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Lefschetz theorems for the integral leaves of a holomorphic one-form

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Introduction

The classic Lefschetz theorems [4] concern intersections of subvarieties of projective space with hyperplanes. This paper is about some similar theorems concerning the integral leaves of holomorphic or real harmonic one forms. Let X be a smooth projective variety, and let $\alpha^h \in H^0(X, \Omega^1_X)$ be a holomorphic one-form on X. We will look at $\alpha = \alpha^h$ or the real part $\alpha = \Re\alpha^h$. Let Y be any connected covering space of X such that the pullback of α is exact, and let $g: Y \to \mathbf{C}$ (resp. $\mathbf{R}$) denote an integral of α. It is well defined up to addition of a constant. We may think of the fibers $g^{-1}(v)$ as intersections of Y with linear hyperplanes in a vector space (this is precise if we take Y to be the covering Z defined below). We obtain some theorems about connectivity of the pairs $(Y, g^{-1}(v))$, analogues of the classical Lefschetz theorems.

The Lefschetz theorems have been interpreted in terms of Morse theory, and this is the basis for expecting the theorems we discuss below. The idea is that Y is obtained by starting with one of the fibers $g^{-1}(v)$ and then expanding by continuously adding other fibers, which changes the topology by attaching various cells along the way. The cells come from singular points in the fibers. By analyzing what can happen at the singular points, we obtain bounds on the connectivity of the pair $(Y, g^{-1}(v))$. This simple description would be enough to justify the theorems, except that the covering Y is not compact (and not even topologically finite). For example, the set of values of g at singular points can be a countable dense set in $\mathbf{C}$ or $\mathbf{R}$. In the end it turns out to work as expected with no additional complications but we treat the argument with some caution because the situation is slightly unusual.

This question arises when considering equivariant harmonic maps from the universal covering of X to trees [3]. The fibers of the harmonic map are more or less unions of connected components of the integral leaves of a harmonic one form, so questions of connectivity of these leaves are important for knowing what can happen. We obtain here a statement which is a technical step used in a

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classification of irreducible two dimensional representations of $\pi_1(X)$, to be discussed in a forthcoming joint paper with K. Corlette [1]. Since the question of 1-connectivity of $(Y, g^{-1}(v))$ is what is at issue there, that aspect is stressed here. We also indicate how one can obtain statements about higher connectivity, and give a sample. To give a more complete discussion is not so easy, as one encounters some problems about which types of singularities can arise in the fibers.

The statement about 1-connectivity for the case of a real harmonic one-form α is as follows.

THEOREM 1. *Suppose X is a connected smooth projective variety, α is a real valued harmonic one form (resp. complex valued holomorphic one form) on X, and $D \subset X$ is a closed subvariety such that $\alpha|_{D^{reg}} = 0$ (resp. D is empty in the case of a complex valued holomorphic form). Let $p\colon Y \to (X-D)$ be any covering space such that the function $g(y) = \int_{y_0}^{y} p^*(\alpha)$ is well defined. There are three possibilities.*

1. *The one form α is identically zero.*
2. *There exists a smooth projective algebraic curve C, a morphism $f\colon X \to C$ with connected fibers, and a harmonic one-form β on C such that $\alpha = f^*(\beta)$.*
3. *For any $v \in \mathbf{R}$ (resp. $v \in \mathbf{C}$), the fiber $g^{-1}(v)$ is connected and the map $\pi_1(g^{-1}(v)) \to \pi_1(Y)$ is surjective.*

Here is how one can see the analogy with the usual Lefschetz theorems. In the present case, the one form α determines a map to an abelian variety $X \to A$ (cf. below). Let Z be the covering of X corresponding to the vector space covering A. The fibers $g^{-1}(v)$ are the intersections of Z with linear hyperplanes in the vector space. Conclusion (3) is analogous to the conclusion of the classical Lefschetz theorem for hyperplane sections of varieties of dimension greater than or equal to two in projective space. The other possibilities correspond to the fact that the image of Z might have dimension zero or one.

Preliminary discussion

Let X be a connected smooth projective variety. Suppose $D \subset X$ is a closed reduced subvariety; let D^{reg} denote the open set of smooth points in D. Put $X^* = X - D$. We will treat two similar situations at the same time; we call these the *complex case* and the *real case*. In the complex case we suppose $\alpha \in H^0(X, \Omega_X^1)$ is a holomorphic one-form such that $\alpha|_{D^{reg}} = 0$. In the real case we suppose that α is a real valued harmonic one-form such that $\alpha|_{D^{reg}} = 0$. In the real case, there is a unique holomorphic one-form $\alpha^h \in H^0(X, \Omega_X^1)$ such that $\alpha = \Re\alpha^h$. In the complex case we set $\alpha^h = \alpha$. Let $K = \mathbf{C}$ in the complex case, and $K = \mathbf{R}$ in the real case. Let $B(x, \varepsilon) \subset K$ denote the set of points $y \in K$ such that $|x-y| < \varepsilon$, and $\bar{B}(x, \varepsilon)$ its closure. Let $\partial B(x, \varepsilon)$ denote the boundary, which is a circle of radius ε in the complex case, two points in the real case. Let $B_{\mathbf{C}}(x, \varepsilon)$ denote the disc of radius ε around x in $\mathbf{C}$.

