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The diameter function on the space of space forms*

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0. Introduction

A manifold whose fundamental group is finite and whose sectional curvature is bounded cannot have an arbitrarily small diameter. This follows, for example, from Gromov's result that for every dimension n, there exists $\varepsilon(n) > 0$ such that if the sectional curvature k_M of an n-dimensional manifold M is bounded, $|k_M| \leqslant 1$, and the diameter d_M of M satisfies $d_M < \varepsilon(n)$, M is finitely covered by a nilmanifold [3]. In particular, the fundamental group $\pi_1(M)$ is infinite. Since a manifold of positive Ricci curvature has finite fundamental group [7], such a manifold cannot have a diameter less than $\varepsilon(n)$. A spherical space form, that is, a manifold for which $k_M \equiv 1$, must therefore have a diameter greater than or equal to Gromov's $\varepsilon(n)$.

Gromov's result, of course, has far more general implications than simply forcing a lower bound on the diameter of space forms. Among other things, it forces a lower bound on the diameter of any manifold with bounded curvature and finite fundamental group. In spite of the wide range of manifolds to which these lower diameter bounds apply, the values of those lower bounds were not known, even in the simplest case – the case of manifolds of constant positive curvature. In this paper, we determine lower bounds on the diameters of spherical spaces forms and prove the following theorem:

THEOREM 0.1. *The infimum of diameters for spherical space forms is* $\frac{1}{2}$ arc-$\cos \dfrac{1}{\sqrt{3}}\left(\tan \dfrac{3\pi}{10}\right)$. *(It is not, however, the actual diameter of any space form.)*

This is an optimal lower bound, since it is the actual diameter of a space in the Hausdorff closure of space forms. It occurs in dimension three. Dimension three is special for several reasons; not only does the minimal lower bound on the diameter of space forms occur with respect to three-dimensional space forms, but the groups that act on S^3 to produce these space forms are the building blocks for groups that produce space forms of all other dimensions. These groups have been completely described by J. Wolf. Except for reducible actions

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and two other types of groups, all the elements of these groups may be represented as block matrices, in which each block is a two-by-two complex matrix that appears as an element of a fundamental group of a three-dimensional space form. In each such group, at least one generator permutes the complex coordinates in blocks of two. These groups only occur as subgroups of $SO(4n+4)$ and act on S^{4n+3}. In effect, the group permutes the coordinate three-spheres (S^3's) in S^{4n+3} and acts on each three-sphere as one of the groups acting on S^3.

The lower bounds we have determined for space forms are dependent on dimension, and increase as dimension increases. In every case, these lower bounds increase to $\pi/2$ as the dimension increases without bound. (For a complete list of the lower bounds on space forms, see p. 19.)

In the even dimensions, the only spherical space forms are the sphere and projective space. The smallest diameter that occurs when the curvature is normalized to one is therefore $\pi/2$. As a consequence of Klingenberg's injectivity radius estimate, this lower bound applies also for positively curved manifolds whose curvature is bounded above by one, not just to those of constant positive curvature. Since the even-dimensional lower bounds apply to manifolds of variable curvature, we are presented with the question: Does the lower bound determined here apply to manifolds of variable curvature also?

1. Limit spaces of space forms

Joseph Wolf has classified spherical space forms. In each odd dimension, there are infinitely many isometry types. They are of the form S^n/Γ, where Γ is a finite subgroup of $SO(n+1)$. Wolf described these space forms by classifying the underlying groups Γ. He enumerated six types of groups. While each group is finite, groups of each type may be found that are arbitrarily large [11]. (For a complete list of the groups, see p. 16.)

Let K be the set of compact subsets of S^n. If G is a closed subgroup of $SO(n+1)$, the orbits Gx in S^n are compact and therefore are elements of K. We define distance on K as the Hausdorff distance, $d_H(X, Y) = \inf\{\varepsilon > 0$: the ε-neighborhood of X contains Y and the ε-neighborhood of Y contains $X\}$. Under this Hausdorff distance, K is a compact set [5]. Observe that we may consider the space forms S^n/Γ as subsets of K. Relying on this observation, we need not expand the definition of Hausdorff distance beyond the simple one given above. We would like to determine when a sequence of these space forms, viewed as a subset of K, converges in the Hausdorff sense. We rely on the following easy result:

THEOREM 1.1. *If Γ_1 is close to Γ_2 in the Hausdorff sense in $SO(n+1)$, S^n/Γ_1 is close to S^n/Γ_2 in the Hausdorff sense in K.*

If Γ_i converges to a set G in $SO(n+1)$, G is a subgroup of $O(n+1)$. From Theorem 1.1, we can conclude that if Γ_i converges to G in the Hausdorff sense, S^n/Γ_i converges to S^n/G in the Hausdorff sense. Moreover, if M is a limit point in the set of space forms, then $M = \lim_{i\to\infty} S^n/\Gamma_i$, where Γ_i is an infinite sequence of groups at a type enumerated by Wolf. This fact comprises the easier part of our second theorem:

THEOREM 1.2. *If M is in the Hausdorff closure of the set of space forms, then $M = S^n/G$, where G is a subgroup of $O(n+1)$ whose identity component is a q-dimensional torus T^q and whose quotient by the identity component G/G^0 is finite, or else G itself is a finite group.*

Proof. We use the following theorem of C. Jordan: If S is a connected Lie group, then there exists an integer $r = r(S)$ with the property that if $F \subset S$ is a finite subgroup of S, then F admits an abelian normal subgroup F_0 such that $[F: F_0] \leqslant r(S)$ [8].

In our case, S is $SO(n+1)$. We have a sequence of finite groups $F_n \subset S$ such that F_n converges to a group $G \subset S$. By the theorem of Jordan, for each F_n, there exists F_{n0}, an abelian normal subgroup in F_n. We consider $G_0 = \lim F_{n0}$. We claim that G_0 is an abelian normal subgroup of G and that $[G: G_0] \leqslant r(SO(n+1))$.

G_0 is abelian and normal in G because its elements are limits of elements in the F_{n0} and because multiplication is continuous in $SO(n+1)$.

If G_0 has more than r cosets in G, then we have at least $r+1$ distinct sequences $f_{i_n} \in F_n$ such that $f_{i_n}F_{n0} \to g_iG_0$. Since $g_iG_0 \neq g_jG_0$ for $i \neq j$, there is a positive distance between g_iG_0 and g_jG_0. (This is because G_0 is closed.) But then for n large enough, $f_{i_n}F_{n0} \neq f_{j_n}F_{n0}$. So F_{n0} has at least $r+1$ distinct cosets. This contradicts the theorem. Therefore $[G: G_0] \leqslant r$, so $\dim G = \dim G_0$. Since $I \in G_0$, the identity component of G, G^0, must be contained in the abelian, normal subgroup G_0. We can conclude that G^0 is abelian.

