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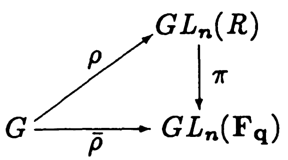
List of notations

$W(\mathbf{F}_q)$: The ring of Witt vectors of the field of q elements $\mathbf{F}_q$ where $q = p^r$.
$\mathscr{C}^0$: The category whose objects are complete Artinian local rings with residue field $\mathbf{F}_q$ and whose morphisms are local homomorphisms that are the identity on the residue field.
K: A finite extension of Q_p with ring of integers A.
G: $\mathrm{Gal}(\bar{K}/K)$.
$\bar{\rho}$: A residual representation, $\bar{\rho}: G \to GL_n(\mathbf{F}_q)$.

REMARKS. All homomorphisms, including $\bar{\rho}$ above, are assumed continuous. We take $p > 2$ throughout this paper. All group schemes are commutative.

Introduction

Let R be in $\mathscr{C}^0$ and m_R be the maximal ideal of R. Let $\Gamma_n(R)$ be the kernel of the reduction map $GL_n(R) \to GL_n(\mathbf{F}_q)$. Let $\rho: G \to GL_n(R)$ be a homomorphism such that $\pi \circ \rho = \bar{\rho}$ where π is the canonical projection $R \to R/m_R = \mathbf{F}_q$.



We call ρ_1 and ρ_2 strictly equivalent if $\rho_1 = Y\rho_2 Y^{-1}$ for some Y in $\Gamma_n(R)$. A strict equivalence class of lifts of $\bar{\rho}$ to R is called a deformation of $\bar{\rho}$ to R.

DEFINITION. Let $\bar{\rho}$ be given. For R in $\mathscr{C}^0$ we define Mazur's functor $F: \mathscr{C}^0 \to \text{Sets}$ by $F(R) = \{\text{the set of deformations of } \bar{\rho} \text{ to } R\}$. Note that F is a functor.

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Mazur has shown that F satisfies the first three Schlessinger criteria given below. He has also shown that when $\bar{\rho}$ is absolutely irreducible, F satisfies the fourth of these criteria. In fact, one needs only that the endomorphism ring of the galois module associated to $\bar{\rho}$ be $\mathbf{F_q}$ to ensure that F satisfies the fourth criterion. The argument used in [B] works with this weaker hypothesis. We let $C(\bar{\rho})$ denote this endomorphism ring. Schlessinger showed that a functor satisfying these four criteria is pro-representable. Thus for such $\bar{\rho}$ there exists a universal deformation ring, $R(\bar{\rho})$. For more on these topics see [B], [M1] and [Sch].

We want to define a modified version of Mazur's functor. We restrict our attention to those elements of $F(R)$ such that the galois modules determined by the deformation to R are the generic fibers of finite flat group schemes over A. The aim of this paper is to do this functorially and in some cases compute the (uni)versal flat deformation rings, $R_{fl}(\bar{\rho})$. Mazur has considered in [M1] a restriction that is similar in the ordinary case. The results here apply in the supersingular case. We find that if $K = Q_p$, $C(\bar{\rho}) = \mathbf{F_q}$ and $\bar{\rho}$ comes from the generic fibre of finite flat group scheme over Z_p then $R_{fl}(\bar{\rho}) = W(\mathbf{F_q})[[T_1, T_2]]$.

Section 1

We let X be a property for finite $W(\mathbf{F_q})[G]$-modules such that the set of finite $W(\mathbf{F_q})[G]$-modules with property X is closed under direct sums, subobjects and quotients. We define a subfunctor F_X of F.

DEFINITION. We define $F_X(R)$ to be those ρ in $F(R)$ such that when viewed as an $W(\mathbf{F_q})[G]$-module, ρ has property X. Note that R has finite cardinality and we assume that $\bar{\rho}$ has property X.

PROPOSITION 1.1. *F_X is a functor.*

Proof. Let R and S be objects in $\mathscr{C}^0$ and $\phi: R \to S$ a morphism in $\mathscr{C}^0$. Then it suffices to show that $\rho \in F_X(R)$ implies $\phi \circ \rho \in F_X(S)$

$$G \xrightarrow{\rho} GL_n(R) \xrightarrow{\phi} GL_n(S)$$

Let $B = R^n$ and $D = S^n$ as rings with G-actions. The map ϕ induces a map $\phi^n: B \to D$ of G-rings. As D is a finite ring, it is finitely generated as a B-module. Let $x_1, x_2, \ldots, x_m$ generate D over B. Then we have a surjection of $W(\mathbf{F_q})[G]$-modules $\Psi: B^m \to D$ given by $\Psi(e_i) = x_i$, where the e_i are canonical basis elements of B^m. As property X is closed under direct sums, B^m has property X. As it is closed under quotients, D has property X, the desired result. Note that for $\phi \circ \rho$ to have property X we do not need R and S to be in $\mathscr{C}^0$. All we really require is a homomorphism $R \to S$ of finite rings. This will come up in Section 4.

DEFINITION. Let $E:\mathscr{C} \to$ Sets be a functor such that $E(\mathbf{F_q})$ is a point. Let R_0, R_1, and $R_2 \in \mathscr{C}^0$ with morphisms $R_i \to R_0$ for $i = 1, 2$. We have the map:

$$E(R_1 \times_{R_0} R_2) \to E(R_1) \times_{E(R_0)} E(R_2) \qquad (*)$$

The Schlessinger criteria are:

H1. $R_2 \to R_0$ small implies (*) is surjective.
H2. $R_0 = \mathbf{F_q}$, $R_2 = \mathbf{F_q}[\varepsilon]$ implies (*) is bijective.
H3. $\dim t_E = \dim(E(\mathbf{F_q}[\varepsilon])) < \infty$.
H4. $R_1 = R_2$, $R_2 \to R_0$ small implies (*) is bijective.

A map $R \to S$ is small if it is surjective and has kernel a principal ideal annihilated by the maximal ideal of R. The dual numbers $\mathbf{F_q}[\varepsilon]$ of $\mathbf{F_q}$ is the ring with elements $a + b\varepsilon$ where a and b are in $\mathbf{F_q}$ and $\varepsilon^2 = 0$. Schlessinger shows that t_E is a vector space over $\mathbf{F_q}$.

THEOREM 1.1. *The functor F_X satisfies the first three Schlessinger criteria. If $C(\bar{\rho}) = \mathbf{F_q}$ then F_X also satisfies the fourth criterion.*

Proof. We make use of the fact that Mazur's functor, F, satisfies the criteria. We first note that if we have **H1** for F_X then the rest follow.

H2. As $\mathbf{F_q}[\varepsilon] \to \mathbf{F_q}$ is small, (*) is surjective for F_X by **H1** for F_X. We know (*) is bijective for F so (*) must be injective for F_X.
H3. From Lemma 2.10 of [Sch] we know t_{F_X} and t_F are $\mathbf{F_q}$ spaces. We have $t_{F_X} \subseteq t_F$ which is finite dimensional and we are done.
H4. Here we suppose $C(\bar{\rho}) = \mathbf{F_q}$. By **H1** for F_X, (*) is surjective for F_X. By **H4** for F, (*) is bijective for F. Thus (*) is bijective for F_X.