The albanese map determined by α

Fix a base point $x_0 \in X^*$, not in the zero set of α. Let $\mathrm{Alb}(X)$ denote the albanese variety $H^0(X, \Omega^1_X)^*/H_0(X, \mathbf{Z})$. Integration from x_0 determines a natural map

$$X \to \mathrm{Alb}(X)$$

which sends x_0 to the origin; and the one-form α is pulled back from a linear one-form which we call α_{Alb} on $\mathrm{Alb}(X)$. Let B be the sum of all abelian subvarieties of $\mathrm{Alb}(X)$ on which α_{Alb} vanishes. Let

$$A = \mathrm{Alb}(X)/B$$

and consider the map $\psi: X \to A$. We will call this the *albanese map determined by* α. Note that α_{Alb} projects to a linear one form α_A on A, and $\alpha = \psi^*(\alpha_A)$. The pair (A, α_A) has the property that if A' is a nontrivial abelian subvariety of A, then the restriction of α_A to A' is nonzero.

Let $\tilde{A}$ denote the universal covering of A (it is a complex vector space). Set

$$Z = X \times_A \tilde{A}.$$

Let $\pi: Z \to X$ denote the projection on the first factor, and $\tilde{\psi}: Z \to \tilde{A}$ the projection on the second factor. Let $Z^* = Z - \pi^{-1}(D)$. Fix a point $z_0 \in Z$ projecting to $x_0 \in X$.

There is a function $g_{\tilde{A}}: \tilde{A} \to K$ defined by the properties that $g_{\tilde{A}}(\tilde{\psi}(z_0)) = 0$ and $d(g_{\tilde{A}}) = \alpha_{\tilde{A}}$. Let $g = g_{\tilde{A}} \circ \tilde{\psi}$. It is characterized by the same properties on Z, and can be given by the integral

$$g(z) = \int_{z_0}^{z} \alpha.$$

The integral is independent of the choice of path.

The critical locus

Let $S \subset X$ be the set of points x such that $\alpha(x) = 0$. Let $\Sigma = S \cup D$.

LEMMA 2. *The set S is a closed algebraic subvariety of X. The image $\psi(\Sigma)$ is a finite set.*

Proof. In the complex case it is clear that S is an algebraic subvariety. For the real case, we claim that the zeros of α are the same as the zeros of α^h. Taking the real part of a linear form gives a map

$$\mathrm{Hom}_{\mathbf{C}}(T(X)_x, \mathbf{C}) \to \mathrm{Hom}_{\mathbf{R}}(T_{\mathbf{R}}(X)_x, \mathbf{R}).$$

This is an isomorphism, so $\alpha(x) = 0$ if and only if $\alpha^h(x) = 0$. Thus S, being the zero set of the holomorphic one form α^h, is an algebraic subvariety.

Suppose $\Sigma' \to \Sigma$ is a resolution of singularities of an irreducible component of Σ. We obtain a map $\mathrm{Alb}(\Sigma') \to \mathrm{Alb}(X)$ (a translate of a morphism of abelian varieties), compatible with the morphisms $\Sigma' \to \mathrm{Alb}(\Sigma')$ and $X \to \mathrm{Alb}(X)$. The pullback of α to Σ' is zero. Hence the image of $\mathrm{Alb}(\Sigma')$ is contained in a translate of the abelian subvariety B defined above. When we project to $A = \mathrm{Alb}(X)/B$, $\mathrm{Alb}(\Sigma')$, and hence Σ', map to a single point. □

Write $\psi(\Sigma) = \{\sigma_1, \ldots, \sigma_r\}$ with σ_i distinct. Let $\Sigma_i = \psi^{-1}(\sigma_i)$, so we have a decomposition

$$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$$

into disjoint pieces. Note that the pieces are not necessarily connected.

Let $d_A(x, y)$ denote a euclidean metric on A. Let $B_A(x, \varepsilon)$ denote the open ball $\{y \in A, d_A(x, y) < \varepsilon\}$, and let $\bar{B}_A(x, \varepsilon)$ denote its closure. Assume that ε_1 is small enough so that $\bar{B}_A(x, \varepsilon_1)$ is simply connected for all $x \in A$, and such that for distinct points $\sigma_i \neq \sigma_j$ in $\psi(\Sigma)$, $d_A(\sigma_i, \sigma_j) \geqslant 4\varepsilon_1$.

Put $M_{A,i} = \bar{B}_A(\sigma_i, \varepsilon_1)$. Let $M^o_{A,i}$ denote the interior (the open ball) and $\partial M_{A,i}$ the boundary. Set $M_i = \psi^{-1}(M_{A,i})$, $M^o_i = \psi^{-1}(M^o_{A,i})$, and $\partial M_i = \psi^{-1}(\partial M_{A,i})$. Note that M^o_i is the interior of M_i and ∂M_i is the boundary. The M^o_i are open neighborhoods of the $\psi^{-1}(\sigma_i)$.

Define functions $g_{A,i}: M_{A,i} \to K$ by $g_{A,i}(\sigma_i) = 0$ and $d(g_{A,i}) = \alpha_A|_{M_{A,i}}$. These functions are well defined because $M_{A,i}$ are connected and simply connected. Set

$$g_i = g_{A,i} \circ \psi: M_i \to K.$$

Similarly, define $g^h_{A,i}: M_{A,i} \to \mathbf{C}$ by $g^h_{A,i}(\sigma_i) = 0$ and $d(g^h_{A,i}) = \alpha^h|_{M_{A,i}}$. Set

$$g^h_i = g^h_{A,i} \circ \psi: M_i \to \mathbf{C}.$$

Note that in the complex case $g_i = g^h_i$ and in the real case $g_i = \Re g^h_i$.

LEMMA 3. *If we choose ε_1 sufficiently small, then ∂M_i is smooth and M_i is a tubular neighborhood of $\psi^{-1}(\sigma_i)$. And there exists $\varepsilon_2 > 0$ such that on the set*

$(g_i^h)^{-1}(B_{\mathbf{C}}(0, 2\varepsilon_2)) \cap \partial M_i$, the differential $d(g_i^h)|_{\partial M_i}$ is nonvanishing.

