence $\Lambda^T E(g,\Phi,\rho,\lambda)$ is rapidly decreasing, and the expression

$$\sum_P n(P)^{-1}\sum_\rho\int_{i\mathfrak{A}_P^*}\sum_\Phi E(g,\ I(f,\ \rho,\ \lambda)\Phi,\ \rho,\ \lambda)\Lambda^T\bar{E}(h,\ \Phi,\ \rho,\ \lambda)\,\mathrm{d}\lambda$$

is convergent and equal to $\Lambda^T K_f(g,h)$. The integral over

$$(g,h)\in\mathbf{N}_0(F)\backslash\mathbf{N}_0(\mathbb{A})\times\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$$

of its product with $\xi(h)\psi(g)$ is equal to

$$\sum_{\mathbf{P}}n(P)^{-1}\sum_\rho\int_{i\mathfrak{A}_P^*}\sum_\Phi E_\psi(I(f,\ \rho,\ \lambda)\Phi,\ \rho,\ \lambda)\int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})}\Lambda^T\bar{E}(h,\ \Phi,\ \rho,\ \lambda)\xi(h)\,\mathrm{d}h\,d\lambda, \tag{3.3}$$

where

$$E_\psi(\Phi,\ \rho,\ \lambda)=\int_{\mathbf{N}_0(F)\backslash\mathbf{N}_0(\mathbb{A})}E(u,\ \Phi,\ \rho,\ \lambda)\psi(u)\,\mathrm{d}u.$$

We shall sketch a proof of the following. Suppose that $n\geqslant 3$ (the case of $n=3$ being trivial, we shall concentrate on $n\geqslant 4$ in the sketch of the proof below).

4. PROPOSITION. *The only possible non-zero contributions to* (3.3) *are parametrized by:*

(a) $\mathbf{P}$ *of type* (n_1, n_2, n_3) *and unitary, one dimensional* $\rho = \rho_1 \times \rho_2 \times \rho_3$ *with* $\rho_1^{n_1}\rho_2^{n_2}\rho_3^{n_3} = 1$;

(b) $\mathbf{P}$ *of type* $(n-2, 2)$ *and* $\rho = \rho_1 \times \rho_2$ *where* ρ_1 *is unitary one dimensional and* ρ_2 *is a cuspidal* $GL(2, \mathbb{A})$*-module whose central character* ω_{ρ_2} *is equal to* ρ_1^{2-n};

(c) $\mathbf{P} = \mathbf{G}$, $n = 3$ *and* ρ *is a cuspidal* $PGL(3, \mathbb{A})$*-module, or* $n = 4$ *and* ρ *is the discrete-series representation of* $PGL(4, \mathbb{A})$ *which is equivalent to the unique subrepresentation of* $I((\rho_1 \times \rho_1)\delta_{P_1}^{-1/2})$, *where* $\mathbf{P}_1$ *is the parabolic of type* $(2, 2)$ *and* ρ_1 *is a cuspidal representation of* $GL(2, \mathbb{A})$.

Sketch of proof. (This is only a sketch since although a few cases of the assertion made in the following sentence are explicitly computed below, the assertion is not proven below in full generality. The assertion is the following).

As a function in T the integral $\int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})} \Lambda^T E(h, \Phi, \rho, \lambda)\xi^{-1}(h)\,dh$ converges to a linear combination of exponentials in linear forms in λ and T divided by such linear forms, in λ. Examples are computed explicitly below, see e.g. Propositions 9 and 11 for a general n, and the complete discussion in the cases of $n = 2$ and $n = 3$. In particular the limit over T cannot be taken inside the integral over $i\mathfrak{A}_P^*$. Instead, the elementary Lemma 10 below implies that the limit of (3.3) as T goes to infinity is equal to

$$\sum_{\mathbf{P}} n(P)^{-1} \sum_{\rho} \int_{i(\mathfrak{A}_P^*)'} \sum_{\Phi} E_\psi(I(f, \rho, \lambda)\Phi, \rho, \lambda) F(\Phi, \rho, \lambda, \xi)\,d\lambda \tag{4.1}$$

where $(\mathfrak{A}_P^*)'$ are the hyperplanes defined by the linear forms in λ in the denominator, and $F(\Phi, \rho, \lambda, \xi)$ are the residues of the

$$\int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})} \Lambda^T E(h, \Phi, \rho, \lambda)\xi^{-1}(h)\,dh$$

on these hyperplanes. By virtue of a standard argument of "generalized linear independence of characters (see, e.g. [FK], Theorem 2), using the absolute convergence of the integrals, the ample supply of the f, unitarity estimates and the Stone-Weierstrass theorem, Corollary 3 would imply, when $n \geqslant 4$, that the coefficient $E_{\psi_\alpha}(\Phi, \rho, \lambda)$ is 0 for every character ψ_α, $\alpha \in F^{n-2}$, unless $\alpha = (0, \ldots, 0)$, for every pair (ρ, λ) which occurs non-trivially in (3.3), and every $\Phi \in H_P^0(\rho)$.

If (π, V) is an irreducible representation of $\mathbf{G}(\mathbb{A})$ and ψ a character of $\mathbf{N}_0(F)\backslash\mathbf{N}_0(\mathbb{A})$, in analogy with [BZ] introduce the $\mathbf{A}_0(\mathbb{A})$-modules of coinvariants

$$(\pi_\psi, V_\psi) \quad \text{by } V_\psi = V/\langle \pi(u)v - \psi(u)v;\ v \in V,\ u \in \mathbf{N}_0(\mathbb{A})\rangle.$$

Any such character ψ is of the form

$$\psi_\beta(u) = \psi\left(\sum_{1 \leqslant i < n} \beta_i u_{i,i+1}\right)$$

for some $\beta = (\beta_1, \ldots, \beta_{n-1}) \in F^{n-1}$; here $u = (u_{ij}) \in \mathbf{N}_0(\mathbb{A})$. The largest number of non-zero components of β such that $V_{\psi_\beta} \neq 0$ is an invariant of the representation π, which we call here the *index* of π. A $\mathbf{G}(\mathbb{A})$-module with (maximal) index $n-1$ is called *generic*, or non-degenerate, and it is said to have a Whittaker model. A discrete-series $\mathbf{G}(\mathbb{A})$-module whose index is 0 is one dimensional.

Moeglin and Waldspurger [MW1] have shown that if π is an irreducible discrete series $\mathbf{G}(\mathbb{A})$-module then there are positive integers m and k with $n = mk$ and a cuspidal $\mathrm{GL}(m, \mathbb{A})$-module ρ, such that π is the unique submodule of the $\mathbf{G}(\mathbb{A})$-module $I((\rho \times \cdots \times \rho)\delta_P^{-1/2})$ which is normalizedly induced from the $\mathbf{G}(\mathbb{A})$-module indicated, where $\mathbf{P} = \mathbf{MN}$ is the standard parabolic of type $(m, \ldots, m)$, and δ_P is its modular function. The index of this π is $j = (m-1)k$. If $\mathbf{M}_1 = \Pi_{1 \leqslant i \leqslant r} \mathrm{GL}(n_i)$ is the Levi subgroup of a standard parabolic, and the $\mathrm{GL}(n_i, \mathbb{A})$-module ρ_i has index j_i, then the induced $\mathbf{G}(\mathbb{A})$-module $I((\rho_1 \times \cdots \times \rho_r)\,\mathrm{e}^{\langle \lambda, H_{P_1}\rangle})$ has the index $(\Sigma_{1 \leqslant i \leqslant r} j_i) + r - 1$, for any $\lambda \in \mathfrak{A}^*_{P_1,\mathbb{C}}$.

The Eisenstein series $E(u, \Phi, \rho, \lambda)$ which occurs in (3.3) is an element in the space of the $\mathbf{G}(\mathbb{A})$-module $\pi = I(\rho \otimes \mathrm{e}^{\langle \lambda, H_P\rangle})$, whose index is 2. On the other hand, if P is of type $(n_1, \ldots, n_r)$, and ρ is a discrete series $\mathbf{M}(\mathbb{A})$-module, then $n_i = m_i k_i$, and the index of π is $r - 1 + \Sigma_{1 \leqslant i \leqslant r}(m_i - 1)k_i$. Since the k_i, m_i and r are positive integers, we conclude that either $r = 3$ and $m_i = 1 (1 \leqslant i \leqslant 3)$, or $r = 2$ and $m_1 = 1, m_2 = 2$ and $k_2 = 1$, or $r = 1$, in which case $m_1 = 3$ and $k_1 = 1$ or $m_1 = k_1 = 2$.

This completes our sketch of the proof of the proposition.

REMARK. Note that in case (b) ρ_2 may not be taken to be one dimensional, as then the index of $\pi = I(\rho \otimes \mathrm{e}^{\langle \lambda, H\rangle})$ be one. In case (c), when $n = 3$ the ρ cannot be one dimensional (the index would then be zero). When $n = 4$ the ρ cannot be cuspidal (index 3) or one dimensional (index 0), nor can ρ_1 be one dimensional, as then the index of π would be 1.

We shall need below several decompositions.

5. PROPOSITION. *Denote by* $\mathbf{P}_\mathbf{x}$ *a parabolic subgroup of type* $\mathbf{x}$ *of* $\mathbf{G}$, *by* I *the identity* $(n-2) \times (n-2)$ *matrix, and by* $r(i,j)$ *the matrix whose entries are* 0 *except for a single* 1 *on each row and column, which represents the reflection* (i,j). *Then*

$$\mathbf{G} = \mathbf{P}_{(n-1,1)}\mathbf{H} \cup \mathbf{P}_{(n-1,1)} r(n-1, n)\mathbf{H} \cup \mathbf{P}_{(n-1,1)} \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{H},$$

where

$$\mathbf{H} = \left\{ \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix};\ * \in \mathrm{GL}(n-1) \right\} \subset \mathbf{G}.$$

Moreover,

$$\mathbf{G} = \mathbf{P}_{(n-2,2)}\mathbf{H} \cup \mathbf{P}_{(n-2,2)} r(n-2,\ n)\mathbf{H} \cup \mathbf{P}_{(n-2,2)} \begin{pmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{H}$$

$$= \mathbf{P}_{(2,n-2)}\mathbf{H} \cup \mathbf{P}_{(2,n-2)} r(1, n)\mathbf{H} \cup \mathbf{P}_{(2,n-2)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{H}.$$

Proof. The map $g \mapsto (0, \ldots, 0, 1)g$ is an isomorphism from $\mathbf{P}_{(n-1,1)} \backslash \mathbf{G}$ to the projective n-space $\mathbb{P}^n$, which decomposes as the disjoint union of three orbits, namely $(0, \ldots, 0, 1)\mathbf{H}$, $(0, \ldots, 0, 1, 0)\mathbf{H}$, and $(0, \ldots, 0, 1, 1)\mathbf{H}$. The first decomposition follows.

Denote by $\mathbf{U}_l$ the group of matrices (u_{ij}) in $\mathbf{G}$ with $u_{ii} = 1$ $(1 \leqslant i \leqslant n)$, and $u_{ij} = 0$ if $i \neq j$ unless $i = l$ and $i \leqslant j < n$. Also put $\mathbf{U}'_l$ for the group of (u_{ij}) in $\mathbf{G}$ with $u_{ii} = 1$ $(1 \leqslant i \leqslant n)$ and $u_{ij} = 0$ if $i \neq j$ unless $(i, j) = (l, n)$. The Bruhat decomposition, with $\mathbf{P}_1 = \mathbf{P}_{(n-1,1)}$ and $\mathbf{P}_{(1,1)} = \mathbf{P}_{(n-2,1,1)}$, asserts

$$\mathbf{P}_1 = \bigcup_{1 \leqslant i < n} \mathbf{P}_{(1,1)} r(i,\ n-1)\mathbf{U}_i.$$

Hence

$$\mathbf{G} = \mathbf{P}_{(1,1)}\mathbf{H} \cup \bigcup_{1 \leqslant i \leqslant n-2} \mathbf{P}_{(1,1)} r(n-1,\ n) r(i,\ n)\mathbf{U}'_i\mathbf{H}$$

$$\cdot\cup \bigcup_{1 \leqslant i < n} \mathbf{P}_{(1,1)} r(i,\ n-1) \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{H}.$$

Then

$$\mathbf{G} = \mathbf{P}_{(n-2,2)}\mathbf{H} \cup \bigcup_{1 \leqslant i \leqslant n-2} \mathbf{P}_{(n-2,2)} r(i,\ n)\mathbf{U}'_i\mathbf{H}$$

$$\cdot\cup \bigcup_{1 \leqslant i \leqslant n-2} P_{(n-2,2)} r(i,\ n-1) \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \mathbf{H}.$$

Note that

$$\mathbf{P}_{(n-2,2)}r(i,\, n)\mathbf{U}_i'\mathbf{H}=\mathbf{P}_{(n-2,2)}r(i,\, n)\mathbf{H}\cup\mathbf{P}_{(n-2,2)}r(1,n)\begin{pmatrix}1&0&1\\0&I&0\\0&0&1\end{pmatrix}\mathbf{H}$$

for any $i(1\leqslant i\leqslant n-2)$. The last double cosets in the two last displayed lines are equal, since

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-1&0&1\end{pmatrix}$$

lies in the bottom right 3×3 corner of $\mathbf{P}_{(n-2,2)}$. Taking $i=n-2$ the second decomposition follows.

To obtain the last decomposition apply to the previous one the automorphism $\sigma(g)=J^tg^{-1}J$, where $J=(a_{ij})$, $a_{i,n+1-i}=1$ and $a_{ij}=0$ if $i+j\neq n+1$. Then $\sigma\mathbf{P}_{(n-2,2)}=\mathbf{P}_{(2,n-2)}$, and $\sigma\mathbf{H}=r(1,n)\mathbf{H}r(1,n)$. Since $\mathbf{G}=\mathbf{G}r(1,n)$, the last decomposition follows, as required.

REMARK. Let I' be the identity $(n-3)\times(n-3)$ matrix and put

$$\kappa=\begin{pmatrix}I'&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&1\end{pmatrix}.$$

Then

$$\mathbf{P}_{(n-2,2)}\begin{pmatrix}0&0&1\\0&I&0\\1&0&1\end{pmatrix}\mathbf{H}=\mathbf{P}_{(n-2,2)}\kappa\mathbf{H}.$$

Put $\mathbf{H}^-=r(1,n)\mathbf{H}r(1,n)$. In the following $\mathbf{P}_{\mathbf{x}}^{\mathbf{H}}$ denotes a standard parabolic subgroup of $\mathbf{H}$ of type $\mathbf{x}$. By A we indicate an $(n-3)\times(n-3)$ matrix, and B, C will be row vectors of length $n-3$; a, c, d are scalars. Proposition 5 has the following

COROLLARY. *We have the disjoint coset decompositions*

$$\mathbf{P}_{(n-2,2)}\backslash\mathbf{G} = \mathbf{P}^{\mathbf{H}}_{(n-2,1)}\backslash\mathbf{H} \cup r(n-2, n)\cdot\mathbf{P}^{\mathbf{H}}_{(n-3,2)}\backslash\mathbf{H}\cdot \cup \kappa \begin{pmatrix} A & {}^tB & {}^tC & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \backslash\mathbf{H} \tag{5.1}$$

and

$$\mathbf{P}_{(2,n-2)}\backslash\mathbf{G} = \mathbf{P}^{\mathbf{H}}_{(2,n-3)}\backslash\mathbf{H} \cup \mathbf{P}^{\mathbf{H}^-}_{(1,n-2)}\backslash\mathbf{H}^- \cdot r(1,n)$$
$$\cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 1 & 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} 1 & 0 & C & 0 \\ c & d & B & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \backslash\mathbf{H}. \tag{5.2}$$

REMARK. Proposition 5 implies that the structure of $\mathbf{H}\backslash\mathbf{G}/\mathbf{H}$ is independent of $n \geqslant 3$. It would be interesting to pursue a comparison theory between $\mathbf{G}_n(\mathbb{A})$-modules with a $\mathbf{H}_n(\mathbb{A})$-invariant form and $\mathbf{G}_3(\mathbb{A})$-modules with a $\mathbf{H}_3(\mathbb{A})$-invariant form (for $n > 3$) on developing and then comparing the (non-Fourier) bi-period summation formulae associated with such double cosets.

6. PROPOSITION. *If* $n \geqslant 3$, *no discrete-series representation occurs in* (3.3).

Proof. We need to show that the terms described by Proposition 4(c) do not occur in (3.3). We give a complete proof in the case of $n = 3$, and a sketch in the case of $n = 4$. Suppose first that $n = 3$ and ρ is a cuspidal PGL(3, $\mathbb{A}$)-module. If $\Phi \in \rho$ then $\Lambda^T\Phi = \Phi$, since $\Phi_{\mathbf{N}} = 0$ for all $\mathbf{P} \neq \mathbf{G}$ by definition of cuspidality.

The Fourier expansion of the cusp form $\bar{\Phi}$ is

$$\bar{\Phi}(g) = \sum_{p \in \mathbf{N}_H(F)\backslash\mathbf{H}(F)} W_{\bar{\Phi},\psi}(pg),$$

where

$$W_{\bar{\Phi},\psi}(g) = \int_{\mathbf{N}_0(F)\backslash\mathbf{N}_0(\mathbb{A})} \bar{\Phi}(ug)\psi(u)^{-1}\,\mathrm{d}u,$$

and

$$\mathbf{N}_H = \left\{ \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathbf{H} = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

The integral of $\bar{\Phi}\xi$ over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ is equal then to

$$\int_{\mathbf{N}_H(F)\backslash\mathbf{H}(\mathbb{A})} W_{\bar{\Phi},\psi}(h)\xi(h)\,\mathrm{d}h = \int_{\mathbf{N}_H(\mathbb{A})\backslash\mathbf{H}(\mathbb{A})}\int_{\mathbf{N}_H(F)\backslash\mathbf{N}_H(\mathbb{A})} W_{\bar{\Phi},\psi}(nh)\xi(h)\,\mathrm{d}n\,\mathrm{d}h.$$

The inner integral here is 0 since ψ is non-trivial on $\mathbf{N}_H(F)\backslash\mathbf{N}_H(\mathbb{A})$ and $W_{\bar{\Phi},\psi}(nh) = \psi(n)W_{\bar{\Phi},\psi}(h)$.

Now the terms associated with the cuspidal ρ in the spectral expression for the kernel have $\mathbf{P} = \mathbf{G}$, $n(P) = 1$, $\mathfrak{A}_P^* = \{0\}$, and these terms are

$$\sum_{\rho}\sum_{\Phi\in\rho}(\rho(f)\Phi)(u)\Lambda^T\bar{\Phi}(h) = \sum_{\rho}\sum_{\Phi\in\rho}(\rho(f)\Phi)(u)\bar{\Phi}(h).$$

The integral of the product of this with $\psi(u)\xi(h)$ over $\mathbf{N}_0(F)\backslash\mathbf{N}_0(\mathbb{A})\times\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ vanishes since $\int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})}\Phi(h)\xi(h)dh = 0$ for all $\Phi\in\rho$, as required.

Of course this proof generalizes to show that when $n \geqslant 3$ no cuspidal representation $(\pi =)\rho$ of $\mathbf{G}(\mathbb{A})$ would contribute a non-zero term to (3.3).