Thus we only need verify **H1** for F_X. Let R_0, R_1, and R_2 be as in **H1** and put $R_3 = R_1 \times_{R_0} R_2$. Let $\rho_1 \times_{\rho_0} \rho_2 \in F_X(R_1) \times_{F_X(R_0)} F_X(R_2)$. By **H1** for F we can choose $\rho \in F(R_3)$ such that $\rho \to \rho_1 \times_{\rho_0} \rho_2$ under the map:

$$F(R_3) \to F_X(R_1) \times_{F_X(R_0)} F_X(R_2).$$

We show ρ has property X. We easily see that the map $R_3 \to R_1 \times R_2$ is injective so the map

$$R_3^n \to R_1^n \times R_2^n$$

is an injective map of $W(\mathbf{F_q})[G]$-modules. As the $W(\mathbf{F_q})[G]$-modules determined by ρ_1 and ρ_2 have property X and the $W(\mathbf{F_q})[G]$-module determined by ρ is a submodule of their direct sum we are done.

PROPOSITION 1.2. *Let $\bar{\rho}$ have property X and suppose $C(\bar{\rho}) = \mathbf{F_q}$. Let $R(\bar{\rho})$ and*

$R_X(\bar{\rho})$ be the pro-representing objects of F and F_X respectively. Then the map $R(\bar{\rho}) \to R_X(\bar{\rho})$ is surjective, i.e. F_X is a closed subfunctor of F.

Proof. Let S be the image of $R(\bar{\rho})$ in $R_X(\bar{\rho})$. Then the universal property X deformation to $R_X(\bar{\rho})$ factors through S. As S is a subring of $R_X(\bar{\rho})$ this deformation to S lies in $\lim F_X(S/m_S^l)$. Thus by universality there is a unique map $R_X(\bar{\rho}) \to S$. Composing this with the injection $S \to R_X(\bar{\rho})$ we recover the deformation to $R_X(\bar{\rho})$. By universality this composed map is the identity so the map $S \to R_X(\bar{\rho})$ is surjective and we have $S = R_X(\bar{\rho})$ the desired result.

Section 2

We now provide two examples of property X which are closed under direct sums, subobjects and quotients. Recall that K is a finite extension of Q_p with ring of integers A. We need the following lemma.

LEMMA 2.1. *Let $0 \to T \to U \to V \to 0$ be an exact sequence of G-modules. Suppose U is the generic fibre of a finite flat group scheme $\mathscr{U}$ over A. Then there are unique finite flat group schemes $\mathscr{T}$ and $\mathscr{V}$ over A such that the above sequence is the generic fibre of the exact sequence*

$$0 \to \mathscr{T} \to \mathscr{U} \to \mathscr{V} \to 0$$

of finite flat group schemes over A.

Proof. The idea is to take for $\mathscr{T}$ the schematic closure of T in $\mathscr{U}$. One then takes $\mathscr{U}/\mathscr{T}$ for $\mathscr{V}$. See [S] and [R] for details.

DEFINITION. Let F_{fl} be the subfunctor of F consisting of those $\rho \in F(R)$ such that the galois module determined by ρ is the generic fiber of a finite flat group scheme over A.

From the above lemma and the fact that a direct sum of finite flat group schemes over A is again a finite flat group scheme over A we see that F_{fl} is a functor satisfying the Schlessinger criteria. Thus when $C(\bar{\rho}) = \mathbf{F}_q$ we have that F_{fl} is pro-representable. Note that there is no assumption on the ramification of A.

For the second example of property X we need to review the modules of [FoLa]. We are only concerned with a special case of their modules and we assume, in their notation, that $\mathscr{D} = Z_p$ throughout. Here we insist that K be unramified over Q_p. Let σ be the absolute Frobenius.

A filtered Dieudonné A-module is an A-module furnished with a decreasing, exhaustive, separated filtration $(M^i)_{i \in Z}$ of sub-A-modules such that for each integer i, we have a σ-semi-linear map $\varphi^i: M^i \to M$. Furthermore it is required that for $x \in M^{i+1}$, $\varphi^{i+1}(x) = p\varphi^i(x)$. These filtered modules form a Z_p-linear additive category, $\underline{MF}$.

We denote by $\underline{MF}^f_{\text{tor}}$ the full subcategory of $\underline{MF}$ whose objects M have underlying spaces that are A-modules of finite length and satisfy $\Sigma \operatorname{Im} \varphi^i = M$. The category $\underline{MF}^{f,j}_{\text{tor}}$ is a full subcategory of $\underline{MF}^f_{\text{tor}}$ whose objects satisfy $M^0 = M$ and $M^j = 0$. We call $\operatorname{Rep}^f_{Z_p}$ the category of finite $Z_p[G]$-modules. In [FoLa] they construct a faithful exact contravariant functor, $U_S: \underline{MF}^{f,q}_{\text{tor}} \to \operatorname{Rep}^f_{Z_p}$. They show that when restricted to certain subcategories of $\underline{MF}^{f,q}_{\text{tor}}$, U_S is fully faithful. All we will need is their result that U_S is fully faithful on $\underline{MF}^{f,j}_{\text{tor}}$ for $j < p$.

Fontaine and Lafaille have shown in Section 5 of [FoLa] that when the residue field of A is algebraically closed the G-modules associated to these filtered modules have the closure properties of property X of the previous section. They do not state this result in the explicit form we want, but they give the explicit galois action for simple filtered modules. In the case of arbitrary residue fields Faltings has shown in [Fa] Theorem 2.6 that the G-modules associated to these filtered modules have the requisite closure properties.

DEFINITION. Let $j < p$. Let F_j be the subfunctor of F such that each ρ comes from a Fontaine-Lafaille module of filtration length equal to j.

From the previous paragraph we know that F_j is a functor satisfying the Schlessinger criteria.

Fontaine and Lafaille have shown in Section 9 of [FoLa] that the category $\underline{MF}^{f,2}_{\text{tor}}$ is antiequivalent to the category of finite flat group schemes over A. For M an object of $\underline{MF}^{f,2}_{\text{tor}}$, $U_S(M)$ is the generic fiber of the corresponding group scheme. (As U_S is fully faithful, they recover the result of Raynaud that a finite group scheme over an unramified extension of Q_p extends to a finite flat group scheme over the ring of integers in at most one way.) We say that a representation has weight j if it comes from an object M of $\underline{MF}^{f,j}_{\text{tor}}$.

Suppose now that K is unramified over Q_p and we are given a representation $\bar{\rho}$: $\operatorname{Gal}(\bar{K}/K) \to GL_2(\mathbf{F}_p)$ such that the galois module $\mathbf{F}_p^2$ is the generic fiber of a finite flat group scheme over A. In this situation the functors F_{fl} and F_2 are the same. The problem of calculating the representing object of these functors is addressed in the following sections.

Section 3

We are now ready to do a computation. We restrict our attention to two dimensional representations as those arising from modular forms are two dimensional. When doing computations in this paper we only mean that the representing object is topologically isomorphic to the ring given. Our results are not explicit as those in [B].

