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Subintegrality, invertible modules and the Picard group*

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Introduction

This work began as an attempt to generalize the following result of Dayton [4]: If A is a reduced G-algebra containing $\mathbb{Q}$ and B is the seminormalization of A then there is a functorial isomorphism $\theta\colon \mathrm{Pic}(A) \to B/A$. (Recall that a G-algebra is a graded commutative ring $A = \bigoplus_{n\geqslant 0} A_n$, where A_0 is a field and A is finitely generated as an A_0-algebra.)

We extend this result to a more general situation, where A may not be graded or finitely generated or even reduced, and B may not be the full seminormalization of A. To describe our result more precisely, let $A \subseteq B$ be an extension of commutative rings containing $\mathbb{Q}$ and suppose that this extension is subintegral (Swan [9, §2]). Our main result is to construct in this situation a natural group homomorphism $\xi_{B/A}\colon B/A \to \mathscr{I}(A, B)$, where $\mathscr{I}(A, B)$ is the group of invertible A-submodules of B, and to prove the following

(5.6) MAIN THEOREM. *Let A be an excellent $\mathbb{Q}$-algebra of finite Krull dimension and let $A \subseteq B$ be a subintegral extension. Then the homomorphism $\xi_{B/A}$: $B/A \to \mathscr{I}(A, B)$ is an isomorphism.*

The assumption of excellence and finite dimension is needed to carry out our proof of the Main Theorem by induction on dimension. However, the conclusion of (5.6) holds without these assumptions. The assumption that A contains $\mathbb{Q}$ is needed because we use the exponential and logarithmic series, and in fact the conclusion of (5.6) need not hold without this assumption. See the Final Remark at the end of the paper for more details.

Suppose A is a reduced G-algebra containing $\mathbb{Q}$ and B is the seminormalization of A. Then A is excellent and of finite Krull dimension, the extension $A \subseteq B$ is subintegral and $\mathscr{I}(A, B) = \mathrm{Pic}(A)$ (2.5). So (5.6) gives $B/A \cong \mathrm{Pic}(A)$ in this situation. Furthermore, our map $\xi_{B/A}$ differs from the map

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θ^{-1} of [4] by a group automorphism of B/A (Section 7), whence (5.6) also yields Dayton's result as a special case.

The grading of A appears to play a crucial role in Dayton's work, and indeed enters explicitly into the definition of θ. We started out by noticing that many of Dayton's introductory remarks do not require the grading, or can be suitably modified. This led us to ask how far we could get without requiring the ring to be graded.

The most natural way to obtain a map from an additive group to a multiplicative group seems to be with an exponential map. Roughly speaking $\xi_{B/A}(\bar{b}) = \hat{A}\, e^b \cap A[b]$, where $b \in B$ and $\hat{A}$ is a suitable completion. The most obvious completion is the b-adic one. However, this completion is not always available, for example, if b is a unit. Therefore, we define $\xi_{B/A}(\bar{b})$ to be the reduction of $A[[T]]\, e^{bT} \cap A[b][T]$ modulo $T = 1$. One might hope that $\xi_{B/A}(\bar{b})$ is invertible with inverse $\xi_{B/A}(-\bar{b})$. Obviously $\xi_{B/A}(\bar{b})\xi_{B/A}(-\bar{b}) \subseteq A$, but in general we do not have equality. For example, if b is an indeterminate over A, then $\xi_{B/A}(\bar{b}) = 0$. The heart of the matter then is to prove the equality $\xi_{B/A}(\bar{b})\xi_{B/A}(-\bar{b}) = A$ for suitable b, and for this it suffices to show that $1 \in \xi_{B/A}(\bar{b})\xi_{B/A}(-\bar{b})$. An important step towards our solution of this problem is to find an elementwise characterization of a subintegral ring extension, which is perhaps a result of independent interest. We define an element b of B to be *subintegral* over A if there exist $c_1, \dots, c_p \in B$ such that $b^n + \Sigma_{i=1}^{p} \binom{n}{i} c_i b^{n-i} \in A$ for all $n \gg 0$. This condition is shown to be independent of the overring B to which b belongs (see (4.3)). We then prove

(4.17) THEOREM. *For an extension $A \subseteq B$ of $\mathbb{Q}$-algebras the following two conditions are equivalent*:

(i) *every element of B is subintegral over A*,
(ii) *the extension $A \subseteq B$ is subintegral.*

In order to prove (4.17) and also to deduce the invertibility of $\xi_{B/A}(\bar{b})$ from (4.17), we first prove in Section 3 the invertibility of $\xi_{B/A}(\bar{b})$ in a universal situation.

Once one knows that $\xi_{B/A}(\bar{b})$ is invertible it is not difficult to prove that $\xi_{B/A}: B/A \to \mathscr{I}(A, B)$ is a homomorphism. The proof that $\xi_{B/A}$ is an isomorphism is then obtained by reducing modulo the conductor and using induction on Krull dimension. Factoring out by the conductor yields in general a nonreduced ring, but this causes us no problems, since we prove our result even for nonreduced rings.

Two easy examples will help motivate our work. First let $A = k[X, Y]/((Y - X^2)(Y - X^3))$, where k is a field of characteristic zero, and let B be the seminormalization of A. Then Spec(A) consists of two affine lines, intersecting normally in one intersection point and tangentially in the other, and Spec(B)

consists of two affine lines meeting normally in each of two intersection points. To describe the extension $A \subseteq B$ algebraically, let $\bar{A}$ be normalization of A. Then $\bar{A} = k[t] \times k[u]$, the inclusion $A \to \bar{A}$ being given by $X \mapsto (t, u)$ and $Y \mapsto (t^2, u^3)$. By [8, 1.3],

$$B = \{(f, g) \in k[t] \times k[u] \mid f(0) = g(0),\ f(1) = g(1)\}.$$

Calculations with the units-Pic sequence [1] (which we leave for the reader) now show that $\mathrm{Pic}(A) = k \oplus k^*$ and $\mathrm{Pic}(B) = k^*$. Furthermore one can show that $B/A \cong k$. Clearly A cannot be graded in any nontrivial way. This suggests that even in the nongraded case, B/A may still be related in some way to Pic. On the other hand, let $A = k[[T^2, T^3]]$, which has seminormalization $B = k[[T]]$. Then $\mathrm{Pic}(A) = 0$, since A is local, but $B/A = k$. However, since the origin is the only singular (or non-seminormal) point of $\mathrm{Spec}(k[T^2, T^3])$, one would expect the isomorphism $\mathrm{Pic}(k[T^2, T^3]) \cong k[T]/k[T^2, T^3]$ to be reflected somehow at the local level. These examples can be understood in terms of our Main Theorem and the exact sequence (2.4). In the first, $A^* = B^* = k^*$, $\mathscr{I}(A, B) \cong B/A = k$, $\mathrm{Pic}(A) = k \oplus k^*$, and $\mathrm{Pic}(B) = k^*$. In the second $B^*/A^* \cong k$, $\mathscr{I}(A, B) \cong B/A = k$, and $\mathrm{Pic}(A) = 0$.

We remark that analysis of all aspects of the problems discussed above is quite easy in the case of an elementary subintegral extension $A \subseteq B$, i.e. in case $B = A[b]$ with $b^2, b^3 \in A$. However, difficulties arise already in the case of a two-step extension, i.e. a composite of two elementary subintegral extensions. A generic two-step example, which was the starting point of and illustrates our theory, is described in Section 6.

1. Notation

The sets of integers, nonnegative integers, positive integers and rational numbers are denoted, respectively, by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$ and $\mathbb{Q}$.

For an indeterminate T and for $d \in \mathbb{Z}^+$ let

$$\binom{T}{d} = (1/d!) \prod_{i=0}^{d-1} (T - i) \in \mathbb{Q}[T].$$

By a ring A we always mean a commutative ring with 1, and A^* denotes the group of units of A.

$\mathrm{Nil}(A)$ denotes the nilradical of A, i.e. the ideal of all nilpotents of A. Put $\mathrm{Uni}(A) = 1 + \mathrm{Nil}(A)$. Then $\mathrm{Uni}(A) \subseteq A^*$ is the group of all unipotents of A.

Suppose now that $\mathbb{Q} \subseteq A$. Then for $a \in \operatorname{Nil}(A)$,

$$\exp(a) = e^a = \sum_{i=0}^{\infty} a^i/i! \quad \text{and} \quad \log(1+a) = \sum_{i=1}^{\infty} (-1)^{i+1} a^i/i$$

are finite sums of elements of A, hence belong to A. Further, we have

$$\log(\exp(a)) = a \quad \text{and} \quad \exp(\log(1+a)) = 1+a.$$

Thus

$$\exp\colon \operatorname{Nil}(A) \to \operatorname{Uni}(A) \quad \text{and} \quad \log\colon \operatorname{Uni}(A) \to \operatorname{Nil}(A)$$

are isomorphisms of groups and are inverses of each other.

Note that if A' is a subring of A containing $\mathbb{Q}$ then

$$\exp(\operatorname{Nil}(A')) = \operatorname{Uni}(A') \quad \text{and} \quad \log(\operatorname{Uni}(A')) = \operatorname{Nil}(A').$$

2. Invertible modules

Let $A \subseteq B$ be an extension of (arbitrary commutative) rings. In this section we discuss the group $\mathcal{I}(A, B)$ of invertible A-submodules of B. This group is well known if $B = S^{-1}A$ with S a multiplicative set of nonzero divisors in A [2, Ch. 2, §5]. We do not know a reference for the more general case that we need in this paper. Some of our lemmas generalize results of [4, §1] to the nongraded case.

(2.1) DEFINITION. The set of all A-submodules of B is a commutative semigroup under multiplication, with identity A. The invertible elements of this semigroup, called *invertible A-submodules* of B, from an abelian group denoted $\mathcal{I}(A, B)$.

Let $A' \subseteq B'$ be an extension of rings and let $\varphi\colon B \to B'$ be a ring homomorphism such that $\varphi(A) \subseteq A'$. If $I \in \mathcal{I}(A, B)$ then clearly $A'\varphi(I) \in \mathcal{I}(A', B')$. Therefore writing $\mathcal{I}(\varphi)(I) = A'\varphi(I)$ the assignment $(A \subseteq B) \mapsto \mathcal{I}(A, B)$ becomes a functor from the category of ring extensions to the category of abelian groups.

(2.2) LEMMA. *Let $I \in \mathcal{I}(A, B)$. Then:*

(1) *There exists $m_1, \ldots, m_r \in I$, $n_1, \ldots, n_r \in I^{-1}$ such that $\Sigma_{i=1}^r m_i n_i = 1$. Consequently, $I = (m_1, \ldots, m_r)A$ and $IB = B$.*

(2) $I^{-1} = (A : I) = \{x \in B \mid xI \subseteq A\}$.

(3) *I is a projective A-module of finite type, and of rank one.*

(4) *If J is any A-submodule of B then the multiplication map $I \otimes_A J \to IJ$ is an isomorphism. In particular, $I \otimes_A B \to IB = B$ is an isomorphism.*

Proof. Parts (1), (2), (3), except for the assertion about rank, are proved as in [2]. We prove (4) first in the case $J = B$. If $\Sigma_{i=1}^r m_i n_i = 1$ as in (1) then we get a surjection $f: A^r \to I$, sending the ith element e_i of the standard basis of A^r to m_i, which is split by $g(m) = (mn_1, mn_2, \ldots, mn_r)$, for $m \in I$. Tensoring with B we get a commutative diagram

$$\begin{array}{ccccc} I \otimes_A B & \xrightarrow{g\otimes 1} & A^r \otimes_A B & \xrightarrow{f\otimes 1} & I \otimes_A B \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \mu} \\ B & \xrightarrow{l} & B^r & \xrightarrow{h} & B \end{array}$$

of B-linear maps, where μ is multiplication, h is defined by $h(e_i) = m_i$ and l is defined by $l(1) = (n_1, \ldots, n_r)$. Clearly $(f \otimes 1)(g \otimes 1) = 1_{I \otimes_A B}$ and $hl = 1_B$, so $g \otimes 1$ and l are monomorphisms. Therefore μ is a monomorphism. The image of μ is IB, which by (1) equals B, so μ is onto. Hence μ is an isomorphism. Now let J be an arbitrary A-submodule of B. Since I is a projective, hence flat, A-module the inclusion $J \to B$ induces an inclusion $I \otimes_A J \to I \otimes_A B$ and it follows that the map $I \otimes_A J \to IJ$ is an isomorphism. In particular $I \otimes I^{-1} \cong A$, showing that $\text{rank}(I) = 1$. □

(2.3) LEMMA. *Let I be an A-submodule of B, I of finite type and projective of rank one as an A-module. Suppose also that $IB = B$. Then $I \in \mathscr{I}(A, B)$.*

