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An analogy of Tian–Todorov theorem on deformations of CR-structures

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Introduction

If X is a compact Kähler manifold with trivial canonical line bundle, then the parameter space of the Kuranishi family of X is locally isomorphic to an open set in $H^{(1)}(X, \Lambda)$, where Λ is the holomorphic tangent bundle ([Ti], [To], [Go-Mi]). On the other hand, for the CR-structure case, the deformation theory was also successfully developed by ([A2], [A3], [Mil]), and the versal family *in the sense of Kuranishi* was established. Therefore it is reasonable to try to obtain a corresponding result, namely, "an analogy of Tian–Todorov theorem".

For this purpose, we recall the Tian–Todorov's approach in ([Ti], [To]). The Tian–Todorov's approach consists of the following two parts.

Part 1. As the canonical line bundle is trivial, the deformation equation, which is holomorphic tangent bundle valued, can be reduced to the equation on scalar valued differential forms.

Part 2. By using the Hodge structure on compact Kähler manifolds, we see that the obstructions vanish.

On the other hand, contrast to compact Kähler manifolds, in our case, *Hodge structure* is not simple. We see this more precisely. Let $(M, {}^0T'')$ be a compact strongly pseudo convex CR-manifold. Furthermore we assume that $(M, {}^0T'')$ admits a normal vector field, namely there is a global vector field ζ on M satisfying;

(1) $\zeta_p \notin {}^0T''_p + {}^0\bar{T}''_p$ for every point p of M,
(2) $[\zeta, \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T'')$.

This manifold $(M, {}^0T'', \zeta)$ is called a normal s.p.c. manifold. For this normal s.p.c. manifold, Tanaka already studied *Hodge structure.* Namely he set

$$B^{p,q} = \Gamma(M, \wedge^p({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*),$$

and he introduced d', d''-operators on these spaces. So he had a double complex

$(B^{p,q}, d', d'')$. However, his double complex does not seem so efficient for our problem. For example, on $B^{p,q}$,

$$d'd'' + d''d' \neq 0.$$

Therefore the Tian–Todorov's argument completely breaks down. Nevertheless, because of ingenuity and simpleness of Tian–Todorov's argument, we would like to adopt their method for our deformation problem. For this purpose, in this paper, we introduce a new subcomplex $(\mathbf{F}^{p,q}, d', d'')$ of $(\Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*), d', d'')$. And we see that our $\mathbf{F}^{n-2,q}$ comes from E_q bundle, which is introduced in the successful line of deformation of CR-structures ([A2], [A3]). Then on $\mathbf{F}^{p,q}$,

$$d'd'' + d''d' = 0.$$

Furthermore for $\mathbf{F}^{p,q}$, we show

(1) the CR-analogue of Tian–Todorov lemma (Section 4 in this paper),
(2) the key equality; for u in $\mathbf{F}^{n-2,q}$,

$$d' \square_{d''} u = (2/3)d''^* d''(d'u) + (1/2)d''d''^*(d'u) \quad \text{(Section 6 in this paper).}$$

With (1) and (2), following the Tian–Todorov method, we have the following theorem.

MAIN THEOREM. *Let $(M, {}^0T'')$ be a normal s.p.c. manifold with $\dim_{\mathbb{R}} M \geqslant 7$. And we assume that its canonical line bundle $K = \wedge^n(T')^*$ is trivial in CR-sense. Then the obstructions of deformations in Z^1 appear in $J^{n-2,2}$. That is, if $J^{n-2,2} = 0$, then for any deformation of CR structures in Z^1 is unobstructed. Here $Z^1 = \{u \in \mathbf{F}^{n-2,1} | d'u = 0\}$ and $J^{n-2,2} = ((\mathrm{Ker}\, d'') \cap d'\mathbf{F}^{n-3,2})/(d'\mathbf{F}^{n-3,2} \cap d''\mathbf{F}^{n-2,1})$.*

Though, at present, we have no example indicating whether $J^{n-2,2} = 0$ is a useful condition, as will be proven in Section 9, if $d'd''$-lemma holds in the double complex $(\mathbf{F}^{p,q}, d', d'')$, the main theorem provides the complete analogue of Tian–Todorov's theorem (i.e. smoothness of the versal family of CR structures in the sense of Kuranishi). Therefore the above theorem seems a CR-analogue of Tian–Todorov's theorem.