Let $K = Q_p$ and $\bar{\rho}: G \to GL_2(\mathbf{F_p})$ be the representation coming from the galois action on the p-division points of an elliptic curve over Q_p with good supersingular reduction. From the Weil pairing we know $\det \bar{\rho} = \chi$ the cyclotomic character and the good reduction implies that $\bar{\rho}$ is of weight 2. We know from Section 1.11 of [Se1] that $C(\bar{\rho}) = \mathbf{F_p}$. Thus in the sense of Section 2 of this paper and Section 2.8 [Se2] $\bar{\rho}$ is weight 2. We show that in this case $R_{fl}(\bar{\rho}) = R_2(\bar{\rho}) = Z_p[[T_1, T_2]]$. In fact all we need to compute $R_{fl}(\bar{\rho})$ is that $\bar{\rho}$ is weight 2 and that $C(\bar{\rho}) = \mathbf{F_p}$. First we compute t_{F_2}.

LEMMA 3.1. $F_2(\mathbf{F_p}[\varepsilon]) = \widetilde{\mathrm{Ext}}^1_2(H, H)$ *where H is the galois module associated to* $\bar{\rho}$. *The extensions on the right are in the category of weight* 2 *representations and the tilde indicates that we only consider those extensions killed by p.*

Proof. We view $(\mathbf{F_p}[\varepsilon])^2$ as a 4-dimensional $\mathbf{F_p}$ space. Then $\rho \in F_2(\mathbf{F_p}[\varepsilon])$ can be written as $\rho(\sigma) = \begin{pmatrix} \bar{\rho}(\sigma) & 0 \\ Z_\sigma & \bar{\rho}(\sigma) \end{pmatrix}$. Such a representation clearly gives rise to an element of $\widetilde{\mathrm{Ext}}^1_2(H, H)$. An element of $\widetilde{\mathrm{Ext}}^1_2(H, H)$ gives such a representation and hence gives an element of $F_2(\mathbf{F_p}[\varepsilon])$. It is a simple calculation to verify that strict equivalence of lifts corresponds to equivalent extensions.

As $H = U_S(M)$ for M an object of $\underline{MF}^{f,2}_{\mathrm{tor}}$, $\widetilde{\mathrm{Ext}}^1_2(H, H) = \widetilde{\mathrm{Ext}}^1_2(M, M)$ where the extensions on the right are in $\underline{MF}^{f,2}_{\mathrm{tor}}$ and are also killed by p. From [FoLa] we know that $lg_A M = lg_{Z_p} U_S(M)$ so M has underlying space a 2 dimensional $\mathbf{F_p}$ space, and $\mathbf{F_p} = \mathrm{End}_{Z_p[G]}(H) = \mathrm{End}(M)$ by full faithfulness. We will see that only the size of the steps in the filtration and knowledge of End(M) are needed to determine $\widetilde{\mathrm{Ext}}^1_2(M, M)$.

To simplify computation, we consolidate the φ^i attached to M into a single notation. Recall that in this case the object M is killed by p and thus a vector space over $\mathbf{F_p}$. We will soon show that $\dim_{\mathbf{F_p}} M^1 = 1$. Thus we can choose a basis (e_1, e_2) over $\mathbf{F_p}$ of M^0 where e_2 spans M^1. We have

$$\varphi^0 = \begin{pmatrix} \alpha & 0 \\ \beta & 0 \end{pmatrix}, \quad \varphi^1 = \begin{pmatrix} * & \gamma \\ * & \delta \end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta \in \mathbf{F_p}$. As $\varphi^1: M^1 \to M$, the $*$'s are to indicate that φ^1 is not defined at e_1. We combine these to get a single matrix: $X_M = \left(\begin{array}{c|c} \alpha & \gamma \\ \beta & \delta \end{array}\right)$. The dashes give the filtration structure. The number of columns to the right of the dashes is the dimension of M^1. So here we have that M^1 is 1-dimensional. We get that an object M of $\underline{MF}^{f,2}_{\mathrm{tor}}$ is determined by a single dashed matrix X_M, where the dashes indicate the size of the steps in the filtration. When we multiply X_M by a matrix R which respects the filtration of M, RX_M denotes the dashed matrix formed by $R\varphi^0$ and $R\varphi^1$. Similarly $X_M R$ denotes the dashed matrix formed from $\varphi^0 R$ and $\varphi^1 R$. We say X_M commutes with R if $X_M R = RX_M$.

LEMMA 3.2. $\widetilde{\mathrm{Ext}}^1_2(M,M) = \mathrm{Hom}(M,M)/([R, X_M] \mid R$ *in* $M_2(\mathbf{F_p})$ *respects the filtration of* M).

Proof. Let $0 \to M \to N \to M \to 0$, $N \in \widetilde{\mathrm{Ext}}^1_2(M,M)$. Then

$$X_N = \begin{pmatrix} X_M & C \\ 0 & X_M \end{pmatrix}$$

where $C \in M_2(\mathbf{F_p})$ is a dashed matrix. Note that X_N has *two* sets of dashes. The second and fourth elements of an ordered basis for the underlying space of N form a basis for N^1. The set of all possible C's corresponds to the $\mathrm{Hom}(M,M)$. We mod out by equivalent extensions. An element $P \in \widetilde{\mathrm{Ext}}^1_2(M,M)$ with corresponding matrix $D \in M_2(\mathbf{F_p})$ is equivalent to N if there is a matrix $\begin{pmatrix} I & R \\ 0 & I \end{pmatrix}$ such that

$$\begin{pmatrix} I & R \\ 0 & I \end{pmatrix}\begin{pmatrix} X_M & C \\ 0 & X_M \end{pmatrix} = \begin{pmatrix} X_M & D \\ 0 & X_M \end{pmatrix}\begin{pmatrix} I & R \\ 0 & I \end{pmatrix}.$$

We get the relation $D = C + [R, X_M]$. Here R must preserve the filtration of M. This is because the isomorphism of N to P must preserve the filtrations. So C and D give equivalent extensions whenever $C - D = [R, X_M]$ for some R respecting the filtration as above, and we are done.

We now use some of our knowledge of $\bar{\rho}$. From [Se1] we know that $C(\bar{\rho}) = \mathbf{F_p}$ so H and thus M have endomorphism ring $\mathbf{F_p}$. We know that $\dim_{\mathbf{F_p}} M^0 = 2$ and $\dim_{\mathbf{F_p}} M^2 = 0$. All that remains is $\dim_{\mathbf{F_p}} M^1$. This equals 1 because otherwise an endomorphism of M would not have to respect any filtration structure and any element of $M_2(\mathbf{F_p})$ that commuted with X_M would do. As the centralizer of every element of $M_2(\mathbf{F_p})$ is at least 2 dimensional we conclude $\dim_{\mathbf{F_p}}(M^1) = 1$. So to compute t_{F_2} we note $\mathrm{Hom}(M,M)$ is 4 dimensional, the set of R that preserve the filtration is 3 dimensional, and $\mathrm{kernel}(R \to [R, X_M])$ is simply $\mathrm{End}(M)$ which is 1 dimensional. Thus t_{F_2} is $4-(3-1) = 2$ dimensional. One of these dimensions is easily seen to come from twisting $\bar{\rho}$ by the étale character $1+\varepsilon\chi$. The other dimension is not so clear.

We now have that $R_2(\bar{\rho}) = Z_p[[T_1, T_2]]/I$ for some ideal I. We show that I is zero. We do this by counting the $Z_p/(p^l)$-valued points of $R_2(\bar{\rho})$. We will show that for every l, there are $p^{2(l-1)}$ of them. This is the number of $Z_p/(p^l)$-valued points of $Z_p[[T_1, T_2]]$. So if $f \in I$ and $(x,y) \in (pZ_p/(p^l))^2$ then $f(x,y) \equiv 0 \bmod p^l$. Lifting to characteristic zero, we see that if $(x,y) \in (pZ_p)^2$ then $f(x,y) = 0$. Without difficulty one can show such an f must be zero. Thus $I = 0$ and $R_2(\bar{\rho}) = Z_p[[T_1, T_2]]$.