Proof. The multiplication map $I \otimes_A B \to IB = B$ is a surjection of projective B-modules of rank one, hence an isomorphism. Since I is projective of finite type, there is a split surjection $f: A^r \to I$. Let the splitting be $g: I \to A^r$. Tensoring with B we obtain $f \otimes 1: B^r \to I \otimes_A B = B$, split by $g \otimes 1$. Suppose $(g \otimes 1)(1) = (n_1, \ldots, n_r)$. Then letting $m_i = f(e_i)$ we have $\Sigma m_i n_i = 1$. The restriction of $g \otimes 1$ to $I \subseteq B$ is just g, so we obtain that $n_i I \subseteq A$ for all i. Now, if we let J be the A-submodule generated by $n_1, \ldots, n_r$, then $IJ = A$, so $I \in \mathscr{I}(A, B)$. □

(2.4) THEOREM. *There is a functorial exact sequence*

$$0 \longrightarrow A^* \xrightarrow{i} B^* \xrightarrow{\theta} \mathscr{I}(A, B) \xrightarrow{\text{cl}} \text{Pic}(A) \xrightarrow{i} \text{Pic}(B).$$

Proof. The maps are the obvious ones: $\theta b = Ab$, $\text{cl}(I)$ is the class of I in $\text{Pic}(A)$ and the maps i are the natural ones. Note that cl is a homomorphism by (2.2). Exactness of the sequence is now proved as in [2]. Functoriality is clear except perhaps for the commutativity of the square

$$\begin{array}{ccc} \mathscr{I}(A, B) & \longrightarrow & \mathrm{Pic}(A) \\ \downarrow & & \downarrow \\ \mathscr{I}(A', B') & \longrightarrow & \mathrm{Pic}(A') \end{array}$$

where $\varphi: B \to B'$ with $\varphi(A) \subseteq A'$. For $I \in \mathscr{I}(A, B)$ the A'-homomorphism $g: I \otimes_A A' \to A'\varphi(I)$ given by $g(m \otimes a') = a'\varphi(m)$ is clearly onto, and both modules are projective of rank one. Therefore g is an isomorphism. Thus $I \otimes_A A' \cong A'\varphi(I) = \mathscr{I}(\varphi)(I)$, proving that the square is commutative. □

(2.5) REMARK. Two extreme cases are as follows:

(1) if $\mathrm{Pic}(A) = 0$ (e.g. A a local ring or a UFD) then $\mathscr{I}(A, B) = B^*/A^*$,
(2) if $A^* = B^*$ and $\mathrm{Pic}(B) = 0$ then $\mathscr{I}(A, B) = \mathrm{Pic}(A)$. This is the case, in particular, if A is a reduced G-algebra and B is the seminormalization of A, thus re-proving Theorem 1.4 of [4].

(2.6) PROPOSITION. *Let $\mathfrak{a}$ be a B-ideal contained in A. Then the natural homomorphism $\phi: \mathscr{I}(A, B) \to \mathscr{I}(A/\mathfrak{a}, B/\mathfrak{a})$ is an isomorphism.*

Proof. Since $\mathfrak{a} \subseteq (A : I^{-1}) = I$, we have $\mathfrak{a} \subseteq I$ for all $I \in \mathscr{I}(A, B)$. Therefore, by definition of ϕ we have $\phi(I) = I/\mathfrak{a}$, which shows that ϕ is a monomorphism. To prove the surjectivity of ϕ, let $I' \in \mathscr{I}(A/\mathfrak{a}, B/\mathfrak{a})$ and let $I = \pi^{-1}(I')$, where $\pi: B \to B/\mathfrak{a}$ is the canonical surjection. Put $J' = I'^{-1}$ and $J = \pi^{-1}J'$. Then $IJ = \pi^{-1}I'\pi^{-1}J' \subseteq \pi^{-1}(I'J') = \pi^{-1}(A/\mathfrak{a}) = A$. We claim now that $IJ = A$. To see this it suffices to show that $1 \in IJ$. Let $m_i' \in I'$, $n_i' \in J'$ be such that $\Sigma\, m_i'n_i' = 1$. If $\pi(m_i) = m_i'$, $\pi(n_i) = n_i'$, then $\Sigma\, m_i n_i = 1 + \gamma$, with $\gamma \in \mathfrak{a}$. Multiplying by $1 - \gamma$ we get $\Sigma\, m_i n_i(1 - \gamma) = 1 - \gamma^2$. Taking into account that $\gamma \in I \cap J$ so that $\gamma^2 \in IJ$, and that $n_i(1 - \gamma) \in J$, we conclude that $1 \in IJ$. Thus $I \in \mathscr{I}(A, B)$ and $I' = I/\mathfrak{a} \in \mathrm{im}(\phi)$.

(2.7) PROPOSITION. *The natural homomorphism $\phi: \mathscr{I}(A, B) \to \mathscr{I}(A_{\mathrm{red}}, B_{\mathrm{red}})$ is onto, with kernel* Uni(B)/Uni(A). *If $\mathbb{Q} \subseteq A$, then the kernel is isomorphic, via* log, *to* Nil(B)/Nil(A).

Proof. Here A_{red} and B_{red} are the reduced rings of A and B. The proof that ϕ is onto and has the indicated kernel is by diagram chasing using the map between the exact sequences of (2.4) for the extensions $A \subseteq B$ and $A_{\mathrm{red}} \subseteq B_{\mathrm{red}}$ and using the fact that $A^* \to (A_{\mathrm{red}})^*$ and $B^* \to (B_{\mathrm{red}})^*$ are onto, and that $\mathrm{Pic}(A) \cong \mathrm{Pic}(A_{\mathrm{red}})$, $\mathrm{Pic}(B) \cong \mathrm{Pic}(B_{\mathrm{red}})$ [1, Ch. 9, Prop. 3.4]. The final assertion is immediate from the remarks made in Section 1. □

The following lemma is used frequently throughout the paper:

(2.8) LEMMA. *Let I, J, J' be A-submodules of B such that $I \in \mathscr{I}(A, B)$, $I \subseteq J$, $I^{-1} \subseteq J'$ and $JJ' \subseteq A$. Then $J = I$ and $J' = I^{-1}$.*

Proof. Since $A = II^{-1} \subseteq JJ' \subseteq A$, $J \in \mathscr{I}(A, B)$ and $J' = J^{-1}$. Now, $I^{-1} \subseteq J' = J^{-1}$ implies $J \subseteq I$ and the lemma follows. □

3. A universal setup

The universal construction described in this section is motivated by systems of subintegrality introduced in the next section. The purpose of this section is to prove Theorem (3.8), which asserts that the "universal" submodules I, I' defined in (3.3) are inverses of each other.

(3.1) NOTATION. Let $C = \mathbb{Q}[x_1, \ldots, x_p, y_1, \ldots, y_q, z, w]$ be the polynomial ring in $p+q+2$ variables over $\mathbb{Q}$ and put $x_0 = y_0 = 1$. Make C a graded ring by defining $\deg(x_i) = i$ for every i, $\deg(y_j) = j$ for every j and $\deg(z) = \deg(w) = 1$. For a graded subring R of C let R_+ denote the ideal of R generated by all elements of positive degree and let $\hat{R}$ denote the completion of R with respect to R_+.

(3.2) LEMMA. *Let $R \subseteq S$ be graded subrings of C. Then $R = \hat{R} \cap S$.*

Proof. Every element f of $\hat{R}$ has a unique expression of the form $f = \Sigma_{n \geqslant 0} f_n$ with $f_n \in R$ homogeneous of degree n. Since this statement also applies to $\hat{S}$, $f \in S$ if and only if $f_n = 0$ for almost all n if and only if $f \in R$. □

(3.3) NOTATION. (1) $e_n(z) = \Sigma_{i=0}^{n-1} z^i/i!$.

(2) $\gamma_n = \Sigma_{i=0}^{p} \binom{n}{i} x_i z^{n-i}$.

(3) Let $s, p, N \in \mathbb{Z}^+$ with $N \geqslant s + p$ and let $R \subseteq S$ be graded $\mathbb{Q}$-subalgebras of C such that $x_1 z^s, \ldots, x_p z^s, z \in S$ and $\gamma_n \in R$ for all $n \geqslant N$.

(4) Let $I = \hat{R} e^z \cap S$, $I' = \hat{R} e^{-z} \cap S$.

(5) Let $\partial = \partial/\partial z$, the partial derivative w.r.t. z, and let $D: \hat{C} \to \hat{C}$ be the differential operator defined by $D = \Sigma_{i=0}^{p} (1/i!) x_i \partial^i$.

(6) Let Δ denote the composite

$$\hat{C} \xrightarrow{e^{-z}} \hat{C} \xrightarrow{D} \hat{C} \xrightarrow{e^{z}} \hat{C}$$

and Δ' the composite

$$\hat{C} \xrightarrow{e^{z}} \hat{C} \xrightarrow{D} \hat{C} \xrightarrow{e^{-z}} \hat{C}.$$

(3.4) LEMMA. (1) $D(z^n) = \gamma_n$.

(2) $\Delta = \Sigma_{i=0}^{p} (1/i!)(-1)^i x_i \Delta_i$ *and* $\Delta' = \Sigma_{i=0}^{p} (1/i!) x_i \Delta'_i$, *where* $\Delta_i = (1-\partial)^i$ *and* $\Delta'_i = (1+\partial)^i$.

(3) *If* $i \geqslant 1$ *then the elements* $\Delta_i(z^n)$, $\Delta_i(e_n(z))$, $\Delta'_i(z^n)$, $\Delta'_i(e_n(-z))$ *belong to* $z^{n-i}\mathbb{Q}[z]$.

(4) $D(\mathbb{Q} \oplus z^N \mathbb{Q}[[z]]) \subseteq \hat{R}$.

(5) *If* $f \in \mathbb{Q}[z]$ *such that* $e^{-z} f \in \mathbb{Q} \oplus z^N \mathbb{Q}[[z]]$ *and* $\Delta_i(f) \in z^s \mathbb{Q}[z]$ *for* $1 \leqslant i \leqslant p$ *then* $\Delta(f) \in I$. *In particular,* $\Delta(z^n)$, $\Delta(e_n(z)) \in I$ *for all* $n \geqslant N$.

(6) *If* $f \in \mathbb{Q}[z]$ *and* $e^z f \in \mathbb{Q} \oplus z^N\mathbb{Q}[[z]]$ *and* $\Delta_i'(f) \in z^s\mathbb{Q}[z]$ *for* $1 \leqslant i \leqslant p$ *then* $\Delta'(f) \in I'$. *In particular,* $\Delta'(z^n)$, $\Delta'(e_n(-z)) \in I'$ *for all* $n \geqslant N$.

Proof. (1) is clear and (2) is a straightforward verification. (3) and (4) are immediate from (1) and the definitions of Δ_i, Δ_i' and R. As for (5), we have $\Delta(f) \in \hat{R}e^z$ by (4) and $\Delta(f) \in S$ by (2) and the hypothesis on $\Delta_i(f)$, so $\Delta(f) \in I$. The last part of (5) follows now from (3) by noting that $e^{-z}e_n(z) \in \mathbb{Q} \oplus z^n\mathbb{Q}[[z]]$. The proof of (6) is similar. □

(3.5) LEMMA. *Let* $g_r(z) = \Sigma_{i=0}^r(-1)^i\binom{r}{i}\Delta(z^{r-i})\Delta'(z^i)$ *and let* $n \geqslant 2p$. *Then* $\Sigma_{r=0}^n(1/r!)g_r(z) = 1$.

Proof. We have

$$\begin{aligned}1 &= \Delta(e^z)\Delta'(e^{-z}) \\ &= \left(\sum_{i=0}^{\infty}(1/i!)\Delta(z^i)\right)\left(\sum_{i=0}^{\infty}(1/i!)(-1)^i\Delta'(z^i)\right) \\ &= \sum_{r=0}^{\infty}(1/r!)g_r(z).\end{aligned}$$

Therefore it is enough to prove that $g_r(z) = 0$ for every $r > 2p$. Let $r > 2p$. Then, for a given i, $r-i > p$ or $i > p$. Therefore, since ∂^p is the highest power of ∂ appearing in Δ (resp. Δ'), z divides $\Delta(z^{r-i})$ or $\Delta'(z^i)$ whence $g_r(0) = 0$. Therefore it is enough to prove that $\partial g_r(z) = 0$. Since ∂ commutes with Δ, Δ' by (3.4)(2), we have (in fact, for every $r \in \mathbb{Z}^+$)

$$\begin{aligned}\partial g_r(z) &= \sum_{i=0}^{r}(-1)^i\binom{r}{i}[\Delta(\partial z^{r-i})\Delta'(z^i) + \Delta(z^{r-i})\Delta'(\partial z^i)] \\ &= \sum_{i=0}^{r}(-1)^i\binom{r}{i}[(r-i)\Delta(z^{r-i-1})\Delta'(z^i) + i\Delta(z^{r-i})\Delta'(z^{i-1})] \\ &= \sum_{i=0}^{r-1}(-1)^i\left[(r-i)\binom{r}{i} - (i+1)\binom{r}{i+1}\right]\Delta(z^{r-i-1})\Delta'(z^i).\end{aligned}$$

Now, since $(r-i)\binom{r}{i} - (i+1)\binom{r}{i+1} = 0$, the lemma is proved. □

(3.6) LEMMA. *Let* H *be an additive subgroup of* C. *Let* T *be an indeterminate, let* $0 \neq F(T) \in C[T]$ *and let* $d = \deg_T F(T)$. *If there exists an integer* m *such that* $F(m+i) \in H$ *for* $i = 0, 1, \ldots, d$, *then* $F(n) \in H$ *for every integer* n.