Since G^0 is a compact, connected, abelian Lie group, it is isomorphic to T^q, the q-dimensional torus, for some $q \in \mathbb{N}$. G/G^0 is finite, because G is compact, and the order of G is the number of connected components of G. This completes the proof.

Often, it is easier to describe the space X/G^0 than X/G. We therefore will make repeated use of the fact that $X/G^0/G/G^0 = X/G$.

Another fact that we will find useful is stated in the following lemma:

LEMMA 1.3 (Inclusion lemma). *If H is a closed subgroup of a closed group of isometries G on M, M/H has a diameter at least as large as M/G.*

In this paper, we will consider only odd-dimensional spheres S^n and only irreducible actions by the groups Γ, since the other results are well-known.

2. Three-dimensional space forms

The spherical space forms of dimension three are of the form S^3/Γ, where $\Gamma \subset SO(4)$ is a finite group that acts freely and properly discontinuously on S^3. We recall from Wolf that the possible irreducible groups Γ may be described as follows [11].

Case 1. The group is generated by

$$A = \begin{bmatrix} e^{2\pi i/m} & \\ & e^{2\pi ik/m} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} & 1 \\ e^{4\pi il/t} & \end{bmatrix}.$$

These generators are subject to the further restrictions that $t \equiv 0 \bmod 4$, $(t(r-1), m) = 1$, $(l, t/2) = 1$ and $r \not\equiv r^2 \equiv 1 \bmod m$.

Case 2. Γ is generated by the two generators above, subject to the same restrictions, and has one additional generator: either

$$\begin{bmatrix} e^{2\pi i/4} & \\ & e^{-2\pi i/4} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} & e^{-2\pi il/t} \\ -e^{2\pi il/t} & \end{bmatrix}.$$

Case 3. Γ is generated by

$$\begin{bmatrix} e^{2\pi i\left(\frac{u+v}{uv}\right)} & \\ & e^{2\pi i\left(\frac{u-v}{uv}\right)} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} & 1 \\ -1 & \end{bmatrix},$$

where $v \equiv 0 \bmod 2$ and $(u, v) = 1$.

Cases 4–6. Γ is $\mathbb{Z}_u \times \mathbf{T}_v^*$, $\mathbb{Z}_u \times \mathbf{0}^*$ or $\mathbb{Z}_u \times \mathbf{I}^*$. These groups, $\mathbf{T}^*$, $\mathbf{0}^*$ and $\mathbf{I}^*$ are the preimages of the tetrahedral, octahedral, and icosahedral groups in the unit quaternious under the cover map $S^3 \to SO(3)$ by $q \to A$, where $q\,x\,q^{-1} = Ax$ for all $x \in \mathbb{R}^3$. (For the product $q\,x\,q^{-1}$, x is treated as a purely imaginary quaternion; that is, (x_1, x_2, x_3) is identified with $x_1 i + x_j j + x_3 k$.) $\mathbf{T}_v^*$ and $\mathbf{0}_v^*$ are augmentations of $\mathbf{T}^*$ and $\mathbf{0}^*$. That is, $\mathbf{T}_v^*$ is formed by substituting a generator of order 3^v for the generator of order 3 in $\mathbf{T}^*$.

THEOREM 2.1. *Let Γ_{mt} be a group of a type described in cases* 1, 2 *and* 3. *Then the diameter of S^3/Γ_{mt} is bounded below by $\pi/4$. If $G = \lim_{m\to\infty, t\to\infty} \Gamma_{mt}$, S^3/G is an interval.*

Proof. We prove the theorem for case 1 groups. The other proofs are similar. We note that the extra generators in case 2 are already present as elements of the limit group in case 1. We observe that B^2 is a scalar matrix, $B^2 = e^{4\pi il/t} \cdot I$. As m and t become infinitely large, $\langle A \rangle$ and $\langle B^2 \rangle$ converge to linearly independent circle actions, since $\langle B^2 \rangle$ converges to $\{e^{i\theta} \cdot I : \theta \in [0, 2\pi)\}$, and $\langle A \rangle$ converges to $\left\{ \begin{bmatrix} e^{i\varphi} & \\ & e^{ic\varphi} \end{bmatrix}, \varphi \in [0, 2\pi), c \neq 1 \right\}$. Therefore, $\langle A, B^2 \rangle$ converges to a two-torus of the form $\begin{bmatrix} e^{i\varphi} & \\ & e^{i\theta} \end{bmatrix}$, where $\varphi \in [0, 2\pi)$ and $\theta \in [0, 2\pi)$. The orbit space S^3/T^2 is therefore one dimensional. Since we can write $x \in S^3$ as $x = (r_2 e^{i\theta_1}, r_2 e^{i\theta_2})$, where $r_1^2 + r_2^2 = 1$, we find that the points for which $r_1 = 1$ or $r_1 = 0$ give rise to isotropy. This tells us that S^3/T^2 is an interval and that $r_1 = 1$ and $r_1 = 0$ correspond to the endpoints of that interval [6]. The diameter of S^3/T^2 is therefore the distance from $(1, 0)$ to $(0, 1)$ in the sphere, i.e., it is $\pi/2$.

If $G = \lim_{m \to \infty, t \to \infty} \Gamma_{mt}$, G has an additional generator outside of this two-torus. It is the limit of B as $t \to \infty$: $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. T^2 is merely the connected component of the identity in G. So $S^3/G = S^3/T^2/G/T^2$. Since $B^2 \in T^2$, $G/T^2 = \left\langle \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\rangle \cong \mathbb{Z}_2$. This last generator transposes $r_1 = 0$ and $r_1 = 1$, keeping the midpoint fixed. In effect, it folds the interval in half. The diameter of S^3/G, therefore, is $\pi/4$.

THEOREM 2.2. *If* $\Gamma = \mathbb{Z}_u \times \mathbf{T}_v^*$, S^3/Γ *has diameter at least* $\arccos\sqrt{2/3}$. *If* $\Gamma = \mathbb{Z}_u \times \mathbf{0}^*$, S^3/Γ *has diameter at least* $\frac{1}{2}\arccos 1/\sqrt{3}$. *If* $\Gamma = \mathbb{Z}_u \times \mathbf{I}^*$, S^3/Γ *has diameter at least* $\frac{1}{2}\arccos\left[\frac{1}{\sqrt{3}}\left(\tan\frac{3\pi}{10}\right)\right]$. *In each case, the limit is topologically* S^2.