Let us now count $Z_p/(p^l)$-valued points of $R_2(\bar{\rho})$. We try to find objects N of $\underline{MF}^{f,2}_{\mathrm{tor}}$ whose underlying spaces are free $Z_p/(p^l)$ modules of rank 2. We denote the kernel of p on N by N_p. So we seek N such that $N_p \simeq M$. This is the same as requiring $X_N \equiv X_M \bmod p$, where X_N has a dashed structure like X_M. As $X_N \in M_2(Z_p/(p^l))$ and is determined mod p there are $p^{4(l-1)}$ possibilities. We now show that there are only $p^{2(l-1)}$ possibilities up to isomorphism. We see that $X_{N_1} \simeq X_{N_2}$ if and only if there is a matrix $R \in M_2(Z_p/(p^l))$ such that:

$$R = \begin{pmatrix} r & 0 \\ s & t \end{pmatrix}, \; R \equiv I_2 \bmod p, \; R \cdot X_{N_1} = X_{N_2} \cdot R.$$

We require R to be lower triangular to preserve the filtration. There are $p^{3(l-1)}$ such R and p^{l-1} lie in the center of $M_2(Z_p/p^l)$ and thus commute with all X_N. We know that no others commute because $C(\bar{\rho}) = \mathbf{F_p}$. Thus up to isomorphism there are $p^{4(l-1)}/(p^{3(l-1)}/p^{(l-1)}) = p^{2(l-1)}$ lifts of M to $Z_p/(p^l)$.

Thus we have proved:

THEOREM 3.1. *Let* $p > 2$ *and* $G = \mathrm{Gal}(\bar{Q}_p/Q_p)$, *and* $\bar{\rho}: G \to GL_2(\mathbf{F_p})$ *be weight* 2 *and suppose* $C(\bar{\rho}) = \mathbf{F_p}$. *Then* $R_{fl}(\bar{\rho}) = R_2(\bar{\rho}) = Z_p[[T_1, T_2]]$.

Section 4

We now generalize the results of the previous section. Let $G = \mathrm{Gal}(\bar{Q}_p/Q_p)$. We consider $\bar{\rho}: G \to GL_2(\mathbf{F_q})$, an irreducible flat representation with $\det \bar{\rho}|_I = \chi$ and $q = p^r$. This is the representation attached to an eigenform of weight 2 on $\Gamma_0(N)$ where $p \nmid N$ at a non-ordinary prime. The following proposition is due to Serre. See [Se2].

PROPOSITION 4 (Serre). *Let* $G = \mathrm{Gal}(\bar{Q}_p/Q_p)$. *Let* $\bar{\rho}: G \to GL_2(\mathbf{F_q})$. *We assume* $\mathbf{F}_{\mathbf{p}^2} \subset \mathbf{F_q}$. *If* $\bar{\rho}$ *is irreducible and flat with* $\det \bar{\rho}|_I = \chi$ *then* $\bar{\rho}|_I = \begin{pmatrix} \psi & 0 \\ 0 & \psi^p \end{pmatrix}$ *where* ψ *is a fundamental character of level* 2. *Furthermore* $C(\bar{\rho}) = \mathbf{F_q}$.

Proof. Let $I_t = I/I_p$ the tame inertia group. Recall that a fundamental character of level s is a homomorphism $I_t \to \mathbf{F}^*_{\mathbf{p}^s} \to \bar{\mathbf{F}}^*_{\mathbf{p}}$ which extends to an embedding of $\mathbf{F}_{\mathbf{p}^s}$ into $\bar{\mathbf{F}}_{\mathbf{p}}$ as fields. Let V be the 2 dimensional $\mathbf{F_q}$ space on which G acts. As $\bar{\rho}$ is irreducible it follows from Proposition 4 of [Se1] that the wild inertia group I_p acts trivially on V. Thus I_t acts on V and this action is semisimple. By Proposition 1 of [Se2] this action is via two characters of level 1 or 2. As $\det \bar{\rho}|_I = \chi$ it follows they are both level 1 or both level 2. If they are both level 1, Serre shows in Sections 2.3 and 2.4 of [Se2] that $\bar{\rho}|_I = \begin{pmatrix} \chi^a & 0 \\ 0 & \chi^b \end{pmatrix}$ where $0 \leqslant a \leqslant b \leqslant p-2$. As $\det \bar{\rho}|_I = \chi$ and $\bar{\rho}$ is flat we must have by Theorem 3.4.3 of [R] $a = 0$ and $b = 1$. Thus $\bar{\rho}|_I = \begin{pmatrix} 1 & 0 \\ 0 & \chi \end{pmatrix}$. Then by local class field theory the image of $\bar{\rho}$ is abelian and one sees by considering centralizers that the full image of $\bar{\rho}$ would be contained in $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ and $\bar{\rho}$ would not be irreducible. Thus we have that $\bar{\rho}|_I = \begin{pmatrix} \psi^a & 0 \\ 0 & \psi^b \end{pmatrix}$ where ψ is a fundamental character of level 2 and $a+b=p+1$ as $\det \bar{\rho}|_I = \chi$. But by the flatness and Theorem 3.4.3 of [R] we must have, without loss of generality, $a=0$, and $b=p$, the desired result. It remains to show $C(\bar{\rho}) = \mathbf{F_q}$. One easily sees that those elements of $GL_2(\mathbf{F_q})$ that commute with the image of $\bar{\rho}|_I$ are of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. If in fact all such elements commute with the full image of $\bar{\rho}$ then we find without difficulty that the full image of $\bar{\rho}$ is contained in the set of matrices of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. In particular it is abelian so by local class field theory the order of the image of tame inertia divides $p-1$. But this cannot be true as $\bar{\rho}|_I$ acts via fundamental characters of level 2 so not all diagonal elements commute with the full image of $\bar{\rho}$. Thus $\mathrm{End}_{\mathbf{F_q}}(H)$ is 1 dimensional, and we are done.

We want to compute $R(\bar{\rho})$, the universal deformation ring with no restrictions. We recall from [M1] and [B] that the tangent space is given by $H^1(G, \mathrm{Ad}\,\bar{\rho})$. Furthermore if $H^2(G, \mathrm{Ad}\,\bar{\rho})$ is trivial then $R(\bar{\rho})$ is a power series ring in $\dim_{\mathbf{F_q}} H^1(G, \mathrm{Ad}\,\bar{\rho})$ variables. We show that for the $\bar{\rho}$ above, $H^2(G, \mathrm{Ad}\,\bar{\rho})$ is trivial. It will then follow from the Euler characteristic of local galois cohomology that $\dim_{\mathbf{F_q}} H^1(G, \mathrm{Ad}\,\bar{\rho}) = 5$. See [Se3] for the necessary results in galois cohomology.