Proof. Induction on d. If $d = 0$ then F is constant w.r.t. T, so that $F(n) = F(m) \in H$ for every n. Now, let $d > 0$ and put $G(T) = F(T) - F(T-1)$. Then $\deg_T G(T) = d-1$ and $G(m+1+i) = F(m+1+i) - F(m+i) \in H$ for $i = 0, 1, \ldots, d-1$. So by induction $G(n) \in H$ for every n. Now, since $F(m) \in H$, it follows that $F(n) \in H$ for every n. □

(3.7) LEMMA. *Let* $r \in \mathbb{Z}$ *with* $r \geqslant 2(p+N)$. *Then* $\Delta(z^n)\Delta'(z^{r-n}) \in II'$ *for all* $n \in \mathbb{Z}$.

Proof. Let

$$G(T) = \sum_{i=0}^{p} x_i \sum_{j=0}^{i} (1/(i-j)!)(-1)^{i+j} \binom{T}{j} z^{-j},$$

$$G'(T) = \sum_{i=0}^{p} x_i \sum_{j=0}^{i} (1/(i-j)!) \binom{-T}{j} z^{-j},$$

and $F(T) = G(T)G'(T-r)z^r$. Then $F(n) = \Delta(z^n)\Delta'(z^{r-n})$ by (3.4)(2). So we have to show that $F(n) \in II'$ for all n. Let i be an integer with $0 \leqslant i \leqslant 2p$. Then $r - N - i \geqslant N$ whence $F(N+i) = \Delta(z^{N+i})\Delta'(z^{r-N-i}) \in II'$ by (3.4). Therefore, since $\deg_T F(T) = 2p$, it follows from (3.6) that $F(n) \in II'$ for all n. □

(3.8) THEOREM. *$I, I' \in \mathscr{I}(R, S)$ and $I' = I^{-1}$.*

Proof. We must show that $II' = R$. Since

$$II' = (\hat{R}e^z \cap S)(\hat{R}e^{-z} \cap S) \subseteq \hat{R} \cap S = R$$

by (3.2), it is enough to prove that $1 \in II'$. Let $n \in \mathbb{N}$ with $n \geqslant 2(p+N)$. We have

$$\Delta(e_n(z))\Delta'(e_n(-z)) = \left(\sum_{i=0}^{n-1} (1/i!)\Delta(z^i)\right)\left(\sum_{i=0}^{n-1} (1/i!)(-1)^i\Delta'(z^i)\right)$$
$$= \sum_{r=0}^{n-1} (1/r!)g_r(z) + \sum_{r=n}^{2n-2} (1/r!)h_r(z),$$

where

$$g_r(z) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \Delta(z^{r-i})\Delta'(z^i)$$

and

$$h_r(z) = \sum_{i=r-n+1}^{n-1} (-1)^i \binom{r}{i} \Delta(z^{r-i})\Delta'(z^i).$$

We have $\Delta(e_n(z))\Delta'(e_n(-z)) \in II'$ by (3.4), $h_r(z) \in II'$ for $r \geqslant n$ by (3.7) and $\Sigma_{r=0}^{n-1} (1/r!)g_r(z) = 1$ by (3.5). Therefore $1 \in II'$. □

4. Subintegral elements

Recall that an extension $A \subseteq B$ of rings is an *elementary subintegral extension* if $B = A[b]$ with $b^2, b^3 \in A$. In general, the extension $A \subseteq B$ is said to be *subintegral* if B is a union of subrings which are obtained from A by a finite succession of elementary subintegral extensions, i.e. a union of subrings B' for which there exists a finite sequence $A=A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r = B'$ of rings with $A_{i-1} \subseteq A_i$ an elementary subintegral extension for every i, $1 \leqslant i \leqslant r$ (cf. [9, 2.8]).

In order to define the map $\xi_{B/A}: B/A \to \mathscr{I}(A,B)$ (Section 5) in the case of a subintegral extension $A \subseteq B$ of $\mathbb{Q}$-algebras, we need to prove that the $A[T]$-module $A[[T]]e^{bT} \cap A[b][T]$ is invertible. As a crucial step towards doing this, we find an elementwise characterization of a subintegral extension. As remarked in the Introduction, this characterization may be of some independent interest.

So let $A \subseteq B$ be $\mathbb{Q}$-algebras. We define an element b of B to be *subintegral* over A if there exist $c_1, \ldots, c_p \in B$ such that $b^n + \Sigma_{i=1}^{p} \binom{n}{i} c_i b^{n-i} \in A$ for all $n \gg 0$. In (4.2) we show that this condition on b is equivalent to several other conditions, and is consequently independent of the overring B to which b belongs. In (4.8) we show that the set of elements of B which are subintegral over A form a subring of B. Then, after proving several technical lemmas, we show that an extension $A \subseteq B$ is subintegral if and only if every element of B is subintegral over A (Theorem (4.17)). For a single element b this result means that b is subintegral over A if and only if the extension $A \subseteq A[b]$ is subintegral (Corollary (4.18)).

Let $b \in B$. By a *system of subintegrality* for b in the extension $A \subseteq B$ we mean a tuple $(s, p, N; c_0, \ldots, c_p)$ with $s, p, N \in \mathbb{Z}^+$ and $c_0, c_1, \ldots, c_p \in B$ such that

$$N \geqslant s + p, \; c_0 = 1 \quad \text{and} \quad b^n + \sum_{i=1}^{p} \binom{n}{i} c_i b^{n-s-i} \in A \quad \text{for all } n \geqslant N.$$

We call s the *exponent* of this system.

If $p = 0$ then the system reduces to $(s, 0, N; 1)$ with the familiar condition that $b^n \in A$ for all $n \geqslant N$. A general system of subintegrality is thus an extension of this special condition and is also in the spirit of an equation of integral dependence. In the case of integral dependence over A the coefficients are required to lie in A, whereas the "coefficients" c_i in a system of subintegrality are allowed to be in the overring B. However, these can be chosen to lie in $A[b]$ (see (4.2) below).

We find it necessary to introduce the exponent s as a device to prove the crucial technical lemma (4.12). The role of the exponent can be understood as follows: If we invert b then a system $(s, p, N; c_0, c_1, \ldots, c_p)$ can immediately be converted to one of exponent zero by replacing c_i by c_i/b^s. However, we do not wish to invert b, since $B \to B[b^{-1}]$ need not be an inclusion in general. The exponent s is a formal way of allowing b^s in the denominator.

Let $B[[T]]$ be the formal power series ring in one variable over B and put $W(B) = 1 + TB[[T]]$. Let $I(b)$ denote the $A[T]$-submodule of $A[b][T]$ defined by $I(b) = A[[T]]e^{bT} \cap A[b][T]$.

(4.1) LEMMA. (1) $I(b) = A[[T]]e^{bT} \cap B[T]$.

(2) $I(a) = A[T]$ *for all* $a \in A$. *In particular*, $I(0) = A[T]$.

(3) $I(b)I(b') \subseteq I(b+b')$. *In particular*, $I(b)I(-b) \subseteq A[T]$.

Proof. Obviously, $I(b) \subseteq A[[T]]e^{bT} \cap B[T]$. For the reverse inclusion, it suffices to note that

$$A[[T]]e^{bT} \cap B[T] \subseteq A[b][[T]] \cap B[T] = A[b][T].$$

This proves (1). (2) is clear, and (3) is immediate from (1) and (2). □

(4.2) PROPOSITION. *For an element* b *of* B *the following five conditions are equivalent*:

(i) *there exists a system of subintegrality for* b *in the extension* $A \subseteq A[b]$ *of exponent zero,*

(ii) *there exists a system of subintegrality for* b *in the extension* $A \subseteq B$.

(iii) $I(b)I(-b) = A[T]$,

(iv) $I(b) \cap W(A[b]) \neq \varnothing$,

(v) $I(b) \cap W(B) \neq \varnothing$.

Proof. (i) ⇒ (ii). Trivial.

(ii) ⇒ (iii). Let $(s, p, N; c_0, c_1, \ldots, c_p)$ be a system of subintegrality for b in the extension $A \subseteq B$. In the notation of (3.3) let $S = \mathbb{Q}[x_1z^s, \ldots, x_pz^s, z]$ and $R = \mathbb{Q}[\{\gamma_n \mid n \geqslant N\}]$. Then $R \subseteq S$ are graded $\mathbb{Q}$-subalgebras of C. Let $\varphi: \hat{S} \to B[[T]]$ be the $\mathbb{Q}$-algebra homomorphism given by $\varphi(x_iz^s) = c_iT^{i+s}$ for $1 \leqslant i \leqslant p$ and $\varphi(z) = bT$. Then $\varphi(S) \subseteq B[T]$. Further, since

$$\varphi(\gamma_n) = \left(b^n + \sum_{i=1}^{p} \binom{n}{i} c_i b^{n-s-i}\right) T^n \in A[T] \quad \text{for } n \geqslant N,$$

we have

$$\varphi(R) \subseteq A[T] \quad \text{and} \quad \varphi(\hat{R}) \subseteq A[[T]].$$

Let $I = \hat{R}e^z \cap S$. Then by (3.8) $I \in \mathscr{I}(R, S)$ and $I^{-1} = \hat{R}e^{-z} \cap S$. Now,

$$\varphi(I)A[T] \subseteq A[[T]]e^{bT} \cap B[T] = I(b)$$

by (4.1). Similarly, $\varphi(I^{-1})A[T] \subseteq I(-b)$. Therefore, since $I(b)I(-b) \subseteq A[T]$ by (4.1), we get $\varphi(I)A[T] = I(b)$ and $I(b)I(-b) = A[T]$ by (2.8).

(iii) ⇒ (iv). Choose $f_i(T)$, $g_i(T) \in A[[T]]$ such that

$$f_i(T)e^{bT} \in I(b), \qquad g_i(T)e^{-bT} \in I(-b) \quad \text{and} \quad \sum_{i=1}^{r} f_i(T)e^{bT}g_i(T)e^{-bT} = 1.$$

Then

$$\sum_{i=1}^{r} f_i(T)g_i(0)e^{bT} \in I(b) \cap W(A[b]).$$

(iv) $\Rightarrow$ (v). *Trivial.*

(v) $\Rightarrow$ (iv). Clear, since $I(b) \subseteq A[b][T]$.

(iv) $\Rightarrow$ (i). Let $f \in I(b) \cap W(A[b])$ and put $g = fe^{-bT}$. Then $g \in A[[T]]$. Write

$$f = \sum_{i=0}^{p} f_i T^i \quad \text{with} \quad f_i \in A[b],\ f_0 = 1 \quad \text{and} \quad g = \sum_{n=0}^{\infty} g_n T^n \quad \text{with} \quad g_n \in A.$$

Then we have

$$(-1)^n n!\, g_n = b^n + \sum_{i=1}^{p} \binom{n}{i} c_i b^{n-i} \quad \text{with} \quad c_i = (-1)^i i! f_i.$$

It follows that $(0, p, p; c_0, c_1, \ldots, c_p)$ is a system of subintegrality for b in the extension $A \subseteq A[b]$. □

(4.3) COROLLARY. *An element b of B is subintegral over A if and only if b satisfies any one of the equivalent conditions of the above proposition. Moreover, the definition of subintegrality of b over A is independent of the overring to which b belongs.*

Proof. The condition defining subintegrality of b over A implies (ii) and is implied by (i), which is a condition independent of the overring to which b belongs. □

(4.4) LEMMA. *Let $A' \subseteq B'$ be an extension of $\mathbb{Q}$-algebras and let $\varphi: B \to B'$ be a $\mathbb{Q}$-algebra homomorphism such that $\varphi(A) \subseteq A'$. If $b \in B$ and $(s, p, N; c_0, c_1, \ldots, c_p)$ is a system of subintegrality for b in the extension $A \subseteq B$ then $(s, p, N; \varphi(c_0), \varphi(c_1), \ldots, \varphi(c_p))$ is a system of subintegrality for $\varphi(b)$ in the extension $A' \subseteq B'$. In particular, if b is subintegral over A then $\varphi(b)$ is subintegral over A'.*

Proof. Clear. □

(4.5) LEMMA. *If $b, b' \in B$ are subintegral over A then so is $b + b'$.*

Proof. If $f \in I(b) \cap W(B)$ and $f' \in I(b') \cap W(B)$ then $ff' \in I(b+b') \cap W(B)$ by (4.1). Therefore $b + b'$ is subintegral over A by (4.3). □

We would like to show next that if b, $b' \in B$ are subintegral over A then so is bb'. To do this and also to prove some other properties of subintegral elements and systems of subintegrality, we find it convenient to first do the same in a universal setup and then specialize to the given situation. We have already done this in the proof of (ii) $\Rightarrow$ (iii) of the above proposition. Note that if $R \supseteq \mathbb{Q}[\{y_n \mid n \geqslant N\}]$ as in (3.3) then z is subintegral over R. The universal setup in Section 3 is motivated in fact by systems of subintegrality as discussed above. Thus in the following notation z and w are two generic subintegral elements.