Proof. Now we consider space forms of the type S^3/Γ, where Γ is the image of $\beta \otimes \alpha$, for some representation β of $\mathbb{Z}_u$ and some representation α of $\mathbf{T}_v^*$, $\mathbf{0}^*$ or $\mathbf{I}^*$. For any such representation, $\left\langle \begin{bmatrix} e^{2\pi i/u} & \\ & e^{2\pi i/u} \end{bmatrix} \right\rangle$ is a subgroup. As u becomes very large, this converges to the Hopf action on S^3: $(z_1, z_2) \to (e^{i\theta} z_1 \quad e^{i\theta} z_2)$. The limit space is therefore S^2 (of diameter $\pi/2$). The finite group (G/S^1) that acts on this S^2 is then $\mathbf{T}$, $\mathbf{0}$, or $\mathbf{I}$, respectively. That is, since $-I \in \{e^{i\theta} \cdot I, \theta \in [0, 2\pi)\}$, the actions of A and $-A$ in $\mathbf{T}_v^*, \mathbf{0}^*$, and $\mathbf{I}^*$ are identified on S^3/S^1. These are the only actions that are identified, since $A_1 \hat{z} = A_2 \hat{z}$ for all $\hat{z} \in S^3/S^1$ only if for any $z \in S^3$ there exists some $\theta \in [0, 2\pi)$ such that $A_1 z = A_2 e^{i\theta} z$. But then $A_2^{-1} A_1 z = e^{i\theta} z$, so z is an eigenvector for all $z \in S^3$ and $e^{i\theta}$ is an eigenvalue. The only elements of $\mathbf{T}_v^*$, $\mathbf{0}^*$ and $\mathbf{I}^*$ with this property are $\pm I$, since no other elements of the groups have two equal eigenvalues. Thus, $A_2 = \pm A_1$, and $S^3/S^1 \times \mathbf{T}^* = S^2/\mathbf{T}$. Therefore, the diameter of the limit space is the diameter of $S^2/\mathbf{T}$, where S^2 is the two-sphere of radius $\frac{1}{2}$.

$S^2/\mathbf{T}$ has fundamental domain shown in Fig. 1. Under the action of the tetrahedral group, AB is identified with AC, and BD with DC. The resulting limit space is (topologically) again S^2, but not a symmetric sphere. Its diameter is the

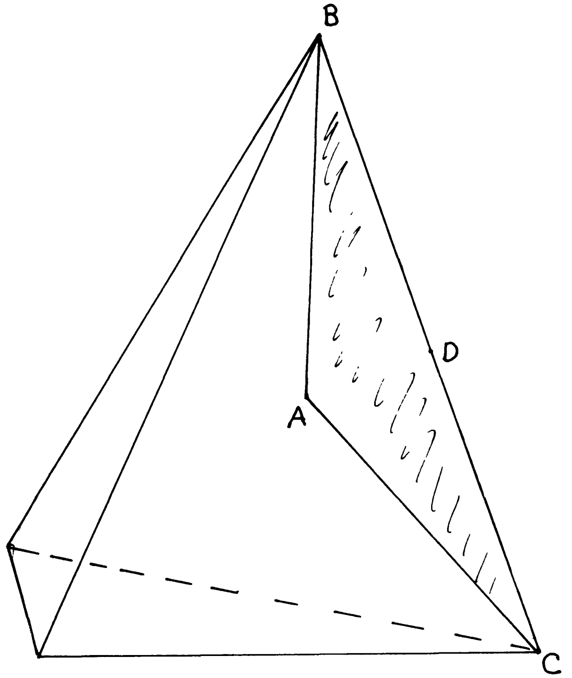

Fig. 1.

length of the curve AB. In the sphere of radius $\frac{1}{2}$, this is $\arccos\sqrt{\frac{2}{3}}$. Similarly, in $S^2/\mathbf{0}$, we have the fundamental domain on the sphere corresponding to the triangle shaded in Fig. 2. Again, AB and AC are identified, as are BD and CD. The diameter of this triangular "envelope" is the distance AC. In the sphere of diameter $\pi/2$, the distance AC is $\frac{1}{2}\arcsin\sqrt{\frac{2}{3}}$ or $\frac{1}{2}\arccos\frac{1}{\sqrt{3}}$. Similarly, the fundamental domain in S^2 under the action by $\mathbf{I}$ corresponds to the shaded triangle in Fig. 3. AB and AC are identified, as are BD and CD. The diameter of $S^2/\mathbf{I}$ therefore corresponds to the distance AC, which in the sphere of radius $\frac{1}{2}$ is

$$\frac{1}{2}\arcsin\frac{2}{\sqrt{3}}\sqrt{1-\frac{1}{4\cos^2(3\pi)/10}}=\frac{1}{2}\arccos\frac{1}{\sqrt{3}}\left(\tan\frac{3\pi}{10}\right).$$

We have omitted discussing the difference between $\mathbf{T}_v^*$ and $\mathbf{T}^*$. $\mathbf{T}^*$ has generators $X=\begin{bmatrix} e^{2\pi i/3} & \\ & e^{-2\pi i/3}\end{bmatrix}$, and P and Q, where P and Q correspond to flips across two edges of the tetrahedron in the tetrahedral group. $\mathbf{T}_v^*$ has

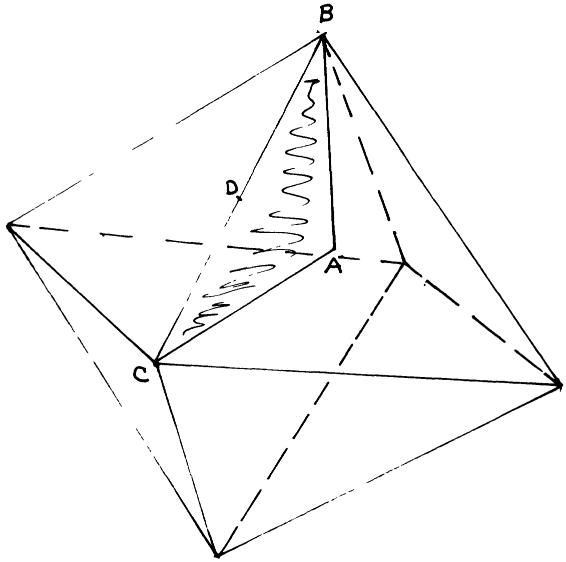

Fig. 2.