Proof. The distance function $d_A(\psi(x), \sigma_i)^2$ is real analytic and proper on the inverse image of a neighborhood of σ_i. The set of values of ε_1 such that ∂M_i is not smooth is the image of the closed real analytic critical point set of this function, and by Sard's theorem it is countable. Hence it is finite. Thus if ε_1 is small enough then ∂M_i is always smooth. Morse theory implies that there is a deformation retraction from M_i to $\psi^{-1}(\sigma_i)$ (we take this as the definition of tubular neighborhood).

Similarly if ε_1 is sufficiently small then $\partial M_i \cap (g_i^h)^{-1}(0)$ is smooth. Thus we may choose ε_2 so that $\partial M_i \cap (g_i^h)^{-1}(v)$ is smooth for $v \in B_{\mathbf{C}}(0, \varepsilon_2)$. This smoothness is equivalent to the nonvanishing of $(dg_i^h)|_{\partial M_i}$. □

Fix ε_1 and ε_2 as in the lemma, simultaneously for all M_i. Put

$$N_i = (g_i^h)^{-1}(\bar{B}_{\mathbf{C}}(0, \varepsilon_2)) \cap M_i.$$

LEMMA 4. *N_i is a smooth manifold with corners. Its boundary decomposes*

$$\partial N_i = T_i \cup R_i \cup C_i$$

where $T_i = (g_i^h)^{-1}(B_{\mathbf{C}}(0, \varepsilon_2)) \cap \partial M_i$ and $R_i = (g_i^h)^{-1}(\partial B_{\mathbf{C}}(0, \varepsilon_2)) \cap M_i^\circ$ are smooth pieces, and $C_i = (g_i^h)^{-1}(\partial B_{\mathbf{C}}(0, \varepsilon_2)) \cap \partial M_i$ is a smooth corner.

Proof. The function g_i^h is smooth outside of Σ_i, in particular it is smooth near the boundary of N_i. This implies that the R_i are smooth. The previous lemma implies that T_i is smooth, and the nonvanishing of the differential dg_i^h restricted to ∂M_i implies that C_i is a smooth corner. □

Let $F_i = g_i^{-1}(0) \cap N_i$. Put $F_i^\circ = F_i \cap N_i^\circ$ and $\partial F_i = F_i \cap \partial N_i$. Note that Σ_i is a compact subvariety of F_i°.

Some homotopy theory

A *pair* (U, V) consists of a topological space U and a subspace $V \subset U$. Recall that (U, V) is *k-connected* if, for any $l \leqslant k$ and any continuous map of the l-dimensional ball $h: \bar{B}^l \to A$ such that $h(\partial B^l) \subset V$, there exists a continuous map $\varphi: \bar{B}^l \times [0, 1] \to U$ such that $\varphi(x, 0) = h(x)$, $\varphi(\bar{B}^l \times \{1\}) \subset V$, and $\varphi(\partial B^l \times [0, 1]) \subset V$.

We have the following properties, which are either standard or easy:

5.1. *If φ exists as required in the definition, then we may choose it in such a way that $\varphi(y, t) = y$ for $y \in \partial B^l$.*

5.2 (Transitivity). *Suppose $W \subset V \subset U$, and (V, W) is k-connected. Then (U, V) is k-connected if and only if (U, W) is k-connected.*

5.3 (Deformation). *Suppose (U, V) is a pair, and $U' \subset U$, $V' \subset U' \cap V$. Suppose $f: U \times [0,1] \to U$ is a continuous map such that $f(u,0) = u$, $f(V \times [0,1]) \subset V$, $f(U' \times [0,1]) \subset U'$, $f(U, \{1\}) \subset U'$ and $f(V, \{1\}) \subset V'$. Then (U, V) is k-connected if and only if (U', V') is k-connected.*

5.4 (Excision I). *Suppose $W \subset V \subset U$. Suppose that the U-closures $\bar{W}$ and $\overline{U-V}$ are disjoint. Then (U, V) is k-connected if and only if $(U-W, V-W)$ is k-connected.*

One uses the trick of dividing up the ball $\bar{B}^l$ into small pieces, each of which maps either into $U-W$ or V, then treating the resulting complex inductively one skeleton at a time.

5.5 (Excision II). *Suppose $W \subset V \subset U$. Suppose that there exists $W' \subset W$ such that $\overline{W'} \cap \overline{U-V} = \varnothing$ and there is a deformation from the pair $(U-W', V-W')$ to the pair $(U-W, V-W)$ (in the sense of* (3) *above). Then (U, V) is k-connected if and only if $(U-W, V-W)$ is k-connected.*

This is a combination of 5.3 and 5.4.

5.6 (Exhaustion). *Suppose U (resp. V) is an infinite union of open subsets U_i (resp. V_i), with $V_i \subset U_i$. If the pairs (U_i, V_i) are k-connected then (U, V) is k-connected.*

This follows from compactness of $\bar{B}^l$ and ∂B^l–the image of any map h is contained in one of the U_i or V_i.

5.7 (Expression in terms of homotopy groups). *The pair (U, V) is k-connected if and only if the inclusion induces isomorphisms $\pi_i(V) \cong \pi_i(U)$ for $i < k$ and a surjection $\pi_i(V) \to \pi_i(U)$ for $i = k$.*

This comes from the definition of homotopy groups. Note that the same result holds for homology groups, but that property does not characterize k-connectedness.

5.8 (Covering spaces). *Suppose $p: Y \to U$ is a covering space, and $V \subset U$ is a subspace. Then (U, V) is k-connected if and only if $(Y, p^{-1}(V))$ is k-connected.*

This is because the covering map p has the homotopy lifting property.