(4.6) NOTATION. Let $C = \mathbb{Q}[x_1, \ldots, x_p, y_1, \ldots, y_q, z, w]$ as in (3.1) and let $\gamma_n = \Sigma_{i=0}^{p} \binom{n}{i} x_i z^{n-i}$ as in (3.3). Put $\delta_n = \Sigma_{j=0}^{q} \binom{n}{j} y_j w^{n-j}$. Let $N \geqslant p$, $M \geqslant q$ be integers and let $R = \mathbb{Q}[\{\gamma_n \mid n \geqslant N\} \cup \{\delta_n \mid n \geqslant M\}] \subseteq C$.

(4.7) LEMMA. *With the above notation zw is subintegral over R.*

Proof. Let

$$F(T) = \sum_{i=0}^{p} x_i z^{-i} \binom{T}{i}, \quad G(T) = \sum_{j=0}^{q} y_j w^{-j} \binom{T}{j} \quad \text{and} \quad H(T) = F(T)G(T).$$

Then

$$\gamma_n = F(n) z^n, \qquad \delta_n = G(n) w^n$$

and

$$H(T) = \sum_{k=0}^{p+q} \sum_{i+j=k} x_i y_j z^{-i} w^{-j} H_{ij}(T) \quad \text{with} \quad H_{ij}(T) = \binom{T}{i}\binom{T}{j}.$$

If $i+j=k$ then $\deg_T H_{ij} = k$ and so we can write $H_{ij}(T)$ uniquely in the form $H_{ij}(T) = \sum_{h=0}^{k} a_h \binom{T}{h}$ with $a_h \in \mathbb{Q}$. Since $\binom{r}{h} = 0$ for nonnegative integers $r < h$, we see that $a_h = 0$ for $h < \max(i,j)$. It follows that $H(T) = \Sigma_{h=0}^{p+q} u_h z^{-h} w^{-h} \binom{T}{h}$ with $u_h \in C$ for all h and $u_0 = H(0) = 1$. Now,

$$(zw)^n + \sum_{h=1}^{p+q} \binom{n}{h} u_h (zw)^{n-h} = H(n) z^n w^n = \gamma_n \delta_n \in R$$

for all $n \geqslant \max(N, M)$, which shows that zw is subintegral over R. □

(4.8) PROPOSITION. *Let B′ be the set of all elements of B which are subintegral over A. Then B′ is a subring of B containing A.*

Proof. Since A is clearly contained in B', it is enough to prove that if b, b' are subintegral over A then so are $b+b'$ and bb'. The assertion about the sum is in (4.5). To prove it for the product, let $(0, p, N; c_0, c_1, \ldots, c_p)$, $(0, q, M; d_0, d_1, \ldots, d_q)$ be, respectively, systems of subintegrality for b, b' in the extension $A \subseteq B$. In the notation of (4.6) let $\varphi: C \to B$ be the $\mathbb{Q}$-algebra homomorphism given by $\varphi(x_i) = c_i$ for every i, $\varphi(z) = b$, $\varphi(y_j) = d_j$ for every j and $\varphi(w) = b'$. Then $\varphi(R) \subseteq A$ and $\varphi(zw) = bb'$ whence it follows from (4.7) and (4.4) that bb' is subintegral over A. □

(4.9) COROLLARY. *Let $s \in A$, $b \in B$. If b is subintegral over A then the element b/s of $B[s^{-1}]$ is subintegral over $A[s^{-1}]$.*

Proof. Since $b/1$ is subintegral over $A[s^{-1}]$ by (4.4) and $1/s \in A[s^{-1}]$, b/s is subintegral over $A[s^{-1}]$ by the above proposition. □

The next three lemmas are of a technical nature and are needed only to prove Proposition 4.13.

Let L be the quotient field of C. For $e \in \mathbb{N}$ define an operator ϑ_e on $L(T)$ by $\vartheta_e F(T) = F(T) - F(T-e)$. For $k \in \mathbb{Z}^+$ let ϑ_e^k denote the application of ϑ_e k-times with $\vartheta_e^0 = \text{identity}$. Define the T-degree of the zero polynomial in $L[T]$ to be -1. For $d \in \mathbb{Z}^+$ put $\beta_d(T) = \binom{T}{d}$. Note that $\deg_T \beta_d(T) = d$ and $\beta_d(k) = 0$ for nonnegative integers $k < d$.

(4.10) LEMMA. (1) *If $F(T) \in L[T]$ and $d = \deg_T F(T) \geqslant 0$ then $\vartheta_e F(T) \in L[T]$ and $\deg_T \vartheta_e F(T) = d-1$. In particular, $\vartheta_e^k \beta_d(T) = 0$ for all $k > d$.*

(2) *$\vartheta_e^k F(T) = \Sigma_{j=0}^k (-1)^j \binom{k}{j} F(T-je)$ for all $k \geqslant 0$.*

Proof. (1) is clear and (2) is immediate by induction on k. □

(4.11) LEMMA. *Let d, e, r be fixed positive integers and let $\vartheta = \vartheta_e$. Put $\mu(T) = (r - dT)/e$. For $k \in \mathbb{Z}^+$ let $\pi_k(T) = \Pi_{j=0}^k \mu(T-je)$ and $F_k(T) = \beta_k(T)/\mu(T)$. Then for all $k \geqslant i \geqslant 0$ we have*

$$E(k, i, T) \qquad \vartheta^k F_i(T) = k!\, d^k \beta_i(r/d)/\pi_k(T)$$

Proof. Induction on i. First, let $i = 0$ in which case $\beta_i(T) = 1$ and we have to show that $\vartheta^k(1/\mu(T)) = k!\, d^k/\pi_k(T)$ for all $k \geqslant 0$. We do this by induction on k. The assertion is clear for $k = 0$. Let $k > 0$. Then

$$\begin{aligned}
\vartheta^k(1/\mu(T)) &= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (1/\mu(T-je)) \quad \text{by (4.10)} \\
&= \sum_{j=0}^{k} (-1)^j \left(\binom{k-1}{j} + \binom{k-1}{j-1}\right) (1/\mu(T-je)) \quad \text{where } \binom{k-1}{-1} = 0 \\
&= \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (1/\mu(T-je)) \\
&\quad - \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (1/\mu(T-e-je)) \\
&= (k-1)!\, d^{k-1}/\pi_{k-1}(T) - (k-1)!\, d^{k-1}/\pi_{k-1}(T-e) \\
&\quad \text{by (4.10), } E(k-1, 0, T) \text{ and } E(k-1, 0, T-e) \\
&= k!\, d^k/\pi_k(T).
\end{aligned}$$

This proves $E(k, 0, T)$. Now, let $i > 0$. We have

$$\vartheta^k(F_i(T)) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} F_i(T-je)$$

by (4.10). Therefore, since

$$\beta_i(T-je) = i^{-1}(T-i+1)\beta_{i-1}(T-je) - i^{-1}je\beta_{i-1}(T-je)$$

and

$$j\binom{k}{j} = k\binom{k-1}{h} \quad \text{with} \quad h = j-1,$$

we get

$$\begin{aligned}\vartheta^k(F_i(T)) &= i^{-1}(T-i+1) \sum_{j=0}^{k} (-1)^j \binom{k}{j} F_{i-1}(T-je) \\ &\quad + i^{-1}ek \sum_{h=0}^{k-1} (-1)^h \binom{k-1}{h} F_{i-1}(T-e-he) \\ &= i^{-1}(T-i+1)k!\, d^k\beta_{i-1}(r/d)/\pi_k(T) \\ &\qquad + i^{-1}ek(k-1)!\, d^{k-1}\beta_{i-1}(r/d)/\pi_{k-1}(T-e) \\ &\qquad \text{by (4.10), } E(k, i-1, T) \text{ and } E(k-1, i-1, T-e) \\ &= k!\, d^k\beta_i(r/d)/\pi_k(T).\end{aligned}$$

□

(4.12) LEMMA. *Let $N \geqslant p$, $M \geqslant q$ be integers. Put*

$$\sigma_n = \sum_{i=0}^{p} \binom{n}{i} x_i z^{2n-2i}, \qquad \tau_m = \sum_{j=0}^{q} \binom{m}{j} y_j z^{3m-3j}$$

and let

$$R' = \mathbb{Q}[\{\sigma_n \mid n \geqslant N\} \cup \{\tau_m \mid m \geqslant M\}] \subseteq C.$$

Then z is subintegral over R'.

Proof. Replacing M by $M+1$, if necessary, we may assume that M is odd. Put

$$\rho(n, m) = \sum_{j=0}^{q} \sum_{i=0}^{p} \binom{n}{i}\binom{m}{j} x_i y_j z^{2n+3m-2i-3j}.$$

Then $\rho(n, m) = \sigma_n \tau_m$ whence $\rho(n, m) \in R'$ for integers $n \geqslant N$, $m \geqslant M$. Let us now use the notation of (4.11) with $d = 2$, $e = 3$ and $r \geqslant 2N + 3M + 6(p+q)$ a fixed odd integer. Then

$$\vartheta = \vartheta_3, \qquad \mu(T) = (r - 2T)/3, \qquad \pi_k(T) = \prod_{j=0}^{k} \mu(T - 3j)$$

and

$$F_k(T) = \beta_k(T)/\mu(T).$$

Put $G_k(T) = \beta_k(\mu(T))/\mu(T)$. Then for $k \geqslant 1$ we have

$$G_k(T) \in C[T] \quad \text{and} \quad \deg_T G_k(T) = k - 1.$$

Let

$$H(T) = \sum_{j=1}^{q} \sum_{i=0}^{p} x_i y_j \beta_i(T) G_j(T) z^{r-2i-3j}.$$

Then

$$H(T) \in C[T] \quad \text{and} \quad \deg_T H(T) = p + q - 1.$$

Let

$$J(T) = \sum_{i=0}^{p} x_i F_i(T) z^{r-2i}, \qquad K(T) = J(T) + H(T) \quad \text{and} \quad P(T) = \vartheta^{p+q} K(T).$$

Then $P(T) = \vartheta^{p+q} J(T)$, since $\vartheta^{p+q} H(T) = 0$ by (4.10). Since r, M are odd, there exists an integer n such that $r = 2n + 3M$, i.e. $M = \mu(n)$. It follows that $n - 3k$ and $\mu(n - 3k)$ are integers with $n - 3k \geqslant N$ and $\mu(n - 3k) \geqslant M$ for $k = 0, 1, \ldots, p + q$. Therefore

$$K(n - 3k) = \rho(n - 3k, \mu(n - 3k))/\mu(n - 3k) \in R'$$

for these values of k, whence by (4.10)

$$P(n) = \sum_{k=0}^{p+q} (-1)^k \binom{p+q}{k} K(n - 3k) \in R'.$$

Now,

$$\begin{aligned}P(T) &= \vartheta^{p+q}J(T)\\ &= \sum_{i=0}^{p} x_i z^{r-2i}\vartheta^{p+q}F_i(T)\\ &= \sum_{i=0}^{p} x_i z^{r-2i}(p+q)!\,2^{p+q}\beta_i(r/2)/\pi_{p+q}(T)\end{aligned}$$

by (4.11), which shows that

$$\sigma_{r/2} = \sum_{i=0}^{p} \beta_i(r/2)x_i z^{r-2i} \in \mathbb{Q}P(n) \subseteq R'.$$

Thus we have shown that $\sigma_{r/2} \in R'$ for every odd integer $r \geqslant 2N+3M+6(p+q)$. Therefore, since $\sigma_{r/2} \in R'$ for every even integer $r \geqslant 2N$ by hypothesis, we get

$$\sigma_{r/2} \in R' \quad \text{for every integer } r \geqslant 2N+3M+6(p+q). \tag{*}$$

Now, for $i \geqslant 1$ we can write $\beta_i(T) = \Sigma_{j=1}^{i} a_{ij}\beta_j(2T)$ with $a_{ij} \in \mathbb{Q}$ and we get

$$\begin{aligned}\sigma_{r/2} &= z^r + \sum_{i=1}^{p} \beta_i(r/2)x_i z^{r-2i}\\ &= z^r + \sum_{i=1}^{p}\left(\sum_{j=1}^{i} a_{ij}\beta_j(r)\right) x_i z^{r-2i}\\ &= z^r + \sum_{j=1}^{p}\binom{r}{j} u_j z^{r-s-j},\end{aligned}$$

where $s = 2p$ and $u_j = \Sigma_{i=j}^{p} a_{ij}x_i z^{j+s-2i}$. Now, it follows from (*) and (4.3) that z is subintegral over R'. □

(4.13) PROPOSITION. *Let $b \in B$. If b^2, b^3 are subintegral over A then so is b.*

Proof. Let $(0, p, N; c_0, c_1, \ldots, c_p)$, $(0, q, M; d_0, d_1, \ldots, d_q)$ be, respectively, systems of subintegrality for b^2, b^3 in the extension $A \subseteq B$. With the notation of the above proposition define a $\mathbb{Q}$-algebra homomorphism $\varphi: C \to B$ by $\varphi(x_i) = c_i$ for every i, $\varphi(y_j) = d_j$ for every j, $\varphi(z) = b$ and $\varphi(w) = 0$. Then $\varphi(R') \subseteq A$. By the above proposition z is subintegral over R'. Therefore by (4.4) b is subintegral over A. □

The next three results are needed mainly to prove Theorem (4.17).