In order to deal with the remaining case of Proposition 4(c) suppose that $n = 4$ and ρ is the discrete series PGL$(4,\mathbb{A})$-module which is equivalent to the unique irreducible subrepresentation of $I((\rho_1\times\rho_1)\delta_{P_{(2,2)}}^{-1/2})$, where ρ_1 is a cuspidal GL$(2,\mathbb{A})$-module, and $\mathbf{P}_{(2,2)}$ is the standard parabolic of type $(2,2)$. Note that the space of this ρ is spanned by residues of some Eisenstein series, which are automorphic functions; the spectral expression for the kernel does not use the realization of ρ as a subrepresentation of $I((\rho_1\times\rho_1)\delta_{P_{(2,2)}}^{-1/2})$. The coefficient $\Phi_{\mathbf{N}}$ of $\Phi\in\rho$ is 0 if $\mathbf{N}$ is the unipotent radical of a parabolic subgroup of type $(1,3)$ or $(3,1)$, since the GL$(2,\mathbb{A})$-module ρ_1 is cuspidal. Indeed, the integral over $\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})$, of the Eisenstein series whose residue is Φ, vanishes, since $w\mathbf{N}w^{-1}\cap\mathbf{M}_{(2,2)}$ is non-trivial for every element w of the Weyl group. Hence

$$\Lambda^T\Phi(g) = \Phi(g) - \sum_{\delta\in\mathbf{P}_{(2,2)}(F)\backslash\mathbf{G}(F)}\hat{\tau}_{P_{(2,2)}}(H(\delta g) - T)\Phi_{\mathbf{N}_{(2,2)}}(\delta g);$$

note that the dimension of the center of the Levi subgroup $\mathbf{M}_{(2,2)}$ of $\mathbf{P}_{(2,2)}$ is 1.

We need to show that $\int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})}\Lambda^T\Phi(h)\xi^{-1}(h)\,dh$ is zero, where $\mathbf{H}\simeq$ GL(3) embeds in $\mathbf{G} =$ PGL(4) as $\mathbf{H} = \left\{\begin{pmatrix} g & 0\\ 0 & 1\end{pmatrix}; g\in\mathrm{GL}(3)\right\}$. To compute this integral we need to rewrite $\Lambda^T\Phi(g)$. Since $\Phi_{\mathbf{N}_{(3,1)}}$ is zero if $\mathbf{N}_{(3,1)}$ is the unipotent radical of the parabolic of type $(3,1)$, the Fourier expansion of Φ along $\mathbf{N}_{(3,1)}$ is

$$\Phi(g) = \sum_{p\in\mathbf{P}_H(F)\backslash\mathbf{H}(F)}\Phi_{\mathbf{N}_{(3,1)},\psi}(pg),$$

where

$$\Phi_{\mathbf{N}_{(3,1)},\psi}(g) = \int_{\mathbf{N}_{(3,1)}(F)\backslash\mathbf{N}_{(3,1)}(\mathbb{A})} \Phi(ug)\psi(x)^{-1}du,$$

$$u = \begin{pmatrix} 1 & 0 & 0 & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\mathbf{P}_H = \left\{ \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \subset \mathbf{H} = \left\{ \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

To continue we need a special case of the Corollary to Proposition 5, namely the decomposition

$$\mathbf{P}_{(2,2)}\backslash\mathbf{G} = \mathbf{P}_{(2,1)}\backslash\mathbf{H} \cup r(2, 4)\cdot\mathbf{P}_{(1,2)}\backslash\mathbf{H} \cup \kappa\cdot\mathbf{B}'_H\backslash\mathbf{H},$$

where

$$B'_H = \left\{ \begin{pmatrix} u & v & w & 0 \\ 0 & d & c & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \right\}.$$

Here κ is as defined above (5.1); $\mathbf{P}_{(2,1)}$ and $\mathbf{P}_{(1,2)}$ are the parabolic subgroups of $\mathbf{H}$ of types (2, 1), (1, 2); $r(2, 4)$ is an elementary matrix in $\mathbf{G}(F)$ representing the reflection (2, 4). The sum over δ in $\mathbf{P}_{(2,1)}(F)\backslash\mathbf{H}(F)$ is expressed compatibly with the sum representing $\Phi(g)$, as follows.

The Fourier expansion of $\Phi_{\mathbf{N}_{(2,2)}}$ along $\mathbf{N}_{(3,1)} \cap \mathbf{M}_{(2,2)}$ is

$$\Phi_{\mathbf{N}_{(2,2)}}(g) = \sum_{t\in F^\times} \Phi_{\mathbf{N}_{(2,2)},(\mathbf{N}_{(3,1)},\psi)}(a(t)g), \quad a(t) = \mathrm{diag}(1, 1, t, 1).$$

Hence

$$\Phi(g) - \sum_{\delta\in\mathbf{P}_{(2,1)}(F)\backslash\mathbf{H}(F)} \tilde{\tau}_{P_{(2,2)}}(H(\delta g) - T)\Phi_{\mathbf{N}_{(2,2)}}(\delta g)$$
$$= \sum_{p\in\mathbf{P}_H(F)\backslash\mathbf{H}(F)} [\Phi_{\mathbf{N}_{(3,1)},\psi}(pg) - \hat{\tau}_{P_{(2,2)}}(H(pg) - T)(\Phi_{\mathbf{N}_{(2,2)}})_{(\mathbf{N}_{(3,1)},\psi)}(pg)].$$

The integral of the product of this by $\xi^{-1}(g)$ over $g \in \mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ is equal to

$$\int_{\mathbf{M}_{(2,1)}(F)\backslash\mathbf{M}_{(2,1)}(\mathbb{A})} (1 - \hat{\tau}_{P_{(2,2)}}(H(m) - T)) \times \int (\Phi_{\mathbf{N}_{(2,2)}})_{(\mathbf{N}_{(3,1)},\psi)}(mk)\xi^{-1}(mk)\delta(m)^{-1}\, dm\, dk,$$

since $\mathbf{P}_H(\mathbb{A}) = \mathbf{M}_{(2,1)}(\mathbb{A})(\mathbf{N}_{(2,2)} \cap \mathbf{H})(\mathbb{A})$ and $\mathbf{H}(\mathbb{A}) = \mathbf{P}_H(\mathbb{A})(\mathbb{K} \cap \mathbf{H}(\mathbb{A}))$. This integral factorizes through $\mathbf{N}_{(1,3)}(F)\backslash\mathbf{N}_{(1,3)}(\mathbb{A})$, namely through $\Phi_{\mathbf{N}_{(1,3)}}$, which is zero as observed above. Hence the integral over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ is zero.

The second coset, $r(2,4)\cdot\mathbf{P}_{(1,2)}\backslash\mathbf{H}$, in $\mathbf{P}_{(2,2)}\backslash\mathbf{G}$, parametrizes the terms in the sum in $\Lambda^T\Phi(g)$, which – multiplied by $\xi^{-1}(g)$ and integrated over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ – yield

$$-\int_{\mathbf{P}_{(1,2)}(F)\backslash\mathbf{H}(\mathbb{A})} \hat{\tau}_{P_{(2,2)}}(H(r(2,4)h) - T)\Phi_{\mathbf{N}_{(2,2)}}(r(2,4)h)\xi^{-1}(h)\, dh.$$

Note that $r(2,4)\begin{pmatrix} * & * & * & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} r(2,4) = \begin{pmatrix} * & 0 & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$, and the product of this with $\mathbf{N}_{(2,2)}$ contains $\mathbf{N}_{(3,1)}$. Hence this last integral factorizes through $\Phi_{\mathbf{N}_{(3,1)}}$, which is zero.

The last coset, $\kappa B'_H\backslash\mathbf{H}$, in $\mathbf{P}_{(2,2)}\backslash\mathbf{G}$, after multiplication by $\xi^{-1}(g)$ and integration over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$, yields

$$-\int_{\mathbf{B}'_H(F)\backslash\mathbf{H}(\mathbb{A})} \hat{\tau}_{P_{(2,2)}}(H(\kappa h) - T)\Phi_{\mathbf{N}_{(2,2)}}(\kappa h)\xi^{-1}(h)\, dh.$$

Since $\kappa\begin{pmatrix} 1 & v & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\kappa^{-1} = \begin{pmatrix} 1 & -v & 0 & v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and the product of this with $\mathbf{N}_{(2,2)}$ contains $\mathbf{N}_{(1,3)}$, the integral factorizes through $\Phi_{\mathbf{N}_{(1,3)}}$, and this is zero.

It follows that the integral of $(\Lambda^T\Phi)(h)\xi^{-1}(h)$ over h in $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ is zero. This completes the (sketch of) proof of Proposition 5. Of course to complete the proof it has to be shown that each of the three terms associated with the three cosets is integrable, not only factorizes through an integral which vanishes.

Denote by $\mathbf{P}_1$ the parabolic subgroup of type $(n-2,2)$, by ρ_2 a cuspidal representation of GL(2, $\mathbb{A}$) with central character ω_{ρ_2}, by ρ_1 a unitary character of $\mathbb{A}^\times/F^\times$ with $\rho_1^{2-n} = \omega_{\rho_2}$ and also the character $\rho_1(g) = \rho_1(\det g)$ of GL($n-2$, $\mathbb{A}$). Let $\rho = \rho_1 \times \rho_2$ be the $\mathbf{P}_1(\mathbb{A})$-module defined by ρ_1 and ρ_2 on the Levi factor, and extended trivially across the unipotent radical. For any $\lambda \in \mathbb{C}$ put $\rho_\lambda = \rho_1 \otimes \nu^{\lambda/(n-2)} \times \rho_2 \otimes \nu^{-\lambda/2}$, where $\nu(x) = |x|$, $x \in \mathbb{A}^\times$. Denote by τ_λ the vector in the one dimensional space $\mathfrak{A}_1^* = \mathfrak{A}_{P_1}^*$ with $\rho_\lambda = \rho \otimes e^{\langle \tau_\lambda, H \rangle}$.

As in [A1], p. 917, for any F-parabolic subgroup $\mathbf{P}_2$ let $W(\mathfrak{A}_1, \mathfrak{A}_2)$, $\mathfrak{A}_2 = \mathfrak{A}_{P_2}$, denote the set of (distinct) isomorphisms from $\mathfrak{A}_1$ to $\mathfrak{A}_2$ obtained by restricting to $\mathfrak{A}_1$ elements of the Weyl group W. Note that when $n \neq 4$, the set $W(\mathfrak{A}_1, \mathfrak{A}_2)$ is empty unless $\mathbf{P}_2 = \mathbf{P}_1$ or $\mathbf{P}_{(2,n-2)}$, in which case it consists of s = identity or of $s = s_2$, where $s_2 = \begin{pmatrix} 0 & I_2 \\ I_{n-2} & 0 \end{pmatrix}$, respectively. If $n = 4$, $W(\mathfrak{A}_1, \mathfrak{A}_2)$ is empty unless $\mathbf{P}_2 = \mathbf{P}_1$, and then it consists of $s = 1$ and $s = s_2$.

As in [A2], p. 113, for any F-parabolic subgroup $\mathbf{P}$ define $W(\mathfrak{A}_1; \mathbf{P})$ to be the union over all $\mathfrak{A}_2$ of the $s \in W(\mathfrak{A}_1, \mathfrak{A}_2)$ such that $s\mathfrak{A}_1 = \mathfrak{A}_2$ contains $\mathfrak{A} = \mathfrak{A}_P$, and $s^{-1}\alpha > 0$ for all $\alpha \in \Delta_2^P$. Then $W(\mathfrak{A}_1; \mathbf{P})$ is empty unless $\mathbf{P} = \mathbf{G}$, when it consists of the identity, or $\mathbf{P} = \mathbf{P}_1$, when it consists of the identity if $n \neq 4$ and of the identity and s_2 if $n = 4$, or $\mathbf{P} = \mathbf{P}_{(2,n-2)}$, $n \neq 4$, when it consists of s_2.

We shall use the following analogue of the formula (4.1) of [A2], p. 113.

7. PROPOSITION. *We have*

$$\int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} E(ng, \Phi, \rho, \tau_\lambda)dn = \sum_{s \in W(\mathfrak{A}_1; \mathbf{P})} E_{\mathbf{P}}(g, M(s, \tau_\lambda)\Phi, s\rho, s\tau_\lambda),$$

where

$$E_{\mathbf{P}}(g, \Phi, \rho, \tau) = \sum_{\delta \in \mathbf{P}_1(F)\backslash\mathbf{P}(F)} \Phi(\gamma g, \tau) \quad (cf.\ [A1], \text{p. } 927).$$

Proof. The equality is a tautology for $\mathbf{P} = \mathbf{G}$, so we assume that $\mathbf{P} \neq \mathbf{G}$. This identity is asserted in (4.1), [A2], p. 113, when ρ is a *cuspidal* representation (of the Levi factor of an F-parabolic). But the ρ in the proposition is not cuspidal. The trivial representation $\mathbb{1}$ of GL($n-2$, $\mathbb{A}$) is obtained as the residue at $\left(\frac{n-3}{2}, \frac{n-5}{2}, \ldots, \frac{3-n}{2}\right)$ of the Eisenstein series on GL($n-2$, $\mathbb{A}$) induced from the upper triangular subgroup and the parameter $\tau' = (\lambda_1, \ldots, \lambda_{n-2})$, $\sum_{1 \leqslant i < n-1} \lambda_i = 0$, in $\mathbb{C}^{n-2}$. The Eisenstein series in the proposition is also obtained as a residue. Denote by $\mathbf{P}_3$ the F-parabolic of type $(1, \ldots, 1, 2)$. The space $\mathfrak{A}_3$ is $(n-1)$-dimensional, represented by $\tau + \tau_\lambda$, $\tau = (\lambda_1, \ldots, \lambda_{n-2}, 0, 0)$ with $\lambda_1 + \cdots + \lambda_{n-2} = 0$, and $\tau_\lambda = \left(\frac{\lambda}{n-2}, \ldots, \frac{\lambda}{n-2}, -\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$. Denote by

$\rho_3 = \rho_1 \times \cdots \times \rho_1 \times \rho_2$ the representation of $\mathbf{M}_3(\mathbb{A})$, where $\rho_1 \times \cdots \times \rho_1$ is a character of $(\mathbb{A}^\times)^{n-2}$. For $\Phi_3 \in \rho_3$, consider the Eisenstein series $E(g, \Phi_3, \rho_3, \tau + \tau_\lambda)$ on $(g \in)\mathbf{G}(\mathbb{A})$. The series $E(g, \Phi, \rho, \tau_\lambda)$ is obtained as the highest residue (of degree $n-3$), for some Φ_3 (which is in fact the restriction of Φ to $\mathbf{M}_3(\mathbb{A})$), namely

$$E(g, \Phi, \rho, \tau_\lambda) = \lim_{\substack{\lambda_i - \lambda_{i-1} \to 1 \\ 1 \leqslant i < n-2}} \left(\prod_{1 \leqslant i < n-2} (\lambda_i - \lambda_{i-1}) \right) E(g, \Phi_3, \rho_3, \tau + \tau_\lambda).$$

Since ρ_3 is cuspidal, (4.1) of [A2], p. 113, applies:

$$\int_{\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})} E(ng, \Phi_3, \rho_3, \tau + \tau_\lambda) dn$$

$$= \sum_{s \in W(\mathfrak{A}_3; \mathbf{P})} E_{\mathbf{P}}(g, M(s, \tau + \tau_\lambda)\Phi_3, s\rho_3, s(\tau + \tau_\lambda)).$$

Any of the Eisenstein series on the right can have a pole of (the maximal) order $n-3$ only when $\mathbf{P}(\neq \mathbf{G})$ is of type $(n-2, 2)$ or $(2, n-2)$, and such a pole is attained only at $\tau^0 = \left(\frac{n-3}{2}, \frac{n-5}{2}, \ldots, \frac{3-n}{2}, 0, 0\right)$, precisely when $s\tau^0 = \tau^0$ or $s\tau^0 = \left(0, 0, \frac{n-3}{2}, \frac{n-5}{2}, \ldots, \frac{3-n}{2}\right)$, namely when s is the identity or $s_2 = \begin{pmatrix} 0 & I_2 \\ I_{n-2} & 0 \end{pmatrix}$, respectively. Multiplying by $\prod_{1 \leqslant i < n-2} (\lambda_i - \lambda_{i-1})$ and taking the $n-3$ limits as $\lambda_i - \lambda_{i-1} \to 1$, we obtain 0 unless $\mathbf{P}(\neq \mathbf{G})$ is $\mathbf{P}_1$ or $\mathbf{P}_{(2n-2)}$, in which cases we obtain

$$E_{\mathbf{P}_{(n-2,2)}}(g, \Phi, \rho, \tau_\lambda) \text{ or } E_{\mathbf{P}_{(2,n-2)}}(g, M(s_2, \tau_\lambda)\Phi, s_2\rho, s_2\tau_\lambda),$$

respectively, if $n \neq 4$, and their sum if $n = 4$. This is the expression asserted in the proposition.

In [A2], the identity (4.1) is used in the proof [A2], Lemma 4.1, on p. 115, ℓ. 2. The proof of that Lemma then applies without a change in our situation too, to yield

8. PROPOSITION. *For a sufficiently large λ (i.e., $\mathrm{Re}(\lambda) \geqslant 1$), the truncated Eisenstein series $\Lambda^T E(g, \rho, \tau_\lambda)$ is equal to*

$$\sum_{\mathbf{P}_2} \sum_{\gamma \in \mathbf{P}_2(F)\backslash\mathbf{G}(F)} \sum_{s \in W(\mathfrak{A}_1, \mathfrak{A}_2)} \varepsilon_2(s\tau_\lambda)\phi_2(s\tau_\lambda, H_0(\gamma g) - T)$$
$$e^{\langle s\tau_\lambda + \rho_2, H_0(\gamma g)\rangle}(M(s, \tau_\lambda)\Phi)(\gamma g),$$

with the sum over γ converging absolutely.