LEMMA 4.1. *For the $\bar{\rho}$ given above, $H^2(G, \mathrm{Ad}\,\bar{\rho})$ is trivial.*

Proof. By Tate duality, $H^2(G, \mathrm{Ad}\,\bar{\rho})$ is dual to $H^0(G, (\mathrm{Ad}\,\bar{\rho})^*)$. We show that $H^0(G, (\mathrm{Ad}\,\bar{\rho}^*) = ((\mathrm{Ad}\,\bar{\rho})^*)^G$ is trivial. Let $\phi \in ((\mathrm{Ad}\,\bar{\rho})^*)^G$. We show ϕ has 4 dimensional kernel so $\phi = 0$ and we will be done. We have that $\phi: \mathrm{Ad}\,\bar{\rho} \to \mathbf{F_p}$ where the G-action on $\mathrm{Ad}\,\bar{\rho}$ is conjugation by $\bar{\rho}$ and the G-action on $\mathbf{F_p}$ is by the cyclotomic character which is $\det \bar{\rho}|_I$. For $Z \in \mathrm{Ad}\,\bar{\rho}$, $\phi(g.Z) = g.\phi(Z)$. For all $g \in I$ we see $\bar{\rho}(g)Z\bar{\rho}(g)^{-1} - (\det \bar{\rho}(g))Z \in \mathrm{kernel}\ \phi$. It suffices to show for some $g \in I$ that the image of the map $T_g: Z \to \bar{\rho}(g)Z\bar{\rho}(g)^{-1} - (\det \bar{\rho}(g))Z$ is 4 dimensional. We find a $g \in I$ such that the kernel of this map is trivial, an equivalent result. Choose $g \in I$ so that $\psi(g) = \alpha$, an element of order p^2-1 in F_{p^2}. (Recall ψ is a fundamental character of level 2). If we put $Z = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ then $\bar{\rho}(g)Z\bar{\rho}(g)^{-1} - (\det \bar{\rho}(g))Z$ simplifies to

$$\begin{pmatrix} x(1-\alpha^{p+1}) & y(\alpha^{1-p} - \alpha^{p+1}) \\ z(\alpha^{p-1} - \alpha^{p+1}) & w(1-\alpha^{p+1}) \end{pmatrix}.$$

One easily sees that this last matrix equals 0 only if $Z = 0$. So we have that kernel ϕ is 4 dimensional so $\phi = 0$, the desired result.

LEMMA 4.2. *For the $\bar{\rho}$ given, $H^1(G, \mathrm{Ad}\,\bar{\rho})$ is 5 dimensional.*

Proof. We have $\dim H^0 - \dim H^1 + \dim H^2 = -\dim \mathrm{Ad}\,\bar{\rho}$ by the Euler characteristic of $\mathrm{Ad}\,\bar{\rho}$. From the above lemma that H^2 is trivial. H^0 is 1 dimensional because $C(\bar{\rho}) = \mathbf{F_q}$ and $\mathrm{Ad}\,\bar{\rho}$ is 4 dimensional.

We now obtain similar results for the case $\mathbf{F_{p^2}} \not\subseteq \mathbf{F_q}$.

PROPOSITION 4.2. *Let $G = \mathrm{Gal}(\bar{Q}_p/Q_p)$. Let $\bar{\rho}: G \to GL_2(\mathbf{F_q})$. If $\bar{\rho}$ is irreducible, flat and $\det \bar{\rho}|_I = \chi$ the cyclotomic character, then $C(\bar{\rho}) = \mathbf{F_q}$.*

Proof. Let $\tilde{\rho}: G \xrightarrow{\bar{\rho}} GL_2(\mathbf{F_q}) \xrightarrow{i} GL_2(\mathbf{F_{q^2}})$ where i is the injection $i: \mathbf{F_q} \to F_{q^2}$. We have $\mathbf{F_{p^2}} \subseteq \mathbf{F_{q^2}}$ and $\det \tilde{\rho}|_I = \chi$. Furthermore $\tilde{\rho}$ is flat by the argument used in Proposition 1.1. We show $\tilde{\rho}$ is irreducible. Then we can apply Proposition 4.1.

As $\bar{\rho}$ is irreducible the wild inertia acts trivially through $\bar{\rho}$ as before. Thus $\tilde{\rho}(I_p)$ is trivial. We suppose $\tilde{\rho}$ is reducible and arrive at a contradiction.

If $\tilde{\rho}$ is reducible then the image of $\tilde{\rho}$ is contained in matrices of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. As I_p acts trivially we see that $\tilde{\rho}|_I = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}$ where the γ_i are characters. Since $\tilde{\rho}$ is flat we have by Theorem 3.4.3 of [R] that γ_1 and γ_2 are products of distinct fundamental characters of level $2r$. We show $\gamma_1 \neq \gamma_2$. Let ψ be a fundamental character of level $2r$ and let $a = 1 + p + p^2 + \cdots + p^{2r-1}$. As $\det \tilde{\rho}|_I = \chi$, and $\chi = \psi^a$ we see that if $\gamma_1 = \gamma_2$ then they both equal $\psi^{a/2}$ or $-\psi^{a/2}$. But by Theorem 3.4.3 of [R], $\psi^{a/2}$ and $-\psi^{a/2}$ are not flat. Thus $\gamma_1 \neq \gamma_2$.

We now show $\tilde{\rho}$ must be semisimple. If it is not, there is a $\sigma \in G$ such that

$$\tilde{\rho}(\sigma) = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \quad \text{where } y \neq 0.$$

As $\gamma_1 \neq \gamma_2$, there is a $\tau \in I$ such that

$$\tilde{\rho}(\tau) = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \quad \text{where } s \neq t.$$

We see that

$$\tilde{\rho}(\sigma)\tilde{\rho}(\tau)\tilde{\rho}(\sigma)^{-1} = \begin{pmatrix} s & y(s-t)z^{-1} \\ 0 & t \end{pmatrix}.$$

But this must lie in the Image $\tilde{\rho}|_I$ as inertia is normal. As $y(s-t)z^{-1} \neq 0$ we have a contradiction so no such σ exists and $\tilde{\rho}$ is semisimple and thus abelian.

As $\tilde{\rho}$ is abelian we see that the order of the image of inertia must divide $p-1$. By flatness we see $\tilde{\rho}|_I = \begin{pmatrix} 1 & 0 \\ 0 & \chi \end{pmatrix}$. Thus $\bar{\rho}|_I = \begin{pmatrix} 1 & 0 \\ 0 & \chi \end{pmatrix}$ and as $\bar{\rho}$ is abelian we easily see that the image of $\bar{\rho}$ is contained in the diagonal matrices. This contradicts the fact that $\bar{\rho}$ is irreducible. Thus $\tilde{\rho}$ is irreducible.

By Proposition 4.1 we see $C(\tilde{\rho}) = \mathbf{F}_{q^2}$. As the map $C(\bar{\rho}) \otimes_{\mathbf{F}_q} \mathbf{F}_{q^2} \to C(\tilde{\rho})$ is injective we see $C(\bar{\rho}) = \mathbf{F}_q$.

PROPOSITION 4.3. *For $\bar{\rho}$ as in Proposition* 4.2, $H^2(G, \mathrm{Ad}\,\bar{\rho}) = 0$.

Proof. We know $C(\bar{\rho}) = \mathbf{F}_q$ and this is equivalent to $H^0(G, \mathrm{Ad}\,\bar{\rho})$ being 1 dimensional. From the Euler characteristic we see

$$\dim_{\mathbf{F}_q} H^1(G, \mathrm{Ad}\,\bar{\rho}) = 5 + \dim_{\mathbf{F}_q} H^2(G, \mathrm{Ad}\,\bar{\rho}).$$

It is thus enough to show $H^1(G, \mathrm{Ad}\,\bar{\rho})$ is 5 dimensional.