Local topology near Σ_i

Fix a point $\eta \in B(0, \varepsilon_2)$, $\eta \neq 0$. Let $E_i = g_i^{-1}(\eta) \subset N_i$ be the *nearby fiber* at Σ_i. Let $D_i = D \cap \Sigma_i$, and let $N_i^* = N_i - D_i$. Let $F_i^* = F_i - D_i$.

The map $g_i^h: N_i \to B_{\mathbf{C}}(0, \varepsilon_2)$ is a fibration outside of $(g_i^h)^{-1}(0)$ (and D_i is in the fiber over zero, so the same is true of the function restricted to N_i^*). The proof of this well known statement is similar to the proof of Theorem 14 below, so we do not give the details.

In the real case, E_i is equal to the inverse image by g_i^h of a line segment not containing the origin in $B_{\mathbf{C}}(0, \varepsilon_2)$. Hence if we set E_i^h equal to the inverse image by g_i^h of a nonzero point, the pair (N_i^*, E_i) is homotopically equivalent to the pair (N_i^*, E_i^h) arising in the corresponding complex case.

LEMMA 6. *Suppose $V \subset B(0, \varepsilon_2)$ is a nonempty open contractible subset such that $0 \in V$. Then $(N_i^*, g_i^{-1}(V) \cap N_i^*)$ is k-connected for all k.*

Proof. In the real case, let $V' \subset B_{\mathbf{C}}(0, \varepsilon_2)$ be the set of points y such that $\Re y \in V$. Then $g_i^{-1}(V) = (g_i^h)^{-1}(V')$, and V' is a nonempty open contractible subset containing 0. Thus we may reduce to the complex case, which we now suppose. Choose a deformation retraction from $B(0, \varepsilon_2)$ to V obtained by flowing along a vector field which vanishes in the neighborhood of the origin, and is transverse to the boundary pointing inward. This can be lifted to a vector field on N_i, vanishing in a neighborhood of Σ_i. The flow along the vector field is a retraction from N_i^* to $g_i^{-1}(V) \cap N_i^*$. This implies the k-connectedness for all k. □

LEMMA 7. *Suppose $V \subset B(0, \varepsilon_2)$ is a nonempty open contractible subset. Suppose that (N_i^*, E_i) is k-connected. Then $(N_i^*, g_i^{-1}(V) \cap N_i^*)$ is k-connected.*

Proof. The previous lemma treats the case where V contains the origin, so suppose $0 \notin V$. As in the proof of the previous lemma, we may assume we are in the complex case. The map $g: N_i^* \to B(0, \varepsilon_2)$ is a fiber bundle with fiber E_i outside of the fiber over the origin. If we choose $v \in V$, the pair $(N_i^*, g_i^{-1}(v))$ is homotopic to the pair (N_i^*, E_i) and thus k-connected. The pair $(g_i^{-1}(V), g_i^{-1}(v))$ is k-connected, since it is isomorphic to $E_i \times (V, v)$. Apply Property 5.2 to complete the proof. □

LEMMA 8. *Suppose that $K = \mathbf{C}$ and the dimension of the image $\psi(X) \subset A$ is $\geqslant 2$. Then the pairs (N_i^*, E_i) are 1-connected.*

Proof. The connected components of N_i^* and N_i correspond. Choose one of these connected components $N_{i,1}$, and let $T_{i,1}$ be the corresponding part of the boundary piece T_i.

We claim that $T_{i,1}$ is nonempty. Suppose the contrary. Then every fiber of $g_{i,1}: N_{i,1} \to B(0, \varepsilon_2)$ is compact without boundary. These fibers map to open balls in A, hence they must map to finite sets. In other words, $\psi(g_{i,1}^{-1}(v))$ is a finite set for any $v \in B(0, \varepsilon_2)$. This implies that $\dim \psi(N_{i,1}) \leqslant 1$. Hence $d\psi$ has rank $\leqslant 1$ on $N_{i,1}$. As this is an open subset of the connected variety X, the rank of $d\psi$ is $\leqslant 1$ everywhere, contradicting the hypothesis that $\dim \psi(X) \geqslant 2$. This proves that $T_{i,1}$ is nonempty.

By Lemma 3, the map $g_{i,1}: T_{i,1} \to B(0, \varepsilon_2)$ is smooth. Since the ball is contractible, this implies that there exists a C^∞ section $s: B(0, \varepsilon_2) \to T_{i,1}$. In particular, the fiber E_i contains a point $s(\eta)$ in our component $N_{i,1}^*$. To complete the proof of 1-connectedness, it suffices to show that if $\gamma: [0,1] \to N_i^*$ is a path with $\gamma(0) = s(\eta)$ and $\gamma(1) \in E_i$ then γ can be deformed to a path in E_i while leaving fixed the endpoints. First note that by deforming slightly we may assume γ does not meet the fiber $g_i^{-1}(0)$. Now γ is a path in $N_i^{**} = N_i - g_i^{-1}(0)$, such that $g_i(\gamma)$ is a loop in $B^*(0, \varepsilon_2) = B(0, \varepsilon_2) - \{0\}$ based at η. Let $\xi = sg_i(\gamma)$. This is a loop in N_i^{**} based at $s(\eta)$, and it is contractible in T_i, hence in N_i^*. The product $\gamma\xi^{-1}$ projects to a contractible loop in $B^*(0, \varepsilon_2)$. The map $g_i: N_i^{**} \to B^*(0, \varepsilon_2)$ is a fibration, so we may lift the contraction of $g_i(\gamma\xi^{-1})$ to a homotopy from $\gamma\xi^{-1}$ to a path γ' in E_i (fixing the endpoints). Thus γ is homotopic to $\gamma\xi^{-1}\xi$ which is homotopic to $\gamma'\xi$ in N_i^{**}, and $\gamma'\xi$ is homotopic to γ' in N_i^* – all of this with the endpoints fixed. This completes the proof of the 1-connectedness. □