1. Deformation theory of CR-structures

Let X be a complex manifold and let r be a C^∞ exhaustion function which is strictly pluri-subharmonic except a compact subset. Let

$$\Omega = \{x : x \text{ in } X,\ r(x) < 0\}$$

and we assume that the boundary of Ω, $b\Omega$ is smooth. Then, naturally we can put a CR-structure over $b\Omega$. Namely we set

$$ {}^0T'' = \mathbb{C} \otimes T(b\Omega) \cap T''X|_{b\Omega}. $$

Then we have

(1) ${}^0T'' \cap {}^0\bar{T}'' = 0$, $f-\dim_{\mathbb{C}}(\mathbb{C} \otimes T(b\Omega)/({}^0T'' + {}^0\bar{T}'')) = 1$,
(2) $[\Gamma(b\Omega, {}^0T''), \Gamma(b\Omega, {}^0T'')] \subset \Gamma(b\Omega, {}^0T'')$.

This notion is generalized as follows. Let M be a C^∞ orientable real odd dimensional manifold. Let E be a subbundle of the complexified tangent bundle $\mathbb{C} \otimes TM$ satisfying:

(1)′ $E \cap \bar{E} = 0$, $f - \dim_{\mathbb{C}}(\mathbb{C} \otimes T(b\Omega)/(E + \bar{E})) = 1$,
(2)′ $[\Gamma(M, E), \Gamma(M, E)] \subset \Gamma(M, E)$.

This pair (M, E) is called an abstract CR-structure or simply a CR-structure. For our pair $(b\Omega, {}^0T'')$, we set a C^∞ vector bundle isomorphism

$$ \mathbb{C} \otimes T(b\Omega) = {}^0T'' + {}^0\bar{T}'' + \mathbb{C}\zeta, \tag{1.1} $$

where ζ is a real vector field supplement to ${}^0T'' + {}^0\bar{T}''$. And by using this decomposition, we set a Levi form L by: for X, Y in ${}^0T''$,

$$ L(X, Y) = -\sqrt{-1}[X, \bar{Y}]_{\mathbb{C}\zeta}, $$

where $[X, \bar{Y}]_{\mathbb{C}\zeta}$ means the $\mathbb{C}\zeta$ part with respect to (1.1). We recall deformation theory of CR-structures which is developed in [A2]–[A4], [A-M], [Mil]. We assume that we are given a complex manifold X, a strongly pseudo convex domain Ω, and the boundary $b\Omega$, a CR-structure ${}^0T''$ induced from X over M, and we set a C^∞ vector bundle decomposition (1.1).

DEFINITION 1.1. Let E be a subbundle of the complexified tangent bundle $\mathbb{C} \otimes TM$ satisfying:

$$ E \cap \bar{E} = 0, \qquad f - \dim_{\mathbb{C}}(\mathbb{C} \otimes TM/(E + \bar{E})) = 1. \tag{1.1.1} $$

Then, the pair (M, E) is called an almost CR-structure. As E is a subbundle of $\mathbb{C} \otimes TM$, we have a projection map from E to ${}^0T''$ according to (1.1). If this projection map is isomorphism, then we call (M, E) an almost CR-structure which is at a finite distance from $(M, {}^0T'')$ or simply an almost CR-structure. Then, we, immediately, have the following proposition.

PROPOSITION 1.2 (Proposition 1.6.1 in [A2]). *An almost CR-structure* ${}^\phi T''$

corresponds to an element ϕ *of* $\Gamma(M, T' \otimes ({}^0T'')^*)$ *bijectively. The correspondence is that: for* ϕ *in* $\Gamma(M, T' \otimes ({}^0T'')^*)$,

$$ {}^{\phi}T'' = \{X' : X' = X + \phi(X),\ X \text{ in } {}^0T''\}, $$

where $T' = {}^0\bar{T}'' + \mathbb{C}\zeta$. *And we have*

PROPOSITION 1.3 (Proposition 1.6.2 in [A2]). *An almost CR-structure* ${}^{\phi}T''$ *is an actual CR-structure if and only if* ϕ *satisfies the non-linear partial differential equation* $P(\phi) = 0$.