(4.14) LEMMA. *Let* $F(T)=\Sigma_{i=0}^{p} x_i z^{-i}\beta_i(T)$ *and for* $r\in\mathbb{Z}^+$ *let* $G_r(T)=F(T)F(r-T)z^r$. *Let* $\vartheta=\vartheta_1$. *Then*

(1) $\vartheta^{2p}G_r(T)=ax_p^2z^{r-2p}$ *with* $a\in\mathbb{Q}$, $a\neq 0$.

(2) *If* $\lambda\in C$ *then* $\vartheta^p\lambda F(T)=\lambda x_p z^{-p}$.

Proof. We have $G_r(T)=x_p^2z^{r-2p}H(T)+K(T)$ with $H(T)=\beta_p(T)\beta_p(r-T)$, $K(T)\in L[T]$ and $\deg_T K(T)<2p$. Let $a=\vartheta^{2p}H(T)$. Since $H(T)\in\mathbb{Q}[T]$ and $\deg_T H(T)=2p$, we have $a\in\mathbb{Q}[T]$ and $\deg_T a=0$ by (4.10), i.e. $a\in\mathbb{Q}$ and $a\neq 0$. Now, since ϑ is L-linear, (1) follows from (4.10), and so does (2) by observing that $\vartheta^p\binom{T}{p}=1$. □

(4.15) LEMMA. *In the notation of* (4.6), *let* $R'=\mathbb{Q}[\{\gamma_n\mid n\geqslant N\}]$. *Then* $x_p^2z^n\in R'$ *for all* $n\geqslant 2N$ *and* $x_p^3z^n\in R'$ *for all* $n\geqslant 3N$.

Proof. Let $n\geqslant 2N$. Put $r=n+2p$. Then, with ϑ and $G_r(T)$ as in (4.14), we have

$$x_p^2z^n=x_p^2z^{r-2p}=a^{-1}\vartheta^{2p}G_r(T)=a^{-1}\sum_{j=0}^{2p}(-1)^j\binom{2p}{j}G_r(T-j) \qquad (*)$$

by (4.14) and (4.10). Let $m=N+2p$. Then for $j=0,1,\dots,2p$, we have $m-j\geqslant N$ and $r-m+j\geqslant N$ whence

$$G_r(m-j)=F(m-j)z^{m-j}F(r-m+j)z^{r-m+j}=\gamma_{m-j}\gamma_{r-m+j}\in R'.$$

Therefore, substituting $T=m$ in (*), we get $x_p^2z^n\in R'$. Now, let $n\geqslant 3N$ and put $r=n+p$. Then

$$x_p^3z^n=x_p^3z^{r-p}=\vartheta^p x_p^2z^rF(T)=\sum_{j=0}^{p}(-1)^j\binom{p}{j}x_p^2z^rF(T-j) \qquad (**)$$

by (4.14) and (4.10). Let $m=N+p$. Then for $j=0,1,\dots,p$, we have $r-m+j\geqslant 2N$ and $m-j\geqslant N$ whence

$$x_p^2z^rF(m-j)=x_p^2z^{r-m+j}F(m-j)z^{m-j}=x_p^2z^{r-m+j}\gamma_{m-j}\in R'$$

by the first part. Therefore, putting $T=m$ in (**), we get $x_p^3z^n\in R'$. □

(4.16) PROPOSITION. *Let* $b\in B$ *and let* $(0,p,N;c_0,c_1,\dots,c_p)$ *be a system of subintegrality for* b *in the extension* $A\subseteq B$. *Then* $c_p^2b^n\in A$ *for all* $n\geqslant 2N$ *and* $c_p^3b^n\in A$ *for all* $n\geqslant 3N$.

Proof. In the notation of (4.6) let $S=\mathbb{Q}[x_1,\dots,x_p,z]$, $R'=\mathbb{Q}[\{\gamma_n\mid n\geqslant N\}]$ and let $\varphi\colon S\to B$ be the $\mathbb{Q}$-algebra homomorphism given by $\varphi(x_i)=c_i$ for every i

and $\varphi(z) = b$. Then $\varphi(R') \subseteq A$, $c_p^2 b^n = \varphi(x_p^2 z^n)$ and $c_p^3 b^n = \varphi(x_p^3 z^n)$. So the assertion follows from (4.15). □

Now we are ready to prove that an extension $A \subseteq B$ of $\mathbb{Q}$-algebras is subintegral in the sense described at the beginning of this section if and only if every element of B is subintegral over A.

(4.17) THEOREM. *For an extension $A \subseteq B$ of $\mathbb{Q}$-algebras the following two conditions are equivalent*:

(i) *every element of B is subintegral over A,*
(ii) *the extension $A \subseteq B$ is subintegral.*

Proof. (i) $\Rightarrow$ (ii). Let $b \in B$ and let $(0, p, N; c_0, c_1, \ldots, c_p)$ be a system of subintegrality for b in the extension $A \subseteq B$. Call p the *degree* of this system. We prove by induction on p that b belongs to a subring B' of B containing A such that the extension $A \subseteq B'$ is subintegral. If $p = 0$ then $b^n \in B$ for $n \gg 0$ and the assertion is immediate by taking $B' = A[b]$. Now, let $p > 0$. By (4.16) $c_p^2 b^n \in A$ for all $n \geqslant 2N$ and $c_p^3 b^n \in A$ for all $n \geqslant 3N$. Let $A' = A[\{c_p b^n \mid n \geqslant N\}]$. Given $\alpha \in A'$ there exists finitely many elements, say $u_1, \ldots, u_m$, in the set $\{c_p b^n \mid n \geqslant N\}$ such that $\alpha \in A[u_1, \ldots, u_m]$. Since $u_i^2, u_i^3 \in A$ for every i, $A[u_1, \ldots, u_m]$ is obtained from A by a finite succession of elementary subintegral extensions. This shows that the extension $A \subseteq A'$ is subintegral. Now, $b^n + \Sigma_{i=1}^{p-1} \binom{n}{i} c_i b^{n-i} \in A'$ for $n \gg 0$ whence b has in the extension $A' \subseteq B$ a system of subintegrality of degree less than p. Therefore, by induction, b belongs to a subring B' of B containing A' such that the extension $A' \subseteq B'$ is subintegral. Now, the extension $A \subseteq B'$ is subintegral by [9, Lemma 2.3] and we are done.

(ii) $\Rightarrow$ (i). Let B' be the set of all elements of B which are subintegral over A. It is enough to prove that if $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r \subseteq B$ is a sequence of elementary subintegral extensions then $A_r \subseteq B'$. We do this by induction on r. For $r = 0$ we have $A_0 = A \subseteq B'$ by (4.8). Let $r > 0$ and let $A_r = A_{r-1}[b]$ with $b^2, b^3 \in A_{r-1}$. By induction $A_{r-1} \subseteq B'$. In particular, $b^2, b^3 \in B'$ whence $b \in B'$ by (4.13). Therefore, since B' is a ring by (4.8), $A_r \subseteq B'$. □

(4.18) COROLLARY. *Let $A \subseteq B$ be an extension of $\mathbb{Q}$-algebras. Then for an element b of B the following three conditions are equivalent*:

(i) *b is subintegral over A,*
(ii) *the extension $A \subseteq A[b]$ is subintegral,*
(iii) *there exists a finite sequence $A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r$ of subrings of B such that $A_{i-1} \subseteq A_i$ is an elementary subintegral extension for every i, $1 \leqslant i \leqslant r$, and $b \in A_r$.*

Proof. (i) $\Rightarrow$ (ii). By (4.8) every element of $A[b]$ is subintegral over A. Therefore the extension $A \subseteq A[b]$ is subintegral by the above theorem.

(ii) $\Rightarrow$ (iii). Immediate from the definition.

(iii) $\Rightarrow$ (i). Apply the theorem to the extension $A \subseteq A_r$. □

(4.19) COROLLARY. *Let $A \subseteq B$ be an extension of $\mathbb{Q}$-algebras and let $b \in B$. If b is subintegral over A then b is integral over A.*

Proof. Immediate from (4.18). □

(4.20) COROLLARY. *The seminormalization of a reduced $\mathbb{Q}$-algebra A is the set of all elements of its total quotient ring which are subintegral over A.*

Proof. (4.17) and [9, 2.8]. □

5. Main theorem

Let $A \subseteq B$ be an extension of $\mathbb{Q}$-algebras. Assume that this extension is subintegral. Then by (4.17) and (4.3) $I(b)I(-b) = A[T]$ for every $b \in B$ whence

$$I(b) \in \mathscr{I}(A[T], A[b][T]) \subseteq \mathscr{I}(A[T], B[T]) \quad \text{and} \quad I(b)^{-1} = I(-b).$$

Thus we get a map $I_{B/A}: B \to \mathscr{I}(A[T], B[T])$ given by $I_{B/A}(b) = I(b)$.

(5.1) LEMMA. *$I_{B/A}$ is a homomorphism of groups and $A \subseteq \ker(I_{B/A})$.*

Proof. By (4.1)

$$I(b)I(b') \subseteq I(b + b')$$

and

$$I(b)^{-1}I(b')^{-1} = I(-b)I(-b') \subseteq I(-b-b') = I(b + b')^{-1}.$$

Therefore $I(b)I(b') = I(b + b')$ by (2.8). This proves that $I_{B/A}$ is a homomorphism. The last assertion is given by (4.1). □

Let $\sigma: B[T] \to B$ be the B-algebra homomorphism given by $\sigma(T) = 1$. Then $\sigma(A[T]) = A$ whence we have the group homomorphism

$$\mathscr{I}(\sigma): \mathscr{I}(A[T], B[T]) \to \mathscr{I}(A, B).$$

Let $\eta_{B/A}: B \to \mathscr{I}(A, B)$ be the homomorphism obtained by composing $\mathscr{I}(\sigma)$ and $I_{B/A}$, i.e. $\eta_{B/A} = \mathscr{I}(\sigma) \circ I_{B/A}$. Then $A \subseteq \ker(\eta_{B/A})$ by the above lemma whence $\eta_{B/A}$ induces a homomorphism $\xi_{B/A}: B/A \to \mathscr{I}(A, B)$.

To recollect the definition of $\xi_{B/A}$, let $\bar{b} \in B/A$ with representative $b \in B$. Then

$\xi_{B/A}(\bar{b})$ is the invertible A-submodule of B obtained by reducing $A[[T]]e^{bT} \cap A[b][T]$ modulo $T - 1$.

(5.2) REMARKS. (1) $\xi_{B/A}(\bar{b})$ belongs to the subgroup $\mathscr{I}(A, A[b])$ of $\mathscr{I}(A, B)$.