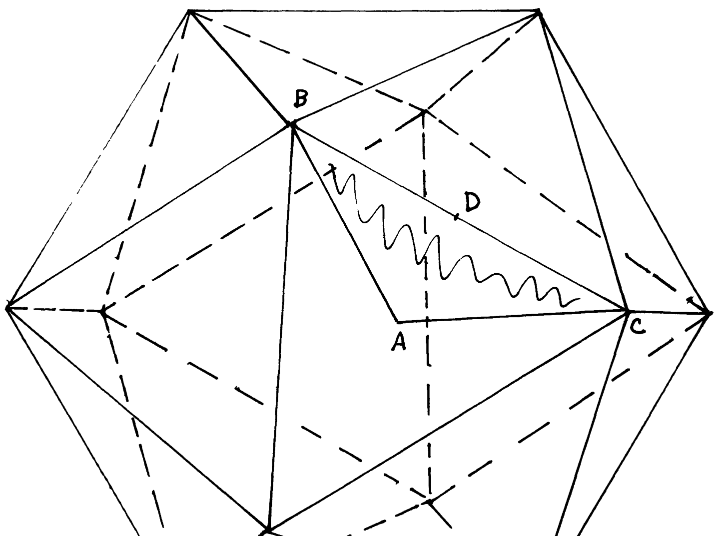

Fig. 3.

generators P, Q and X_v, where

$$X_v = \begin{bmatrix} e^{2\pi i(k+3^{v-1})/3^v} & \\ & e^{2\pi i(k-3^{v-1})/3^v} \end{bmatrix} = e^{2\pi ik/3^v} \cdot I \cdot X.$$

Since $e^{2\pi ik/3^v} \cdot I \in \{e^{i\theta} I,\ \theta \in [0, 2\pi)\}$,

$$S^3/S^1 \times \mathbf{T}_v^* = S^2/\mathbf{T}_v^* = S^2/\mathbf{T}.$$

In other words, the substitution of X_v for X does not affect the limiting case.

3. Space forms of dimension $n = 1 \bmod 4$

In this section, we will prove the following theorem:

THEOREM 3.1. *The diameter of a space form S^n/Γ, where $n \equiv 1 \bmod 4$, is bounded below by* $\arcsin \sqrt{\frac{n-1}{n+1}}$.

We find this limit by exploring the action of one type of group on the sphere S^n. As previously noted, Wolf enumerated six types of groups in the classification of space forms. (The list of the groups is located in Section 4.) However, only one type of group acts on S^n, where $n \equiv 1 \bmod 4$: groups of type I.

By a theorem of Vincent, a group Γ that is the fundamental group of a space form of dimension $n \equiv 1 \bmod 4$ is a group with two generators, A and B, and relations $A^m = B^t = 1$ and $BAB^{-1} = A^r$ [10]. The order of Γ satisfies $|\Gamma| = mt = mt'd$, and, in addition, $((r-1)t, m) = 1$, and where d is the order of r in the multiplicative group of units in $\mathbb{Z}_m$. Wolf showed that under any irreducible representation $\pi_{k,l}$ of a group of type I, the images of A and B are as follows:

$$\pi_{k,l}(A) = \begin{bmatrix} e^{2\pi ik/m} & & & \\ & e^{2\pi ikr/m} & & \\ & & \ddots & \\ & & & e^{2\pi ikr^{d-1}/m} \end{bmatrix}$$

and

$$\pi_{k,l}(B) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ e^{2\pi il/t'} & \cdots & & & 0 \end{bmatrix}$$

where $(k,m)=1=(l,t')$ [11]. A and B are $d \times d$ matrices, so $n+1=2d$.

Hence, these are the only irreducible actions on S^n, where $n \equiv 1 \bmod 4$. We note that B^d is a scalar matrix: $(e^{2\pi i/t'}) \cdot I$. As m and t become infinitely large, the subgroup generated by $\pi_{k,l}(A)$ and $\pi_{k,l}(B^d)$ converges to a torus action on the sphere, the torus having dimension at least two, since $(e^{i\theta}, e^{i\theta}, \ldots, e^{i\theta})$ and $(e^{i\varphi}, e^{ir\varphi}, \ldots, e^{ir^{d-1}\varphi})$ will represent independent vectors.

In fact, as m increases without bound, the condition that d be the order of r in the multiplicative group of units in $\mathbb{Z}_m$ can result in the $(d-1)$-tuple $(e^{2\pi i/m}, e^{2\pi ir/m}, \ldots, e^{2\pi ir^{d-2}/m})$ coming arbitrarily close to any point in the $d-1$ torus $(e^{i\alpha_0}, e^{i\alpha_1}, \ldots, e^{i\alpha_{d-2}})$. The limit group generated by these points will never be a d-torus, since we have the relation $m \mid (1+r+r^2+\cdots+r^{d-1})$. The following theorem is due to Lawrence Washington.

THEOREM 3.2. *For some sequences of Γ and m, $\langle \pi_{k,l}(A) \rangle$ converges to a $d-1$ torus as m increases without bound. (For the proof, see* [4].)

We do not answer the question of whether $\langle \pi_{k,l}(A) \rangle$ can converge to a smaller dimensional torus through a suitably chosen sequence of r's and m's.

We also note that

$$\pi_{k,l}(B^d) = \begin{bmatrix} e^{2\pi il/t'} & & \\ & \ddots & \\ & & e^{2\pi il/t'} \end{bmatrix}$$

converges to an independent circle action. In the limit, then, $\langle \pi_{k,l}(A), \pi_{k,l}(B^d) \rangle$ can converge to a d-torus.

LEMMA 3.3. *The diameter of the limit space S^n/G, where $G = \lim \Gamma_i$, is bounded below by* $\arcsin \sqrt{\frac{d-1}{d}}$, *where $2d = n+1$.*

Proof. Regardless of the dimension of the torus acting on S^n, however, the distance between the orbits of $(1, 0, \ldots, 0)$ and $\left(\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}, \ldots, \frac{1}{\sqrt{d}}\right)$ remains the same. Clearly, no change of direction $(e^{i\theta}, 0, \ldots, 0)$ brings the orbits any closer together than when $\theta = 0$, nor does any shifting through the d-cycle generated by

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{bmatrix}, \text{ the limit of } \pi_{k,l}(B)$$

affect their distance, since the element $\left(\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}, \ldots, \frac{1}{\sqrt{d}}\right)$ is unaffected by this subgroup. So with respect to any limit space S^n/G, where

$$G = \left\langle T^k, \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{bmatrix} \right\rangle,$$

$k = 2, \ldots,$ or d, we have the following lower bound on diameter

$$2 \arcsin \frac{1}{2} \sqrt{2 - \frac{2}{\sqrt{d}}} = \arcsin \sqrt{\frac{d-1}{d}}.$$

LEMMA 3.4. *The diameter of any space form*

$$S^n/\Gamma, \quad \text{with } \Gamma = \langle \pi_{k,l}(A), \pi_{k,l}(B) \rangle,$$

is bounded below by $\arcsin \sqrt{\frac{d-1}{d}}$.