Here $P(\psi)$ is defined as follows. For ψ in $\Gamma(M, T' \otimes ({}^0T'')^*)$,

$$ P(\psi) = \bar{\partial}_b\psi + (1/2)[\psi, \psi], $$

where $\bar{\partial}_b$ means the T'-valued tangential Cauchy–Riemann operator on M (for precise definition, see [A1], [A2]) and $(1/2)[\psi, \psi]$ corresponds to $R_2(\psi)$ in [A1], [A2].

2. The complex $(\Sigma_{p+q=n}\Gamma(M, \wedge^p(T')^* \wedge \wedge^q({}^0T'')^*), d)$ on a s.p.c. manifold

Let (M, E) be an abstract strongly pseudo convex CR-structure with $\dim_{\mathbb{R}} M = 2n-1 \geqslant 7$. Then as is proved in [A1], the vector bundle T' is a CR-holomorphic vector bundle, and so we can introduce the canonical line bundle $\wedge^p(T')^*$ like in the complex manifold case.

We study $\Gamma(M, \wedge^p(T')^* \wedge \wedge^q({}^0T'')^*)$. First, we can introduce d''-operator from $\Gamma(M, \wedge^p(T')^* \wedge \wedge^q({}^0T'')^*)$ to $\Gamma(M, \wedge^p(T')^* \wedge \wedge^{q+1}({}^0T'')^*)$ by: for u in $\Gamma(M, \wedge^p(T')^* \wedge \wedge^q({}^0T'')^*)$,

$$ d''u = (du)_{\wedge^p(T')^* \wedge \wedge^{q+1}({}^0T'')^*}, $$

where $(du)_{\wedge^p(T')^* \wedge \wedge^{q+1}({}^0T'')^*}$ means the $\wedge^p(T')^* \wedge \wedge^{q+1}({}^0T'')^*$ part of du according to the following canonical decomposition.

$$ \wedge^{p+q+1}(\mathbb{C}\mathrm{TM})^* = \sum_{\substack{r+s=p+q+1 \\ r,s \geqslant 0}} \wedge^r(T')^* \wedge \wedge^s({}^0T'')^*. $$

As is shown in [T] p. 64,

$$ du = (du)_{\wedge^{p+2}(T')^* \wedge \wedge^{q-1}({}^0T'')^*} + (du)_{\wedge^{p+1}(T')^* \wedge \wedge^q({}^0T'')^*} + (du)_{\wedge^p(T')^* \wedge \wedge^{q+1}({}^0T'')}. $$

We use the notation

$$Du = (du)_{\wedge^{p+2}(T')^* \wedge \wedge^{q-1}({}^0T'')^*} + (du)_{\wedge^{p+1}(T')^* \wedge \wedge^{q}({}^0T'')^*}.$$

So

$$du = Du + d''u.$$

Therefore

$$ddu = DDu + Dd''u + d''Du + d''d''u.$$

If we see $\wedge^p(T')^* \wedge \wedge^{q+2}({}^0T'')^*$ part, we have

$$d''d''u = 0.$$

Hence we have a d''-differential complex

$$\Gamma(M, \wedge^p(T')^* \wedge \wedge^{q-1}({}^0T'')^*) \xrightarrow{d''} \Gamma(M, \wedge^p(T')^* \wedge \wedge^{q}({}^0T'')^*) \xrightarrow{d''} \Gamma(M, \wedge^p(T')^* \wedge \wedge^{q+1}({}^0T'')^*).$$

And we set $\Gamma^{p,q} = \Gamma(M, \wedge^p(T')^* \wedge \wedge^q({}^0T'')^*)$, and

$$H^q_{d''}(M, \wedge^p(T')^*) = \mathrm{Ker}\, d'' \cap \Gamma^{p,q}/d''\Gamma^{p,q-1}.$$

Let ζ be a supplement real vector field to ${}^0T'' + {}^0\bar{T}''$. Then by using this vector, we have a C^∞ vector bundle decomposition

$$\wedge^{n-1}(T')^* \wedge \wedge^p({}^0T'')^* = (\mathbb{C}\zeta)^* \wedge \wedge^{n-2}({}^0\bar{T}'') \wedge \wedge^p({}^0T'')^* + \wedge^{n-1}({}^0\bar{T}'')^* \wedge \wedge^p({}^0T'')^*.$$

By the same way, we can introduce a d'' operator on

$$\Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p(T')^* \wedge {}^q({}^0T'')^*).$$