LEMMA 9. *Suppose that $K = \mathbf{R}$ and the dimension of the image $\psi(X) \subset A$ is $\geqslant 2$. Then the pairs (N_i^*, E_i) and (N_i^*, F_i^*) are 1-connected.*

Proof. As described previously, the pair (N_i^*, E_i) is equivalent to the pair (N_i^*, E_i^h) which arises in the complex case. Thus the first statement follows from the previous lemma. For the second statement, let $I \subset B_{\mathbf{C}}(0, \varepsilon_2)$ denote the imaginary axis. Decompose into a disjoint union $I = I^+ \cup \{0\} \cup I^-$ where I^+ and I^- are the positive and negative parts. Note that $F_i^* = (g_i^h)^{-1}(I) - D_i$, so

$$F_i^* = (g_i^h)^{-1}(I^+) \cup (F_i^h - D_i) \cup (g_i^h)^{-1}(I^-),$$

where $F_i^h = (g_i^h)^{-1}(0)$. The pair $(N_i^*, (g_i^h)^{-1}(I^+))$ is 1-connected, by Lemma 7 and the 1-connectivity of (N_i^*, E_i^h). This proves that the inclusion $F_i^* \to N_i^*$ induces surjections on π_0 and π_1. By Property 5.7, it suffices to show that it induces an injection on π_0. For this, it suffices to show that every connected component of F_i^* contains a component of $(g_i^h)^{-1}(I^+)$. If $x \in F_i^h$ then we can choose a holomorphic map from a complex disc into X, with the origin mapped to x, and the disc not mapped into F_i^h. The composition of this map with g_i^h is a holomorphic map from a disc to $B_{\mathbf{C}}(0, \varepsilon_2)$. The inverse image of I^+ in the disc contains the origin in its closure, so x is in the closure of $(g_i^h)^{-1}(I^+)$. Suppose C is a component of $(g_i^h)^{-1}(I^-)$. Then C contains a connected component of some fiber of the form E_i^h. From the proof of Lemma 8, it follows that C contains a point $y \in T_i \subset \partial N_i$. But the map $g_i^h: T_i \to B_{\mathbf{C}}(0, \varepsilon_2)$ is smooth and proper, so we can lift the segment I to a path in $(g_i^h)^{-1}(I) \cap T_i$ containing y. This path also contains a point of $(g_i^h)^{-1}(I^+)$. Hence the component of F_i^* containing C also contains a component of $(g_i^h)^{-1}(I^+)$. This completes the proof. □

LEMMA 10. *Suppose that D is empty. Then the pairs (N_i^*, F_i^*) are k-connected for all k.*

Proof. One can show that $N_i^* = N_i$ retracts onto $F_i^* = F_i$ (cf. [2]). □

We also recall a result of Milnor.

PROPOSITION 11. *Suppose D is empty. Suppose that the map $\psi: X \to A$ is finite, and* $\dim(X) = k$. *Then the pair (N_i, E_i) is $k-1$-connected.*

Proof. The singular set Σ_i is finite. The neighborhood N_i is a disjoint union of contractible neighborhoods of points of Σ_i. On each of these components, E_i is the nearby fiber of a function with an isolated singularity. Milnor shows that E_i has the homotopy type of a bouquet of $k-1$-spheres [5]. By looking at homotopy groups this implies that (N_i, E_i) are $k-1$-connected. □

Finally we have to collect these results in the form they will be used in below.

LEMMA 12. *Suppose $V \subset U \subset B(0, \varepsilon_2)$, where U is an open ball and V is nonempty, either a contractible open subset or a single point. If $(N_i^*, g_i^{-1}(V) \cap N_i^*)$ is k-connected then so is the pair $(g_i^{-1}(U) \cap N_i^*, g_i^{-1}(V) \cap N_i^*)$.*

Proof. If U does not contain the origin, then this follows from the fact that g_i is a fibration outside of $g_i^{-1}(0)$. Thus we may assume $0 \in U$. By Property 5.6, we may replace V by a subset which is relatively compact in U. Then we can construct a deformation from $B(0, \varepsilon_2)$ into U which fixes V and a neighborhood of the origin, and maintains U mapping into itself. This can be lifted to a deformation from $(N_i^*, g_i^{-1}(V) \cap N_i^*)$ to $(g_i^{-1}(U) \cap N_i^*, g_i^{-1}(V) \cap N_i^*)$. We may then apply Property 5.3. □

COROLLARY 13. *Suppose $V \subset U \subset B(0, \varepsilon_2)$, where U is an open ball and V is nonempty, either a contractible open subset or a single point. We obtain the following statements about connectedness of the pair $(g_i^{-1}(U) \cap N_i^*, g_i^{-1}(V) \cap N_i^*)$. If $K = \mathbf{R}$ and* $\dim \psi(X) \geq 2$ *then it is 1-connected. If $K = \mathbf{C}$,* $\dim \psi(X) \geq 2$, *and D is empty then it is 1-connected. If $K = \mathbf{C}$,* $\dim \psi(X) \geq 2$, *and V is open, then the pair is 1-connected. If D is empty, the map $\psi: X \to A$ is finite, and* $\dim(X) = k$ *then the pair is $k-1$-connected.*

Proof. This results from combining the above statements. Note that if V is a single point, then the fiber over V is either E_i or F_i^*. □

Main theorem

For each $i = 1, \ldots, r$ let J_i be an index set whose elements correspond to the points of $\tilde{A}$ lying over $\sigma_i \in A$. Label those points by σ_j. The inverse image of $M_{A,i}$ in $\tilde{A}$ decomposes as a disjoint union $\bigcup_{j \in J_i} M_{\tilde{A},j}$ of components $M_{\tilde{A},j}$ which project isomorphically to $M_{A,i}$. Let $M_j = \tilde{\psi}^{-1}(M_{\tilde{A},j}) \subset Z$. The projection $\pi: Z \to X$ provides an isomorphism of M_j onto its image M_i. Keep the same notations as above, but with subscripts j, for the corresponding subsets of M_j: M_j^o, ∂M_j, Σ_j, F_j, N_j, T_j, R_j, C_j, *et cetera.* Combine the index sets into a disjoint union $J = J_1 \cup \cdots \cup J_r$.