Proof. Any finite group Γ will be contained in a group

$$G = \left\langle T^d, \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{bmatrix} \right\rangle.$$

With respect to the elements generated $\pi_{k,l}(A)$ and $\pi_{k,l}(B^d)$, the containment is clear, since these are diagonal matrices whose entries are of the form $(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_d})$ for some $(\theta_1, \ldots, \theta_d) \in [0, 2\pi)^d$. For any representation $\pi_{k,l}$, the image $\pi_{k,l}(B)$ is a product

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & e^{2\pi i l/t'} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{bmatrix}.$$

The left multiplier is an element of T^d; the right is included explicitly as a generator of G. Since the generators of every $\Gamma = \operatorname{Im} \pi_{k,l}$ are contained in G, $\Gamma \subset G$. Since we have containment of the groups $\Gamma \subset G$, by the inclusion lemma (Lemma 1.3), the diameter of S^n/Γ is bounded below by that of S^n/G. Although the representations $\pi_{k,l}$ of type I groups are the only irreducible actions on the spheres S^n with $n \equiv 1 \bmod 4$, these groups also serve as fundamental groups of spheres of other dimensions. All the same conclusions reached in this section with respect to the limits of groups of type I and the lower bound on diameter of the resulting space forms apply in all dimensions.

4. Representations induced from $\pi_{k,l}$ on spheres of dimension 3 mod 4

Within the six types of fundamental groups of spherical space forms, Wolf defined several cases. Using $\mathscr{F}_c(G)$ to denote the set of equivalence classes of irreducible, complex, fixed-point free representations of a group G, Wolf describes 13 general classes $\mathscr{F}_c(G)$ from which all representations of fundamental groups of space forms are derived. The classes $\mathscr{F}_c(G)$ and general information about the groups are given in the following table.

For the purposes of this paper, it shall be convenient to deal with the representations $\alpha_{k,l}$ and $\beta_{k,l}$ as a unit, and separately from the representations of groups of types III, IV, V, and VI. These latter groups may be thought to act on pairs of complex coordinates, as will be explained further in the next section. It is less convenient to think of $\alpha_{k,l}$ in this way, and not useful to consider $\beta_{k,l}$ in this fashion. Moreover, $\alpha_{k,l}$ and $\beta_{k,l}$ are both built on the representations $\pi_{k,l}$.

Table I

Group type	*Generators*	*Relations*	*Elements of* $\mathscr{F}_c(G)$
I	A, B	$A^m = B^t = I$, $BAB^{-1} = A^r$	$\pi_{k,l}$
II	A, B, R	Same as type I, and	$\alpha_{k,l}$
		$RAR^{-1} = A^l$, $RBR^{-1} = B^k$	$\beta_{k,l}$
		$R^2 = B^{t/2}$	
III	A, B, P, Q	Same as type I, and	$\nu_{k,l}$
		$P^4 = 1$, $P^2 = Q^2 = (PQ)^2$	$\nu_{k,l_1,j}$
		$AP = PA$, $AQ = QA$	$\mu_{k,l}$
		$BPB^{-1} = Q$, $BQB^{-1} = PQ$	
IV	A, B, P, Q, R	Same as type III	$\psi_{k,l}$
		$R^2 = p^2$	$\gamma_{k,l}$
		$RPR^{-1} = QP$	$\xi_{k,l}$
		$RQR^{-1} = Q^{-1}$	$\gamma_{k,l,j}$
		$RAR^{-1} = A^{-1}$	$\eta_{k,l}$
		$RBR^{-1} = B^k$	
V	A, B, T, U, V	A, B as in type I	$\iota_{k,l,j}$
		T, U, V generate $\mathbf{I}^*$	
		and $G = \langle A, B \rangle \times \mathbf{I}^*$	
VI	A, B, T, U, V, S	$G = \langle \Gamma_1, S \rangle$, where Γ_1	$\kappa_{k,l,j}$
		is type V, $S^2 = -1$	
		$S\mathbf{I}^*S^{-1} = \mathbf{I}^*$ and	
		$SKS^{-1} = K$, where $K = \langle A, B \rangle$	

Representations $\alpha_{k,l}$ are induced from $\pi_{k,l}$. By "induced representation," we mean the representation $\tilde{\sigma}$ given in block form by $\tilde{\sigma}_{ij}(g)=\sigma(b_i^{-1}gb_j)$, where σ is defined on a normal subgroup, the b_i's constitute a complete set of coset representatives of that subgroup in the whole group, and $\sigma(b_i^{-1}gbj)$ is taken to be zero when $b_i^{-1}gb_j$ is not in the subgroup. Under this definition, $\alpha_{k,l}=\tilde{\pi}_{k,l}$, where the subgroup is $\langle A, B\rangle$. Since the whole group is $\langle A, B, R\rangle$ and $R=B^{t/2}$, there are only two cosets: $\langle A, B\rangle$ and $R\langle A, B\rangle$. So

$$\alpha_{k,l}(g)=\begin{bmatrix}\pi_{k,l}(g) & \pi_{k,l}(gR^{-1})\\ \pi_{k,l}(Rg) & \pi_{k,l}(RgR^{-1})\end{bmatrix}.$$

Note that in each row and in each column, there is only one nonzero entry. If $g\in\langle A, B\rangle$, then $Rg\notin\langle A, B\rangle$, and so on.

THEOREM 4.1. Arcsin $\sqrt{\dfrac{n-1}{n+1}}$ *is a lower bound on the diameters of space forms of the type* S^n/Γ, *where* Γ *is the image of a finite group under* $\alpha_{k,l}$ *or* $\beta_{k,l}$.

Proof. We have seen that for all k and l,

$$\langle\pi_{k,l}(A),\ \pi_{k,l}(B)\rangle\subset\left\langle T^d, \begin{bmatrix}0 & 1 & & \\ & & \ddots & \\ & & & 1\\ 1 & & & 0\end{bmatrix}\right\rangle.$$

Groups with representations $\alpha_{k,l}$ have an additional generator, R. We have

$$\alpha_{k,l}(A)=\begin{bmatrix}\pi_{k,l}(A) & \\ & \pi_{k,l}(A)^l\end{bmatrix},\quad \alpha_{k,l}(B)=\begin{bmatrix}\pi_{k,l}(B) & \\ & \pi_{k,l}(B)^k\end{bmatrix},$$

and

$$\alpha_{k,l}(R)=\begin{bmatrix} & I_d\\ \pi_{k,l}(B)^{t/2} & \end{bmatrix}.$$

Therefore, if $\Gamma=\operatorname{Im}\alpha_{k,l}$,

$$\Gamma\subset\left\langle \mathbf{T}^{2d}, \begin{bmatrix}\begin{bmatrix}0 & 1 & & \\ & & \ddots & \\ & & & 1\\ 1 & & & 0\end{bmatrix} & \\ & I_d\end{bmatrix}, \begin{bmatrix} & I_d\\ I_d & \end{bmatrix}\right\rangle=G_\alpha.$$

When $\alpha_{k,l}$ is reducible, $\alpha_{k,l}=\beta_{k,l}\oplus\beta_{k,l}$, a representation of half the degree. Therefore, if $\Gamma=\mathrm{Im}\,\beta_{k,l}$, $\Gamma\subset\langle \mathbf{T}^d, P(d)\rangle=G_\beta$, where $P(d)$ is a subgroup of the symmetric group on d symbols, represented by elementary matrices which permute d complex coordinates.