Namely, for u in

$$\Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p(T')^* \wedge {}^q({}^0T'')^*),$$

$$d''u = (du)_{(\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^{q+1}({}^0T'')^*},$$

where $(du)_{(\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^{q+1}({}^0T'')^*}$ means the $(\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^{q+1}({}^0T'')^*$ part of du according to the vector bundle decomposition

$$\wedge^{p+q+1}(\mathbb{C} \otimes \mathrm{TM})^* = \sum_{\substack{h+i+j=p+q+1 \\ h,i,j \geqslant 0}} \wedge^h(\mathbb{C}\zeta)^* \wedge \wedge^i({}^0\bar{T}'')^* \wedge \wedge^j({}^0T'')^*.$$

Similarly, for u in $\Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*)$, we set

$$d'u = (du)_{(\mathbb{C}\zeta)^* \wedge \wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*},$$

where $(du)_{(\mathbb{C}\zeta)^* \wedge \wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*}$ means the $(\mathbb{C}\zeta)^* \wedge \wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*$ part of du according to the above decomposition. So we have that for u in $\Gamma(M,(\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*)$,

$$du = d''u + d'u + (du)_{\wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^{q+1}({}^0T'')^*}.$$

By a direct computation, we have

$$(du)_{\wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^{q+1}({}^0T'')^*} = -d\theta \wedge (\zeta \,\lrcorner\, u),$$

where θ is a 1-form defined by $\theta|_{{}^0T'' + {}^0\bar{T}''} = 0$ and $\theta(\zeta) = 1$. We see the relation between these operators. For u in $\Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p({}^0T'')^* \wedge \wedge^q({}^0\bar{T}'')^*)$,

$$du = d'u + d''u - d\theta \wedge (\zeta \,\lrcorner\, u).$$

And so

$$\begin{aligned} ddu &= d'd'u + d''d'u + d'd''u \\ &\quad - (d(d\theta \wedge (\zeta \,\lrcorner\, u)))_{(\mathbb{C}\zeta)^* \wedge \wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^{q+1}({}^0T'')^*} \\ &\quad + d''d''u - d\theta \wedge (\zeta \,\lrcorner\, d'u) - d'(d\theta \wedge (\zeta \,\lrcorner\, u)) \\ &\quad - d\theta \wedge (\zeta \,\lrcorner\, d''u) - d''(d\theta \wedge (\zeta \,\lrcorner\, u)). \end{aligned}$$

By comparing the type, we have the following relations. Namely, from $(\mathbb{C}\zeta)^* \wedge \wedge^{p+2}({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*$ part,

$$d'd'u = 0. \tag{2.1}$$

From $(\mathbb{C}\zeta)^* \wedge \wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^{q+1}({}^0T'')^*$ part,

$$d''d'u + d'd''u - (d(d\theta \wedge (\zeta \,\lrcorner\, u)))_{(\mathbb{C}\zeta)^* \wedge \wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^{q+1}({}^0T'')^*} = 0. \tag{2.2}$$

From $(\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^{q+2}({}^0T'')^*$ part,

$$d''d''u = 0. \tag{2.3}$$

From $\wedge^{p+2}({}^0\bar{T}'')^* \wedge \wedge^{q+1}({}^0T'')^*$ part,

$$d\theta \wedge (\zeta \rfloor d'u) + d'(d\theta \wedge (\zeta \rfloor u)) = 0. \tag{2.4}$$

From $\wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^{q+2}({}^0T'')^*$ part,

$$d\theta \wedge (\zeta \rfloor d''u) + d''(d\theta \wedge (\zeta \rfloor u)) = 0. \tag{2.5}$$

A CR manifold is called a normal s.p.c. manifold if and only if there is a vector field ζ on M satisfying

$$\zeta_p \notin {}^0T''_p + {}^0\bar{T}''_p \quad \text{for every point } p \text{ of } M, \tag{2.6.1}$$

$$[\zeta, \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T''). \tag{2.6.2}$$

In this paper, we assume that $(M, {}^0T'')$ is a normal s.p.c. manifold and we adopt this vector ζ as a supplement vector to ${}^0T'' + {}^0\bar{T}''$. And so from this, it follows that $-d\theta$ is an element of $\Gamma(M, ({}^0\bar{T}'')^* \wedge ({}^0T'')^*)$.