Let $g_j = g_i \circ \pi \colon M_j \to K$. Let $a_j = g_{\tilde{A}}(\sigma_j)$. The function $g\colon Z \to K$ restricts to a translate

$$g|_{M_j}(z) = g_j(z) + a_j.$$

Fix a number δ with $0 < 5\delta < \varepsilon_2$. For each $b \in K$, let $J(b) \subset J$ denote the set of indices j such that $|b - a_j| < 3\delta$. Let $U_b = B(b, \delta)$ be the open disc of radius δ around b.

Suppose $b \in K$, and suppose $V \subset U_b$ is a *nonempty contractible subset.* Suppose that V satisfies the following condition.

(*) There exists a map $\zeta\colon U \times [0,1] \to U$ such that $\zeta(x,0) = x$, $\zeta(U \times \{1\}) \subset V$, and $\zeta(V \times [0,1]) \subset V$.

Define a subset of Z

$$P(b, V) = g^{-1}(V) \cup \bigcup_{j \in J(b)} (g^{-1}(U_b) \cap N_j).$$

Recall that $Z^* = Z - \pi^{-1}(D)$. Put $P^*(b, V) = P(b, V) \cap Z^*$. Put $Q(b) = g^{-1}(U_b)$ and $Q^*(b) = Q(b) \cap Z^*$.

THEOREM 14. *The pair $(Q^*(b), P^*(b, V))$ is k-connected for all k.*

Proof. Let $Q^L(b) = Q(b) - \bigcup_{j \in J(b)} (g^{-1}(U_b) \cap N_j^o)$. Let $F(b) = g^{-1}(b)$ and $F^L(b) = F(b) \cap Q^L(b)$. Note that the closure of $Q^L(b)$ is a manifold with corners. Its boundary consists of some pieces of the form $\partial N_j \cap Q(b)$ (for $j \in J(b)$), and one piece of the form $g^{-1}(\partial U_b)$. They meet in smooth corners. Similarly $F^L(b)$ is a smooth manifold with boundary, the boundary consisting of pieces of the form $\partial F(b) \cap \partial N_j$. We claim that there exists a trivialization

$$\Phi\colon Q^L(b) \cong F^L(b) \times U_b$$

such that $g = p_2 \circ \Phi$ (where p_2 is projection on the second factor in the product) and

$$\Phi(\partial N_j \cap Q(b)) = (\partial N_j \cap F(b)) \times U_b$$

for $j \in J(b)$.

We will prove the claim separately in the real and complex cases; consider first the complex case. Choose two vector fields $\mathbf{u}$ and $\mathbf{v}$ on X with the following properties. If x is not in any $N_i^o \cap (g_i^h)^{-1}(B_{\mathbf{C}}(0, \delta))$, then $dg(\mathbf{u}_x)$ and $dg(\mathbf{v}_x)$ should be the unit vectors in the real and imaginary directions respectively. So if $x \in T_i \subset \partial N_i$ then $\mathbf{u}_x$ and $\mathbf{v}_x$ should be tangent to T_i. It is possible to choose these vector fields because dg is nonzero outside $N_i^o \cap (g_i^h)^{-1}(B_{\mathbf{C}}(0, \delta))$, and has maximal rank when restricted to the boundary pieces T_i. Lift these to vector fields $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ on Z. They have uniformly bounded length measured with respect to the complete metric d_Z which is pulled back from X.

Construct Φ as follows. Let $U_{b,\mathbf{R}} = \{x \in U_b : x - b \in \mathbf{R}\}$. There is a deformation retraction from U_b to $U_{b,\mathbf{R}}$ obtained by making points flow in the positive or negative imaginary direction until reaching $U_{b,\mathbf{R}}$. Lift this to a retraction from $Q^L(b)$ to $Q^L(b) \cap g^{-1}(U_{b,\mathbf{R}})$, by flowing along the vector field $\tilde{\mathbf{v}}$. The fact that $\tilde{\mathbf{v}}$ is bounded with respect to the complete metric d_Z implies that the flow along the vector field exists. Note that if $j \in J(b)$ then $\partial N_j \cap Q(b) = T_j \cap Q(b)$, and the vector field is tangent to T_j, so the flow does not cross into the interior of N_j. If $j \notin J(b)$ then $Q(b)$ does not meet $N_i^o \cap (g_i^h)^{-1}(B_{\mathbf{C}}(0, \delta))$, by the definitions of $J(b)$, $Q(b)$, and U_b. In particular, the flow is always a lifting of the flow in U_b, so we obtain a lifting of the retraction. There is a deformation retraction from $U_{b,\mathbf{R}}$ to $\{b\}$ where points flow in the positive or negative real direction. Similarly this can be lifted to a deformation retraction from $Q^L(b) \cap g^{-1}(U_{b,\mathbf{R}})$ to $F^L(b)$, by flowing along the vector field $\tilde{\mathbf{u}}$. The composition gives a map $Q^L(b) \to F^L(b)$ which serves as the projection of Φ on the first factor. The projection on the second factor is given by g. The fact that the vector fields are tangent to T_j implies that the map $p_1 \circ \Phi$ preserves $\partial N_j \cap Q(b)$.