The orbit of $(1,0,\ldots,0)$ under G_α is $\{e^{i\theta}\cdot\mathbf{e}_s\colon \theta\in[0,2\pi)$, $\mathbf{e}_s$ is the sth standard normal coordinate vector, $s\in\{1,2,\ldots,2d\}\}$. This is because the torus action serves to multiply the given vector by $e^{i\theta}$, and the permutation subgroup,

$$\left\langle \begin{bmatrix} \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{bmatrix} & \\ & I_d \end{bmatrix}, \begin{bmatrix} & I_d \\ I_d & \end{bmatrix} \right\rangle,$$

is a transitive action on the coordinate axes. The distance between a diagonal point

$$\left(\frac{1}{\sqrt{2d}},\frac{1}{\sqrt{2d}},\ldots,\frac{1}{\sqrt{2d}}\right)$$

and the orbit of $(1,0,\ldots,0)$ is (as before) $\arcsin\sqrt{\dfrac{2d-1}{2d}}$. Here $n+1=4d$, so $\arcsin\sqrt{\dfrac{2d-1}{2d}}=\arcsin\sqrt{\dfrac{n-1}{n+1}}$. Therefore, diameter $S^n/G_\alpha\geqslant\arcsin\sqrt{\dfrac{n-1}{n+1}}$.

Since all the finite groups Γ are contained in G_α, the diameter of S^n/Γ is bounded by this number by the inclusion lemma.

The argument for G_β follows exactly the same reasoning, except that if $\Gamma=\mathrm{Im}\,\beta_{k,l}$, Γ has degree d, so the diagonal element is

$$\left(\frac{1}{\sqrt{d}},\frac{1}{\sqrt{d}},\ldots,\frac{1}{\sqrt{d}}\right),$$

and the distance to the orbit of $(1,0,\ldots,0)$ is

$$\arcsin\sqrt{\frac{d-1}{d}}=\arcsin\sqrt{\frac{n-1}{n+1}}.$$

5. Actions by groups of types III, IV, V and VI on spheres of dimension 3 mod 4

Our strategy for determining lower bounds on the diameters of space forms with fundamental groups of types III, IV, V and VI is to consider the action of the representations on pairs of coordinate axes. That is, any point $x \in S^{4n+3}$, written as a point in $\mathbb{C}^{2n+2}$, has an even number of coordinates. If all but one pair of coordinates (say (z_1, z_2)) are zero, the nonzero pair of coordinates may be thought of as a point in S^3, since then $z_1\bar{z}_1 + z_2\bar{z}_2 = 1$. Pairing the coordinates this way, we think of each pair as a coordinate S^3 in S^{4n+3}. The actions under groups of types III through VI treat these coordinates in pairs. That is, the subgroups of the torus actions on these spheres are actually scalar multiplication on each coordinate S^3; they converge to the Hopf action on each coordinate S^3 in the limit. Moreover, the permutation elements of the groups permute the complex coordinates in pairs; they are no longer transitive actions on the complex coordinate axes of $\mathbb{C}^{2n+2}$, but only on the pairs of coordinate axes. For instance,

$$\begin{bmatrix} 0 & I_2 & & \\ & & \ddots & \\ & & & I_2 \\ I_2 & & & 0 \end{bmatrix} \text{ replaces } \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & 0 & & 0 \end{bmatrix}$$

as an element of the permutation subgroup in the limit groups.

In fact, as we shall see, the groups all converge to a group of the form $\langle T^d, P(d), F\rangle$. Here, T^d is a d-dimensional torus, $P(d)$ is a transitive subgroup of the symmetric group on d symbols, and F is a finite group that acts on each of the coordinate 3-spheres, without interchanging them. F has the same restriction, $\hat{F}$, to any coordinate 3-sphere. The representation has degree $2d$ over $\mathbb{C}$. To estimate the diameters of space forms S^n/Γ, where $\Gamma \subset \langle T^d, P(d), F\rangle$, we use the following theorem.

THEOREM 5.1. *$S^n/\langle T^d, P(d), F\rangle$ has diameter at least* $\arccos \dfrac{1}{\sqrt{d}} \cos B$, *where $B =$ diameter $S^3/S^1 \times \hat{F}$, and $\hat{F}$ is the restriction of F to any coordinate S^3. $S^n/\langle T^{2d}, P(2d), F\rangle$ has diameter at least* $\arccos\left(\dfrac{1}{\sqrt{2d}} \cos B\right)$.

Proof. We use the law of cosines for spheres. We wish to compute the distance between a well-chosen point in the diagonal

$$\Delta = \left\{\left(\frac{1}{\sqrt{d}}q, \frac{1}{\sqrt{d}}q, \ldots, \frac{1}{\sqrt{d}}q\right) \middle| q = (z_1, z_2) z_1\bar{z}_1 + z_2\bar{z}_2 = 1, \text{ and } 4d = n+1\right\}$$

and a "farthest" orbit in a coordinate three sphere $= \{(q', 0, \ldots, 0) \mid q' = (z_3, z_4) z_3\bar{z}_3 + z_4\bar{z}_4 = 1\}$. Refer to Fig. 4.

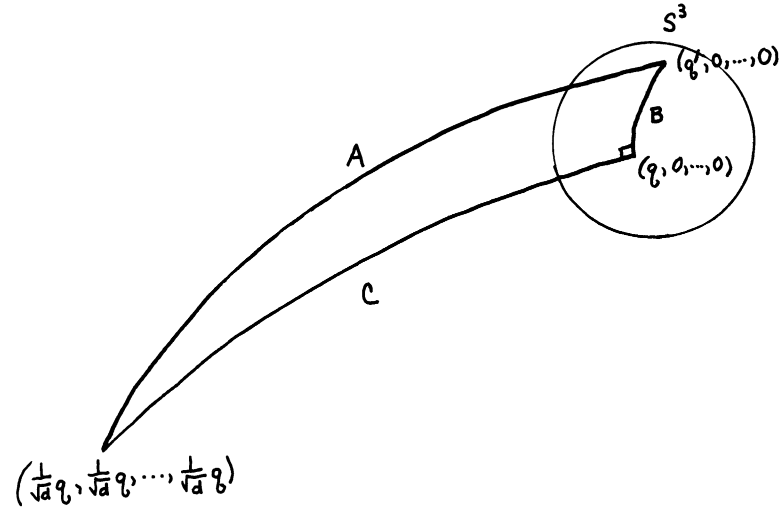

Fig. 4.