Recall that $\varepsilon_2(\Lambda)$, for $\Lambda \in \mathfrak{A}_0^*$, is defined in [A1], p. 940, to be 1 if the set of $\alpha \in \Delta_2$ with $\langle \Lambda, \alpha^\vee \rangle \leqslant 0$ is even, and -1 otherwise. The function $\phi_2(\Lambda, H)$ on $(\Lambda, H) \in \mathfrak{A}_0^* \times \mathfrak{A}_0$ takes the values 0 and 1. It is equal to 1 precisely when for every $\alpha \in \Delta_2$, we have $\langle \Lambda, \alpha^\vee \rangle \leqslant 0$ and $\langle \tilde{\omega}_\alpha, H \rangle > 0$ or $\langle \Lambda, \alpha^\vee \rangle > 0$ and $\langle \tilde{\omega}_\alpha, H \rangle \leqslant 0$. As noted in Proposition 7, $\mathbf{P}_2$ ranges over the set

$$\{\mathbf{P}_1 = \mathbf{P}_{(n-2,2)}, \mathbf{P}_2 = \mathbf{P}_{(2n-2)}\},$$

and $s = 1$ or s_2. It is clear that $\varepsilon_1(\tau_\lambda) = 1$ and $\varepsilon_2(s_2\tau_\lambda) = -1$. When $s = 1$, the characteristic function $\phi_1(\tau_\lambda, H_0(g) - T)$ can be expressed as $\chi(\delta_{\mathbf{P}_1}(g)^{1/2} < t_1)$, the characteristic function of the g such that $\delta_{\mathbf{P}_1}(g)^{1/2} < t_1$, where $t_1(>0)$ depends linearly on $T \in \mathfrak{A}_0^+$ and $t_1 \to \infty$ as $T \to \infty$. When $s = s_2$ the characteristic function $\phi_2(s_2\tau_\lambda, H_0(g) - T)$ can be written as $\chi(\delta_{\mathbf{P}_2}(g)^{1/2} \geqslant t_2)$, the characteristic function of the g such that $\delta_{\mathbf{P}_2}(g)^{1/2} \geqslant t_2$, where $t_2(>0)$ depends linearly on T and $t_2 \to \infty$ as $T \to \infty$. Further, the exponential $e^{\langle \tau_\lambda + \rho_1, H_0(g) \rangle}$ is equal to $\delta_{\mathbf{P}_1}(g)^{(\lambda+1)/2}$, while $e^{\langle s_2\tau_\lambda + \rho_2, H_0(g) \rangle}$ is equal to $\delta_{\mathbf{P}_2}(g)^{(\lambda-1)/2}$. In summary, the identity of Proposition 8 can be rewritten as follows:

COROLLARY. *The truncated Eisenstein series $\Lambda^T E(h, \Phi, \rho, \tau_\lambda)$ is equal to the difference between*

$$\sum_{\gamma \in \mathbf{P}_1(F)\backslash \mathbf{G}(F)} \chi(\delta_1(\gamma h)^{1/2} < t_1)\delta_1(\gamma h)^{(1+\lambda)/2}\Phi(\gamma h) \tag{8.1}$$

and

$$\sum_{\gamma \in \mathbf{P}_2(F)\backslash \mathbf{G}(F)} \chi(\delta_2(\gamma h)^{1/2} \geqslant t_2)\delta_2(\gamma h)^{(1-\lambda)/2}(M(s_2, \rho, \lambda)\Phi)(\gamma h). \tag{8.2}$$

We use this Corollary to prove, with $\rho = \rho_1 \times \rho_2$, the following

9. PROPOSITION. *The integral of the product of* (8.1) *and* $\xi^{-1}(h)$ *over h in* $\mathbf{H}(F)\backslash \mathbf{H}(\mathbb{A})$ *is equal to* $\delta(\rho_1/\xi)L_{\Phi_{\mathbb{K},\xi}}(\rho_2 \otimes \xi^{-1}, (n-1)/2)t_1^\lambda/\lambda$.

As usual, if χ is a character of $\mathbb{A}^\times/F^\times$ we put $\delta(\chi) = 1$ if χ is 1 on every $a \in \mathbb{A}^\times$ with $|a| = 1$, and $\delta(\chi) = 0$ if not. The L-function is the one associated in [JPS] to the cusp form $\Phi_{\mathbb{K},\xi}(A) = \int_{\mathbb{K}^{\mathbf{H}}} \Phi\left(\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} k\right)\xi^{-1}(k)dk$, $A \in \mathrm{GL}(2, \mathbb{A})$, in ρ_2, twisted by ξ^{-1}.

Proof. We use the Corollary to Proposition 5 to express the integral of (8.1) as a sum of three integrals, corresponding to the three cosets in (5.1). Corresponding to the second coset in (5.1), we obtain the integral

$$\int_{\mathbf{P}^{\mathbf{H}}_{(n-3,2)}(F)\backslash \mathbf{H}(\mathbb{A})} \chi(\delta_1(rh)^{1/2} < t_1)\delta_1(rh)^{(1+\lambda)/2}\Phi(rh)\xi^{-1}(h)dh, \tag{8.1.2}$$

where $r = r(n-2, n)$. By the Iwasawa decomposition $\mathbf{H}(\mathbb{A}) = \mathbf{P}^{\mathbf{H}}_{(n-3,2)}(\mathbb{A})\mathbb{K}^{\mathbf{H}}$ we write $h = mnk$, and note that rmr^{-1} ranges over $\mathbf{L}(F)\backslash\mathbf{L}(\mathbb{A})$, where $\mathbf{L}$ is the Levi subgroup of type $(n-3, 1, 2)$ of $\mathbf{G}$. Note that $\mathrm{GL}(2, F)\backslash\mathrm{GL}(2, \mathbb{A})$ can be expressed in the form $\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})\cdot S$ for some Siegel domain S, where $\mathbf{N}$ denotes here the upper triangular unipotent subgroup of GL(2). But $\Phi_g(a) = \Phi\left(\begin{pmatrix} I & 0 \\ 0 & a \end{pmatrix} g\right)$ is a cusp form on $\mathrm{GL}(2, \mathbb{A})$, for any g in $\mathbf{G}(\mathbb{A})$. Consequently the integral factorizes through an integral over $u \in \mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})$ of the cusp form Φ_g, and this inner integral is zero, as is (8.1.2).

Corresponding to the third coset in (5.1) we obtain the integral

$$\int \chi(\delta_1(\kappa \mathrm{h}))^{1/2}\delta_1(\kappa h)^{(1+\lambda)/2}\Phi(\kappa h)\xi^{-1}(h)dh, \qquad (8.1.3)$$

where $\kappa = \begin{pmatrix} I' & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. It ranges over $\left\{\begin{pmatrix} A & {}^tB & {}^tC & 0 \\ 0 & d & c & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & d \end{pmatrix}\right\}(F)\backslash\mathbf{H}(\mathbb{A})$. Since

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix},$$

using the Iwasawa decomposition $\mathbf{H}(\mathbb{A}) = \mathbf{P}^{\mathbf{H}}_{(n-3,1,1)}(\mathbb{A})\mathbb{K}^{\mathbf{H}}$ it is clear that the integral (8.1.3) factorizes through the integral over ${}^t\mathbf{N}(F)\backslash{}^t\mathbf{N}(\mathbb{A})$, where ${}^t\mathbf{N}$ is the lower triangular unipotent subgroup of GL(2), of the cusp form $\Phi_g(a) = \Phi\left(\begin{pmatrix} I & 0 \\ 0 & a \end{pmatrix} g\right)$ on $\mathrm{GL}(2, \mathbb{A})$. This inner integral is zero, and so is – consequently – (8.1.3).

Corresponding to the first coset on the right side of (8.1) we obtain

$$\int_{\mathbf{P}_{(n-2,1)}(F)\backslash\mathbf{H}(\mathbb{A})} \chi(\delta_1(h)^{1/2} < t_1)\delta_1(h)^{(1+\lambda)/2}\Phi(h)\xi^{-1}(h)\,dh. \qquad (8.1.1)$$

By the Iwasawa decomposition $\mathbf{H}(\mathbb{A}) = \mathbf{N}_{(n-2,1)}(\mathbb{A})\mathbf{M}_{(n-2,1)}(\mathbb{A})\mathbb{K}^{\mathbf{H}}$ we may write $h = nmk$, and $m = \mathrm{diag}(a, b, c)$, where a is in $\mathrm{GL}(n-2)$, and b, c in GL(1). Note that $\delta_1(h) = |\det(a)^2/(bc)^{n-2}|$, and the modular function δ with respect to $\mathbf{N}_{(n-2,1)}$, which occurs in the integration formula $dh = \delta^{-1}(m)dn\,dm\,dk$, is $\delta(h) = |\det(a)/b^{n-2}|$. Note also that

$$\Phi(nmk) = \rho_1(\det a)\Phi\left(\begin{pmatrix} I & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} k\right) = \rho_1(\det a)\Phi_k\left(\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}\right),$$

where $\Phi_k(A) = \Phi\left(\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} k\right)$ is a cusp form in ρ_2 (A in $\mathrm{GL}(2, \mathbb{A})$). Denote by $\Phi_{\mathbb{K},\xi}(A)$ the integral over $k \in \mathbb{K}^{\mathrm{H}}$ of $\Phi\left(\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} k\right)\xi^{-1}(k)$. It is again a cusp form in ρ_2, on $\mathrm{GL}(2, \mathbb{A})$.

Consequently (8.1.1) is equal to the product by the volume

$$1 = |\mathrm{SL}(n-2, F)\backslash\mathrm{SL}(n-2, \mathbb{A})|,$$

of the integral

$$\int |a^2/(bc)^{n-2}|^{(1+\lambda)/2}|a/b^{n-2}|^{-1}\Phi_{\mathbb{K},\xi}\left(\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}\right)$$
$$\times\, \xi^{-1}(c^{1-n}b \det a)\rho_1(\det a)d^\times a d^\times b d^\times c.$$

Here a, b, c range over the quotient of $(\mathbb{A}^\times/F^\times)^3$ by the equivalence relation $(z^{n-2}, z, z) \equiv (1, 1, 1)$. Since ρ_2 has the central character $\omega_{\rho_2} = \rho_1^{2-n}$, we have $\Phi_{\mathbb{K},\xi}\left(\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}\right) = \rho_1(c^{2-n})\Phi_{\mathbb{K},\xi}\left(\begin{pmatrix} b/c & 0 \\ 0 & 1 \end{pmatrix}\right)$. The integral ranges over the domain $|a^2/(bc)^{n-2}|^{1/2} < t_1$. Write $u = b/c$ and $v = a/b^{n-2}$. Then the range of integration is $|v|\,|u|^{(n-2)/2} = |a/b^{n-2}||b/c|^{(n-2)/2} < t_1$. The integral takes the form

$$\int_{|v||u|^{(n-2)/2}<t_1} |u|^{(n-2)/2}(|u|^{(n-2)/2}|v|)^\lambda \Phi_{\mathbb{K},\xi}\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right)$$
$$\times\, \rho_1(vu^{n-2})\xi(vu^{n-1})^{-1}d^\times v d^\times u$$

(note that u, v range over $\mathbb{A}^\times/F^\times$).

Integrating out v, and noting that $\mathrm{Re}(\lambda) > 0$, we obtain

$$\delta(\rho_1/\xi)\frac{t_1^\lambda}{\lambda}\int_{\mathbb{A}^\times/F^\times} \Phi_{\mathbb{K},\xi}\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right) |u|^{(n-2)/2}\xi^{-1}(u)d^\times u.$$

Since $\Phi_{\mathbb{K},\xi}$ is a cusp form, it is rapidly decreasing, and the last integral converges. It is a "Tate integral" for the L-function of ρ_2. Namely for any $\Phi_2 \in \rho_2$, the integral

$$\int_{\mathbb{A}^\times/F^\times} \Phi_2\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right) |u|^{s-(1/2)}\xi^{-1}(u)d^\times u = L_{\Phi_2}(\rho_2 \otimes \xi^{-1}, s)$$

coincides – up to a finite number of factors – with the Euler product which defines the L-function of $\rho_2 \otimes \xi^{-1}$; for further details we refer to [JPS]. The Proposition follows.

Before we proceed to integrate (8.2) over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$, note that the result of Proposition 9, and 11 below, will be used in conjunction with the following consequence of the Fourier inversion formula.

10. LEMMA. *Let f be a Schwartz (smooth, rapidly decreasing as $|\mu| \to \infty$) function on $i\mathbb{R}$, and signify by $\int_{i\mathbb{R}}$ the principal value integral* $\lim_{\varepsilon\to 0}(\int_\varepsilon^\infty + \int_{-\infty}^{-\varepsilon})$. *Then* $\lim_{t\to\infty} \int_{i\mathbb{R}} f(\mu)\mu^{-1}\exp(\pm\mu t)d\mu = \pm f(0)$.

To complement Proposition 9, we have

11. PROPOSITION. *The integral over g in $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ of the product of $\xi^{-1}(g)$ and (8.2) is equal to*

$$\delta(\rho_1/\xi)L_{(M(s_2,\rho_2,\lambda)\Phi)\mathbb{K},\xi}(\rho_2 \otimes \xi^{n-1}, (n-1)/2)t_2^{-\lambda}/\lambda.$$

Proof. The coset decomposition (5.2) will be used, and as in the discussion of (8.1) using (5.1), we express the integral of (8.2) as a sum of three integrals, corresponding to the three cosets on the right of (5.2), beginning with the second coset. Since $r(1,n)\mathbf{H}(\mathbb{A})r(1,n) = \mathbf{H}^-(\mathbb{A})$, the integral over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ of the summands in (8.2) parametrized by the second coset in (8.2) is equal to

$$\int_{\mathbf{P}^{\mathbf{H}}_{(2,n-3)}(F)\backslash\mathbf{H}(\mathbb{A})} \chi(\delta_2(h)^{1/2} \geqslant t_2)\delta_2(h)^{(1-\lambda)/2}(M(s_2,\rho_2,\lambda)\Phi)(h)\xi^{-1}(h)\,dh.$$

We shall abbreviate here and below and write $M\Phi$ for $M(s_2,\rho_2,\lambda)\Phi$; note that $(M\Phi)_g(A) = (M\Phi)\left(\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} g\right)$ is a cusp form in ρ_2 on $A \in \mathrm{GL}(2,\mathbb{A})$ for every $g \in \mathbf{G}(\mathbb{A})$. The Iwasawa decomposition $\mathbf{H}(\mathbb{A}) = \mathbf{P}^{\mathbf{H}}_{(2,n-3)}(\mathbb{A})\mathbb{K}^{\mathbf{H}}$ can be used to show that the integral factorizes through $h = mnk$, with $m = m_1 m_2$, and $m_1 = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$, with A in $\mathrm{GL}(2,F)\backslash\mathrm{GL}(2,\mathbb{A})$. Writing A as A_1A_2, with A_1 ranging over $\mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})$, $\mathbf{N}$ being the upper triangular unipotent subgroup of $\mathrm{GL}(2)$, and A_2 over a suitable Siegel domain, since $(\xi^{-1}\delta_2)(h)$ is independent of A_1 we conclude that the integral factorizes through $\int (M\Phi)_g(A_1A_2)dA_1$, $A_1 \in \mathbf{N}(F)\backslash\mathbf{N}(\mathbb{A})$, and this is zero since $(M\Phi)_g$ is a cusp form.

The integral over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$ of the terms in (8.2) parametrized by the third coset in (5.2) is equal to

$$\int \chi(\delta_2(rh)^{1/2} \geqslant t_2)\delta_2(rh)^{(1-\lambda)/2}(M\Phi)(rh)\xi^{-1}(h)dh$$

where

$$r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and h ranges over $\left\{ \begin{pmatrix} a & 0 & C & 0 \\ c & d & B & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \right\}(F)\backslash \mathbf{H}(\mathbb{A})$. Applying again the Iwasawa decomposition, and noting that r commutes with $x = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & I \end{pmatrix}$, and that $\xi^{-1}\delta_2(xrh) = (\xi^{-1}\delta_2)(rh)$, it follows that the integral factorizes through the integral

$$\int (M\Phi)_{rh}\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\right)dc, \qquad c \in \mathbb{A} \bmod F,$$

which is zero since $(M\Phi)_{rh}$ is a cusp form in ρ_2.

There remains the first coset in (5.2). The integral over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$, of the terms in (8.2) parametrized by this first coset, is equal to

$$\int_{\mathbf{P}^{\mathbf{H}^-}_{(1,n-2)}(F)\backslash \mathbf{H}^-(\mathbb{A})} \chi(\delta_2(hr)^{1/2} \geqslant t_2)\delta_2(hr)^{(1-\lambda)/2}(M\Phi)(hr)\xi^{-1}(h)dh.$$

Here $r = r(1, n)$, and we used the fact that $r\mathbf{H}r = \mathbf{H}^-$. The Iwasawa decomposition $\mathbf{H}^-(\mathbb{A}) = \mathbf{N}^{H^-}_{(1,n-2)}(\mathbb{A})\mathbf{M}^{H^-}_{(1,n-2)}\mathbb{K}^{\mathbf{H}^-}$ can be used to write h as nmk, and we use the change of variables formula $dh = \delta(m)^{-1}dndmdk$, with $\delta(m) = |b^{n-2}/\det(c)|$ if $m = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$; a, b in $\mathbb{A}^\times$, c in $\mathrm{GL}(n-2, \mathbb{A})$. Note that

$$(M\Phi)(nmk) = \rho_1(\det c)(M\Phi)\left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & I \end{pmatrix}k\right),$$

and the function $A \mapsto (M\Phi)\left(\begin{pmatrix} A & 0 \\ 0 & c \end{pmatrix} g\right) = \rho_1(\det c)(M\Phi)\left(\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} g\right)$ for any $c \in \mathrm{GL}(n-2, \mathbb{A})$ and $g \in \mathbf{G}(\mathbb{A})$, is a cusp form in $A \in \mathrm{GL}(2, \mathbb{A})$ in the space of the cuspidal representation ρ_2 of $\mathrm{GL}(2, \mathbb{A})$. Put

$$(M\Phi)_{\mathbb{K},\xi}(A) = \int_{\mathbb{K}^{\mathbf{H}^-}} (M\Phi)\left(\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} k\right) \xi^{-1}(k)\,dk.$$

Since $\delta_2(hr) = |(ab)^{n-2}/\det(c)^2|$, our integral takes the form

$$\int |(ab)^{n-2}/\det(c)^2|^{(1-\lambda)/2}|b^{n-2}/\det(c)|^{-1}(M\Phi)_{\mathbb{K},\xi}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right)$$
$$\times \rho_1(\det c)\xi(a^{n-1}/b \det c)d^\times a d^\times b d^\times c.$$

It ranges over the $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ in $\mathbf{M}_{(1,1,n-2)}(F)\backslash\mathbf{M}_{(1,1,n-2)}(\mathbb{A})$, with $|(ab)^{n-2}/\det(c)^2|^{1/2} \geqslant t_2$. Integrating over c in $\mathrm{SL}(n-2, F)\backslash\mathrm{SL}(n-2, \mathbb{A})$ we earn a volume factor which is equal to 1, and we may assume that c lies in $\mathbb{A}^\times/F^\times$, as do a, b, and (a, b, c) are taken modulo the equivalence relation $(z, z, z^{n-2}) \equiv (1, 1, 1)$. Write $u = a/b$ and $v = b^{n-2}/c$. Then the integral ranges over $|v|\,|u|^{(n-2)/2} \geqslant t_2$, and it takes the form

$$\int_{|v|\,|u|^{(n-2)/2} \geqslant t_2} |u|^{(n-2)/2}(|u|^{(n-2)/2}|v|)^{-\lambda}(M\Phi)_{\mathbb{K},\xi}\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right)$$
$$\times\ \rho_1(v)^{-1}\xi(u^{n-1}v)d^\times u d^\times v.$$

Integrating out v we obtain

$$\delta(\xi/\rho_1)\frac{t_2^{-\lambda}}{\lambda}\int (M\Phi)_{\mathbb{K},\xi}\left(\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\right) \xi(u)^{n-1}|u|^{(n-2)/2}d^\times u$$
$$= \delta(\rho_1/\xi)\lambda^{-1}t_2^{-\lambda}L_{(M\Phi)_{\mathbb{K},\xi}}\left(\rho_2 \otimes \xi^{n-1}, \frac{n-1}{2}\right),$$

where $L_{(M\Phi)_{\mathbb{K},\xi}}\left(\rho_2 \otimes \xi^{n-1}, \frac{n-1}{2}\right)$ is the value at $(n-1)/2$ of the L-function of $\rho_2 \otimes \xi^{n-1}$ associated with the cusp form $\xi^{n-1}(M\Phi)_{\mathbb{K},\xi}$ in $\rho_2 \otimes \xi^{n-1}$. This completes the proof of our proposition.