We treat the real case in a similar fashion. Choose a vector field $\mathbf{u}$ on X such that if x is not in any $N_i^o \cap (g_i^h)^{-1}(B_{\mathbf{C}}(0, \delta))$, then $dg(\mathbf{u}_x)$ is a unit vector in $T(K) = \mathbf{R}$, and if $x \in \partial N_i \cap g_i^{-1}(B(0, 4\delta))$ then $\mathbf{u}_x$ is tangent to ∂N_i. Again note that dg is nonzero outside $N_i^o \cap (g_i^h)^{-1}(B_{\mathbf{C}}(0, \delta))$. We have to check that dg has maximal rank when restricted to the boundary $\partial N_i \cap g_i^{-1}(B(0, 4\delta))$. There is no problem on the pieces T_i. For the pieces R_i and the corners C_i note that, since $5\delta < \varepsilon_2$, the region indicated lies over the part of $\partial B_{\mathbf{C}}(0, \varepsilon_2)$ which projects submersively on the real axis. This pulls back to the desired property of g. Hence we can construct $\mathbf{u}$. Lift it to a vector field $\tilde{\mathbf{u}}$ on Z, again having bounded length with respect to d_Z. Use the flow along $\tilde{\mathbf{u}}$ to lift the standard retraction from $U_b \subset \mathbf{R}$ to $\{b\}$, to a retraction from $Q^L(b)$ to $F^L(b)$. This works as before, in view of the definition of $J(b)$. The retraction preserves $\partial N_j \cap Q(b)$, and gives the first projection of Φ. Again g serves as the second projection. This completes the proof of the claim.

To prove the theorem, note that

$$Q^*(b) = Q^L(b) \cup \bigcup_{j \in J(b)} (g^{-1}(U_b) \cap N_j^*),$$

with the boundary between the two pieces being $\bigcup_{j \in J(b)} (g^{-1}(U_b) \cap \partial N_j)$. This boundary corresponds to $\partial F^L(b) \times U_b$ via the identification Φ. This boundary has a collar which retracts to it in the complement of the interior of $Q^L(b)$. The second piece of this union is contained entirely inside $P(b, V)$. Hence we may apply Property 5.5 (Excision II). Set $P^L(b, V) = P(b, V) \cap Q^L(b)$. Then the connectedness of $(Q^*(b), P^*(b, V))$ is the same as that of $(Q^L(b), P^L(b, V))$. But our identification gives

$$\Phi: P^L(b, V) \cong F^L(b) \times V \cup \partial F^L(b) \times U_b.$$

If we apply the deformation given by condition (*) in the second variable, we see that the pair

$$(F^L(b) \times U_b, F^L(b) \times V \cup \partial F^L(b) \times U_b)$$

is k-connected for all k. Via the homeomorphism Φ we obtain the same result for $(Q^L(b), P^L(b, V))$, and hence the theorem. □

COROLLARY 15. *Suppose that $V \subset U_b$ is an open, nonempty contractible subset. Suppose that all the pairs (N_i^*, E_i) are k-connected. Then the pair $(Q(b), g^{-1}(V) \cap Z^*)$ is k-connected.*

Proof. By Lemmas 7 and 12, the pairs $(g^{-1}(U_b) \cap N_j^*, g^{-1}(V) \cap N_j^*)$ are k-connected. By Property 5.5, this implies that the pair $(P^*(b, V), g^{-1}(V) \cap Z^*)$ is k-connected (it is easy to see that there exist the collars of $\partial F^L(b) \times V$ in $F^L(b) \times V$ required to satisfy the hypothesis needed for Excision II). Property 5.2 and Theorem 14 now imply that $(Q(b), g^{-1}(V) \cap Z^*)$ is k-connected. □

COROLLARY 16. *Suppose that $V = \{v\} \subset U_b$ consists of a single point. Suppose that all the pairs (N_i^*, E_i) and (N_i^*, F_i^*) are k-connected. Then the pair $(Q(b), g^{-1}(V) \cap Z^*)$ is k-connected.*

Proof. This is the same as the previous proof, referring at the beginning to Lemma 12 and the remark in the proof of Corollary 13. □

The global results

THEOREM 17. *Suppose that all the pairs (N_i^*, E_i) are k-connected. Suppose that $V \subset U_b \subset K$ is a contractible open subset satisfying condition (*). Then the pair $(Z^*, g^{-1}(V) \cap Z^*)$ is k-connected. Suppose in addition that the pairs (N_i^*, F_i^*) are k-connected. Then for any $v \in K$ the pair $(Z^*, g^{-1}(v) \cap Z^*)$ is k-connected.*

Proof. Choose a sequence of subsets $W_0, W_1, \ldots \subset K$, starting with $W_0 = V$, with the following properties. There exists a sequence of points $b_1, b_2, \ldots \in K$ such that $W_i = W_{i-1} \cup U_{b_i}$, and such that $U_{b_i} \cap W_{i-1}$ is an open contractible subset of U_{b_i} satisfying condition (*). Finally, $K = \bigcup_{i=0}^{\infty} W_i$. This is easy to do if $K = \mathbf{R}$, and can be done with a clever picture if $K = \mathbf{C}$ (beginning with $W_1 = U_b$). Corollary 15 implies that the pairs $(Z^* \cap g^{-1}(U_{b_i}), Z^* \cap g^{-1}(U_{b_i} \cap W_{i-1}))$ are k-connected. Since W_{i-1} is open and the complement of U_b is closed, we can apply

Excision (I) to conclude that the pairs $(Z^* \cap g^{-1}(W_i), Z^* \cap g^{-1}(W_{i-1}))$ are k-connected. By transitivity, the pairs $(Z^* \cap g^{-1}(W_i), Z^* \cap g^{-1}(W_0))$ are k-connected. This implies that $(Z^*, Z^* \cap g^{-1}(V))$ is k-connected (by 5.6).