3. The double complex $(A^{p,q}(M), d', d'')$

We use the notation $A^{p,q}(M)$ by

$$A^{p,q}(M) = \Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*).$$

Then, as is shown, we have d', d''-operators on $A^{p,q}(M)$, and we have a double complex $(A^{p,q}(M), d', d'')$. On $A^{p,q}(M)$, we put an inner product $\langle , \rangle$. Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal base of ${}^0T''_x$ where x in M, with respect to the Levi metric. Then

$$\langle \phi, \psi \rangle = (1/(p!q!)) \sum_{\substack{i_1, \ldots, i_p \\ j_1, \ldots, j_q}} \phi(\zeta, \bar{e}_{i_1}, \bar{e}_{i_p}, e_{j_1}, e_{j_q})\overline{\psi(\zeta, \bar{e}_{i_1}, \bar{e}_{i_p}, e_{j_1}, e_{j_q})}.$$

Like the case for $B^{p,q}$ in [T], there is a unique $*$ operator on this complex. Namely,

$$*: A^{p,q}(M) \to A^{n-1-q,n-1-p}(M)$$

satisfies

(1) $*$ is a real operator,
(2) $\langle\phi, \psi\rangle(d\theta)^{n-1} = (n-1)!(\zeta \rfloor \phi)\wedge(\zeta \rfloor *\bar{\psi})$ for ϕ, ψ in $A^{p,q}(M)$,
(3) $**\phi = (-1)^{p+q}\phi$.

And by the same way as for $B^{p,q}$ in [T], we introduce operators L and Λ. We define $L: A^{p,q}(M) \to A^{p+1,q+1}(M)$ by $L\phi = -d\theta \wedge \phi$, for ϕ in $A^{p,q}(M)$. Let Λ be the adjoint operator of L with respect to $\langle\, ,\rangle$. Let δ' be the adjoint operator d' and let δ'' be the adjoint operator of d''. Then we have

$$d''\Lambda - \Lambda d'' = -\sqrt{-1}\,\delta', \qquad d'\Lambda - \Lambda d' = \sqrt{-1}\,\delta'', \tag{3.1}$$

$$\delta'' L - L\delta'' = -\sqrt{-1}\,d', \qquad \delta' L - L\delta' = \sqrt{-1}\,d'', \tag{3.2}$$

$$d''\delta' + \delta' d'' = 0, \qquad d'\delta'' + \delta'' d' = 0, \tag{3.3}$$

(see Lemma 12.3 in [T]).

$$\Lambda L\phi = L\Lambda\phi + (n-k-1)\phi, \ \phi \in A^{p,q}(M), \quad k = p+q, \tag{3.4}$$

(see Lemma 12.1 in [T]).

And like the case for $B^{p,q}$,

$$d'd'' + d''d'$$

does not vanish.

4. A CR analogue of Tian-Todorov's lemma

Let $(M, {}^0T'')$ be a normal s.p.c. manifold with a real vector field ζ satisfying (2.6.1) and (2.6.2) and with $\dim_{\mathbb{R}} M = 2n-1 \geqslant 7$. In this section, we will assume that the canonical line bundle $K_M = \wedge^n(T')^*$ is trivial in CR-sense, that is there exists a nowhere vanishing section $\omega \in \Gamma(M, \wedge^n(T')^*)$ satisfying $d''\omega = 0$. If M is a real hypersurface of a complex manifold X, then the above assumption implies the existence of a nowhere vanishing CR-section $\Omega \in \Gamma(M, \wedge^n(T'X|_M)^*)$, since the projection operator $\rho': \mathbb{C}TX|_M \to T'X|_M$ induces an isomorphism

$$(\rho')^*: \Gamma(M, \wedge^n(T'X|_M)^* \wedge \wedge^q({}^0T'')^*) \to \Gamma(M, \wedge^n(T')^* \wedge \wedge^q({}^0T'')^*)$$

satisfying $d'' \circ (\rho')^* = (\rho')^* \circ \bar{\partial}_b$. And moreover, by the Lewy extension theorem, it is equivalent to the existence of a nowhere vanishing holomorphic n-form on an inward collar neighbourhood of M, if M is a strongly pseudo-convex boundary of an isolated singularity. Obviously Gorenstein singularity is this case.