Bernstein's center

It remains to compute the contribution to the spectral side of the summation formula of the terms parameterized by the data of Proposition 4(a). It might be true that for a general $n \geqslant 3$ the only terms described by Proposition 4(a) which contribute to the summation formula are associated with a parabolic subgroup of type (n_1, n_2, n_3) where at least two of the n_1, n_2, n_3 are equal to 1, but we do not pursue this question here. In fact we shall discuss below the contributions of the terms of Proposition 4(a) only in the special case where $n = 3$ (and $n_1 = n_2 = n_3 = 1$). Before embarking on this computation for $n = 3$ we shall complete a special form of the summation formula for a general $n \geqslant 3$, which does not involve the terms of Proposition 4(a). This special case, for a general $n \geqslant 3$, depends on a choice of the test function f.

Let v be a non-archimedean place of F. A *cuspidal pair* in G_v is a pair (M_v, ρ_v) consisting of a (standard) Levi subgroup M_v and a supercuspidal (irreducible) M_v-module ρ_v. The pairs (M_v, ρ_v), (M'_v, ρ_v) are *equivalent* if there is g in G_v with $M'_v = g^{-1}M_v g$ and ρ_v equivalent to $m \mapsto \rho'_v(g^{-1}mg)$. An equivalence class is called an *infinitesimal character* (of G_v). For every irreducible G_v-module π_v there exists a cuspidal pair (M_v, ρ_v) such that π_v is a constituent of the composition series of the G_v-module $I(\rho_v; G_v, P_v)$ normalizedly (= "unitarily") induced from the P_v-module extended from ρ_v on M_v by 1 on the unipotent radical of the (standard) parabolic $P_v = M_v N_v$ defined by M_v. The *infinitesimal* character $\chi(\pi_v)$ of π_v is defined to be the infinitesimal character of (M_v, ρ_v); it is uniquely determined (see [BZ]).

The set $\Theta(G_v)$ of infinitesimal characters has the structure of a complex algebraic variety. Indeed, the group $X(M_v)$ of unramified characters $\mu: M_v \to \mathbb{C}^\times$ of M_v acts on the set $\operatorname{Irr} M_v$ of irreducible M_v-modules by $\mu: \rho_v \to \mu\rho_v$. For any cuspidal pair (M_v, ρ_v), the image of the map $X(M_v) \to \Theta(G_v)$, $\mu \mapsto (M_v, \mu\rho_v)$, is called a *connected component* of $\Theta(G_v)$. This component has the natural structure of an affine complex algebraic variety as a quotient of $X(M_v)$ ($\simeq \mathbb{C}^{\times d}$ for some $d = d(M_v) \geqslant 0$), by a finite group. The $\Theta(G_v)$ is a complex algebraic variety equal to the disjoint union of infinitely many connected components Θ.

As a consequence of the theory of the Bernstein center (see [BD] for a preliminary draft, and the forthcoming work [B]), one has the following

12. PROPOSITION. *Let Θ be a connected component in $\Theta(G_v)$. Then for any $f_v \in C_c^\infty(G_v)$ there exists $f_{v,\Theta} \in C_c^\infty(G_v)$ such that for any $\pi_v \in \operatorname{Irr} G_v$ we have $\pi_v(f_{v,\Theta}) = 0$ if $\chi(\pi_v) \notin \Theta$, and $\pi_v(f_{v,\Theta}) = \pi_v(f_v)$ if $\chi(\pi_v) \in \Theta$.*

We use this Proposition 12 as follows. Fix a non-archimedean place u of F, a unitary character ρ_{1u}^0 of $F_u^\times$ and a supercuspidal (irreducible) representation ρ_{2u}^0 of $\mathrm{GL}(2, F_u)$ with central character $\omega_{2n}^0 = (\rho_{1u}^0)^{2-n}$. Denote by M_u the standard Levi subgroup of type $(1, \ldots, 1, 2)$, and by ρ_u^0 the supercuspidal representation

$\rho_{1u}^0 \times \cdots \times \rho_{1n}^0 \times \rho_{2n}^0$ of M_u. Denote by Θ_u^0 the connected component of the infinitesimal character of ρ_u^0.

We shall derive the summation formula for a function f which is a (finite linear combination of) product(s) over all places v of F of the form $\otimes f_v$, where $f_v \in C_c^\infty(G_v)$ for all v, $f_v = f_v^0$ is the quotient by vol(K_v) of the characteristic function of K_v in G_v for almost all v, and f_u has the property that $f_u = f_{u,\Theta_u^0}$. For any such f we have that $\pi(f) = 0$ for every representation π of $\mathbf{G}(\mathbb{A})$ of the form $I(\rho; \mathbf{G}(\mathbb{A}), \mathbf{P}(\mathbb{A}))$, normalizedly induced from a pair $(\mathbf{P}, \rho)$ described in Proposition 4(a).

For such f, the summation formula is obtained from (3.3), where the sums over $\mathbf{P}$ and ρ range over the connected components of pairs $(\mathbf{P}, \rho)$ (up to conjugation) as listed in Proposition 4(b). The factor $\int \xi^{-1}\Lambda^T E$ in (3.3) is equal – by virtue of the Propositions 9 and 11 – to the difference

$$\delta(\rho_1/\xi)[L_{\Phi_{\mathbb{K},\xi}}(\rho_2 \otimes \xi^{-1}, (n-1)/2)t_1^\lambda/\lambda \\ - L_{(M(s_2,\rho_2,\lambda)\Phi)_{\mathbb{K},\xi}}(\rho_2^\vee \otimes \xi, (n-1)/2)t_2^{-\lambda}/\lambda],$$

since $\xi^{n-1} = \rho_1^{n-1} = \omega_2^{-1}\xi$ if $\rho_1 = \xi$, and $\rho_2^\vee$, the contragredient of ρ_2, is equivalent to $\rho_2 \otimes \omega_2$, where ω_2 is the central character of ρ_2. More precisely we need the complex conjugate of this. Of course on $\lambda \in i\mathbb{R}$ we have $\bar{\lambda} = -\lambda$. The factor is then

$$\delta(\rho_1/\xi)[L_{(M(s_2,\rho_2^\vee,-\lambda)\bar{\Phi})_{\mathbb{K},\xi^{-1}}}(\rho_2 \otimes \xi^{-1}, (n-1)/2)t_2^\lambda/\lambda \\ - L_{\bar{\Phi}_{\mathbb{K},\xi^{-1}}}(\rho_2^\vee \otimes \xi, (n-1)/2)t_1^{-\lambda}/\lambda].$$

For the given smooth function f, the sum over Φ in (3.3) is finite, and the function $E_\psi(I(f, \rho, \lambda)\Phi, \rho, \lambda)$ is holomorphic and rapidly decreasing in $\lambda \in i\mathfrak{A}_{\mathbf{P}}^\times (\simeq i\mathbb{R})$ as $|\lambda| \to \infty$.

By virtue of Lemma 10 we may take the limit as $t_1 \to \infty$ (and so $t_2 \to \infty$) to obtain the required result, namely that when $T \to \infty$ the limit of (3.3) is

$$\sum_{(\mathbf{P},\rho)} n(P)^{-1} \sum_{\Phi} E_\psi(I(f, \rho, 0)\Phi, \rho, 0)\delta(\rho_1/\xi) \\ \times [L_{(M(s_2,\rho_2^\vee,0)\bar{\Phi})_{\mathbb{K},\xi^{-1}}}(\rho_2 \otimes \xi^{-1}, (n-1)/2) + L_{\bar{\Phi}_{\mathbb{K},\xi^{-1}}}(\rho_2^\vee \otimes \xi, (n-1)/2)].$$

F. The case of PGL(3)

It remains to compute the contributions to the summation formula from the terms parametrized by the data described by Proposition 4(a). We shall do this only in the case where $n = 3$, and then $\mathbf{P} = \mathbf{B}$ is the upper triangular subgroup of $\mathbf{G} = \mathrm{PGL}(3)$, and $\rho = \rho_1 \times \rho_2 \times \rho_3$ is a character of $\mathbf{B}(\mathbb{A})/\mathbf{B}(F)$, which is trivial on $\mathbf{N}(\mathbb{A})/\mathbf{N}(F)$, $\mathbf{N}$ being the unipotent radical of $\mathbf{B}$.

Assume then that $n = 3$, and put $\mathbf{P}_1 = \mathbf{P}_{(2,1)}$. To integrate the automorphic function $\Lambda^T E(g, \Phi, \rho, \lambda)$ over $g \in \mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A})$, we note that we may – as we will – integrate instead over $\mathbf{H}^0(F)\backslash\mathbf{H}^0(\mathbb{A})$, where

$$\mathbf{H}^0 = r(2, 3)\mathbf{H}r(2, 3) = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} \right\}.$$

Indeed, $\Lambda^T E(r'gr, \Phi) = \Lambda^T E(g, \Phi^r)$ with $\Phi^r(g) = \Phi(gr)$ if $r' \in \mathbf{G}(F)$, since $\Lambda^T E$ is automorphic, and we may replace the orthonormal basis $\{\Phi\}$ by $\{\Phi^r\}$. We need a coset decomposition analogous to that of Proposition 5, with $\mathbf{H}^0$ replacing $\mathbf{H}$, and with respect to $\mathbf{B}$. Put

$$\varepsilon_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \varepsilon_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \varepsilon_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

13. PROPOSITION. *If* $\mathbf{G} = \mathrm{GL}(3)$ *and* $\mathbf{P}_1 = \mathbf{P}_{(2,1)}$, *then we have the disjoint union*

$$\begin{aligned} \mathbf{G} &= \mathbf{P}_1\mathbf{H}^0 \cup \mathbf{P}_1 r(23)\mathbf{H}^0 \cup \mathbf{P}_1\varepsilon_2 r(23)\mathbf{H}^0 \\ &= \mathbf{B}\mathbf{H}^0 \cup \mathbf{B}r(12)\mathbf{H}^0 \cup \mathbf{B}r(23)\mathbf{H}^0 \cup \mathbf{B}\varepsilon_1 r(12)\mathbf{H}^0 \\ &\cup \mathbf{B}\varepsilon_2 r(23)\mathbf{H}^0 \cup \mathbf{B}\varepsilon_3 r(23)\mathbf{H}^0. \end{aligned}$$

Consequently, if

$$\mathbf{B}^0 = \mathbf{B} \cap \mathbf{H} = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\},$$

then we have the disjoint coset decomposition

$$\begin{aligned} \mathbf{B}\backslash\mathbf{G} = \mathbf{B}^0\backslash\mathbf{H}^0 &\cup r(12)\cdot\mathbf{B}^0\backslash\mathbf{H}^0 \cup r(23)\cdot\mathbf{B}^0\backslash\mathbf{H}^0 \\ &\cup \varepsilon_1 r(12)\cdot \left\{ \begin{pmatrix} a & 0 & z \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \right\} \backslash\mathbf{H}^0 \cup \varepsilon_2 r(23)\cdot \left\{ \begin{pmatrix} a & 0 & z \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \right\} \backslash\mathbf{H}^0 \\ &\cup \varepsilon_3 r(23)\cdot \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \right\} \backslash\mathbf{H}^0. \end{aligned}$$

DEFINITION. Below we refer to the six cosets in the last decomposition as "the first coset", ..., "the sixth coset".

Proof. The homogeneous space $\mathbf{P}_1\backslash\mathbf{G}$ is isomorphic to the projective 3-space via the isomorphism $g\mapsto(0,0,1)g$. The orbit $(0,0,1)\mathbf{H}^0$ consists of the vectors (a,b,c) with $b=0$, that of $r(23)\mathbf{H}^0$ consists of (a,b,c) with $a=c=0$, and the orbit $(0,0,1)\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}r(23)\mathbf{H}^0$ consists of (a,b,c) with $b\neq 0$, and $a\neq 0$ or $c\neq 0$. When the first decomposition.

To deduce from it the second decomposition, recall the Bruhat decomposition

$$\mathbf{P}_1=\mathbf{B}\cup\mathbf{B}r(1,2)\mathbf{N}_1=\mathbf{B}\cup\mathbf{B}r(1,2)\cup\mathbf{B}\varepsilon_1 r(12)\mathbf{A},$$

where $\mathbf{A}$ is the diagonal subgroup and

$$\mathbf{N}_1=\left\{\begin{pmatrix}1&*&0\\0&1&0\\0&0&1\end{pmatrix}\right\}=\{a^{-1}\varepsilon_1 a;\ a\in\mathbf{A}\}.$$

Then

$$\mathbf{P}_1\mathbf{H}^0=\mathbf{B}\mathbf{H}^0\cup\mathbf{B}r(12)\mathbf{H}^0\cup\mathbf{B}\varepsilon_1 r(12)\mathbf{H}^0.$$

Moreover

$$\mathbf{P}_1 r(23)\mathbf{H}^0=\mathbf{B}r(23)\mathbf{H}^0,$$

since $r(23)r(12)r(23)\in\mathbf{H}^0$ and

$$\mathbf{B}\varepsilon_1 r(12)r(23)\mathbf{H}^0=\mathbf{B}r(23)\mathbf{H}^0.$$

Finally

$$\mathbf{P}_1\varepsilon_2 r(23)\mathbf{H}^0=\mathbf{B}\varepsilon_2 r(23)\mathbf{H}^0\cup\mathbf{B}\varepsilon_3 r(23)\mathbf{H}^0,$$

since

$$\mathbf{B}\varepsilon_1 r(12)\varepsilon_2 r(23)\mathbf{H}^0=\mathbf{B}\varepsilon_3\varepsilon_1 r(12)r(23)\mathbf{H}^0=\mathbf{B}\varepsilon_3 r(23)\mathbf{H}^0$$
$$(r(23)\varepsilon_1 r(23)\in\mathbf{H}^0).$$

To obtain the coset decomposition it suffices to note that

$$\mathbf{B}\backslash\mathbf{B}r\mathbf{H}^0 = r\cdot(r^{-1}\mathbf{B}r \cap \mathbf{H}^0\backslash\mathbf{H}^0).$$

The proposition follows.

REMARK. (1) Since

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \varepsilon_3 r(23) = \varepsilon_3 r(12) \begin{pmatrix} 0 & 0 & b \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

we have that $\mathbf{B}\varepsilon_3 r(23)\mathbf{H}^0 = \mathbf{B}\varepsilon_3 r(12)\mathbf{H}^0$. (2) It is easy to see that $\mathbf{B}\varepsilon_1 r(12)\mathbf{H}^0 = \mathbf{B}\varepsilon_1\mathbf{H}^0$, and that $\mathbf{B}\varepsilon_2 r(23)\mathbf{H}^0 = \mathbf{B}\varepsilon_2\mathbf{H}^0$.

Since the character $\rho = \rho_1 \times \rho_2 \times \rho_3$ is a cuspidal representation of the diagonal subgroup $\mathbf{A}(\mathbb{A}) = \mathbf{B}(\mathbb{A})/\mathbf{N}(\mathbb{A})$, Lemma 4.1 of [A2], p. 114, applies. It asserts, in our case, the following.

14. PROPOSITION. *The truncated Eisenstein series* $\Lambda^T E(h, \Phi, \rho, \lambda)$, *where* $\lambda \in \mathfrak{A}^*_{0,\mathbb{C}}$ *has real part* $\mathrm{Re}(\lambda)$ *in* $\rho_0 + (\mathfrak{A}^*_0)^+$, *and T is sufficiently large in the positive Weyl chamber* $\mathfrak{A}_0$, *is equal to*

$$\sum_{s\in W} \sum_{\gamma\in\mathbf{B}(F)\backslash\mathbf{G}(F)} \varepsilon_0(s\lambda)\phi_0(s\lambda, H_0(\gamma h) - T)e^{\langle s\lambda+\rho_0, H_0(\gamma h)\rangle}(M(s, \rho, \lambda)\Phi)(\gamma h). \qquad (14.1)$$

We may identify the two dimensional spaces $\mathfrak{A}_0$ and $\mathfrak{A}^*_0$ with the space of the vectors (x_1, x_2, x_3) in $\mathbb{R}^3$ with $x_1+x_2+x_3=0$. The simple roots are $\alpha_1=(1, -1, 0)$ and $\alpha_2=(0, 1, -1)$, and a dual basis is given by $\mu_1=(2/3, -1/3, -1/3)$, $\mu_2=(1/3, 1/3, -2/3)$ ($\langle\alpha_i, \mu_j\rangle = \delta_{ij}$). If $a = \mathrm{diag}(a_1, a_2, a_3)$ then $\alpha_1(a) = |a_1/a_2| = e^{\langle\alpha_1, H(a)\rangle}$ and $\alpha_2(a) = |a_2/a_3| = e^{\langle\alpha_2, H(a)\rangle}$, thus $H(a) = \ln|a_1/a_2|\mu_1 + \ln|a_2/a_3|\mu_3$. Hence if $a = \mathrm{diag}(x_1, x_2/x_1, x_2^{-1})$, then $H(a) = \ln|x_1|\alpha_1 + \ln|x_2|\alpha_2$. We shall also write $\lambda = \lambda\mu_1 + \lambda_2\mu_2$, and note that $\lambda \in (\mathfrak{A}^*_0)^+$ if $\lambda_1 > 0$ and $\lambda_2 > 0$. Recall that $\varepsilon_0(\Lambda)$ is defined for $\Lambda \in \mathfrak{A}^*_0$ in [A1], p. 940, to be 1 if $\langle\Lambda, \alpha^\vee\rangle \leqslant 0$ for an even number of $\alpha \in \Delta_0 = \{\alpha_1, \alpha_2\}$, and it is -1 otherwise. The function $\Phi_0(\Lambda, H)$ on $(\Lambda, h) \in \mathfrak{A}^*_0 \times \mathfrak{A}_0$ is defined there to be equal to 1 if $\langle\Lambda, \alpha_i^\vee\rangle \leqslant 0$ and $\langle\mu_i, H\rangle > 0$, or $\langle\Lambda, \alpha_i^\vee\rangle > 0$ and $\langle\mu_i, H\rangle \leqslant 0$, for both $i = 1, 2$; it is 0 otherwise.

First Coset

We are to consider the integral over $\mathbf{H}^0(F)\backslash\mathbf{H}^0(\mathbb{A})$ of the product by $\xi^{-1}(h)$ $(=\xi^{-1}((ad-bc)/e^2)$ if $h = \begin{pmatrix} a & 0 & b \\ 0 & e & 0 \\ c & 0 & d \end{pmatrix} \in \mathbf{H}^0(\mathbb{A}))$ of the expression displayed in

Proposition 14. Using the coset decomposition of Proposition 13, we consider first the coset $\mathbf{B}^0(F)\backslash\mathbf{H}^0(F)$. Applying the Iwasawa decomposition

$$\mathbf{H}^0(\mathbb{A}) = \mathbf{N}^0(\mathbb{A})\mathbf{A}^0(\mathbb{A})\mathbb{K}^0,$$

and noting the change of variables formula $dh = \delta^{-1}(a)dn\,da\,dk$, where $\delta(a) = e^{\langle\rho_0, H(a)\rangle}(=|a_1/a_3|$ if $a = (a_1, a_2, a_3)$, $\rho_0 = \alpha_1 + \alpha_2 = \mu_1 + \mu_2$), our integral takes the form

$$\sum_{s\in W} \varepsilon_0(s\lambda) \int_{\mathbf{A}(F)\backslash\mathbf{A}(\mathbb{A})} \phi_0(s\lambda, H(a) - T)e^{\langle s\lambda, H(a)\rangle}(M(s, \rho, \lambda)\Phi)_{\mathbb{K}^0,\xi}(a)\xi^{-1}(a)da, \tag{14.2}$$

where $\xi(\mathrm{diag}(a, b, c)) = \xi(ac/b^2)$. Write $T = \ln t_1\cdot\alpha_1 + \ln t_2\cdot\alpha_2$. Note that $(M(s, \rho, \lambda)\Phi)_{\mathbb{K}^0,\xi}(a) = \int_{\mathbb{K}^0}(M(s, \rho, \lambda)\Phi(ak)\xi^{-1}(k)dk$ is zero unless ${}^s\rho = \xi({}^s\rho(a) = \rho(s(a)))$ on $\mathbf{A}(\mathbb{A}) \cap \mathbb{K}^0$. We may choose ρ in its connected component with ${}^s\rho = \xi$ on $\mathbf{A}(\mathbb{A})$ if ${}^s\rho = \xi$ on $\mathbf{A}(\mathbb{A}) \cap \mathbb{K}^0$. Then $(M(s, \rho, \lambda)\Phi)_{\mathbb{K}^0,\xi}(a)\xi^{-1}(a)$ is independent of $a\in\mathbf{A}(\mathbb{A})$, and is equal to its value at $a = 1$. Also we write $\delta(\rho/\xi) = 1$ if $\rho = \xi$ on $\mathbf{A}(\mathbb{A}) \cap \mathbb{K}^0$, and $\delta(\rho/\xi) = 0$ otherwise.