We know $H^1(G, \mathrm{Ad}\,\bar{\rho})$ is at least 5 dimensional and we show that the map $H^1(G, \mathrm{Ad}\,\bar{\rho}) \otimes_{\mathbf{F}_q} \mathbf{F}_{q^2} \to H^1(G, \mathrm{Ad}\,\tilde{\rho})$ is injective, where $\tilde{\rho}$ is as before. Applying Lemma 4.2 will finish the proof.

We show that no nontrivial deformation ρ of $\bar{\rho}$ to $\mathbf{F}_q[\varepsilon]$ becomes trivial when considered as a deformation of $\tilde{\rho}$ to $\mathbf{F}_{q^2}[\varepsilon]$.

Suppose ρ is such a deformation. Then $\rho(\sigma) = A_\sigma + \varepsilon B_\sigma$. We claim there is an $X \in M_2(F_{q^2})$ such that $(I + \varepsilon X)\rho(\sigma)(I - \varepsilon X) = A_\sigma$. Simplifying, we see that $[A_\sigma, X] = B_\sigma$. We show that there exists a $Y \in M_2(\mathbf{F}_q)$ such that $[A_\sigma, Y] = B_\sigma$, i.e. that ρ is the trivial deformation to $\mathbf{F}_q[\varepsilon]$.

Let τ be the automorphism of $\mathbf{F}_{q^2}$ given by raising to the qth power. Then τ fixes $\mathbf{F}_q$. We have

$$[A_\sigma, X] = B_\sigma, \quad [A_\sigma, X^\tau] = B_\sigma$$

as A_σ and B_σ lie in $M_2(\mathbf{F}_q)$. Subtracting one from the other we see $A_\sigma(X - X^\tau) = (X - X^\tau)A_\sigma$. As $C(\tilde{\rho}) = \mathbf{F}_{q^2}$ we see that $X - X^\tau$ lies in the center of $M_2(F_{q^2})$. Thus if $X = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$ we have $t = t^\tau$, $u = u^\tau$ and $(s - v) = (s - v)^\tau$. Thus we have

$$X = \begin{pmatrix} s-v & t \\ u & s-v \end{pmatrix} + \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}.$$

Choosing $Y = \begin{pmatrix} s-v & t \\ u & s-v \end{pmatrix}$ we are done.

From the preceding discussion we have proved the following theorem.

THEOREM 4.1. *Let* $p > 2$, $G = \mathrm{Gal}(\bar{Q}_p/Q_p)$ *and* $\bar{\rho}\colon G \to GL_2(\mathbf{F}_q)$ *be an irreducible weight* 2 *representation such that* $\det \bar{\rho}|_I = \chi$. *Then the universal deformation ring* $R(\bar{\rho}) = W(\mathbf{F}_q)[[T_1, T_2, T_3, T_4, T_5]]$.

We now compare $R(\bar{\rho})$ to $R_{fl}(\bar{\rho})$. The ideas are the same as in Section 3 but a little more care must be taken. We again let H be the galois module and M the Fontaine-Lafaille module associated to H. Now $\dim_{\mathbf{F}_p} M^0 = 2r$ and we have a map $\mathbf{F_q} \to \mathrm{End}(M)$. This map gives M an $\mathbf{F_q}$ structure. The underlying space of M can now be realized as a 2-dimensional $\mathbf{F_q}$ space. As endomorphisms of a Fontaine-Lafaille module preserve the filtration we see that M^1 is also an $\mathbf{F_q}$ space. Since $C(\bar{\rho}) = \mathbf{F_q}$ the same argument as in Section 3 shows that $\dim_{\mathbf{F_q}} M^1 = 1$. We also note that if N is the Fontaine-Lafaille module associated to a $W(\mathbf{F_q})/(p^l)$-valued point of $R_{fl}(\bar{\rho})$ then the underlying space of N is a free $W(\mathbf{F_q})/(p^l)$-module of rank 2 and N^1 is direct summand free of rank 1 over $W(\mathbf{F_q})/(p^l)$. To find t_{F_2} we cannot merely compute $\widetilde{\mathrm{Ext}}^1_2(M, M)$. Instead we have to compute $\widetilde{\mathrm{Ext}}^1_{2,\mathbf{F_q}}(M, M)$ the extensions of M by M killed by p having an $\mathbf{F_q}$ structure.

LEMMA 4.3. *$F_2(\mathbf{F_q}[\varepsilon]) = \widetilde{\mathrm{Ext}}^1_{2,\mathbf{F_q}}(H, H)$ where H is the galois module associated to $\bar{\rho}$ and the extensions on the right are in the category of weight 2 representations with an $\mathbf{F_q}$ structure.*

Proof. As before.

LEMMA 4.4. *$\widetilde{\mathrm{Ext}}^1_{2,\mathbf{F_q}}(M, M) = \mathrm{Hom}_{F_q}(M, M)/([R, X_M] \mid R$ in $M_2(\mathbf{F_q})$ respects the filtration).*

Proof. Note that the Fontaine-Lafaille modules here are Z_p-modules, but we give them an $\mathbf{F_q}$-structure. We can do this as the σ semilinear structure vanishes for modules over Z_p so there is no difficulty in making these two structures compatible. For $N \in \widetilde{\mathrm{Ext}}^1_{2,\mathbf{F_q}}(M, M)$ we have $X_N = \begin{pmatrix} X_M & C \\ 0 & X_M \end{pmatrix}$ where C is a dashed matrix in $M_2(\mathbf{F_q})$. That is, the set of allowable C's corresponds to $\mathrm{Hom}_{\mathbf{F_q}}(M, M)$. An element $P \in \widetilde{\mathrm{Ext}}^1_{2,\mathbf{F_q}}(M, M)$ with corresponding matrix D is equivalent to N if there is a matrix $\begin{pmatrix} I & R \\ 0 & I \end{pmatrix}$ such that

$$\begin{pmatrix} I & R \\ 0 & I \end{pmatrix}\begin{pmatrix} X_M & C \\ 0 & X_M \end{pmatrix} = \begin{pmatrix} X_M & D \\ 0 & X_M \end{pmatrix}\begin{pmatrix} I & R \\ 0 & I \end{pmatrix}.$$

Here the matrices I, R, C, D and X_M are all in $M_2(\mathbf{F_q})$. Furthermore not only does R preserve the filtration as before, but to ensure that $\begin{pmatrix} I & R \\ 0 & I \end{pmatrix}$ is an isomorphism respecting the $\mathbf{F_q}$ structure, we see R must commute with the $\mathbf{F_q}$ structure. We again get $D = C + [R, X_M]$, and the result follows.

So the set of R that preserve the filtration is the set of $R \in M_2(\mathbf{F_q})$ that preserve a 1 dimensional subspace of a 2 dimensional $\mathbf{F_q}$ space. The set of such R is a 3 dimensional $\mathbf{F_q}$ space. We get that

$$\dim_{\mathbf{F_q}} t_{F_2} = 4 - (3 - \dim_{\mathbf{F_q}} C(\bar{\rho})) = 4 - (3 - 1) = 2,$$

just as before.