In this situation, we will consider a bundle isomorphism

$$i_q\colon T' \otimes \wedge^q({}^0T'')^* \to \wedge^{n-1}(T')^* \wedge \wedge^q({}^0T'')^* \text{ given by } i_q(u) = u \rfloor \omega.$$

Note that

$$i_q({}^0\bar{T}'' \otimes \wedge^q({}^0T'')^*) = (\mathbb{C}\zeta)^* \wedge \wedge^{n-2}({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*$$

holds. We will denote $i_{q|{}^0\bar{T}''\otimes\wedge^q({}^0T'')^*}$ by the same symbol i_q.

Now we have the Lie bracket, on $\Gamma(M, {}^0\bar{T}'' \otimes ({}^0T'')^*)$, given by

$$[\phi, \psi](X, Y) := [\phi(X), \psi(Y)] + [\psi(X), \phi(Y)]$$
$$-\phi([X, \psi(Y)]_{{}^0T''} + [\psi(X), Y]_{{}^0T''}) - \psi([\phi(X), Y]_{{}^0T''} + [X, \phi(Y)]_{{}^0T''})$$

for X, $Y \in \Gamma(M, {}^0T'')$. And then, a Lie bracket is induced on $\Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^{n-2}({}^0\bar{T}'')^* \wedge ({}^0T'')^*)$ by $[\alpha, \beta] := i_2[i_1^{-1}\alpha, i_1^{-1}\beta]$.

The main purpose of this section is to obtain a CR-analogue of Tian–Todorov's lemma (cf. [Ti], [To]) analyzing this induced Lie bracket.

First of all, in order to simplify the argument, we will obtain a local frame of $\mathbb{C}TM$ normalized at a reference point in M.

LEMMA 4.1. *For any point $p \in M$, there exists a local frame $e_1, e_2, \ldots, e_{n-1}$ of ${}^0T''$ around p satisfying*

(1) $[e_i, e_j](p) = 0$ $(i, j = 1, 2, \ldots, n-1)$,

(2) $[\bar{e}_i, e_k](p) = \sqrt{-1}\,\delta_{ik}\zeta_p$ $(i, k = 1, 2, \ldots, n-1)$.

Proof. Let $\langle f_1, f_2, \ldots, f_{n-1}\rangle$ be a moving frame of ${}^0T''$ such that $[f_i, f_j] = 0$ $(i, j = 1, 2, \ldots, n-1)$. Note that such a local frame always exists because M is realized locally as a real hypersurface of a complex manifold (cf. [A3] and [Ku]). Set $e_i := \Sigma_{s=1}^{n-1} q_{si} f_s$ where q_{si} is a C^∞-function. Then (1) holds if and only if

$$\sum_{s=1}^{n-1}\sum_{i=1}^{n-1} \{q_{si}(p) f_s(q_{tj})(p) - q_{sj}(p) f_s(q_{ti})(p)\} = 0 \quad (i, j = 1, 2, \ldots, n-1).$$

Hence, if q_{ij} satisfies

$$f_s(q_{ij})(p) = 0 \quad (s = 1, 2, \ldots, n-1) \tag{4.1.1}$$

then (1) holds.

Next, let $[\bar{f}_i, f_k] = \Sigma_{r=1}^{n-1} a_{ik}^r f_r - \Sigma_{r=1}^{n-1} \bar{a}_{ki}^r \bar{f}_r + \sqrt{-1}\, c_{ik}\zeta$, where a_{ik}^r and c_{ik} are C^∞-functions satisfying $\bar{c}_{ik} = c_{ki}$. Then (2) holds if and only if

$$\sum_{s=1}^{n-1}\sum_{l=1}^{n-1} \bar{q}_{si}(p) q_{lk}(p) c_{sl}(p) = \delta_{ik} \quad (i, k = 1, 2, \ldots, n-1)$$

and

$$\sum_{l=1}^{n-1} q_{lk}(p) f_l(\bar{q}_{ri})(p) + \sum_{s=1}^{n-1}\sum_{l=1}^{n-1} q_{lk}(p)\bar{a}_{ls}^r(p)\bar{q}_{si}(p) = 0 \quad (i, k, r = 1, \ldots, n-1).$$