The following table lists the various functions in the integral.

s	$s\lambda$	$\varepsilon_0(s\lambda)$	$\phi_0(s\lambda, H(a) - T)$	$e^{\langle s\lambda, H(a)\rangle}$
id	$\lambda_1\mu_1+\lambda_2\mu_2$	1	$\lvert x_1\rvert<t_1$, $\lvert x_2\rvert<t_2$	$\lvert x_1\rvert^{\lambda_1}\lvert x_2\rvert^{\lambda_2}$
$r(12)$	$-\lambda_1\mu_1+(\lambda_1+\lambda_2)\mu_2$	-1	$t_1<\lvert x_1\rvert$, $\lvert x_2\rvert<t_2$	$\lvert x_1\rvert^{-\lambda_1}\lvert x_2\rvert^{\lambda_1+\lambda_2}$
$r(23)$	$(\lambda_1+\lambda_2)\mu_1-\lambda_2\mu_2$	-1	$\lvert x_1\rvert<t_1$, $t_2<\lvert x_2\rvert$	$\lvert x_1\rvert^{\lambda_1+\lambda_2}\lvert x_2\rvert^{-\lambda_2}$
$r(23)r(12)$	$\lambda_2\mu_1-(\lambda_1+\lambda_2)\mu_2$	-1	$\lvert x_1\rvert<t_1$, $t_2<\lvert x_2\rvert$	$\lvert x_1\rvert^{\lambda_2}\lvert x_2\rvert^{-\lambda_1-\lambda_2}$
$r(12)r(23)$	$-(\lambda_1+\lambda_2)\mu_1+\lambda_1\mu_2$	-1	$t_1<\lvert x_1\rvert$, $\lvert x_2\rvert<t_2$	$\lvert x_1\rvert^{-\lambda_1-\lambda_2}\lvert x_2\rvert^{\lambda_1}$
$r(13)$	$-\lambda_2\mu_1-\lambda_1\mu_2$	1	$t_1<\lvert x_1\rvert$, $t_2<\lvert x_2\rvert$	$\lvert x_1\rvert^{-\lambda_2}\lvert x_2\rvert^{-\lambda_1}$

We shall label below by ((3)i) the various terms in the integral $\int\xi^{-1}\Lambda^T E$ to be substituted in (3.3), in our present case of PGL(3), where $n = 3$, and the character ρ of the minimal parabolic subgroup.

Since

$$\int_{|x|>t} |x|^{-\lambda}d^\times x = \int_{|x|<t^{-1}} |x|^{\lambda}d^\times x = \frac{t^{-\lambda}}{\lambda}, \quad \int_{|x|<t} |x|^{\lambda}d^\times x = \frac{t^{\lambda}}{\lambda},$$

our integral is equal to

$$\frac{t_1^{\lambda_1}t_2^{\lambda_2}}{\lambda_1\lambda_2}\,\Phi_{\mathbb{K}^0}(1)\delta(\xi/\rho)$$

$$+\frac{t_1^{-\lambda_1}}{\lambda_1}\,\frac{t_2^{\lambda_1+\lambda_2}}{\lambda_1+\lambda_2}\,(M(r(12), \rho, \lambda)\Phi)_{\mathbb{K}^0,\xi}(1)\delta(\xi/{}^{r(12)}\rho)$$

$$+\frac{t_1^{\lambda_1+\lambda_2}}{\lambda_1+\lambda_2}\frac{t_2^{-\lambda_2}}{\lambda_2}(M(r(23),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(1)\delta(\xi/^{r(23)}\rho)$$

$$+\frac{t_1^{\lambda_2}}{\lambda_2}\frac{t_2^{-\lambda_1-\lambda_2}}{\lambda_1+\lambda_2}(M(r(23)r(12),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(1)\delta(\xi/^{r(23)r(12)}\rho)$$

$$+\frac{t_1^{-\lambda_1-\lambda_2}}{\lambda_1+\lambda_2}\frac{t_2^{\lambda_1}}{\lambda_1}(M(r(12)r(23),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(1)\delta(\xi/^{r(12)r(23)}\rho)$$

$$+\frac{t_1^{-\lambda_2}}{\lambda_2}\frac{t_2^{-\lambda_1}}{\lambda_1}(M(r(13),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(1)\delta(\xi/^{r(13)}\rho). \tag{(3)1}$$

Second Coset

Next we consider the coset $r(12)\cdot\mathbf{B}^0(F)\backslash\mathbf{H}^0(F)$, and again integrate over $\mathbf{H}^0(F)/\mathbf{H}^0(\mathbb{A})$ the corresponding partial sum of (14.1), multiplied by $\xi^{-1}(h)$. Applying the Iwasawa decomposition $\mathbf{H}^0(\mathbb{A})=\mathbf{N}^0(\mathbb{A})\mathbf{A}(\mathbb{A})\mathbb{K}^0$, noting that $r(12)\mathbf{N}^0(\mathbb{A})r(12)$ consists of upper triangular unipotent matrices, and that $dh=e^{-\langle\rho_0,H_0(a)\rangle}dn\,da\,dk$, and making the change $a\mapsto r(12)ar(12)$ of variables on $\mathbf{A}(\mathbb{A})$, we obtain the integral

$$\sum_{s\in W}\varepsilon_0(s\lambda)\int_{\mathbf{A}(F)\backslash\mathbf{A}(\mathbb{A})}\phi_0(s\lambda,H(a)-T)e^{\langle s\lambda,H(a)\rangle}$$
$$\times(M(s,\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(12))\delta(^{r(12)}\xi/^{s}\rho)e^{\langle\rho_0-r(12)\rho_0,H(a)\rangle}da. \tag{14.3}$$

The argument used in the case of (13.2) implies that $(M(s,\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(ar(12))\xi^{-1}(r(12)ar(12))$ is zero unless $^{s}\rho={}^{r(12)}\xi$ on $\mathbf{A}(\mathbb{A})\cap\mathbb{K}^0$, but then we may choose ρ in its connected component to satisfy $^{s}\rho={}^{r(12)}\xi$ on $\mathbf{A}(\mathbb{A})$, and our function is independent of a. If $H(a)=\ln|x_1|\alpha_1+\ln|x_2|\alpha_2$, since $\rho_0-r(12)\rho_0=(1,\ 0,\ -1)-(0,\ 1,\ -1)=(1,\ -1,\ 0)$, the new factor in the integrand of (14.3) (as compared with that of (14.2)) is $|x_1|^2/|x_2|$.

The corresponding table for (14.3) is the same as for (14.2), except that the 6 entries in the last column are multiplied by $|x_1|^2/|x_2|$. Consequently (14.3) is equal to (put $r=r(12)$, $s=r(23)$, for brevity)

$$\frac{t_1^{\lambda_1+2}}{\lambda_1+2}\frac{t_2^{\lambda_2-1}}{\lambda_2-1}\Phi_{\mathbb{K}^0,\xi}(r(12))\delta(\rho/^{r}\xi)$$

$$-\frac{t_1^{2-\lambda_1}}{2-\lambda_1}\frac{t_2^{\lambda_1+\lambda_2-1}}{\lambda_1+\lambda_2-1}(M(r(12),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(12))\delta(^{r}\rho/^{r}\xi)$$

$$+\frac{t_1^{2+\lambda_1+\lambda_2}}{2+\lambda_1+\lambda_2}\frac{t_2^{-\lambda_2-1}}{\lambda_2+1}(M(r(23),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(12))\delta(^{s}\rho/^{r}\xi)$$

$$+\frac{t_1^{2+\lambda_2}}{2+\lambda_2}\frac{t_2^{-\lambda_1-\lambda_2-1}}{\lambda_1+\lambda_2+1}(M(r(23)r(12),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(12))\delta(^{sr}\rho/^{r}\xi)$$

$$-\frac{t_1^{2-\lambda_1-\lambda_2}}{2-\lambda_1-\lambda_2}\frac{t_2^{\lambda_1-1}}{\lambda_1-1}(M(r(12)r(23),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(12))\delta({}^{rs}\rho/{}^{r}\xi)$$

$$-\frac{t_1^{2-\lambda_2}}{2-\lambda_2}\frac{t_2^{-\lambda_1-1}}{\lambda_1+1}(M(r(13),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(12))\delta({}^{srs}\rho/{}^{r}\xi). \qquad ((3)2)$$

Third Coset

The subsum parametrized by $r(23)\cdot \mathbf{B}^0(F)\backslash\mathbf{H}^0(F)$ in (14.1), or rather its integral over $\mathbf{H}^0(F)\backslash\mathbf{H}^0(\mathbb{A})$, can be treated analogously. Applying again the Iwasawa decomposition and making the change $a\mapsto r(23)ar(23)$ of variables on $\mathbf{A}(\mathbb{A})$, an integral analogous to (14.2) and (14.3) is obtained, namely $r(12)$ has to be replaced by $r(23)$ in (14.3). Note that $\rho_0 - r(23)\rho_0 = (1, 0, -1) - (1, -1, 0) = (0, 1, -1)$, and $e^{\langle\rho_0 - r(23)\rho_0, H(a)\rangle} = |x_2^2/x_1|$. Hence the last column in the table for (14.2) has to be multiplied by $|x_2^2/x_1|$ to obtain the analogous table, for the coset $r(23)\cdot\mathbf{B}^0(F)\backslash\mathbf{H}^0(F)$. Integrating we obtain (put $r = r(23)$, $s = r(12)$, in the following expression)

$$\frac{t_1^{\lambda_1-1}}{\lambda_1-1}\frac{t_2^{\lambda_2+2}}{\lambda_2+2}\Phi_{\mathbb{K}^0,\xi}(r(23))\delta(\rho/{}^{r}\xi)$$

$$+\frac{t_1^{-\lambda_1-1}}{\lambda_1+1}\frac{t_2^{\lambda_1+\lambda_2+2}}{\lambda_1+\lambda_2+2}\cdot(M(r(12),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(23))\delta({}^{s}\rho/{}^{r}\xi)$$

$$-\frac{t_1^{\lambda_1+\lambda_2-1}}{\lambda_1+\lambda_2-1}\frac{t_2^{2-\lambda_2}}{2-\lambda_2}(M(r(23),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(23))\delta({}^{r}\rho/{}^{r}\xi)$$

$$-\frac{t_1^{\lambda_2-1}}{\lambda_2-1}\frac{t_2^{2-\lambda_1-\lambda_2}}{2-\lambda_1-\lambda_2}(M(r(23)r(12),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(23))\delta({}^{rs}\rho/{}^{r}\xi)$$

$$+\frac{t_1^{-\lambda_1-\lambda_2-1}}{\lambda_1+\lambda_2+1}\frac{t_2^{2+\lambda_1}}{2+\lambda_1}(M(r(12)r(23),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(23))\delta({}^{sr}\rho/{}^{r}\xi)$$

$$-\frac{t_1^{-\lambda_2-1}}{\lambda_2+1}\frac{t_2^{2-\lambda_1}}{2-\lambda_1}(M(r(13),\rho,\lambda)\Phi)_{\mathbb{K}^0,\xi}(r(23))\delta({}^{srs}\rho/{}^{r}\xi). \qquad ((3)3)$$

Fourth Coset

The next coset of $\mathbf{B}(F)\backslash\mathbf{G}(F)$ to be considered is

$$\varepsilon_1 r(12)\cdot\left\{\begin{pmatrix} a & 0 & z\\ 0 & a & 0\\ 0 & 0 & b\end{pmatrix}\right\}(F)\backslash\mathbf{H}^0(F).$$

The integral over $\mathbf{H}^0(F)\backslash\mathbf{H}^0(\mathbb{A})$ of the product by $\xi^{-1}(h)$ of the subsum parameterized by this coset in (14.1) is the sum over $s \in W$ of the integral over $u = b/s \in \mathbb{A}^\times$ and $w = c/b \in \mathbb{A}^\times/F^\times$, of the product with $\varepsilon_0(s\lambda)$ of

$$\phi_0(s\lambda, H(h) - T)(M\Phi)(h)e^{\langle s\lambda + \rho_0, H(h)\rangle}\xi(b^2/ac)|c/a|d^\times(a/b)d^\times(c/b),$$

where

$$(M\Phi)(g) = \int_{\mathbb{K}^0} (M(s, \rho, \lambda)\Phi)(gr(12)k(\xi^{-1}(k)dk,$$

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/u & 0 \\ 0 & 0 & w \end{pmatrix}.$$

As usual, the Iwasawa decomposition was used, and it was noted that ε_1 commutes with

$$r(12)\mathbf{N}^0 r(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $(M\Phi)(h)$ is zero unless ${}^s\rho_3(w) = {}^s\rho(\mathrm{diag}(1, 1, w))$ is equal to $\xi(w) = \xi(\mathrm{diag}(1, 1, w))$ on all $w \in \mathbb{A}^\times$ with $|w| = 1$. We may choose ρ in its connected component, when $\delta({}^s\rho_3/\xi) = 1$, such that ${}^s\rho_3 = \xi$ on $\mathbb{A}^\times$. With this choice, $(M\Phi)(h)\xi^{-1}(w)$ is independent of w. The integrand can therefore be expressed in the form

$$\delta({}^s\mu_3/\xi)\phi_0(s\lambda, H - T)(M\Phi)\left(\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/u & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) e^{\langle s\lambda + \rho_0, H\rangle}|uw|\xi(u)d^\times u d^\times w,$$

$$H = H(h).$$

Note that if $g = (g_v) \in \mathbf{G}(\mathbb{A})$ where $g_v = n_v a_v k_v \in \mathbf{G}(F_v)$, the $H(g) = \sum_v H_v(g_v)$, where $H_v(g_v)$ is defined to be $H_v(a_v)$, and $\ln|\chi(a_v)| = \langle \chi, H_v(a_v)\rangle$ for any $\chi \in X(\mathbf{A}) = \mathrm{Hom}(\mathbf{A}, \mathrm{GL}(1))$. For $x = (x_v)$, $y = (y_v)$ in $\mathbb{A}^\times$, put $|x| = \Pi_v |x_v|_v$, and $\|(x, y)\| = \Pi_v \|(x_v, y_v)\|_v$, where $\|(x_v, y_v)\|_v$ is $\max(|x_v|_v, |y_v|_v)$ in the non-archimedean case and $(|x_v|_v^2 + |y_v|_v^2)^{1/2}$ in the archimedean case. Also we write (x, y) for an element in $\mathbb{A}^\times$ with $|(x, y)| = \|(x, y)\|$. Then

$$H\left(\begin{pmatrix} 1 & 0 & 0 \\ 1 & u^{-1} & 0 \\ 0 & 0 & w \end{pmatrix}\right) = H(\mathrm{diag}((u(1, u^{-1}))^{-1}, (1, u^{-1}), w))$$

$$= -\tfrac{1}{3}\ln(|wu^2|\,\|(1, u^{-1})\|^3)\alpha_1 - \tfrac{1}{3}\ln(|uw^2|)\alpha_2$$
$$= -\tfrac{1}{3}\ln|w|(\alpha_1 + 2\alpha_2) - \tfrac{1}{3}\ln|u|(2a_1 + \alpha_2) - \ln\|(1, \mathrm{u}^{-1})\|\alpha_1$$
$$= -\ln|w|\cdot\mu_2 - \ln|u|\cdot\mu_1 - \ln\|(1, u^{-1})\|\cdot\alpha_1.$$

IV(1). We shall consider each of the summands indexed by $s \in W$. When $s = 1$, the characteristic function $\phi_0(\lambda, H - T)$ is non-zero when

$$\tfrac{1}{3}\ln|w| + \tfrac{2}{3}\ln|u| + \ln\|(1, u^{-1})\| \geqslant -\ln t_1,$$
$$\text{or } \ln|w| \geqslant A = -3\ln t_1 - 2\ln|u| - 3\ln\|(1, u^{-1})\|,$$

and

$$\tfrac{2}{3}\ln|w| + \tfrac{1}{3}\ln|u| \geqslant -\ln t_2, \quad \text{or } \ln|w| \geqslant B = -\tfrac{3}{2}\ln t_2 - \tfrac{1}{2}\ln|u|.$$

Note that

$$-\ln|u_v(1, u_v^{-1})^2|_v = \begin{cases} -\ln|u_v|_v \leqslant 0, & \text{if } |u_v|_v \geqslant 1, \\ \ln|u_v|_v \leqslant 0, & \text{if } |u_v|_v \leqslant 1, \end{cases}$$

is always non-positive, and consequently so is

$$-\ln|u| - 2\ln\|(1, u^{-1})\| = -\sum_v \ln|u_v(1, u_v^{-1})^2|_v,$$

which is therefore less than $\ln(t_1^2/t_2)$ if we choose t_1 and t_2 with $t_2 < t_1^2$ (later we also require that $t_1 < t_2^2$). It follows that $B > A$, namely the integral ranges, when $s = 1$, over the $u \in \mathbb{A}^\times$ and $w \in \mathbb{A}^\times/F^\times$ with $|w|^{-1} \leqslant t_2^{3/2}|u|^{1/2}$.