First, we determine the diameter of the orbit space of a coordinate S^3 under the action. This will be the diameter of $S^3/\langle \hat{F}, S^1 \rangle$. We generally computed these diameters in Section 2. We take two points $q \in S^3$ and $q' \in S^3$ on a coordinate three-sphere that realize this diameter. Then the closest point in the diagonal to $(q, 0, \ldots, 0)$ is clearly

$$\left(\frac{1}{\sqrt{d}}\,q, \frac{1}{\sqrt{d}}\,q, \ldots, \frac{1}{\sqrt{q}}\,q\right).$$

The distance between $(q, 0, \ldots, 0)$ and this diagonal point is easily seen to be $\arccos \frac{1}{\sqrt{d}}$. Call this distance C. No other point on any coordinate three-sphere is any closer to this diagonal point. No other point in the orbit of $(q, 0, \ldots, 0)$ is any closer to this diagonal point, either. If we take $T \in \langle T^d, F \rangle$, $T(q, 0, \ldots, 0) = (q'', 0, \ldots, 0)$, where $q'' \in S^3$ is another point in the coordinate 3-sphere. Moreover, if

$$R \in P(d), \quad R(q, 0, \ldots, 0) = (0, \ldots, 0, q, 0, \ldots, 0),$$

where q has been moved to another coordinate three sphere. Neither of these transformations decreases distance. Together, they generate all the transformations of the group. So the distance between the orbits of $\left(\frac{1}{\sqrt{d}}q,\dots,\frac{1}{\sqrt{d}}q\right)$ and $(q, 0,\dots,0)$ is exactly $\arccos\frac{1}{\sqrt{d}}$. By the law of cosines for spheres, we know that A, the distance from

$$\Delta=\left(\frac{1}{\sqrt{d}}q,\dots,\frac{1}{\sqrt{d}}q,\dots,\frac{1}{\sqrt{d}}q\right) \quad \text{to} \quad (q', 0,\dots,0)$$

satisfies

$$\cos A=\cos B\cos C+\sin B\sin C\cos\theta.$$

The angle θ is that between the minimal geodesic from Δ to the coordinate S^3 and the tangent space to the S^3. Therefore, $\theta=\pi/2$, and we can write $\cos A=\cos B\cos C$.

As we have already determined,

$$C=\arccos\frac{1}{\sqrt{d}}, \quad \text{so } A=\arccos\left(\frac{1}{\sqrt{d}}\cos B\right),$$

where B is the diameter of the coordinate S^3 after identifications under $\hat{F}$. No element in the orbit of $(q',0,\dots,0)$ is any closer to Δ, since for

$$T\in\langle F, T^d\rangle,\ T(q', 0,\dots,0)=(q''', 0, 0,\dots,0)$$

and

$$d_{S^3}(q,q''')\geqslant d_{S^3}(q,q'),$$

because q and q' realize the distance between the respective orbits. Since $P(d)$ fixes the diagonal, it has no effect on the distance from Δ to $(q',0,\dots,0)$. Therefore, $S^n/\langle T^d, P(d), F\rangle$ has diameter at least $\arccos\frac{1}{\sqrt{d}}\cos B$.

Where the representation has degree $4d$, the limit group is contained in $\langle T^{2d}, P(2d), F\rangle$, and the distance C equals $\arccos\frac{1}{\sqrt{2d}}$, so $A=\arccos\frac{1}{\sqrt{2d}}\cos B$.

Wolf's groups give rise to four basic actions on the coordinate three-spheres: the quaternion group, and the binary tetrahedral, octahedral, and icosahedral groups. We therefore have four estimates for diameters.

THEOREM 5.2. *The quaternionic, tetrahedral, octahedral, and icosahedral actions on the coordinate three-spheres give rise to the following lower bounds on the diameters of S^n/Γ (respectively):*

$$\arccos\sqrt{\frac{2}{n+1}},\ \arccos\sqrt{\frac{8}{(3(n+1)}},\ \arccos\sqrt{\frac{2}{n+1}+\frac{2}{\sqrt{3}(n+1)}}$$

and

$$\arccos\sqrt{\frac{6+2\sqrt{3}\tan(3\pi)/10}{3(n+1)}}.$$

Proof. First we consider those representations that give rise to the quaternionic action on the coordinate three-spheres: $\mu_{k,l}$ and $\eta_{k,l}$. Representations of the first type, $\mu_{k,l}$ are of degree $2d$. (The action interchanges d coordinate three-spheres.) Representations of type $\eta_{k,l}$ are of degree $4d$. If $\Gamma = \operatorname{Im}\mu_{k,l}$, the diameter of S^n/Γ is at least $\arccos\sqrt{\frac{2}{n+1}}$. The finite group acting on each coordinate three-sphere is generated by $P=\begin{bmatrix} i & \\ & -i\end{bmatrix}$ and $Q=\begin{bmatrix} & 1\\ -1 & \end{bmatrix}$. So each coordinate three-sphere is subject to the circle action $e^{i\theta}I_2$ and to the action by $\langle P, Q\rangle$. The diameter of S^3/S is $\pi/2$. The action of $\langle P, Q\rangle$ on this S^2 is reduced by half, since $-I\in S^1$. Therefore, the effective action on the orbit space S^2 is a rotation group of order four, i.e., the dihedral group of degree 2. It interchanges the positive and negative rays of each of the three coordinate axes. The orbit space is therefore one octant of the two-sphere of radius one-half. Its diameter is $\pi/4$.

Applying Theorem 5.1, we find that diameter

$$S^n\Bigg/\left\langle T^d, \begin{bmatrix} 0 & I_2 & & \\ & & \ddots & \\ & & & I_2 \\ I_2 & & & 0\end{bmatrix}, P, Q\right\rangle$$

is at least $\arccos\left(\frac{1}{\sqrt{d}}\cos\frac{\pi}{4}\right)=\arccos\frac{1}{\sqrt{2d}}$. Here $4d=n+1$, so $\arccos\sqrt{\frac{2}{n+1}}$. By Lemma 1.3, diameter $S^n/\operatorname{Im}\mu_{k,l}$ is also bounded by this number. Similarly, if $\Gamma=\operatorname{Im}\eta_{k,l}$ the diameter of S^n/Γ is bounded by $\arccos\sqrt{\frac{2}{n+1}}$. The degree of $\eta_{k,l}$

is $4d$. Applying Theorem 5.1 and Lemma 1.3, we find that the diameter of S^n/Γ is at least

$$\arccos\left(\frac{1}{\sqrt{2d}}\cos\frac{\pi}{4}\right) = \arccos\frac{1}{2\sqrt{d}}.$$

Here, $8d = n+1$, so the diameter is bounded by $\arccos\sqrt{\dfrac{2}{n+1}}$.

Now we consider representations whose actions on the coordinate S^3's are generated by subgroups of the circle and the binary tetrahedral group. These are $\nu_{k,l}$, $\nu_{k,l,j}$, $\gamma_{k,l}$, $\xi_{k,l,j}$, and $\gamma_{k,l,j}$. The first two, $\nu_{k,l}$ and $\nu_{k,l,j}$, have degree $2d$. The others have degree $4d$. Although $\operatorname{Im}\xi_{k,l,j}$ contains $\mathbf{0}_v^*$ as a subgroup, $\mathbf{0}_v^*$ is induced from $\mathbf{T}_v^*$, and the additional generator does not appear as part of the action on S^3, but as part of the permutation subgroup $P(2d)$.