(2) Another description of $\xi_{B/A}(\bar{b})$: Let $b \in B$. Choose a system $(s, p, N; c_0, \ldots, c_p)$ of subintegrality for b in the extension $A \subseteq B$, and in the notation of (3.3) let $S = \mathbb{Q}[x_1 z^s, \ldots, x_p z^s, z]$ and $R = \mathbb{Q}[\{\gamma_n \mid n \geqslant N\}]$. Let $\psi: S \to B$ be the $\mathbb{Q}$-algebra homomorphism given by $\psi(x_i z^s) = c_i$ for $1 \leqslant i \leqslant p$ and $\psi(z) = b$. Then $\psi(R) \subseteq A$ and so we have the map $\mathscr{I}(\psi): \mathscr{I}(R, S) \to \mathscr{I}(A, B)$. Let $I = \hat{R}e^z \cap S$, as in (3.3). Then $I \in \mathscr{I}(R, S)$ by (3.8) whence $\mathscr{I}(\psi)(I) \in \mathscr{I}(A, B)$. We claim that $\xi_{B/A}(\bar{b}) = \mathscr{I}(\psi)(I)$. To see this, let $\varphi: \hat{S} \to B[[T]]$ be the $\mathbb{Q}$-algebra homomorphism given by $\varphi(x_i z^s) = c_i T^{i+s}$ for $1 \leqslant i \leqslant p$ and $\varphi(z) = bT$. Then $\varphi(S) \subseteq B[T]$, $\varphi(R) \subseteq A[T]$ and, as seen in the proof of (4.2) (ii) $\Rightarrow$ (iii), we have $\varphi(I)A[T] = I(b)$. Now, from the commutativity of the diagram

$$\begin{array}{ccc} & S & \\ \swarrow^{\varphi} & & \searrow^{\psi} \\ B[T] & \xrightarrow{\sigma} & B \end{array}$$

we get $\mathscr{I}(\psi)(I) = \mathscr{I}(\sigma)(\varphi(I)A[T]) = \mathscr{I}(\sigma)(I(b)) = \xi_{B/A}(\bar{b})$.

(3) In the case when A, B are G-algebras and $b \in B_+$, we can give yet another description of $\xi_{B/A}(\bar{b})$, namely $\xi_{B/A}(\bar{b}) = \hat{A}e^b \cap B$, where $\hat{}$ denotes completion with respect to the ideal generated by all elements of positive degree. Let us show this first for the case when b is homogeneous of positive degree. In this case, we may replace the c_i of (2) by its homogeneous component of degree $(s + i)\deg(b)$ to assume that c_i is homogeneous of positive degree for $1 \leqslant i \leqslant p$. Then ψ extends to a $\mathbb{Q}$-algebra homomorphism $\hat{\psi}: \hat{S} \to \hat{B}$ with $\hat{\psi}(\hat{R}) \subseteq \hat{A}$. So by (2) we get $\xi_{B/A}(\bar{b}) = \mathscr{I}(\psi)(\hat{R}e^z \cap S) \subseteq \hat{A}e^b \cap B$. Similarly, $\xi_{B/A}(-\bar{b}) \subseteq \hat{A}e^{-b} \cap B$. Now, since $(\hat{A}e^b \cap B)(\hat{A}e^{-b} \cap B) \subseteq \hat{A} \cap B = A$, we get $\xi_{B/A}(\bar{b}) = \hat{A}e^b \cap B$ by (2.8). This proves the assertion for the case when b is homogeneous of positive degree. The general case follows now by induction on the number of nonzero homogeneous components of b and another application of (2.8).

(5.3) LEMMA. *$\xi_{B/A}$ is functorial, i.e. if $A \subseteq B$, $A' \subseteq B'$ are subintegral extensions of $\mathbb{Q}$-algebras and $\varphi: B \to B'$ is a homomorphism of $\mathbb{Q}$-algebras with $\varphi(A) \subseteq A'$ then the diagram*

$$\begin{array}{ccc} B/A & \xrightarrow{\xi_{B/A}} & \mathscr{I}(A, B) \\ \downarrow{\scriptstyle \bar{\varphi}} & & \downarrow{\scriptstyle \mathscr{I}(\varphi)} \\ B'/A' & \xrightarrow{\xi_{B'/A'}} & \mathscr{I}(A', B') \end{array}$$

where $\bar{\varphi}$ is induced by φ, is commutative.

Proof. Let $\bar{b} \in B/A$ with representative $b \in B$. It is checked easily that

$$\mathscr{I}(\varphi)(\xi_{B/A}(\bar{b})) \subseteq \xi_{B'/A'}(\bar{\varphi}(\bar{b})) \quad \text{and} \quad \mathscr{I}(\varphi)(\xi_{B/A}(-\bar{b})) \subseteq \xi_{B'/A'}(\bar{\varphi}(-\bar{b}))$$

whence we get $\mathscr{I}(\varphi)(\xi_{B/A}(\bar{b})) = \xi_{B'/A'}(\bar{\varphi}(\bar{b}))$ by (2.8). □

(5.4) LEMMA. *If N is a nonnegative integer and $b^n \in A$ for all $n \geqslant N$ then $\xi_{B/A}(\bar{b}) = A(e_N(b), b^N)$, where $e_N(b) = \Sigma_{i=0}^{N-1} b^i/i!$. In particular, if b is nilpotent then $\xi_{B/A}(\bar{b}) = A\exp(b)$.*

Proof. The second part is immediate from the first. To prove the first part, note that the assumption means that $(0, 0, N; 1)$ is a system of subintegrality for b in the extension $A \subseteq B$. The corresponding universal setup is $S = \mathbb{Q}[z]$ and $R = \mathbb{Q}[\{z^n \mid n \geqslant N\}]$. Let $\psi: S \to B$ be the $\mathbb{Q}$-algebra homomorphism given by $\psi(z) = b$. Then $\psi(R) \subseteq A$ and by (5.2)(2) $\xi_{B/A}(\bar{b}) = \psi(I)A$, where $I = \hat{R}e^z \cap S$. So it is enough to prove that $I = R(e_N(z), z^N)$. Writing $J = R(e_N(z), z^N)$ we note first that, since $e_N(z)e^{-z} \in \hat{R}$ and $z^N e^{-z} \in \hat{R}$, we have $J \subseteq I$. Next, we show that $z^n \in J$ for all $n \geqslant N$. This is clear for $n \geqslant 2N$ and follows for the remaining values by descending induction on n, since $z^n e_N(z) \in J$ for $n \geqslant N$. Similarly, writing $I' = \hat{R}e^{-z} \cap S$ and $J' = R(e_N(-z), z^N)$, we have $J' \subseteq I'$ and $z^n \in J'$ for all $n \geqslant N$. It follows that $e_{2N}(z)e_{2N}(-z) \in 1 + JJ'$ and that $e_{2N}(z) \in J$ and $e_{2N}(-z) \in J'$. Thus $1 \in JJ' \subseteq II' = R$ by (3.8) whence the equality $J = I$ follows from (2.8). □

(5.5) LEMMA. *Let $\mathfrak{a} \subseteq \operatorname{Nil}(B)$ be an ideal of B, $B' = B/\mathfrak{a}$ and $A' = A/A \cap \mathfrak{a}$. If $\xi_{B'/A'}$ is an isomorphism then so is $\xi_{B/A}$.*

Proof. Note that, since $A \subseteq B$ is a subintegral extension, so is $A' \subseteq B'$ by (4.17) and (4.4). Put $\xi = \xi_{B/A}$ and $\xi' = \xi_{B'/A'}$. Then, denoting by $\varphi: B \to B'$ the natural surjection, we have the commutative diagram

$$\begin{array}{ccc} B/A & \xrightarrow{\ \xi\ } & \mathscr{I}(A, B) \\ \Big\downarrow{\scriptstyle \bar{\varphi}} & & \Big\downarrow{\scriptstyle \mathscr{I}(\varphi)} \\ B'/A' & \xrightarrow{\ \xi'\ } & \mathscr{I}(A', B') \end{array}$$

given by (5.3). Assume that ξ' is an isomorphism.

Injectivity of ξ. From the diagram we get

$$\ker(\xi) \subseteq \ker(\bar{\varphi}) = \mathfrak{a}/A \cap \mathfrak{a}.$$

Therefore it is enough to show that

$$\ker(\xi) \cap (\mathfrak{a}/A \cap \mathfrak{a}) = 0.$$

Let $\bar{b} \in \ker(\xi) \cap (\mathfrak{a}/A \cap \mathfrak{a})$ where $\bar{b}$ is the class of $b \in \mathfrak{a}$. Since b is nilpotent, we have $A = \xi(\bar{b}) = A\exp(b)$ by (5.4) which shows that $\exp(b) \in A$. Therefore $b = \log(\exp(b)) \in A$ whence $\bar{b} = 0$.

Surjectivity of ξ. Since ξ', $\bar{\varphi}$ are surjective, it is enough to prove that $\ker(\mathscr{I}(\varphi)) \subseteq \operatorname{im}(\xi)$. Let $I \in \ker(\mathscr{I}(\varphi))$. Then $\varphi(I) = A' = \varphi(I^{-1})$ whence there exist $c, c' \in \mathfrak{a}$ such that $1 + c \in I$ and $1 + c' \in I^{-1}$. Let $b = \log(1+c)$, $b' = \log(1+c')$. Then $b, b' \in \operatorname{Nil}(B)$, $\exp(b) = 1 + c$ and $\exp(b') = 1 + c'$. By (5.4)

$$\xi(\bar{b}) = A\exp(b) = A(1+c) \subseteq I \quad \text{and} \quad \xi(\bar{b}') = A\exp(b') = A(1+c') \subseteq I^{-1}. \qquad (*)$$

Further, since

$$(1+c)(1+c') \in II^{-1} = A,$$

we have

$$b + b' = \log((1+c)(1+c')) \in A.$$

Therefore

$$\xi(\bar{b})\xi(\bar{b}') = A\exp(b+b') = A$$

showing that

$$\xi(\bar{b}') = \xi(\bar{b})^{-1}.$$

Now it follows from (*) and (2.8) that $\xi(\bar{b}) = I$, proving that $I \in \operatorname{im}(\xi)$. □

We are now ready for the main result, which we prove here under the assumptions that A be excellent and of finite Krull dimension. As indicated in the Final Remark at the end of the paper, the conclusion of the Main Theorem holds without these assumptions. For the definition and properties of excellent rings see [5] or [6].

(5.6) MAIN THEOREM. *Let A be an excellent $\mathbb{Q}$-algebra of finite Krull dimension and let $A \subseteq B$ be a subintegral extension. Then the homomorphism $\xi_{B/A}: B/A \to \mathscr{I}(A, B)$ is an isomorphism.*

Proof. Put $A' = A_{\text{red}}$, $B' = B_{\text{red}}$. Since $A \subseteq B$ is a subintegral extension, so is $A' \subseteq B'$ by (4.17) and (4.4). So by [9, 4.1] B' is contained in the seminormalization of A' (in its total quotient ring). By (5.5) it is enough to prove that $\xi_{B'A'}$ is an isomorphism. We use induction on $\dim(A)$. If $\dim(A) = 0$ then A' is its own total quotient ring whence $B' = A'$ and the assertion holds trivially in this case. Now, let $\dim(A) > 0$. Since A is excellent (or pseudo-geometric), the normalization of A', hence also its seminormalization, is a finite A'-module. Consequently, B' is a finite A'-submodule of the total quotient ring of A'. Therefore the conductor $\mathscr{C}$ of A' in B' contains a nonzero divisor of A' whence $\dim(A'/\mathscr{C}) < \dim(A') = \dim(A)$. Put $A'' = A'/\mathscr{C}$, $B'' = B'/\mathscr{C}$ and let $\varphi: B' \to B''$ be the natural surjection. Then the extension $A'' \subseteq B''$ is subintegral by (4.17) and (4.4) whence $\xi_{B''/A''}$ is an isomorphism by induction. Now, since $\mathscr{I}(\varphi)$ is an isomorphism by (2.6), the commutative diagram

$$\begin{array}{ccc} B'/A' & \xrightarrow{\xi_{B'/A'}} & \mathscr{I}(A',B') \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{I}(\varphi)} \\ B''/A'' & \xrightarrow{\xi_{B''/A''}} & \mathscr{I}(A'',B'') \end{array}$$

shows that $\xi_{B'/A'}$ is an isomorphism. □

(5.7) COROLLARY. *The homomorphism* $\xi_{B/A}: B/A \to \mathscr{I}(A,B)$ *is an isomorphism in each of the following two cases*:

(1) *A is a finitely generated algebra over a field of characteristic zero.*

(2) *A is an excellent local ring containing $\mathbb{Q}$ (e.g. a complete local ring containing $\mathbb{Q}$ or a localization of a finitely generated algebra over a field of characteristic zero).*

Proof. In each case A is an excellent $\mathbb{Q}$-algebra of finite Krull dimension. □

(5.8) COROLLARY (cf. [4]). *Suppose A is a reduced G-algebra containing $\mathbb{Q}$ and B is the seminormalization of A. Then* $\xi_{B/A}: B/A \to \mathscr{I}(A,B)$ *induces an isomorphism* $B/A \to \mathrm{Pic}(A)$.

Proof. This is a special case of part (1) of the above corollary, since in this case $\mathscr{I}(A,B) = \mathrm{Pic}(A)$ by (2.5). □

(5.9) COROLLARY. *In* (5.6) *the injectivity of* $\xi_{B/A}$ *holds without assuming that* $\dim(A) < \infty$.