On the domain of integration, the integrand is the product of

$$\begin{aligned} |uw|\exp(\langle\lambda + \rho_0, H\rangle) &= |uw|\exp\{-(1 + \lambda_1)\ln(|wu^2|\,\|(1, u^{-1})\|^3)/3 \\ &\quad -(1 + \lambda_2)\ln|u^2w|/3\} \\ &= (\|(1, u^{-1})\|^3|u^2w|)^{-(1+\lambda_1)/3}|uw^2|^{-(1+\lambda_2)/3}|uw| \\ &= \|(1, u^{-1})\|^{-(1+\lambda_1)}|u|^{(-2\lambda_1+\lambda_2)/3}|w|^{-(\lambda_1+2\lambda_2)/3} \end{aligned}$$

and

$$(M\Phi)\left(\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/u & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\xi(u)\delta({}^s\mu_3/\xi).$$

Integrating with respect to $w \in \mathbb{A}^\times/F^\times$ on $|w|^{-1} \leqslant |u|^{1/2}t_2^{3/2}$, we obtain

$$\|(1, u^{-1})\|^{-(1+\lambda_1)}|u|^{-(2\lambda_1+\lambda_2)/3}(|u|^{1/2}t_2^{3/2})^{(\lambda_1+2\lambda_2)/3}/(\tfrac{1}{3}\lambda_1 + \tfrac{2}{3}\lambda_2)$$
$$= \|(1, u^{-1})\|^{-(1+\lambda_1)}|u|^{-\lambda_1/2}t_2^{\lambda_2+\lambda_1/2}/\tfrac{1}{3}(2\lambda_2 + \lambda_1).$$

We need to integrate this over $u = (u_v)$ in $\mathbb{A}^\times$. Note that ρ_v is unramified and that $(M\Phi)(g)$ is right-$GL(3, R_v)$ and left-$\mathbf{A}(F_v)$ invariant, for almost all v. When ${}^s\rho_3 = \xi$ we have that $\xi(u)(M\Phi)\left(\begin{pmatrix}1 & 0 & 0\\ 1 & 1/u & 0\\ 0 & 0 & 1\end{pmatrix}\right)$, as a function in $u_v \in F_v^\times$, is a multiple of ${}^s\rho_v^-(u_v) = {}^s\rho_v(\mathrm{diag}(1, 1/u_v, 1))\xi_v(u_v) = {}^s\rho_v(\mathrm{diag}(1, 1/u_v, u_v))$ if $|u_v|_v \leqslant 1$, and of ${}^s\rho_v^+(u_v) = {}^s\rho_v(\mathrm{diag}(1/u_v, 1, 1))\xi_v(u_v) = {}^s\rho_v(\mathrm{diag}(1/u_v, 1, u_v))$ if $|u_v|_v \geqslant 1$, for almost all v. Note that in the non-archimedean case we have

$$|(1, u_v^{-1})|_v^{-(1+\lambda_1/2)}|u_v|_v^{-\lambda_1/2} = \begin{cases}|u_v|_v^{1+\lambda_1/2} & \text{if } |u_v|_v \leqslant 1,\\ |u_v|_v^{-\lambda_1/2} & \text{if } |u_v|_v \geqslant 1.\end{cases}$$

Hence the integral of the local factor over $F_v^\times$ against $d^\times u_v$ is equal – in the non-archimedean case – to

$$\int_{|u|_v \leqslant 1} (M\Phi)_v \left(\begin{pmatrix}1 & 0 & 0\\ 1 & 1/u & 0\\ 0 & 0 & 1\end{pmatrix}\right) |u|_v^{1+\lambda_1/2}\xi_v(u)d^\times u$$
$$+ \int_{|u|_v > 1} (M\Phi)_v \left(\begin{pmatrix}1 & 0 & 0\\ 1 & 1/u & 0\\ 0 & 0 & 1\end{pmatrix}\right) |u|_v^{-\lambda_1/2}\xi_v(u)d^\times u.$$

At almost all v we put $x = {}^s\rho_v^-(\pi_v)$ and $y = {}^s\rho_v^+(\pi_v)$, where π_v is a uniformizing parameter in F_v. We obtain

$$\sum_0^\infty q_v^{-n(1+\lambda_1/2)}x^n + \sum_1^\infty q_v^{-n\lambda_1/2}y^{-n}$$
$$= (1 - q_v^{-1-\lambda_1/2}x)^{-1} + (1 - q_v^{-\lambda_1/2}/y)^{-1} - 1$$
$$= (1 - xy^{-1}q_v^{-1-\lambda_1})/(1 - q_v^{-\lambda_1/2}/y)(1 - xq_v^{-1-\lambda_1/2})$$
$$= L_v(\lambda_1/2, ({}^s\rho_v^+)^{-1})L_v(1 + \lambda_1/2, {}^s\rho_v^-)/L(1 + \lambda_1, {}^s\rho_v^-/{}^s\rho_v^+).$$

At the remaining finite number of places we obtain a multiple of this product of L-factors by a polynomial in $q_v^{\lambda_1/2}$, or a holomorphic function in λ_1 in the archimedean case. Denote the product over v by

$$L_{M\Phi}(\lambda_1/2, ({}^s\rho^+)^{-1})L_{M\Phi}(1+\lambda_1/2, {}^s\rho^-)/L_{M\Phi}(1+\lambda_1, {}^s\rho^-/{}^s\rho^+)$$
$$=\varepsilon(\lambda_1/2, ({}^s\rho^+)^{-1})L_{M\Phi}(1-\lambda_1/2, {}^s\rho^+)$$
$$\times L_{M\Phi}(1+\lambda_1/2, {}^s\rho^-)/L_{M\Phi}(1+\lambda_1, {}^s\rho^-/{}^s\rho^+). \qquad ((3)4.1)$$

This quotient has a simple pole on the line $\lambda_1 \in i\mathbb{R}$ if ${}^s\rho^-$ (or ${}^s\rho^+$) factorizes through $v(x) = |x|$; it is holomorphic, of polynomial grown as $\lambda_1 \in i\mathbb{R}$, $|\lambda_1| \to \infty$. When the pole exists we may choose ρ in its connected component to satisfy ${}^s\rho^- = 1$ (or ${}^s\rho^+ = 1$). In this case the pole occurs at $\lambda_1 = 0$ (in the number field case, and at $\lambda_1 \in i\mathbb{Z}/\log q$ in the function field case). The result of our computation is of course the product of ((3)4.1) with $t_2^{\lambda_2+\lambda_1/2}/((2\lambda_2+\lambda_1)/3)$.

IV(5). The next summand is that of $s = r(12)$, when $s\lambda = -\lambda_1\mu_1 + (\lambda_1+\lambda_2)\mu_2$. The characteristic function $\phi_0(s\lambda, H-T)$ is 0 unless $\ln|w| \leqslant A$ and $\ln|w| \geqslant B$. But $B > A$ hence the integrand is always zero.

IV(6). Similarly, when $s = r(12)r(23)$, so $s\lambda = -(\lambda_1+\lambda_2)\mu_1 + \lambda_1\mu_2$, the characteristic function vanishes unless $\ln|w| \leqslant A < B \leqslant \ln|w|$, and the integrand is always zero.

The remaining three cases of s are analogously treated. To simplify the notations we consider only the case where $\rho = 1$ and $\xi = 1$. The key ingredients of the computations would then be seen, and the general case can be treated as in the case of $s = 1$ above, with additional notational effort only.

IV(2). When $s = r(23)$, then $s\lambda = (\lambda_1+\lambda_2)\mu_1 - \lambda_2\mu_2$, and the characteristic function is zero unless $\ln|w| \geqslant A$ and $\ln|w| \leqslant B$, namely the integral ranges over the u, w with

$$|u|^{1/2}t_2^{3/2} \leqslant |w|^{-1} \leqslant t_1^3|u|^2\|(1, u^{-1})\|^3.$$

Since

$$e^{\langle s\lambda+\rho_0, H\rangle} = \|(1, u^{-1})\|^{-(1+\lambda_1+\lambda_2)}|wu^2|^{-(1+\lambda_1+\lambda_2)/3}|uw^2|^{-(1-\lambda_2)/3},$$

the integral over $w \in \mathbb{A}^\times/F^\times$ in the designed domain of the product of this with $|uw|$ is

$$\|(1, u^{-1})\|^{-(1+\lambda_1+\lambda_2)}|u|^{-(2\lambda_1+\lambda_2)/3}$$
$$\times[(t_1^3|u|^2\|(1, u^{-1})\|^3)^{(\lambda_1-\lambda_2)/3} - (|u|^{1/2}t_2^{3/2})^{(\lambda_1-\lambda_2)/3}]/((\lambda_1-\lambda_2)/3).$$

The integral of this over u in $\mathbb{A}^\times$ with respect to $d^\times u$ is

$$[t_1^{\lambda_1-\lambda_2}\varepsilon(\lambda_2)L_{M\Phi}(1-\lambda_2)L_{M\Phi}(1+\lambda_2)/L_{M\Phi}(1+2\lambda_2)$$
$$-t_2^{(\lambda_1-\lambda_2)/2}\varepsilon((\lambda_1+\lambda_2)/2)L_{M\Phi}(1-(\lambda_1+\lambda_2)/2)$$
$$\times L_{M\Phi}(1+(\lambda_1+\lambda_2)/2)/L_{M\Phi}(1+\lambda_1+\lambda_2)]/((\lambda_1-\lambda_2)/3); \qquad ((3)4.2)$$

the computation is carried out as in the case where $s = 1$.

IV(3). When $s = r(23)r(12)$ and $s\lambda = \lambda_2\mu_1 - (\lambda_1 + \lambda_2)\mu_2$, the characteristic function specifies the same domain of $|w|$ as in the previous case of $s = r(23)$, and

$$e^{\langle s\lambda+\rho_0,H\rangle} = |wu^2|^{-(1+\lambda_2)/3}\|(1,u^{-1})\|^{-(1+\lambda_2)}|uw^2|^{(\lambda_1+\lambda_2-1)/3}$$
$$= \|(1,u^{-1})\|^{-(1+\lambda_2)}|u|^{(\lambda_1-\lambda_2)/3}|w|^{(2\lambda_1+\lambda_2)/3}|uw|^{-1}.$$

Multiplying this by $|uw|$, and integrating over $w \in \mathbb{A}^\times/F^\times$ in the domain specified by the non-vanishing of the characteristic function, we obtain

$$\|(1,u^{-1})\|^{-(1+\lambda_2)}|u|^{(\lambda_1-\lambda_2)/3}[(t_2^{1/2}|u|^{1/6})^{-2\lambda_1-\lambda_2}$$
$$-(t_1|u|^{2/3}\|(1,u^{-1})\|)^{-2\lambda_1-\lambda_2}]/((2\lambda_1+\lambda_2)/3).$$

The integral of this over $u \in \mathbb{A}^\times$ against $d^\times u$ is equal to

$$[t_2^{-(\lambda_1+\lambda_2/2)}\varepsilon(\lambda_2/2)L_{M\Phi}(1-\lambda_2/2)L_{M\Phi}(1+\lambda_2/2)/L_{M\Phi}(1+\lambda_2)$$
$$-t_1^{-2\lambda_1-\lambda_2}\varepsilon(\lambda_1+\lambda_2)L_{M\Phi}(1-\lambda_1-\lambda_2)$$
$$\times L_{M\Phi}(1+\lambda_1+\lambda_2)/L_{M\Phi}(1+2\lambda_1+2\lambda_2)]/((2\lambda_1+\lambda_2)/3). \qquad ((3)4.3)$$

IV(4). When $s = r(13)$ and $s\lambda = -\lambda_2\mu_1 - \lambda_1\mu_2$, the characteristic function $\phi_0(s\lambda, H - T)$ vanishes unless $\ln|w| \leqslant A$ and $\ln|w| \leqslant B$; but $A < B$, hence the support is specified by

$$|w| \leqslant t_1^{-3}|u|^{-2}\|(1,u^{-1})\|^{-3}.$$

Also

$$e^{\langle s\lambda+\rho_0,H\rangle} = \|(1,u^{-1})\|^{-(1-\lambda_2)}|wu^2|^{-(1-\lambda_2)/3}|uw^2|^{-(1-\lambda_1)/3}$$
$$= \|(1,u^{-1})\|^{-(1-\lambda_2)}|u|^{(\lambda_1+2\lambda_2)/3}|w|^{(2\lambda_1+\lambda_2)/3}|uw|^{-1}.$$

The integral over $w \in \mathbb{A}^\times/F^\times$ (on the specified domain) of the product of this with $|uw|$ is equal to

$$\|(1,u^{-1})\|^{-(1-\lambda_2)}|u|^{(\lambda_1+2\lambda_2)/3}(t_1^{-3}|u|^{-2}\|(1,u^{-1})\|^{-3})^{(2\lambda_1+\lambda_2)/3}/((2\lambda_1+\lambda_2)/3).$$

The integral of this on $u \in \mathbb{A}^\times$, by $d^\times u$, is

$$t_1^{-2\lambda_1-\lambda_2}((2\lambda_1+\lambda_2)/3)^{-1}\varepsilon(\lambda_1)L_{M\Phi}(1-\lambda_1)L_{M\Phi}(1+\lambda_1)/L_{M\Phi}(1+2\lambda_1) \quad ((3)4.4)$$

Fifth Coset

The coset $\varepsilon_2 r(23). \left\{\begin{pmatrix} a & 0 & z \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}\right\} (F)\backslash \mathbf{H}^0(F)$ is treated analogously. Carrying out the computation we would obtain terms ((3)5.i), $1 \leqslant i \leqslant 4$, analogous to ((3)4.1).

Sixth Coset

The remaining subsum of (14.1) to be considered ranges over the coset

$$\varepsilon_3 r(23)\cdot \left\{\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}\right\} (F)\backslash \mathbf{H}^0(F) \quad \text{in } \mathbf{B}(F)\backslash \mathbf{G}(F).$$

The integral over $\mathbf{H}^0(F)\backslash \mathbf{H}^0(\mathbb{A})$ can be expressed – on using the Iwasawa decomposition $\mathbf{H}^0(\mathbb{A}) = \mathbf{N}^0(\mathbb{A})\mathbf{A}(\mathbb{A})\mathbb{K}^0$ – as the sum over $s \in W$ of the product with $\varepsilon_0(s\lambda)$ of the integral of

$$\phi_0(s\lambda, H-T)e^{\langle s\lambda+\rho_0, H\rangle}|a/c|^{-1}(M\Phi)(h)\xi^{-1}(ac/b^2)dz\, d^\times u d^\times v,$$

where

$$H = H(h),\ h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & z & 1 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & b \end{pmatrix} = \begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ u & z & 1 \end{pmatrix},$$

$$(M\Phi)(g) = \int_{\mathbb{K}^0} (M(s,\rho,\lambda)\Phi)(gr(23)k)\xi^{-1}(k)\,dk,$$

over

$$z \in \mathbb{A}, \quad u = a/b \in \mathbb{A}^\times, \quad v = c/b \in \mathbb{A}^\times/F^\times.$$

For $u = (u_v) \in \mathbb{A}$, $z = (z_v) \in \mathbb{A}$, we let $|(1, z_v, u_v)|_v$ be $\max(1, |z_v|_v, |u_v|_v)$ if v is non-archimedean, and $(1 + |z_v|_v^2 + |u_v|_v^2)^{1/2}$ otherwise, put $\|(1, z, u)\|$ for $\Pi_v \|(1, |z_v|_v,$

$|u_v|_v)\|$, and put $(1, z_v, u_v)$ for an element of F_v with absolute value $|(1, z_v, u_v)|_v$, and $(1, z, u)$ for $((1, z_v, u_v)) \in \mathbb{A}^\times$. Then

$$\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ u & z & 1 \end{pmatrix} \in \mathbf{N}(\mathbb{A})\,\mathrm{diag}(u/(1, u),\ v(1, u)/(1, z, u),\ (1, z, u))\mathbb{K},$$

where $\mathbb{K} = \Pi_v K_v$, and K_v is the standard maximal compact subgroup of G_v. Changing variables $z \mapsto z/v$, noting that $\mathbf{G}$ is a projective group, H becomes

$$H\left(\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ u & z & 1 \end{pmatrix}\right) = (\tfrac{1}{3}\ln|u^2/v| - \ln\|(1, u)\|)\alpha_1 + (\tfrac{1}{3}\ln|uv| - \ln\|(1, z, u)\|)\alpha_2.$$

VI(1). We shall consider separately each of the six terms indexed by $s \in W$, with $s = 1$ treated now. As $\lambda = \lambda_1\mu_1 + \lambda_2\mu_2$ with $\lambda_i > 0$, the characteristic function $\phi_0(\lambda, H - T)$ is supported on the set determined by $\langle H - T, \mu_i\rangle < 0$ $(i = 1, 2)$, namely on the u, v, z with

$$|u^2/v|^{1/3}/\|(1, u)\| < t_1, \quad |uv|^{1/3}/\|(1, u, z)\| < t_2,$$

or equivalently

$$t_1^{-3}|u|^2/\|(1, u)\|^3 \leqslant |v| \leqslant t_2^3|u|^{-1}\|(1, u, z)\|^3.$$

The integrand is the product by

$$\xi^{-1}(uv)(M\Phi)\left(\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ u & z & 1 \end{pmatrix}\right) \tag{$*$}$$

$$= \xi^{-1}(uv)^s\rho(\mathrm{diag}(u/(1, u),\ v(1, u)/(1, z, u),\ (1, z, u)))(M\Phi)(k(u, z)),$$

where $k(u, z) \in \mathbb{K}$ is independent of v, of (recall that $c/a = v/u$, and that the change $z \mapsto z/v$ added a factor $|v|^{-1}$)

$$\begin{aligned} e^{\langle s\lambda + \rho_0, H\rangle}|u|^{-1} &= (|u^2/v|/\|(1, u)\|^3)^{(1+\lambda_1)/3}(|uv|/\|(1, z, u)\|^3)^{(1+\lambda_2)/3}|u|^{-1} \\ &= (|u^2/v|/\|(1, u)\|^3)^{\lambda_1/3}(|uv|/\|(1, z, u)\|^3)^{\lambda_2/3}\|(1, z, u)\|^{-1}\|(1, u)\|^{-1} \\ &= |v|^{(\lambda_2 - \lambda_1)/3}|u|^{(2\lambda_1 + \lambda_2)/3}\|(1, z, u)\|^{-1-\lambda_2}\|(1, u)\|^{-1-\lambda_1}. \end{aligned}$$

The integral of this product over v in $\mathbb{A}^0/F^\times$, $\mathbb{A}^0 = \{a \in \mathfrak{A}^\times;\ |a| = 1\}$, is a multiple of $\delta({}^s\rho_2/\xi)$, where ${}^s\rho_2(v) = {}^s\rho(\mathrm{diag}(1, v, 1))$. If ${}^s\rho_2 = \xi$ on $\mathbb{A}^0$ we may choose ρ in its connected component to have ${}^s\rho_2 = \xi$ on $\mathbb{A}^\times$. Then (*) is independent of v.