So we have $R_{fl}(\bar{\rho}) = W(\mathbf{F_q})[[T_1, T_2]]/I$ for some ideal I. The same counting argument as before shows $I = 0$. The only difference is that here we find $R_{fl}(\bar{\rho})$ has $q^{2(l-1)}$ nonisomorphic $W(\mathbf{F_q})/(p^l)$-valued points. We have proved the following theorem.

THEOREM 4.2. *Let* $\bar{\rho}: \mathrm{Gal}(\bar{Q}_p/Q_p) \to GL_2(\mathbf{F_q})$ *be an irreducible weight 2 representation where* $\det \bar{\rho}|_I = \chi$ *the cyclotomic character. Then* $R_{fl}(\bar{\rho}) = R_2(\bar{\rho}) = W(\mathbf{F_q})[[T_1, T_2]]$.

The following corollary, which is similar to Mazur's result in the ordinary case, is immediate.

COROLLARY 4.1. *For* $\bar{\rho}$ *as above, there is a surjective map* $R(\bar{\rho}) \to R_{fl}(\bar{\rho})$. *The kernel of this map is an ideal generated by three elements.*

Section 5

We now make a straightforward generalization of the previous theory.

We concern ourselves first with computing weight j universal deformation rings for $j < p$. The key points here is that $U_S: \underline{MF}^{f,j}_{\mathrm{tor}} \to Z_p[G]$-modules is fully faithful. We assume then that $K = Q_p$ and $\bar{\rho}: G \to GL_2(\mathbf{F_q})$ is a weight j representation whose associated galois module H has endomorphism ring $\mathbf{F_q}$. To H there corresponds to a Fontaine-Lafaille module $M \in \underline{MF}^{f,j}_{\mathrm{tor}}$. Here we assume the filtration length of M is exactly j. We have a map $\mathbf{F_q} \to \mathrm{End}(M)$. The underlying space of M is a $2r$ dimensional $\mathbf{F_p}$ space. As $\mathbf{F_q}$ acts on M, we have $\dim_{\mathbf{F_p}} M^i$ is one of 0, r or $2r$. It cannot be 0 or $2r$ for the same reason as before. Here X_M has a slightly different dashed structure. There are in fact $j-1$ dashed lines. The basis with respect to which X_M is written is found by choosing a basis of M^{j-1} and augmenting it to a basis of M^{j-2} and then augmenting to a basis of M^{j-3} and so on. The number of columns between the ith and $i+1$st dashed lines of X_M is the dimension of M^i/M^{i+1}. For the problem here, each M^i is an $\mathbf{F_q}$ space so most of the dashed lines are adjacent with no columns between them. We again have the following lemma for determining $t_{F_j} = \widetilde{\mathrm{Ext}}^1_{j,\mathbf{F_q}}(M, M)$.

LEMMA 5.1. $\widetilde{\mathrm{Ext}}^1_{j,\mathbf{F_q}}(M, M) = \mathrm{Hom}_{\mathbf{F_q}}(M, M)/([R, X_M] \mid R$ *in* $M_{2r}(\mathbf{F_p})$ *respects the filtration and* $\mathbf{F_q}$ *structure*).

As before we see t_{F_2} is 2 dimensional. The same argument of counting $W(\mathbf{F_q})/(p^l)$ valued points of $R_j(\bar{\rho})$ carries through and we have the following:

THEOREM 5.1. *Let* $p > 2$, $G = \mathrm{Gal}(\bar{Q}_p/Q_p)$ *and* $\bar{\rho}: G \to GL_2(\mathbf{F_q})$ *be weight* j. *Suppose also that* $C(\bar{\rho}) = \mathbf{F_q}$. *Then* $R_j(\bar{\rho}) = W(\mathbf{F_q})[[T_1, T_2]]$.

Section 6

Finally we do a computation for an ordinary flat residual $\mathbf{F_p}$ representation. By ordinary we mean the representation has semisimplification a direct sum of 1 dimensional representations, one of which is unramified. Assuming $\det \bar{\rho} = \chi$, from Section 2.8 of [Se2] we see that this representation restricted to inertia is given by

$$\bar{\rho}|_I = \begin{pmatrix} \chi & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or } \bar{\rho}|_I = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}.$$

We consider the latter case. We see that $\bar{\rho} = \begin{pmatrix} \chi\psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$ where ψ_1 and ψ_2 are étale characters. One easily sees that $C(\bar{\rho}) = \mathbf{F_p}$, so by the methods of Section 4 we see that $R_{fl}(\bar{\rho}) = Z_p[[T_1, T_2]]$.

We now insist that $\psi_1 \neq \psi_2$. That such a representation exists is an easy exercise in computing extensions of Fontaine-Laffaille modules. A Z_p-valued point x of $R_{fl}(\bar{\rho})$ gives the representation $\rho_x = \begin{pmatrix} \chi\Psi_1 & * \\ 0 & \Psi_2 \end{pmatrix}$ where χ is the cyclotomic character and the Ψ_i are unramified characters with values in Z_p^* such that $\Psi_i \equiv \psi_i \bmod p$. This follows as the p-divisible group associated to ρ_x has non-trivial connected-étale sequence on p-torsion and thus itself has non-trivial connected-étale sequence. The characters on the diagonal of ρ_x must be as described because there are only two 1-dimensional p-divisible groups over $W(\bar{\mathbf{F}}_\mathbf{p})$, μ_{p^∞} and the constant p-divisible group Q_p/Z_p, i.e. $\rho_x|_I = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$. We will show that for any pair of étale characters (Ψ_1, Ψ_2) lifting (ψ_1, ψ_2) there is a unique lifting ρ of $\bar{\rho}$ to Z_p such that $\rho = \begin{pmatrix} \chi\Psi_1 & * \\ 0 & \Psi_2 \end{pmatrix}$. A simple counting argument will then show that this lifting is then a Z_p-valued point of $R_{fl}(\bar{\rho})$. Thus we will have shown that any ordinary lift of $\bar{\rho}$ that "looks" p-divisible, i.e. has semisimplification the Tate module of a p-divisible group over Z_p, is in fact the Tate module of a p-divisible group over Z_p.

All that follows can be done more generally, but we treat only the specific case for the $\bar{\rho}$ described above.

Let Ψ_1 and Ψ_2 be étale lifts of ψ_1 and ψ_2 to Z_p. We want to consider for $R \in \mathscr{C}^0$, lifts of $\bar{\rho}$ to R such that $\bar{\rho} = \begin{pmatrix} \chi\Psi_1 & * \\ 0 & \Psi_2 \end{pmatrix}$. The characters $\chi\Psi_1$ and Ψ_2 act on R via the composite of their action on Z_p and the map $Z_p \to R$. We call two lifts ρ_1 and ρ_2 as above very strictly equivalent if there is a matrix $Y = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$

where $\alpha \in m_R$ and $\rho_1 = Y\rho_2 Y^{-1}$. We call a very strict equivalence class of lifts of $\bar{\rho}$ to R a $(\chi\Psi_1, \Psi_2)$-deformation.

DEFINITION. For $R \in \mathscr{C}^0$ we have the functor $F^*: \mathscr{C}^0 \to \text{Sets}$ given by $F^*(R) = \{\text{the } (\chi\Psi_1, \Psi_2)\text{-deformations of } \bar{\rho} \text{ to } R\}$.