For the second part of the theorem, use the same proof but replace V by $\{v\}$. Note that we do not use excision at the first step, so the proof does not need openness of W_0. Use Corollary 16 instead of 15. □

REMARK. The first statement will also hold for larger open sets V which are nice enough. Namely, those expressible as $V = \bigcup_{i=0}^{\infty} V_i$ for an increasing sequence V_i with $V_0 \subset U_{b_1}$ and $V_{i+1} = V_i \cup U_{b_i}$ such that $V_i \cap U_{b_i}$ satisfies (*). For in this case the same proof shows that $(g^{-1}(V) \cap Z^*, g^{-1}(V_0) \cap Z^*)$ is k-connected, and we may apply Property 5.2.

COROLLARY 18. *Suppose that* $\dim \psi(X) \geqslant 2$. *If* $K = \mathbf{C}$, *suppose that* D *is empty. Then for any* $v \in K$ *the pair* $(Z^*, g^{-1}(v) \cap Z^*)$ *is* 1*-connected.*

Proof. This follows from Lemmas 8, 9 and 10, and the theorem. □

COROLLARY 19. *Suppose that* D *is empty,* $\dim \psi(X) = k$, *and* ψ *is finite. Then for any* $v \in K$ *the pair* $(Z, g^{-1}(v))$ *is* $k-1$*-connected.*

Proof. This follows from Lemma 10 and Proposition 11, and the theorem. □

COROLLARY 20. *Suppose* $\pi_Y\colon Y \to X^*$ *is any covering space such that a function* $g_Y\colon Y \to K$ *may be defined with* $dg_Y = \pi_Y^* \alpha$. *Under the hypotheses of the previous two corollaries, the same conclusions hold for the pairs* $(Y, g_Y^{-1}(v))$.

Proof. Let $\tilde{X}$ be the universal covering of X^*. There is a function $g_{\tilde{X}}\colon \tilde{X} \to K$, with $dg_{\tilde{X}} = \pi_{\tilde{X}}^* \alpha$. There are covering maps $\tilde{X} \to Z^*$ and $\tilde{X} \to Y$. We may (by translation of g and g_Y) assume that these maps are compatible with the functions g. By Property 5.8, the k-connectivity of $(Z^*, g^{-1}(v) \cap Z^*)$ is equivalent to that of $(\tilde{X}, g_{\tilde{X}}^{-1}(v))$, which in turn is equivalent to that of $(Y, g_Y^{-1}(v))$. □

COROLLARY 21. *Suppose* A' *is an irreducible abelian variety,* $X \subset A'$ *is a smooth connected closed subvariety of dimension* k, *and* α *is the restriction of a linear holomorphic or real harmonic form on* A'. *Suppose* Y *is a covering space of* X *such that the function* g_Y *is defined. Then for any* $v \in K$ *the pair* $(Y, g_Y^{-1}(v))$ *is* $k-1$*-connected.*

Proof. In this case, A' has no abelian subvarieties so it is equal to the abelian variety A defined above. The map ψ is the closed immersion, so it is finite. Apply Corollary 19. □

A factorization statement

We have obtained a result valid when the dimension of $\psi(X)$ is at least two. If the dimension of the image is zero, then α is identically zero.

LEMMA 22. *Suppose* $\dim \psi(X) = 1$. *Then there exists a smooth projective algebraic curve* C, *a morphism* $f: X \to C$ *with connected fibers, and a holomorphic or harmonic one-form* β *on* C *such that* $\alpha = f^*(\beta)$. *The triple* (C, β, f) *is uniquely determined by* α, *up to unique isomorphism. The image of* D *is a finite set in* C.

Proof. Let C be the normalization of $im(\psi)$ in the function field of X; this gives the Stein factorization $\psi = pf$ where p is the finite map from C to $im(\psi)$ and $f: X \to C$ has connected fibers. The curve C is normal, hence smooth, and projective. Set $\beta = p^*\alpha$: this is a holomorphic one form on C, with $f^*\beta = \alpha$ on X. The image of D is a finite set, by Lemma 2.

To prove uniqueness, suppose (C', β', f') were another such triple. Let $\psi': C' \to A'$ denote the albanese map determined by β', with $A' = \mathrm{Alb}(C')/B'$ as before. By functoriality of the albanese construction we obtain a map $\mathrm{Alb}(X) \to \mathrm{Alb}(C')$, and α on $\mathrm{Alb}(X)$ is the pullback of β' on $\mathrm{Alb}(C')$. The image of $B \subset \mathrm{Alb}(X)$ is equal to $B' \subset \mathrm{Alb}(C')$. Therefore the map of albanese varieties gives an injection $A \to A'$. It is surjective because the map $f': X \to C'$ is surjective (since $\alpha \neq 0$). Hence $A' = A$. The albanese map for (X, α) factors as $\psi = \psi' f'$ since $\alpha = (f')^*\beta'$. But $\psi': C' \to im(C')$ is finite, and f' has connected fibers. The uniqueness of the Stein factorization gives an isomorphism $C' \cong C$ with respect to which $\psi' = p$ and $f' = f$. Finally, f is generically smooth, so if $f^*\beta = f^*\beta'$, then $\beta = \beta'$ (and similarly, the isomorphism $Y \cong Y'$ is unique). □

Proof of Theorem 1. First we treat the real harmonic case. If α is identically zero, we get conclusion (1). If not, then $\dim \psi(X) \geqslant 1$. If $\dim \psi(X) = 1$, we obtain conclusion (2) from the lemma above. If $\dim \psi(X) \geqslant 2$ then we obtain conclusion (3) from Corollary 18, Corollary 20, and Property 5.7. In the complex-valued holomorphic case, the proof is the same. Note however that we must assume D is empty. □

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