Integrating against $d^\times v$ over v in $\mathbb{A}^\times/F^\times$ we obtain

$$|u|^{(2\lambda_1+\lambda_2)/3}\|(1, u)\|^{-1-\lambda_1}\|(1, z, u)\|^{-1-\lambda_2}$$
$$\times [(t_2|u|^{-1/3}\|(1, u, z)\|)^{\lambda_2-\lambda_1} - (t_1^{-1}|u|^{2/3}/\|(1, u)\|)^{\lambda_2-\lambda_1}]/((\lambda_2 - \lambda_1)/3).$$

This is

$$[t_2^{\lambda_2-\lambda_1}|u|^{\lambda_1}(\|(1, u)\|\,\|(1, z, u)\|)^{-1-\lambda_1}$$
$$- t_1^{\lambda_1-\lambda_2}|u|^{\lambda_2}(\|(1, u)\|\,\|(1, z, u)\|)^{-1-\lambda_2}]/((\lambda_2 - \lambda_1)/3).$$

Each term in this difference, multiplied by

$$\delta({}^s\rho_2/\xi)(M\Phi)\left(\begin{pmatrix} u & 0 & 0 \\ 0 & 1 & 0 \\ u & z & 1 \end{pmatrix}\right)\xi(u)^{-1},$$

has to be integrated over z in $\mathbb{A}$ (against dz) and over u in $\mathbb{A}^\times$, against $d^\times u$. These global integrals are products of local integrals. We shall now compute these local integrals for almost all v, where $(M\Phi)_v$ is K_v-invariant (and ρ_v is unramified and v is non-archimedean). We first integrate the first summand against dz, to obtain

$$\int |u|_v^{\lambda_1}(\|(1, u)\|_v\|(1, z, u)\|_v)^{-1-\lambda_1 s}$$
$$\times \rho(\mathrm{diag}(u/(1, u), (1, u)/u(1, z, u), (1, z, u)))dz$$
$$= {}^s\rho(\mathrm{diag}(u/(1, u), 1/u, (1, u))) \int_{|z|_v \leq \|(1,u)\|_v} |u|_v^{\lambda_1}\|(1, u)\|_v^{-2-2\lambda_1}dz$$
$$+ {}^s\rho(\mathrm{diag}(u/(1, u), (1, u)/u, 1)) \int_{|z|_v > \|(1,u)\|v} |u|_v^{\lambda_1}\|(1, u)\|_v^{-1-\lambda_1}|z|_v^{-1-\lambda_1}\mu(z)dz$$
$$= {}^s\rho(\mathrm{diag}(u/(1, u), 1/u, (1, u)))|u|_v^{\lambda_1}\|(1, u)\|_v^{-1-2\lambda_1}$$
$$\times (1 - q_v^{-1-\lambda_1}/\mu(\pi_v))/(1 - q_v^{-\lambda_1}/\mu(\pi_v))$$

where $\mu(z) = {}^s\rho(\mathrm{diag}(1, 1/z, z))$, since

$$\int_{|z|_v \leq |a|} dz = |a|$$

and

$$\int_{|z|_v > q_v^r} |z|_v^t \mu(z) dz = (1 - q_v^{-1}) \sum_{n=r+1}^{\infty} (\mu(\pi_v) q_v^{(t-1)})^{-n}$$
$$= (1 - q_v^{-1})(\mu(\pi_v) q_v^{t-1})^{-(r+1)}/(1 - q_v^{-(t-1)} \mu(\pi_v)^{-1}).$$

Put $\mu_1(u) = {}^s\rho(\mathrm{diag}(u, 1/u, 1))$ and $x = \mu_1(\pi_v)$, and $y = \mu(\pi_v)$. The integration over $u \in F_v^\times$ of the product by $\mu_1(u)$ if $|u|_v \leqslant 1$ and by $\mu(u)$ of $|u|_v > 1$, of $|u|_v^{\lambda_1} \|(1, u)\|_v^{-1-2\lambda_1} d^\times u$, has been carried out above as part of the discussion of other cosets in $\mathbf{B}(F) \backslash \mathbf{G}(F)$. Thus integrating over u we obtain

$$(L_v(\lambda_1/2, \mu^{-1}) L_v(1 + \lambda_1/2, \mu_1)/L_v(1 + \lambda_1, \mu_1/\mu))$$
$$\times (L_v(\lambda_1, \mu^{-1})/L_v(1 + \lambda_1, \mu^{-1})).$$

The computation of the remaining finite number of ramified factors is similarly yielding such factors, which depend however on $(M\Phi)v\left(\begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ u & z & 1 \end{pmatrix}\right)$. The product over all v of these factors is equal to

$$\frac{L_{M\Phi}(\lambda_1/2, \mu^{-1}) L_{M\Phi}(1 + \lambda_1/2, \mu_1) L_{M\Phi}(\lambda_1, \mu^{-1})}{L_{M\Phi}(1 + \lambda_1, \mu_1/\mu) L_{M\Phi}(1 + \lambda_1, \mu^{-1})}$$
$$= \frac{\varepsilon(\lambda_1/2, \mu^{-1}) \varepsilon(\lambda_1, \mu^{-1}) L_{M\Phi}(1 - \lambda_1/2, \mu^{-1}) L_{M\Phi}(1 - \lambda_1, \mu^{-1}) L_{M\Phi}(1 + \lambda_1/2, \mu_1)}{L_{M\Phi}(1 + \lambda_1, \mu^{-1}) L_{M\Phi}(1 + \lambda_1, \mu_1/\mu)}$$

((3)6.1)

This product of L-functions is holomorphic on $\lambda_1 \in i\mathbb{R}$, unless μ or μ_1 factorizes through $u \mapsto |u|$. In this case we may choose ρ in its connected component to have $\mu = 1$ or $\mu_1 = 1$. Then the product of the L-function has a simple pole at $\lambda_1 = 0$ ($\lambda_1 \in i\mathbb{Z}/\log q$ in the function field case), and has a polynomial growth in λ_1 as $|\lambda_1| \to \infty$. The integration of the term subtracted in the difference is identical, except that λ_1 and λ_2 have to be interchanged.

Since the presence of the characters ρ and ξ considerably complicates the notations, and the general case of any ρ and ξ has just been treated in the case of $s = 1$, to simplify the notations in the remaining cases of $s \neq 1$ we restrict our attention only to the case of $\rho = 1 = \xi$. Clearly the general case similarly follows.

VI(2). Next we consider $s = r(12)$ in W. Then $s\lambda = -\lambda_1\mu_1 + (\lambda_1 + \lambda_2)\mu_2$, the characteristic function $\phi_0(s\lambda, H - T)$ is 1 when

$$|v| \leqslant t_1^{-3} |u|^2 / \|(1, u)\|^3 \qquad (\text{and } |v| \leqslant t_2^3 |u|^{-1} \|(1, u, z)\|^3,$$

but this last inequality is implied by the first inequality), and

$$e^{\langle s\lambda+\rho_0,H\rangle}|u|^{-1} = (|u^2/v|/\|(1, u)\|^3)^{(1-\lambda_1)/3}(\|uv|/\|(1, z, u)\|^3)^{(1+\lambda_1+\lambda_2)/3}|u|^{-1}$$
$$= |u|^{(\lambda_2-\lambda_1)/3}|v|^{(2\lambda_1+\lambda_2)/3}\|(1, u)\|^{\lambda_1-1}\|(1, z, u)\|^{-1-\lambda_1-\lambda_2}.$$

The integral of the product of this by $d^\times v$ over $v \in \mathbb{A}^\times/F^\times$ in the specified domain is

$$|u|^{\lambda_1+\lambda_2}\|(1, u)\|^{-1-\lambda_1-\lambda_2}\|(1, u, z)\|^{-1-\lambda_1-\lambda_2}t_1^{-(2\lambda_1+\lambda_2)}/((2\lambda_1+\lambda_2)/3).$$

The integral of this over $z \in \mathbb{A}$ and $u \in \mathbb{A}^\times$ is the same as in the previous case where $s = 1$, with λ_1 (there) replaced by $\lambda_1 + \lambda_2$ (here). We then obtain

$$\varepsilon((\lambda_1 + \lambda_2)/2)\varepsilon(\lambda_1 + \lambda_2)L_{M\Phi}(1 - (\lambda_1 + \lambda_2)/2)L_{M\Phi}(1-\lambda_1-\lambda_2)$$
$$\times L_{M\Phi}(1+(\lambda_1+\lambda_2)/2)L_{M\Phi}(1+\lambda_1+\lambda_2)^{-2}t_1^{-2\lambda_1-\lambda_2}/((2\lambda_1+\lambda_2)/3).$$

((3)6.2)

VI(3). When $s = r(23)$, then $s\lambda = (\lambda_1 + \lambda_2)\mu_1 - \lambda_2\mu_2$, we have

$\phi_0(s\lambda, H - T) \neq 0$ when

$$|v| \geqslant t_2^3|u|^{-1}\|(1, u, z)\|^3(> t_1^{-3}|u|^2/\|(1, u)\|^3),$$

and

$$e^{\langle s\lambda+\rho_0,H\rangle}|u|^{-1} = (|u^2/v|/\|(1, u)\|^3)^{(1+\lambda_1+\lambda_2)/3}(|uv|/\|(1, z, u)\|^3)^{(1-\lambda_2)/3}|u|^{-1}$$
$$= |u|^{(2\lambda_1+\lambda_2)/3}|v|^{-(\lambda_1+2\lambda_2)/3}\|(1, u)\|^{-1-\lambda_1-\lambda_2}\|(1, z, u)\|^{-1-\lambda_2}.$$

Integrating against $d^\times v$ on $\mathbb{A}^\times/F^\times$, obtained is $t_2^{-\lambda_1-2\lambda_2}/((\lambda_1 + 2\lambda_2)/3)$ times

$$|u|^{\lambda_1+\lambda_2}\|(1, u)\|^{-1-\lambda_1-\lambda_2}\|(1, z, u)\|^{-1-\lambda_1-\lambda_2}.$$

This factor, and its integral over $z \in \mathbb{A}$ and $u \in \mathbb{A}^\times$, is identical to the corresponding factor and its integral in the previous case when $s = r(12)$. The result of this computation will take the label ((3)6.3).

VI(4). When $s = r(23)r(12)$, $s\lambda = \lambda_2\mu_1 - (\lambda_1 + \lambda_2)\mu_2$, and $\phi_0(s\lambda, H - T) \neq 0$ on $|v| \geqslant t_2^3|u|^{-1}\|(1, u, z)\|^3$. The integrand contains the term

$$e^{\langle s\lambda+\rho_0,H\rangle}|u|^{-1} = (|u^2/v|/\|(1, u)\|^3)^{(1+\lambda_1)/3}(|uv|/\|(1, u, z)\|^3)^{(1-\lambda_1-\lambda_2)/3}|u|^{-1}$$
$$= |u|^{(\lambda_2-\lambda_1)/3}|v|^{-(\lambda_1+2\lambda_2)/3}\|(1, u)\|^{-1-\lambda_2}\|(1, u, z)\|^{-1+\lambda_1+\lambda_2}.$$

The integral of this (times $d^\times v$) over $v \in \mathbb{A}^\times/F^\times$ is $t_2^{-(\lambda_1+2\lambda_2)}/((\lambda_1+2\lambda_2)/3)$ times

$$|u|^{\lambda_2}\|(1, u)\|^{-1-\lambda_2}\|(1, u, z)\|^{-1-\lambda_2}.$$

The expression (and its integral over $z \in \mathbb{A}$, $u \in \mathbb{A}^\times$) has already appeared in the subtracted term in the difference associated with $s = 1$. In any case, the result of this computation would be labeled ((3)6.4).

VI(5). When $s = r(12)r(23)$, we have

$$s\lambda = -(\lambda_1+\lambda_2)\mu_1 + \lambda_1\mu_2, \quad \text{and} \quad \phi_0(s\lambda, H-T)$$

is 1 when $|v| \leqslant t_1^{-3}|u|^2/\|(1, u)\|^3$. In the integrand we find

$$e^{\langle s\lambda+\rho_0, H\rangle}|u|^{-1} = (|u^2/v|/\|(1, u)\|^3)^{(1-\lambda_1-\lambda_2)/3}(|uv|/\|(1, u, z)\|^3)^{(1+\lambda_1)/3}|u|^{-1}$$
$$= |v|^{(2\lambda_1+\lambda_2)/3}|u|^{-(\lambda_1+2\lambda_2)/3}\|(1, u)\|^{-1+\lambda_1+\lambda_2}\|(1, u, z)\|^{-1-\lambda_1}.$$

The integral over $v \in \mathbb{A}^\times/F^\times$ of the product of this with $d^\times v$ is $t_1^{-2\lambda_1-\lambda_2}/((2\lambda_1+\lambda_2)/3)$ times

$$|u|^{\lambda_1}\|(1, u)\|^{-1-\lambda_1}\|(1, u, z)\|^{-1-\lambda_1}.$$

This expression is equal to that appearing in the first term in the difference associated to $s = 1$. The label in this case would be ((3)6.5).

VI(6). Finally, when $s = r(13)$, $s\lambda = -\lambda_2\mu_1 - \lambda_1\mu_2$, and $\phi_0(s\lambda, H-T) \neq 0$ only when

$$t_2^3|u|^{-1}\|(1, z, u)\|^3 \leqslant |v| \leqslant t_1^{-3}|u|^2/\|(1, u)\|^3.$$

But this domain is empty.

This completes our evaluation of the integral over $\mathbf{H}^0(F)\backslash\mathbf{H}^0(\mathbb{A})$ of the product by $\xi^{-1}(h)$ of the truncated Eisenstein series $\Lambda^T E(h, \Phi, \rho, \lambda)$ of (14.1), when $\mathbf{G} = \mathrm{PGL}(3)$ and ρ is a character of the diagonal subgroup. Namely the result is the sum of (3(1)), ((3)2), ((3)3), ((3)4.i) and ((3)5.(i)) ($1 \leqslant i \leqslant 4$) and ((3)6.j) ($1 \leqslant j \leqslant 5$).

G. Conclusion for PGL(3)

To obtain the terms of our summation formula in the continuous series, namely those which are parametrized in (3.3) by the minimal parabolic subgroup $\mathbf{P} = \mathbf{B}$ and a character ρ of $\mathbf{B}(\mathbb{A})/\mathbf{B}(F)$ (note that $n(B) = 1/6$ in (3.3)), we need to replace

$\int \Lambda^T \bar{E} \cdot \xi$ in (3.3) by the complex conjugate of the sum of ((3)i), $1 \leqslant i \leqslant 3$, ((3)i.j), $i = 4, 5$, $1 \leqslant j \leqslant 4$, and ((3)6.i), $1 \leqslant i \leqslant 5$. Then we need to carry out the integration over λ in the two dimensional (over $\mathbb{R}$) space $i\mathfrak{A}_B^*$, namely over λ_1 and λ_2 in $\mathbb{R}$. Finally we shall take the limit as $T \to \infty$ in the positive Weyl chamber, namely as $t_1 \to \infty$ and $t_2 \to \infty$. Note that for λ in $i\mathfrak{A}_B^*$ the complex conjugate $\bar{\lambda}$ is $-\lambda$. Then the function $\int \xi \Lambda^T \bar{E}$ is analytic in λ on $i\mathfrak{A}_B^*$, and each of the expressions ((3)i.j) has analytic continuation in λ to $\mathfrak{A}_{B,\mathbb{C}}^*$. As functions in λ, the ((3)i.j) are slowly increasing in every band $a_i \leqslant \mathrm{Re}(\lambda_i) \leqslant b_i$, $\lambda_i = \langle \lambda, \alpha_i \rangle$, $i = 1$, 2, while the other factor, $E_\psi(I(f, \rho, \lambda)\Phi, \rho, \lambda)$, in (3.3), is rapidly decreasing there. For any f, the sums over ρ and Φ are finite. With these comments out of the way, we now point out the main features of the computations of the various terms.

I. In the case of (3(1)), as in the case of GL(2) we note (see [Sh2], p. 272) that the intertwining operators $M(s, \rho, \lambda)$ are a product of (i) a scalar valued function, $m(s, \rho, \lambda)$, which is a quotient of products of L-functions in the components of ρ, and is holomorphic on $\lambda_1, \lambda_2 \in i\mathbb{R}$, and of (ii) a normalized intertwining operator $R(s, \rho, \lambda) = \otimes_v R(s, \rho_v, \lambda)$, with properties as listed in the case of GL(2). In particular Lemma 10 applies to each of the six terms listed in ((3)1), and the limit of the integral over $\lambda \in i\mathfrak{A}_B^*$ as $T \to \infty$ would be the value of the integrand at $\lambda_1 = \lambda_2 = 0$ (after the factor of type T^λ/λ is removed). Namely, the limit as $T \to \infty$ of (3.3) with ((3)1) replacing $\int \xi^{-1} \Lambda^T E$, is the sum over ρ, Φ of:

$$E_\psi(I(f, \rho, 0), \rho, 0) \sum_{s \in W} i(s)\overline{(M(s, \rho, 0)\Phi)_{\mathbb{K}^0, \xi}(1)}\delta(\xi/{}^s\rho)$$

with $i(s) = 1$ if $s = 1$ or $s = r(13)$, and $i(s) = -1$ otherwise.

It will be useful to recall the functional equation ([A1], (iii), p. 927)

$$M(s_1 s_2, \rho, \lambda) = M(s_1, s_2\rho, s_2\lambda)M(s_2, \rho, \lambda)$$

for any $s_1, s_2 \in W$. The same functional equation holds for the normalized operator $R(s, \rho, \lambda)$, and the scalar valued function $m(s, \rho, \lambda)$. Thus it suffices to recall the definition of $m(s, \rho, \lambda)$ (from [Sh2], p. 272), when s is a simple reflection, and it is

$$m(s_i, \rho, \lambda) = L(\langle \lambda, \alpha_i \rangle, \rho_i/\rho_{i+1})/(L(1 + \langle \lambda, \alpha_i \rangle, \rho_i/\rho_{i+1})\varepsilon(\langle \lambda, \alpha_i \rangle, \rho_i/\rho_{i+1}))$$

where $s_1 = r(12)$, $s_2 = r(23)$, $\rho = \rho_1 \times \rho_2 \times \rho_3$ and $i = 1, 2$, and by the functional equation $L(t, \mu) = \varepsilon(t, \mu)L(1 - t, \mu^{-1})$ it is

$$m(s_i, \rho, \lambda) = L(1 - \lambda_i, \rho_{i+1}/\rho_i)/L(1 + \lambda_i, \rho_i/\rho_{i+1}), \qquad \lambda_i = \langle \lambda, \alpha_i \rangle.$$

The value of this factor at $\lambda_i = 0$ is 1 if ρ_i/ρ_{i+1} is non-trivial, and -1 if it is. Recall that we choose ρ in its connected component to have that ρ_i/ρ_{i+1} is 1 if it factorizes through the absolute value $x \mapsto |x|$.

II. Next we consider the contribution corresponding to ((3)2). In order to leave ((3(2)) as it is, we consider instead the complex conjugate of (3.3). Thus $E_\psi(I(f,\rho,\lambda)\Phi,\rho,\lambda)$ in (3.3) will be replaced by $E_{\bar\psi}(I(\bar f,\rho^{-1},-\lambda)\bar\Phi,\rho^{-1},-\lambda)$; this is an analytic function in $\lambda\in i\mathfrak{A}_B^*$ (note that $\bar\lambda=-\lambda$ there), which has analytic continuation in λ on $\mathfrak{A}_{B,\mathbb{C}}^*$. The analytic continuation of $E_\psi(\ldots,\lambda)$ is in fact holomorphic in λ. Indeed, the residue of the Eisenstein series at a value of λ where it has a pole, lies in a space of a representation without a Whittaker model, hence the Fourier coefficient $E_\psi(\ldots,\lambda)$ has no pole there. Moreover, as a function in λ this E_ψ is rapidly decreasing as $|\lambda|\to\infty$ in any vertical strip $a_i\leqslant \mathrm{Re}(\lambda_i)\leqslant b_i$ $(i=1,2)$.

We shall substitute each of the six terms of ((3)2) in (the complex conjugate of) (3.3) in place of $\int(\xi^{-1}\Lambda^T E)(h)dh$. In each of the six cases we shall move the line of integration $\lambda_j\in i\mathbb{R}$ to a parallel line. In doing this, we need to watch out for poles of the integrand; these will contribute to the integral, by Cauchy's formula.

II(1). In the case of $s=1$ we move $\lambda_1\in i\mathbb{R}$ to $\varepsilon^{-2}+\lambda_1$, $\lambda_1\in i\mathbb{R}$, small $\varepsilon>0$. As the integrand is holomorphic between these two lines, no residue would turn up. The monomial $t_1^{\lambda_1+2}t_2^{\lambda_2-1}$ would then become $t_1^{\lambda_1+\varepsilon}t_2^{\lambda_2-1}$. When $t_1\to\infty$ and $t_2\to\infty$ (in the domain $t_1^{1/2}<t_2<t_1^2$) the absolute value t_1^ε/t_2 has the limit 0, and so the corresponding contribution to the limit of (3.3) as $T\to\infty$ is 0.