THEOREM 6.1. *The functor F^* satisfies the Schlessinger criteria.*

Proof. The verifications for Mazur's functor easily carry over. One need only pay some attention to the notion of very strict equivalence. See Chapter 1 of [B] for details.

DEFINITION. Let $\text{Ad}^*\bar{\rho}$ denote those elements of $M_2(\mathbf{F_p})$ of the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$.

PROPOSITION 6.1. *The tangent space to F^* is given by $H^1(G, \text{Ad}^*\bar{\rho})$.*

Proof. Let $\rho \in F^*(\mathbf{F_p}[\varepsilon])$. Put $f(\sigma) = \rho(\sigma)\bar{\rho}(\sigma)^{-1} \in I + \varepsilon \otimes \text{Ad}^*\bar{\rho}$. We easily see f is a 1-cocycle and conjugating ρ by an element of the form $\begin{pmatrix} 1 & \alpha\varepsilon \\ 0 & 1 \end{pmatrix}$ corresponds to changing f by a coboundary.

THEOREM 6.2. $R^*(\bar{\rho}) = Z_p[[T]]$.

Proof. As in Section 1.6 of [M1], the obstruction to lifting lies in $H^2(G, \text{Ad}^*\bar{\rho})$. Using the fact that $\psi_1 \neq \psi_2$ and a similar argument as in Lemma 4.3 we see $H^2(G, \text{Ad}^*\bar{\rho}) = 0$. One easily sees that $H^0(G, \text{Ad}^*\bar{\rho}) = 0$ and from the Euler characteristic we have $H^1(G, \text{Ad}^*\bar{\rho})$ is 1 dimensional.

PROPOSITION 6.2. *The $Z_p/(p^l)$-valued points of $R^*(\bar{\rho})$, and hence the Z_p-valued points of $R^*(\bar{\rho})$, give rise to isomorphic galois modules.*

Proof. Let $\rho(\sigma) = \begin{pmatrix} \psi\Psi_1 & f(\sigma) \\ 0 & \Psi_2 \end{pmatrix}$ be a $Z_p/(p^l)$-valued point of $R^*(\bar{\rho})$. For some $pz \in Z_p/(p^l)$ we see that

$$\begin{pmatrix} 1 & pz \\ 0 & 1 \end{pmatrix} \rho(\sigma) \begin{pmatrix} 1 & -pz \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \chi\Psi_1 & f(\sigma) + pz(\Psi_2 - \chi\Psi_1) \\ 0 & \Psi_2 \end{pmatrix}$$

is in the same very strict equivalence class as ρ. For some $py \in Z_p/(p^l)$ we have

$$\begin{pmatrix} 1+py & 0 \\ 0 & 1 \end{pmatrix} \rho(\sigma) \begin{pmatrix} (1+py)^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \chi\Psi_1 & (1+py)f(\sigma) \\ 0 & \Psi_2 \end{pmatrix}.$$

We claim that for each of the p^{l-1} choices of $py \in Z_p/(p^l)$, the isomorphic representations obtained above lie in different very strict equivalence classes. Once we have proved this we are done. If this were not so, there would exist py,

pw, $pz \in Z_p/(p^l)$ such that

$$\begin{pmatrix} \chi\Psi_1 & (1+py)f(\sigma) \\ 0 & \Psi_2 \end{pmatrix} = \begin{pmatrix} 1 & pz \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi\Psi_1 & (1+pw)f(\sigma) \\ 0 & \Psi_2 \end{pmatrix} \begin{pmatrix} 1 & -pz \\ 0 & 1 \end{pmatrix}$$

where $py \not\equiv pw \bmod p^l$, that is $y \not\equiv w \bmod p^{l-1}$. Simplifying we have that

$$(1+py)f(\sigma) \equiv (1+pw)f(\sigma) + pz(\Psi_2(\sigma) - \chi\Psi_1(\sigma)) \bmod p^l.$$

This reduces to

$$(y-w)f(\sigma) \equiv z(\Psi_2(\sigma) - \chi\Psi_1(\sigma)) \bmod p^{l-1}.$$

As $\bar{\rho}|_I = \begin{pmatrix} \chi & f(\sigma) \\ 0 & 1 \end{pmatrix}$ there is a $\sigma_0 \in G$ such that $f(\sigma_0)$ and $\Psi_2(\sigma_0) - \chi\Psi_1(\sigma_0)$ are units in $Z_p/(p^l)$. It follows then by examining the above congruence at σ_0 that $y - w = p^r u$ and $z = p^r v$ where u and v units and $r \leqslant l-2$. We now have that

$$p^r u f(\sigma) \equiv p^r v(\Psi_2(\sigma) - \chi\Psi_1(\sigma)) \bmod p^{l-1}.$$

Thus we see that

$$f(\sigma) \equiv (v/u)(\Psi_2(\sigma) - \chi\Psi_1(\sigma)) \bmod p^{l-1-r}.$$

As $r \leqslant l-2$, upon letting $x = u/v$ we have

$$f(\sigma) \equiv x(\Psi_2(\sigma) - \chi\Psi_1(\sigma)) \bmod p.$$

So we have

$$\bar{\rho}(\sigma) = \begin{pmatrix} \chi\Psi_1(\sigma) & x(\Psi_2(\sigma) - \chi\Psi_1(\sigma)) \\ 0 & \Psi_2(\sigma) \end{pmatrix}$$

where all entries of the above matrix are considered $\bmod p$. Conjugating by $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$ we have

$$\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \bar{\rho} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \chi\Psi_1 & 0 \\ 0 & \Psi_2 \end{pmatrix}.$$

This contradicts the fact that the original $\bar{\rho}$ was not semisimple. So all the

$Z_p/(p^l)$-valued points of $R^*(\bar{\rho})$ give rise to isomorphic galois modules and we are done.

THEOREM 6.3. *Let* Ψ_1 *and* Ψ_2 *be any two étale characters lifting* ψ_1 *and* ψ_2. *Recall* $\psi_1 \neq \psi_2$. *Then there is a unique lift* ρ *of* $\bar{\rho}$ *to* Z_p *such that* $\rho = \begin{pmatrix} \chi\Psi_1 & * \\ 0 & \Psi_2 \end{pmatrix}$. *The galois module associated to* ρ *is the Tate module of a p-divisible group over* Z_p.

Proof. We know from Section 3 that $R_{fl}(\bar{\rho})$ has $p^{2(l-1)}Z_p/(p^l)$-valued points which give rise to nonisomorphic galois modules. Each of these representations looks like $\begin{pmatrix} \chi\gamma_1 & * \\ 0 & \gamma_2 \end{pmatrix}$ where the γ_i are étale characters and $\gamma_i \equiv \psi_i \bmod p$. For each pair (γ_1, γ_2) there is by the previous proposition at most one deformation of $\bar{\rho}$ to $Z_p/(p^l)$ with $\chi\gamma_1$ and γ_2 on the diagonal. As there are $p^{2(l-1)}$ pairs (γ_1, γ_2) and $p^{2(l-1)}$ nonisomorphic $Z_p/(p^l)$-valued points of $R_{fl}(\bar{\rho})$, we see that to each pair (γ_1, γ_2) there is in fact a flat lift ρ of $\bar{\rho}$ to $Z_p/(p^l)$ such that $\rho = \begin{pmatrix} \chi\gamma_1 & * \\ 0 & \gamma_2 \end{pmatrix}$. The result follows by taking the limit.

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