The space forms of the type S^n/Γ, where Γ is the image under $\nu_{k,l}$, $\nu_{k,l,j}$, $\gamma_{k,l}$, $\gamma_{k,l,j}$, or $\xi_{k,l,j}$ of a finite group, have diameter at least $\arccos\sqrt{\dfrac{8}{3(n+1)}}$. We saw in Section 2, the diameter of $S^3/\langle T^*, S^1\rangle$ is $\arccos\sqrt{\dfrac{2}{3}}$. The degree of $\nu_{k,l}$ or $\nu_{k,l,j}$ is $2d$. By Theorem 5.1, the diameter of S^n/Γ is at least $\arccos\sqrt{\dfrac{2}{3d}}$. Here $4d = n+1$, so the diameter is at least $\arccos\sqrt{\dfrac{8}{3(n+1)}}$.

Now let $\Gamma = \operatorname{Im}\gamma_{k,l}, \operatorname{Im}\gamma_{k,l_j}$ of $\operatorname{Im}\xi_{k,l,j}$. The degree of each representation is $4d$. Therefore, the diameter of S^n/Γ is bounded below by

$$\arccos\sqrt{\frac{2}{3}}\sqrt{\frac{1}{2d}} = \arccos\frac{1}{\sqrt{3d}} = \arccos\sqrt{\frac{8}{3(n+1)}},$$

since $8d = n+1$.

Groups Γ whose restrictions to the coordinate S^3's involve octahedral actions have representations $\psi_{k,l}$. Since $\psi_{k,l}$ has degree $2d$ and the diameter of $S^3/\langle S^1, 0^*\rangle$ is $\frac{1}{2}\arccos\dfrac{1}{\sqrt{3}}$, Theorem 5.1 tells us that the diameter of S^n/Γ is at least

$$\arccos\frac{1}{\sqrt{d}}\left(\cos\frac{1}{2}\arccos\frac{1}{\sqrt{3}}\right) = \arccos\sqrt{\frac{2}{n+1}+\frac{2}{\sqrt{3}(n+1)}}.$$

Groups of types V and VI are the only nonsolvable fundamental groups of space forms. They contain $\mathbf{I}^*$ as a subgroup. Any space form S^n/Γ, where $\Gamma = \operatorname{Im} \iota_{k,l,j}$ or $\operatorname{Im} k_{\iota,l}$, has diameter at least

$$\arccos \sqrt{\frac{6+2\sqrt{3}\tan(3\pi)/10}{3(n+1)}}.$$

The restriction of Γ to a coordinate S^3 is contained in $\langle S^1, \mathbf{I}^*\rangle S^3/\langle S', I_n^*\rangle$ has

Table II

Group type	*Elements of* $F_c(G)$	*Dimension 3 diameter of limit space*	*Lower bound on diameter, diameter, all dimensions*
I	$\pi_{k,l}$	$\frac{\pi}{4}$	$\arcsin\sqrt{\frac{n-1}{n+1}}$
II	$\alpha_{k,l}$		$\arcsin\sqrt{\frac{n-1}{n+1}}$
	$\beta_{k,l}$	$\frac{\pi}{4}$	$\arcsin\sqrt{\frac{n-1}{n+1}}$
III	$\nu_{k,l}$	$\arccos\sqrt{\frac{2}{3}}$	$\arccos\sqrt{\frac{8}{3(n+1)}}$
	$\nu_{k,l,j}$	$\arccos\sqrt{\frac{2}{3}}$	$\arccos\sqrt{\frac{8}{3(n+1)}}$
	$\mu_{k,l}$		$\arccos\sqrt{\frac{2}{n+1}}$
IV	$\psi_{k,l,j}$	$\frac{1}{2}\arccos\frac{1}{\sqrt{3}}$	$\arccos\sqrt{\frac{2}{n+1}+\frac{2}{\sqrt{3}(n+1)}}$
	$\gamma_{k,l}$		$\arccos\sqrt{\frac{8}{3(n+1)}}$
	$\xi_{k,l,j}$		$\arccos\sqrt{\frac{8}{3(n+1)}}$
	$\gamma_{k,l,j}$		$\arccos\sqrt{\frac{8}{3(n+1)}}$
	$\eta_{k,l}$		$\arccos\sqrt{\frac{2}{n+1}}$
V	$\iota_{k,l,j}$	$\frac{1}{2}\arccos\frac{1}{\sqrt{3}}\left(\tan\frac{3\pi}{10}\right)$	$\arccos\sqrt{\frac{6+2\sqrt{3}\tan(3\pi)/10}{3(n+1)}}$
VI	$\kappa_{k,l}$		$\arccos\sqrt{\frac{6+2\sqrt{3}\tan(3\pi)/10}{3(n+1)}}$

diameter $\frac{1}{2}\arccos\frac{1}{\sqrt{3}}\left(\tan\frac{3\pi}{10}\right)$. By Theorem 5.1, since the degree of $\iota_{k,l,j}$ is $2d$, we have the diameter of S^n/Γ bounded by

$$\arccos\left[\frac{1}{\sqrt{d}}\cos\left(\frac{1}{2}\arccos\frac{1}{\sqrt{3}}\tan\frac{3\pi}{10}\right)\right]$$

$$= \arccos\sqrt{\frac{6+2\sqrt{3}\tan(3\pi)/10}{3(n+1)}} \quad (4d = n+1).$$

If $\Gamma = \operatorname{Im}\kappa_{k,l}$, its degree is $4d$. Therefore Theorem 5.1 and the result of Section 2 imply that the diameter of S^n/Γ is also bounded below by

$$\arccos\left[\frac{1}{2d}\cos\left(\frac{1}{2}\arccos\frac{1}{\sqrt{3}}\tan\frac{3\pi}{10}\right)\right] = \sqrt{\frac{6+2\sqrt{3}\tan(3\pi)/10}{3(n+1)}} \quad (8d = n+1).$$

A corollary of the theorems of Sections 4 and 5 is Corollary 5.3: Any space form of dimension $n \equiv 3 \bmod 4$ has diameter greater than $\arccos\sqrt{\frac{6+2\sqrt{3}\tan(3\pi)/10}{3(n+1)}}$.

A comparison with the lower bounds for diameters of space forms in all dimensions yields the theorem of the introduction: $\frac{1}{2}\arccos\frac{1}{\sqrt{3}}\left(\tan\frac{3\pi}{10}\right)$ serves as a lower bound on diameters of space forms of all dimensions.

To summarize, we may compile our results in Table II.

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