II(2). In the case of $s=r(12)$ in ((3)2) inserted in (3.3), note that the only singularity of the integrand may be obtained from the normalizing factor

$$m(r(12),\rho,\lambda)=L(1-\lambda_1,\rho_2/\rho_1)/L(1+\lambda_1,\rho_1/\rho_2),$$

which depends only on λ_1, and is holomorphic on $\lambda_1\in i\mathbb{R}$. Moving the line of integration in λ_2 from $i\mathbb{R}$ to $\varepsilon-4+i\mathbb{R}$, the monomial $t_1^{2-\lambda_1}t_2^{\lambda_1+\lambda_2-1}$ would become $t_1^{2-\lambda_1}t_2^{\lambda_1+\lambda_2+\varepsilon-5}$. The limit as $T\to\infty$ in the specified domain of T's is zero, and again no non-zero contribution to the limit of (3.3) as $T\to\infty$ is obtained.

II(3). In the case of $s=r(23)$, analogous change of λ_1 from $i\mathbb{R}$ to $-2+i\mathbb{R}$, would yield the same conclusion. This change is permitted since $m(r(23),\rho,\lambda)$ depends only on λ_2.

II(4). In the next case of $s=r(23)r(12)$, the normalizing factor is

$$\begin{aligned}
&m(r(23)r(12),\rho,\lambda)\\
&\quad=m(r(23),\rho_2\times\rho_1\times\rho_3,((\lambda_2-\lambda_1)/3,(2\lambda_1+\lambda_2)/3,\\
&\qquad-(\lambda_1+2\lambda_2)/3))m(r(12),\rho,\lambda)\\
&\quad=\frac{L(\lambda_1+\lambda_2,\rho_1/\rho_3)}{L(1+\lambda_1+\lambda_2,\rho_1/\rho_3)\varepsilon(\lambda_1+\lambda_2,\rho_1/\rho_3)}\cdot\frac{L(\lambda_1,\rho_1/\rho_2)}{L(1+\lambda_1,\rho_1/\rho_2)\varepsilon(\lambda_1,\rho_1/\rho_2)}.
\end{aligned}$$

The ε-factors have neither zeroes nor poles. Changing variables $\lambda_2 \to \lambda_2 - \lambda_1$ the main part (i.e. up to a holomorphic, slowly increasing in vertical strips, function in λ) of the integrand is the product of the Fourier coefficient

$$E_{\bar{\psi}}(I(\bar{f}, \rho^{-1}, -\lambda')\bar{\Phi}, \rho^{-1}, -\lambda'), \tag{(4)1}$$

where

$$\lambda' = \lambda_1\mu_1 + (\lambda_2 - \lambda_1)\mu_2 = \lambda_1(\mu_1 - \mu_2) + \lambda_2\mu_2,$$

with

$$\frac{L(\lambda_1, \rho_1/\rho_2)}{L(1 + \lambda_1, \rho_1/\rho_2)} \cdot \frac{L(\lambda_2, \rho_1/\rho_3)}{L(1 + \lambda_2, \rho_1/\rho_3)} \cdot \frac{t_1^{2 + \lambda_2 - \lambda_1}}{2 + \lambda_2 - \lambda_1} \cdot \frac{t_2^{-1 - \lambda_2}}{1 + \lambda_2}. \tag{(4(2)}$$

We shall move the line of integration of λ_1 from $i\mathbb{R}$ to $2 + i\mathbb{R}$. The resulting expression is holomorphic and of rapid decay in λ_1 as $|\lambda_1| \to \infty$, and in λ_2, and the absolute value of $t_1^{\lambda_2 - \lambda_1} t_2^{-1 - \lambda_2}$, namely t_2^{-1}, goes to 0 as $T \to \infty$. The only pole encountered as λ_1 moves from $i\mathbb{R}$ to $2 + i\mathbb{R}$ is of $L(\lambda_1, \rho_1/\rho_2)$, when ρ_1/ρ_2 (factorizes through the absolute value and so) is 1 (by our normalization). This pole would occur at $\lambda_1 = 1$ (note that the pole at $\lambda_1 = 0$ is canceled by that of $L(1 + \lambda_1, \rho_1/\rho_2)$ in the denominator. We could take the residue at $\lambda_1 = \langle \lambda', \alpha_1 \rangle = 1$, but this would make our formula longer than necessary for any possible practical applications. Instead, we shall introduce a zero at $\lambda_1 = 1$, and explain why it would not restrict the applicability of the summation formula.

To introduce a zero at $\lambda_1 = 1$, fix a place u of F, and let $f = f^u f_u$ be a product of a function f^u on $\mathbf{G}(\mathbb{A}^u)$, $\mathbb{A}^u$ is the ring of adeles without a component at u, and a function f_u on G_u. We take f_u to be spherical, namely K_u-invariant. Then the trace $\operatorname{tr} \pi_u(f_u)$ is zero for any irreducible G_u-module π_u, unless π_u is unramified, namely has a non-zero K_u-fixed vector. In the latter case π_u is the unique unramified subquotient of a G_u-module of the form $I_u(\lambda)$, normalizedly induced from the unramified character $an \mapsto \lambda(a) = e^{\langle H(a), \lambda \rangle}$ of the upper triangular subgroup $B_u = A_u N_u$ of G_u. Moreover, $\operatorname{tr} \pi_u(f_u) = \operatorname{tr} I_u(f_u, \lambda)$ is denoted by $f_u^\vee(\lambda)$, and named the *Satake transform* of f_u, at $\lambda (\in \mathfrak{A}^*_{B,\mathbb{C}})$. Now $I(\bar{f}, \rho^{-1}, -\lambda') = I(\bar{f}^u, (\rho^u)^{-1}, -\lambda') I(\bar{f}_u, \rho_u^{-1}, -\lambda')$, and $I(\bar{f}_u, \rho_u^{-1}, -\lambda')$ acts as 0 unless ρ_u is unramified, in which case it is the product by the scalar $\operatorname{tr} I(\bar{f}_u, \rho_u^{-1}, -\lambda')$ of the projection on the unique K_u-fixed vector in $I(\rho_u^{-1}, {}^-\lambda')$.

Our assumption on f_u will be that $f_u^\vee(\lambda) = 0$ at λ with $\lambda_1 = \langle \lambda, \alpha_1 \rangle$ equals (1 or) -1).

Now if $L(\lambda_1, \rho_1/\rho_2)$ of ((4)2) has a pole, then $\rho_1 = \rho_2$ (by our normalization), and $I(\rho_u^{-1}, -\lambda') = I(-\lambda' + \lambda_u \mu_2)$ for some λ_u which depends on ρ_u, and

$$\operatorname{tr} I(\bar{f}_u, \rho_u^{-1}, -\lambda') = \bar{f}_u^\vee(-\lambda' + \lambda_u\mu_2)$$

is zero when $\langle -\lambda' + \lambda_u\mu_2, \alpha_1\rangle = -\lambda_1$ equals -1. Hence ((4(1), which is equal to

$$\bar{f}_u^\vee(-\lambda' + \lambda(\rho_u)\mu_2)E_{\bar{\psi}}(I(\bar{f}^u, (\rho^u)^{-1}, -\lambda'), \rho^{-1}, -\lambda'),$$

vanishes at $\lambda_1 = 1$, and cancels the pole, necessarily simple, of $L(\lambda_1, \rho_1/\rho_2)$. The fourth term of ((3)2) will consequently make no non-zero contribution to the summation formula, under our assumption that $f_u^\vee(\lambda) = 0$ at λ with $\lambda_1 = \langle\lambda, \alpha_1\rangle = -1$.

REMARK. This assumption on f_u (and f) does not restrict the applicability of the summation formula. Indeed, the representations π of $\mathbf{G}(\mathbb{A})$ which occur in the space $L^2(\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}))$ are unitary, and so are their components. Almost all local components π_v of $\pi = \otimes\pi_v$ are unramified, and we choose u (for a given π) such that π_u is unramified. Then $\pi_u = I_u(\lambda)$, and it is unitary only for λ with $|\mathrm{Re}\langle\lambda, \alpha\rangle| < 1$ (all roots α). Then our assumptions on f_u implies that $\operatorname{tr} I_u(f_u, \lambda)$ vanishes only at $\pi_u = I_u(\lambda)$ which do not occur in the automorphic (unitary) spectrum, and so no information could be obtained about such π_u from the summation formula even if the assumption was not made. In any case, no information is lost.

We shall have to deal with various other terms, in analogous fashion, and will need the vanishing assumption at $\langle\lambda, \alpha\rangle = 1$ for all roots α.

VANISHING ASSUMPTION. The component of f at u is a spherical function f_u whose Satake transform $f_u^\vee$ is zero at any $\lambda = \lambda_1\mu_1 + \lambda_2\mu_2$ with $\langle\lambda, \alpha\rangle = 1$ for some root α of $\mathbf{A}$ in $\mathbf{G}$ (in other words, at λ with λ_1, λ_2 or $\lambda_1 + \lambda_2$ equals 1 or -1).

II(5). The next, fifth, summand, in ((3)2), and its contribution to (3.3), is similarly treated. The normalizing factor $m(r(12), r(23)\rho, r(23)\lambda)m(r(23), \rho, \lambda)$ is the quotient of

$$(L(\lambda_1 + \lambda_2, \rho_1/\rho_3)/L(1 + \lambda_1 + \lambda_2, \rho_1/\rho_3))(L(\lambda_2, \rho_2/\rho_3)/L(1 + \lambda_2, \rho_2/\rho_3))$$

by the holomorphic never-zero ε-factors. Changing variables $\lambda_1 \to \lambda_1 - \lambda_2$, the product of these L-function with the monomial in T in ((3)2) becomes

$$\frac{L(\lambda_1, \rho_1/\rho_3)}{L(1 + \lambda_1, \rho_1/\rho_3)} \cdot \frac{L(\lambda_2, \rho_2/\rho_3)}{L(1 + \lambda_2, \rho_2/\rho_3)} \cdot \frac{t_1^{2-\lambda_1}}{2 - \lambda_1} \cdot \frac{t_2^{\lambda_1 - \lambda_2 - 1}}{\lambda_1 - \lambda_2 - 1}.$$

Moving the line of integration in λ_2 from $i\mathbb{R}$ to $4 + \lambda_2$, $\lambda_2 \in i\mathbb{R}$, we obtain the monomial with absolute value $t_1^2 t_2^{-5}$, whose limit is 0 as $T \to \infty$ in $t_1 < t_2^2$. The integrand may have a pole in $0 \leqslant \mathrm{Re}(\lambda_2) \leqslant 4$ only when ρ_2/ρ_3 (factorizes through the absolute value and so by our normalization) is equal to 1, at $\lambda_2 = 1$. But this pole is canceled by the zero of $f_u^\vee(\lambda)$ at λ with $\langle \lambda, \alpha_2 \rangle = -1$. No non-zero contribution is then made to the summation formula.

II(6). The last term in ((3)2), parametrized by $s = r(13)$, is the most difficult to handle. The normalizing constant $m(r(13), \rho, \lambda)$ is the quotient of

$$\frac{L(\lambda_1, \rho_1/\rho_2)}{L(1+\lambda_1, \rho_1/\rho_2)} \frac{L(\lambda_1+\lambda_2, \rho_1/\rho_3)}{L(1+\lambda_1+\lambda_2, \rho_1/\rho_3)} \frac{L(\lambda_2, \rho_2/\rho_3)}{L(1+\lambda_2, \rho_2/\rho_3)}$$

by a product of ε-factors. This has to be multiplied by

$$(t_1^{2-\lambda_2}/(2-\lambda_2))(t_2^{-1-\lambda_1}/(1+\lambda_1)).$$

It suffices to move the line of integration in λ_2 from $i\mathbb{R}$ to $2 - \varepsilon + \lambda_2$, $\lambda_2 \in \mathbb{R}$, as then the monomial in T has absolute value $t_1^\varepsilon t_2^{-1}$, and its limit as $T \to \infty$ in the specified domain would be 0. The possible poles of the integrand on $0 \leqslant \mathrm{Re}(\lambda_2) \leqslant 2 - \varepsilon$ are obtained from $L(\lambda_2, \rho_2/\rho_3)$ when $\rho_2/\rho_3 = 1$, at $\lambda_2 = 1$, but this pole is compensated by a zero of $f_u^\vee(\lambda)$ at $\lambda_2 = \langle \lambda, \alpha_2 \rangle = -1$, or from $L(\lambda_1 + \lambda_2, \rho_1/\rho_3)$ when $\rho_1/\rho_3 = 1$ at $\lambda_1 + \lambda_2 = 1$, but this pole is canceled by the zero of $f_u^\vee(\lambda)$ at $\lambda_1 + \lambda_2 = \langle \lambda, \alpha \rangle = -1$, where α is the root $\alpha_1 + \alpha_2$.

To summarize, the six terms of ((3)2), when substituted in (3.3), would give an expression whose limit as $T \to \infty$ is 0. Then there is no non-zero contribution to the summation formula from the second coset.

III. The analysis of ((3)3) and the limit as $T \to \infty$ of its contribution to (3.3) is carried out analogously to that of ((3)2). In fact ((3)3) is obtained from ((3)2) on interchanging $(t_1, \lambda_1, r(12))$ with $(t_2, \lambda_2, r(23))$.

To study the contribution of the remaining three cosets of $\mathbf{B}\backslash\mathbf{G}$ to the summation formula we make the next

VANISHING ASSUMPTION II (VA II). The component f_u of f at some place u is a spherical function whose Satake transform $f_u^\vee$ is zero at $\lambda(=\lambda_1\mu_1 + \lambda_2\mu_2) = 0$ (i.e. when $\lambda_1 = \lambda_2 = 0$).

The place u here may be different than that used in the first Vanishing Assumption. Using a function f with such a component implies that $\mathrm{tr}\,\pi(f) = 0$ for π whose component at u is unramified and of the form $\chi_u \otimes I_u(1)$, where $I_u(1)$ is the unramified irreducible G_u-module normalizedly induced from the trivial representation of B_u, and χ_u is any unramified character of $F_u^\times$ or order 3. Since we can choose u at will, the π affected are those whose components are almost all of the form $\pi_u = \chi_u \otimes I_u(1)$. The π which occur discretely in our summation formula are those of the form $I(\xi \times \rho_2)$, where ρ_2 is a cuspidal $G(2, \mathbb{A})$-module with central character ξ^{-1}. If $I(\xi \times \rho_2)$ has the component $\chi_u \otimes I_u(1)$ for almost all places u of F, then $\xi^3 = 1$, and the component of $\rho_2 \otimes \xi^2$ coincides with that of the induced PGL(2, $\mathbb{A}$)-module $I(\mathbb{1})$ (from the trivial representation of $\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$) at almost all palces of F. But no such cuspidal ρ_2 exists, hence the VA II does not restrict the applicability of the summation formula.

The VA II is used to cancel singularities in the integrand of (3.3) introduced by the L-function of the various ((3)i.j); $i = 4$, 5, 6. We deal with each term separately, and cancel its singularity. However it is possible that adding up this terms their singularities would cancel each other, and then the integral over $\lambda \in i\mathfrak{A}_B^*$ would be taken with no need to introduce zeroes using f. But we have not pursued this line of investigation.

IV(1). Replacing $\int \xi \Lambda^T \bar{E}$ in (3.3) by the complex conjugate of the product of ((3)4.1) and $(3/2)t_2^{\lambda_2 + \lambda_1/2}/(\lambda_2 + \lambda_1/2)$, we first change variables $\lambda_2 \mapsto \lambda_2 - \lambda_1/2$, then apply Lemma 10 to take the limit as $t_2 \to \infty$ of the integral over $\lambda_2 \in i\mathbb{R}$. The result is the value of the integrand at $\lambda_2 = 0$, or if we do not change variables in λ_2, the value of the integrand at $\lambda_2 = -\lambda_1/2$ is obtained. The remaining integrand is a function in λ_1, and its part described in ((3)4.1) will have a pole at $\lambda_1 = 0$ if at least two of the components ρ_1, ρ_2, ρ_3 of ρ are equal. However, the VA II guarantees that the other factor in the integrand of (3.3), namely $E_\psi(I(f, \rho, \lambda)\Phi, \rho, \lambda)$, would vanish on $\lambda_2 = -\lambda_1/2$ at $\lambda_1 = 0$. Hence the integrand is holomorphic and rapidly decreasing as $|\lambda_1| \to \infty$, and the corresponding contribution to the summation formula takes the form

$$\frac{1}{4} \sum_\rho \int_{i\mathbb{R}} \sum_\Phi E_\psi(I(f, \rho, \lambda)\Phi, \rho, \lambda)((3)4.1)(\Phi, \rho, \lambda_1) d\lambda_1$$

(((3)4.1) depends on Φ, ρ and λ_1), where $\lambda = \lambda_1\mu_1 + \lambda_2\mu_2 = \lambda_1(\mu_1 - \frac{1}{2}\mu_2) = \frac{1}{2}\lambda_1\alpha_1$. In other words, the integral is supported on the line of representations of the form $I(\rho, \lambda) = I(\rho_1 v^{\lambda_1/2} \times \rho_2 v^{-\lambda_1/2} \times \rho_3)$.

IV(2). In this case ((3)4.2), or rather its complex conjugate, is put in (3.3) instead of $\int \xi \Lambda^T \bar{E}$. Lemma 10, applied separately to each term in the difference of ((3)4.2), permits taking the limit as $T \to \infty$ of the integral over $\lambda_1 - \lambda_2 \in i\mathbb{R}$. The limit is the value of the integrand at $\lambda_1 = \lambda_2$, and VA II implies that $E_\psi(I(f, \rho, \lambda)\Phi, \rho, \lambda)$ is 0 at $\lambda_1 = \lambda_2 = 0$, where the products of the L-functions of ((3)4.2) may have their poles. The integral thus obtained as $T \to \infty$ is supported on the $I(\rho, \lambda) = I(\rho_1 v^{\lambda_1} \times \rho_2 \times \rho_3 v^{-\lambda_1})$, as

$$\lambda = \lambda_1\mu_1 + \lambda_2\mu_2 = \lambda_1(\mu_1 + \mu_2) = \lambda_1(\alpha_1 + \alpha_2) = \lambda_1(1, 0, -1).$$

IV(3). In this case analogous discussion shows that the limit as $T \to \infty$ of the corresponding part of (3.3) is supported on the $I(\rho, \lambda)$ with $\lambda_2 = -2\lambda_1$, thus $\lambda = \lambda_1(\mu_1 - 2\mu_2) = -\lambda_1\alpha_2$ and $I(\rho, \lambda) = I(\rho_1 \times \rho_2 \nu^{-\lambda_1} \times \rho_3 \nu^{\lambda_1})$.

IV(4). Here the support of the integrand of (3.3) as $T \to \infty$ is as in the previous case of IV(3).

V. This case is entirely analogous to IV, the same results are obtained, except that λ_1 and λ_2 may be interchanged.

VI. Entirely analogous discussion can be carried out in the case of the five non zero terms of the sixth coset. The limit of the contribution to (3.3) as $T \to \infty$ from the term (1) is supported on $\lambda_1 = \lambda_2$, in case (2) the support is on $\lambda_2 = -2\lambda_1$, in case (3) on $\lambda_1 = -2\lambda_2$, in case (4) on $\lambda_1 = -2\lambda_2$, and in case (5) on $\lambda_2 = -2\lambda_1$.

This completes our derivation of the summation formula for the symmetric space PGL(3)/GL(2).

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