Proof. Let $\beta \in \ker(\xi_{B/A})$. It is enough to show that the natural image of β in $(B/A)_\mathfrak{p}$ is zero for every $\mathfrak{p} \in \mathrm{Spec}(A)$. Let $\mathfrak{p} \in \mathrm{Spec}(A)$ and let γ be the natural image of β in $(B/A)_\mathfrak{p} = B_\mathfrak{p}/A_\mathfrak{p}$. By (4.17) and (4.9) $A_\mathfrak{p} \subseteq B_\mathfrak{p}$ is a subintegral extension and by (5.3) $\gamma \in \ker(\xi_{B_\mathfrak{p}/A_\mathfrak{p}})$. Therefore, since $\dim(A_\mathfrak{p}) < \infty$, $\gamma = 0$ by (5.6). □

6. An example

We will illustrate various features of the above theory with the following example:

$$A = \mathbb{Q}[x, y, t^2, t^3, z^2 + xt, z^3 + yt] \subseteq B = \mathbb{Q}[x, y, t, z]$$

where x, y, t, z are indeterminates. We can obtain B from A by two elementary subintegral extensions, first adjoin t, then z, so $A \subseteq B$ is a subintegral extension. The ring A is a graded subring of B if we take x, t, z to be of degree one, and y to be of degree 2. This is a "twisted" (and much more complicated, as we will see) version of the example $k[t^2, t^3, z^2, z^3] \subseteq k[t, z]$ that is used in [4]. In fact, B can be regarded as a universal two-step extension of A, in the following sense: let $A' \subseteq B'$ be any two step extension of $\mathbb{Q}$-algebras. That is, $\mathbb{Q} \subseteq A'$ and $B' = A'[b, c]$ with $b^2, b^3 \in A'$ and $c^2, c^3 \in A'[b]$. Then we have $c^2 = \alpha_0 - \alpha_1 b$, $c^3 = \beta_0 - \beta_1 b$ with $\alpha_0, \alpha_1, \beta_0, \beta_1 \in A'$. Hence setting $\varphi(x) = \alpha_1$, $\varphi(y) = \beta_1$, $\varphi(t) = b$, $\varphi(z) = c$ we obtain a $\mathbb{Q}$-algebra homomorphism $\varphi: B \to B'$ such that $\varphi(A) \subseteq A'$.

Our first goal is to find an explicit system of subintegrality for z in the extension $A \subseteq B$. We begin by proving some elementary properties of A. Let $\mathscr{C}$ be the conductor of A in B.

(6.1) LEMMA. *$t^i z^j \in \mathscr{C}$ for all $i, j \geqslant 2$.*

Proof. We have $t^i(z^2 + xt) \in A$ for $i \geqslant 2$, from which it follows that $t^i z^2 \in A$ for $i \geqslant 2$. Similarly using $z^3 + yt$ we conclude that $t^i z^3 \in A$ for $i \geqslant 2$. If $j > 3$ write $j = 2a + 3b$, $a, b \in \mathbb{Z}^+$, so that $(z^2 + xt)^a (z^3 + yt)^b = z^j +$(lower powers in z, excluding z^1)$\in A$. Multiplying by t^i and using induction on j we conclude that $t^i z^j \in A$, for $i \geqslant 2$. Since B is generated as an A-algebra by t and z, the lemma follows. □

(6.2) LEMMA. *For each way of writing $n = 2a + 3b$, $n \geqslant 2$, $a, b \in \mathbb{Z}^+$, we have $z^n + axtz^{n-2} + bytz^{n-3} \in A$.*

Proof. We have $\alpha = (z^2 + xt)^a (z^3 + yt)^b \in A$. By (6.1) the terms in the expansion of α of t-degree $\geqslant 2$ are elements of A. This leaves the sum of terms of t-degree $\leqslant 1$ in A, i.e. $z^n + axtz^{n-2} + bytz^{n-3} \in A$, as desired. □

Now we will make a definite choice for the a and b of (6.2). Namely, for $n \geqslant 2$ we will write $n = 2q_n + 3\varepsilon_n$ with $q_n, \varepsilon_n \in \mathbb{Z}^+$, ε_n as small as possible. That is, $\varepsilon_n = 0$ if n is even and $\varepsilon_n = 1$ if n is odd. Then $q_n = n/2$ if n is even, and $q_n = (n-3)/2$ if n is odd.

(6.3) DEFINITION. With the above notation let

$$\lambda_n = z^n + q_n xtz^{n-2} + \varepsilon_n ytz^{n-3}, \quad n \geqslant 2,$$

$$\mu_n = 3xtz^{n-2} - 2ytz^{n-3}, \quad n \geqslant 4.$$

These elements are homogeneous of degree n and $\lambda_n \in A$ by (6.2).

(6.4) LEMMA. *Let $n \geqslant 2$, and let*

$$F = \alpha z^n + \beta xtz^{n-2} + \gamma ytz^{n-3}, \quad \alpha,\ \beta,\ \gamma \in \mathbb{Q}.$$

Then $F \in A$ if and only if F is a $\mathbb{Q}$-linear combination of expressions of the form $z^n + axtz^{n-2} + bytz^{n-3}$ with $a, b \in \mathbb{Z}^+$, $2a + 3b = n$.

Proof. If $n = 2$, then $F \in A$ if and only if F is a $\mathbb{Q}$-multiple of λ_2, so the lemma is true in this case. Assume now that $n \geqslant 3$. The "if" implication follows from (6.2). To prove the converse, let $\mathfrak{a}$ be the B-ideal generated by $\{x^2, xy, y^2, t^2, xz^{n-1}, yz^{n-2}, tz^{n-1}\}$. Let $\bar{B} = B/\mathfrak{a}$. Then $\bar{B}_n$, the degree n part of $\bar{B}$, has $\mathbb{Q}$-basis $\{z^n, xtz^{n-2}, ytz^{n-3}\}$. All the monomials of degree n in the generators of A vanish in $\bar{B}_n$, except $(z^2+xt)^a(z^3+yt)^b$ for the various ways of writing $n = 2a+3b$, with $a, b \in \mathbb{Z}^+$. Therefore the image of A_n in $\bar{B}_n$ is spanned by the images of the latter, namely by

$$\{z^n + axtz^{n-2} + bytz^{n-3} \mid a,\ b \in \mathbb{Z}^+,\ 2a + 3b = n\}.$$

The "only if" part now follows, and the proof is complete. □

(6.5) COROLLARY. (1) *$z^n \notin A$, for all $n \geqslant 1$.*

(2) *$\mu_n \in A$ for $n = 6$, and $n \geqslant 8$, but μ_4, μ_5, $\mu_7 \notin A$.*

(3) *$\mu_n \in \mathscr{C}$ for $n \geqslant 8$, but $\mu_6 \notin \mathscr{C}$.*

Proof. Let $S = \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid n = 2a+3b\}$. Then it follows from (6.4) by elementary linear algebra that if $\mathrm{Card}(S) = 1$ then the image of A_n in $\bar{B}_n$ is of dimension one, with basis $z^n + q_n xtz^{n-2} + \varepsilon_n ytz^{n-3}$, whereas if $\mathrm{Card}(S) \geqslant 2$ then the image of A_n in $\bar{B}_n$ is of dimension two with basis $\{z^n + q_n xtz^{n-2} + \varepsilon_n ytz^{n-3}, 3xtz^{n-2} - 2ytz^{n-3}\}$. Parts (1) and (2) now follow from (6.4). Since $z\mu_6 = \mu_7 \notin A$, we have $\mu_6 \notin \mathscr{C}$. If $n \geqslant 8$ then for $i, j \geqslant 0$ $t^i z^j \mu_n = t^i \mu_{n+j} \in A$ by (2) and (6.1). Therefore, since t and z generate B as an A-algebra, it follows that $\mu_n \in \mathscr{C}$ for $n \geqslant 8$, completing the proof. □

From (6.5) we conclude that in the extension $A \subseteq B$ there is no system of subintegrality for z of the form $(s, 0, N; 1)$ (i.e. with $p = 0$). However, the next lemma shows that there is a system of subintegrality with $p = 1$.

(6.6) LEMMA. (1) *If $n \geqslant 8$ then $\lambda_n \equiv z^n + (nxt/2)z^{n-2} \bmod \mathscr{C}$.*

(2) *$z^n + (nxt/2)z^{n-2} \in A$ for all $n \geqslant 8$, i.e. $(1, 1, 8;\ 1,\ xt/2)$ is a system of subintegrality for z in the extension $A \subseteq B$. However, $z^7 + (7xt/2)z^5 \notin A$.*

Proof. (1) If n is even we have equality. If n is odd use (6.5) part (3) to replace ytz^{n-3} by $(3/2)xtz^{n-2}$ (mod $\mathscr{C}$).

(2) The first part of (2) follows from (1). The last part follows from the fact that $\mu_7 \notin A$ (6.5). □

(6.7) GENERATORS FOR $\xi_{B/A}(\bar{z})$. Now we are ready to describe the invertible ideal $\xi_{B/A}(\bar{z})$. First we will work out the generic invertible ideal I in the case $s=1$, $p=1$ and $N=8$. In the notation of (3.3) let $S=\mathbb{Q}[x_1 z, z]$ and let $R=\mathbb{Q}[\{\gamma_n \mid n \geqslant 8\}]$. By (3.4) we have $\Delta = 1 + x_1(\partial - 1)$ and $\Delta(e_n(z)) \in I$ for $n \geqslant 8$. Similarly $\Delta' = 1 + x_1(1+\partial)$ and $\Delta'(e_n(-z)) \in I^{-1}$ for $n \geqslant 8$. To find generators for I, it suffices, as was noted in the proof of (2.2), to find elements $m_i \in I$, $n_i \in I^{-1}$ such that $1 = \Sigma m_i n_i$. Such elements are provided by the proof of (3.8), but this gives a rather large number of generators. In fact, three generators suffice, as we obtain by the following somewhat different argument: define $\sigma \in z^8\mathbb{Q}[z]$ by the equation $fg + \rho\sigma = 1$, where $f = e_{16}(z)$, $g = e_{16}(-z)$ and $\rho = z^8$. Then

$$\begin{aligned}
\Delta f \Delta' g + \Delta\rho\Delta'\sigma &= (f + x_1(f'-f))(g + x_1(g'+g)) + (\rho + x_1(\rho'-\rho))(\sigma + x_1(\sigma'+\sigma)) \\
&= fg + x_1(f'g + fg') + x_1^2(f'-f)(g'+g) + \rho\sigma + x_1(\rho'\sigma + \rho\sigma') \\
&\quad + x_1^2(\rho'-\rho)(\sigma'+\sigma) \\
&= 1 + x_1((fg)' + (\rho\sigma)') + x_1^2 H \\
&= 1 + \alpha,
\end{aligned}$$

where $f' = \partial f/\partial z$, etc. and $\alpha = x_1^2 H \in z^{14}\mathbb{Q}[x_1, z]$ (since $f'-f$, $g'+g$ are in $z^{15}\mathbb{Q}[z]$ and $\rho'-\rho$, $\sigma'+\sigma$ are in $z^7\mathbb{Q}[z]$). Now we claim that $x_1^2 z^i \in R$ for $i \geqslant 14$. For we have

$$(z^8 + 8x_1 z^7)(z^{i-6} + (i-6)x_1 z^{i-7}) = z^{i+2} + (i+2)x_1 z^{i+1} + 8(i-6)x_1^2 z^i \in R$$

and $z^{i+2} + (i+2)x_1 z^{i+1} \in R$. Subtracting and dividing by $8(i-6)$, we get $x_1^2 z^i \in R$, for $i \geqslant 14$. Now we have $x_1^2 z^i e^{-z} \in \hat{R}$, so $x_1^2 z^i \in (\hat{R}e^z) \cap S = I$, for $i \geqslant 14$. Similarly $x_1^2 z^i \in (\hat{R}e^{-z}) \cap S = I^{-1}$ for $i \geqslant 14$. Hence $\alpha \in I \cap I^{-1} \cap R$. Multiplying both sides of $\Delta f \Delta' g + \Delta\rho\Delta'\sigma = 1 + \alpha$ by $1-\alpha$ we conclude that $\Delta f[(\Delta' g)(1-\alpha)] + \Delta\rho[(\Delta'\sigma)(1-\alpha)] + \alpha^2 = 1$. Now, Δf, $\Delta\rho$, $\alpha \in I$ and $(\Delta' g)(1-\alpha)$, $(\Delta'\sigma)(1-\alpha)$, $\alpha \in I^{-1}$ by (3.4), so $\{\Delta f, \Delta\rho, \alpha\}$ is the desired set of generators for I. Of course, in the above discussion, 8 can be replaced by any integer $N \geqslant 8$.

The homomorphism $\varphi: \hat{S} \to B[[T]]$ of (4.2) is given in the present situation by $\varphi(x_1 z) = (xt/2)T^2$ and $\varphi(z) = zT$, and we have $\varphi(R) \subseteq A[T]$. We obtain $\xi_{B/A}(\bar{z})$ by applying φ and then setting $T = 1$. This yields 3 generators for $\xi_{B/A}(\bar{z})$.

(6.8) REMARK. If $n \geqslant 8$ then, in the notation of (6.7),

$$\Delta(e_n(z)) = e_n(z) + x_1 e_{n-1}(z) - x_1 e_n(z) = e_n(z) - x_1 z^{n-1}/(n-1)!$$

is an explicit element of $I \cap W(R[z])$. Applying φ and recalling from the proof of (4.2) (ii) $\Rightarrow$ (iii) that $I(z) = A[T]\varphi(I)$ we obtain

$$e_n(zT) - (xtz^{n-2})T^n/(2(n-1)!) \in I(z) \cap W(B) \quad \text{for } n \geqslant 8.$$

If we apply the proof of (4.2) (iv) $\Rightarrow$ (i) to this element of $I(z) \cap W(B)$ (taking $n = 8$), we obtain the system of subintegrality $(0, 8, 8; c_0, \ldots, c_8)$ for z in the extension $A \subseteq B$, where $c_i = (-1)^i z^i$ $(0 \leqslant i \leqslant 7)$ and $c_8 = -4xtz^6$. In (6.6) the use of exponent of subintegrality 1 permitted us to find a system of subintegrality for z in the extension $A \subseteq B$ with $p = 1$ rather than $p = 8$. However, one can check that the elements $z^n + \Sigma_{i=1}^{p} \binom{n}{i} c_i z^{n-s-i}$, $n \geqslant 8$, are the same for the two systems. The referee has pointed out that another system of subintegrality of exponent 0 for z is $(0, 2, 8; 1, -z, -xt)$, which can be obtained by multiplying (6.6)(2) by $1-n$.

The following result is a refinement of (6.5) part (1):

(6.9) LEMMA. *There is no way to write* $z = \Sigma_{i=1}^{m} \zeta_i$, *with* $\zeta_i \in B$ *and* $\zeta_i^n \in A$ *for* $n \gg 0$.

Proof. Indeed, we prove the stronger assertion that there is no way to write $z = \Sigma_{i=1}^{m} \zeta_i$, with $\zeta_i \in B$, such that for each i there exists $n_i \in \mathbb{N}$ with $\zeta_i^{n_i} \in A$. First we show that if $f \in B$, $f = \Sigma_{i \geqslant 0} f_i$, ($f_i$ homogeneous of degree i) and $f^n \in A$ then also $f_1^n \in A$. If $f_0 \neq 0$ we see by induction on i that $f_i \in A$ for all i, in particular $f_1 \in A$. If $f_0 = 0$ then looking at the term of degree n in f^n we conclude that $f_1^n \in A$. Now suppose that $z = \Sigma_{i=1}^{m} \zeta_i$, with $\zeta_i \in B$ and $\zeta_i^{n_i} \in A$ for some $n_i \in \mathbb{N}$, $1 \leqslant i \leqslant m$. Obviously $z = \Sigma_{i=1}^{m} \zeta_{i1}$ (ζ_{i1} being the degree 1 part of ζ_i), so by the first part of the proof we are reduced to the case where all the ζ_i are homogeneous of degree 1. The only variables of degree 1 are x, t, z. We will complete the proof by showing that if $a, b, c \in \mathbb{Q}$, and $(ax + bt + cz)^n \in A$ for some $n \geqslant 1$, then we must have $c = 0$. Suppose that $c \neq 0$. Then we can assume that $c = 1$. Since $A \subseteq A[t] = \mathbb{Q}[x, y, t, z^2, z^3]$, $(ax + bt + z)^n \in A$ for some $n \geqslant 1 \Rightarrow n(ax + bt)^{n-1} z \in A[t] \Rightarrow ax + bt = 0 \Rightarrow z^n \in A$, which contradicts (6.5). □

The relevance of (6.9) to the present work is that if one had a subintegral extension $R \subseteq S$, in which every element $z \in S$ could be written in the form $z = \Sigma_{i=1}^{m} \zeta_i$ with $\zeta_i^n \in R$ for $n \gg 0$ then one could define a homomorphism $S/R \to \mathscr{I}(R, S)$ in a more elementary fashion by applying the formula of (5.4) to each of the ζ_i separately.

It may be of some interest to note that the conductor $\mathscr{C}$ of A in B equals $(t^2z^2, 3xz^6 - 2yz^5 + 6x^2tz^4 - 2y^2tz^2)B$ and that $\sqrt{\mathscr{C}} = (tz, 3xz^2 - 2yz)B$. (The details of the computation can be found in the original version of this paper, available as a preprint from the authors.) This is to be contrasted with the conductor of $k[t^2, t^3, z^2, z^3]$ in $k[t, z]$, which is $t^2z^2k[t, z]$, with radical $tzk[t, z]$.

7. Comparison with work of Dayton

We now compare our results with those of [4] in the reduced G-algebra case. Let A be a reduced G-algebra with seminormalization B (which is again a G-algebra). As in [4] let A_+, B_+ be the ideals generated by elements of positive degree. In this situation we have $A^* = B^* = k^*$, where $k = A_0$, $\mathrm{Pic}(B) = 0$ and by (2.5) $\mathrm{Pic}(A) = \mathscr{I}(A, B)$.

In [4] Dayton proves (with notation as above) that if $\mathbb{Q} \subseteq A$, then there is an isomorphism $\theta\colon \mathrm{Pic}(A) \to B/A$. First let us recall Dayton's definition of θ: if $M \in \mathrm{Pic}(A) = \mathscr{I}(A, B)$ and $f \in M \cap (1 + B_+)$ we will say that f represents M. (Dayton shows that $M^{-1} = (A : f)$ so f determines M.) For any G-algebra R, Dayton defines $GW(R)$ to be

$$\{1 + a_1T + a_2T^2 + \cdots \in R[[T]] \mid a_i \in R_i\}.$$

If $f = 1 + \Sigma_{i=1}^n b_i$ $(b_i \in B_i)$ he defines $f^\tau \in GW(B)$ to be $1 + \Sigma_{i=1}^n b_iT^i$. If $M \in \mathrm{Pic}(A)$, represented by $f \in 1 + B_+$, then sending M to the class of $f^\tau \in GW(B)/GW(A)$ gives a well-defined homomorphism of abelian groups $\tau\colon \mathrm{Pic}(A) \to GW(B)/GW(A)$ [4, 1.11]. As explained in [4] the ghost map

$$gh\colon W(B) = 1 + TB[[T]] \to TB[[T]] \equiv \prod_{i \geq 1} B$$

induces a homomorphism of abelian groups

$$GW(B)/GW(A) \to \prod_{i \geq 1} (B_i/A_i).$$

A key point of Dayton's approach is to show that gh actually maps the image of τ into $\bigoplus_{i \geq 1} (B_i/A_i) \cong B/A$. (Note that $B_0 = A_0 = k$.) The homomorphism θ is then defined to be $gh \circ \tau$. Theorem 3.6 of [4] then shows that $\theta\colon \mathrm{Pic}(A) \to B/A$ is an isomorphism. The homomorphism gh is given by $f(T) \mapsto -Td(\log f(T))/dT$ and hence has inverse $Tg(T) \mapsto \exp(\int -g(T)\,dT)$ ($\int$ means take the antiderivative of the power series, without putting in a constant term), where $f(T) \in GW(B)$, $g(T) \in TB[[T]]$.

We will now compare Dayton's θ^{-1} with our homomorphism

$$\xi_{B/A}\colon B/A \to \mathrm{Pic}(A).$$

If $b \in B_d, d \geqslant 1$, represents an element of B/A then in Dayton's θ^{-1}, b is first identified with $bT^d \in TB[[T]]$. Applying gh^{-1} as described above we obtain $\exp(-d^{-1}bT^d) \in GW(B)$. To interpret this as an element of $\tau\,\mathrm{Pic}(A)$ one has to find $M \in \mathrm{Pic}(A)$ and $f \in M \cap (1+B_+)$ such that f^τ represents $\exp(-d^{-1}bT^d)$ in $GW(B)/GW(A)$. Then $\theta^{-1}b = M$. To compare with our homomorphism, first note that $GW(A)$ and $GW(B)$ can be identified (by setting $T = 1$) with the multiplicative groups of elements of $\hat{A}$, $\hat{B}$ respectively, that are $\equiv 1$ mod the maximal ideal ($\hat{}$ denoting completion with respect to A_+, B_+ respectively, as in Section 3). By (5.2)

$$M := \hat{A}\exp(-d^{-1}b) \cap B \in \mathscr{I}(A, B) = \mathrm{Pic}(A).$$

If $f \in M \cap (1+B_+)$ then f^τ represents $\exp(-d^{-1}bT^d)$ in $GW(B)/GW(A)$ so $M = \hat{A}\exp(-d^{-1}b) \cap B = \theta^{-1}(b)$. By (5.2) our homomorphism sends b to $\hat{A}\exp(b) \cap B$. Thus our homomorphism differs from Dayton's θ^{-1} by the group automorphism $\Sigma_{i\geqslant 1} b_i \mapsto -\Sigma_{i\geqslant 1} ib_i$ of B/A, so we have re-proved Theorem 3.6 of [4]. Our construction of ξ gives the invertible module M and a system of generators for it quite explicitly.

The sign is quite harmless. However the i factors are a more serious difference between our homomorphisms. There seems to be no way to eliminate this difference and still have a natural homomorphism $B/A \to \mathscr{I}(A, B)$ even in the nongraded case.

The following discussion indicates how ξ compares with θ^{-1} as used in some of Dayton's inductive arguments:

(7.1) PROPOSITION. *Let $\mathfrak{a}$ be a B-ideal of A and let $\varphi: B \to B/\mathfrak{a}$ be the canonical surjection. Consider the following diagram:*

$$\begin{array}{ccc} \mathrm{Nil}(B/\mathfrak{a}) & \xrightarrow{\;j\;} & B/A \\ {\scriptstyle \exp}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \xi} \\ \mathrm{Uni}(B/\mathfrak{a}) & \xrightarrow{\;\delta\;} & \mathscr{I}(A, B) \end{array}$$

where $\xi = \xi_{B/A}$, j is the natural map, and δ is defined to make the diagram commutative. Then $\delta(\varphi(b)) = (A:b)^{-1}$ for $b \in B$ with $\varphi(b) \in \mathrm{Uni}(B/\mathfrak{a})$.

Proof. Let $y \in \mathrm{Nil}(B/\mathfrak{a})$ and let $x \in B$ such that $\varphi(x) = y$. Then there exists N such that $x^n \in \mathfrak{a} \subseteq A$ for $n \geqslant N$ so by (5.4) we have $\xi(\bar{x}) = A(e_N(x), x^N)$. It follows that $\xi(\bar{x}) \subseteq (A : e_N(-x))$. Similarly $\xi(-\bar{x}) \subseteq (A : e_N(x))$. Also it is easily checked that $(A : e_N(-x))(A : e_N(x)) \subseteq A$. Therefore by (2.8) we have $\xi(\bar{x}) = (A : e_N(-x)) = (A : e_N(x))^{-1}$. On the other hand $\exp(y) = e_N(y) = \varphi(e_N(x))$, completing the proof. □

We remark that $(A{:}e_N(x))$ is isomorphic to the rank one projective A module obtained by Milnor's patching construction ([7, §2]) using the unit $e_N(\varphi(x))$ of $B/\mathfrak{a}$.

This result is to be compared with the commutative diagram in the proof of [4, 3.6] where a similar commutative square is obtained, with our exp replaced by a more complicated homomorphism involving the ghost map, and our δ replaced by $-\delta$. Also our $\xi^{-1}_{+A/A}$ gives a natural homomorphism fitting into the diagram in the proof of [3, 6.3].

FINAL REMARK. The assumption in our Main Theorem (5.6) that A be excellent and of finite Krull dimension can be dropped. We do this in a forthcoming paper with Les Reid where we derive the more general result from (5.6) by proving that subintegrality of an element is essentially a finite condition. On the other hand, the assumption that A contain $\mathbb{Q}$ cannot be dropped. As a simple example, let K be a field of characteristic 2 and let

$$A = K \oplus t^3K[[t]] \subseteq B = K[[t]].$$

Then for

$$I = (1 + t)A,$$

we have

$$I^2 = (1 + t^2)A \neq A.$$

This shows that $\mathscr{I}(A, B)$ is not killed by 2, hence cannot be isomorphic to B/A, which is killed by 2. In this connection the referee pointed out to us the paper "On the Picard Group of a Class of Non-Seminormal Domains" by Gilmer and Martin in [*Comm. Alg.* 18(10) (1990) 3263–3293], where more examples of this nature can be found.

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