Since $(M, {}^0T'')$ is strongly pseudo-convex, (c_{ik}) is a positive-definite hermitian $(n-1)$-matrix. Hence we can find a unitary matrix (u_{ik}) satisfying $\Sigma_{s=1}^{n-1}\Sigma_{l=1}^{n-1} u_{si} c_{sl} \bar{u}_{lk} = \delta_{ik}$ $(i, k = 1, \ldots, n-1)$. (2) holds if $q_{si}(p) = \bar{u}_{si}$ $(s, i = 1, 2, \ldots, n-1)$ and $f_t(\bar{q}_{ri})(p) + \Sigma_{s=1}^{n-1} \bar{a}_{ts}(p) u_{si} = 0$ $(i, t, r = 1, 2, \ldots, n-1)$. Therefore (1) and (2) hold if q_{ik} $(i, k = 1, 2, \ldots, n-1)$ satisfy $q_{ik}(p) = \bar{u}_{ik}$, $f_s(q_{ik})(p) = 0$, $\bar{f}_s(q_{ik})(p) = -\Sigma_{j=1}^{n-1} a_{sj}^i(p)\bar{u}_{jk}$ $(s = 1, 2, \ldots, n-1)$. Thus Lemma 4.1 follows from the following:

CLAIM. For given $\chi_1, \chi_2, \ldots, \chi_{2n-1}$, there exists a local C^∞-function q satisfying $e_i(q)(p) = \chi_i$ $(i = 1, 2, \ldots, n-1)$, $\bar{e}_i(q)(p) = \chi_{n+k-1}$ $(k = 1, 2, \ldots, n-1)$ and $\zeta(q)(p) = \chi_{2n-1}$.

In fact, since $e_1, e_2, \ldots, e_{n-1}, \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1}, \zeta$ form a local frame of $\mathbb{C}TM$, $\left(\frac{\partial}{\partial x_i}\right)_p$ $(i = 1, 2, \ldots, 2n-1)$ is a linear combination of $(e_1)_p, (e_2)_p, \ldots, (e_{n-1})_p, (\bar{e}_1)_p, (\bar{e}_2)_p, \ldots, (\bar{e}_{n-1})_p, \zeta_p$. Hence it is enough to find a C^∞-function q with preassigned derivatives $\frac{\partial q}{\partial x_i}(p)$ $(i = 1, 2, \ldots, 2n-1)$, and clearly it is possible. □

If we write $\omega = \omega(\mathbb{C}\zeta)^* \wedge \bar{e}_1^* \wedge \bar{e}_2^* \wedge \cdots \wedge \bar{e}_{n-1}^*$ with this local frame, then we have

LEMMA 4.2. $e_k\omega(p) = 0$ $(k = 1, 2, \ldots, n-1)$.

Proof.

$$\begin{aligned} d''\omega(\zeta, \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1}, e_k) &= (-1)^n e_k\omega(\zeta, \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1}) \\ &+ (-1)^n\omega([\zeta, e_k]_{T'}, \bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1}) \\ &+ \sum_{i=1}^{n-1} (-1)^{n+i}\omega(\zeta, [\bar{e}_i, e_k]_{T'}, \bar{e}_1, \ldots \overset{i}{\vee} \ldots, \bar{e}_{n-1}). \end{aligned}$$

Because of (2.6.2) and Lemma 4.1, we have Lemma 4.2. □

Throughout this section, by $e_1, e_2, \ldots, e_{n-1}$ we denote a local frame of ${}^0T''$ as in Lemma 4.1, and use the abbreviations $\bar{e}_I^* = \bar{e}_{i_1}^* \wedge \bar{e}_{i_2}^* \wedge \cdots \wedge \bar{e}_{i_p}^*$ and $e_K^* = e_{k_1}^* \wedge e_{k_2}^* \wedge \cdots \wedge e_{k_q}^*$ for $I = (i_1, i_2, \ldots, i_p)$ and $K = (k_1, k_2, \ldots, k_q)$ respectively.

LEMMA 4.3. *If*

$$\phi = \frac{1}{q!} \sum_{|K|=q} \left\{ \phi_K \zeta \otimes e_K^* + \sum_{i=1}^{n-1} \phi_K^i \bar{e}_i \otimes e_K^* \right\} \in \Gamma(M, T' \otimes ({}^0T'')^*),$$

then

$$i_q\phi = \frac{(-1)^{n-1}}{q!} \sum_{|K|=q} \left\{ \omega \phi_K \bar{e}_N^* \otimes e_K^* \right.$$
$$\left. + \sum_{i=1}^{n-1} (-1)^i \omega \phi_K^i \zeta^* \wedge \bar{e}_1^* \wedge \bar{e}_2^* \wedge \cdots \vee^i \cdots \wedge \bar{e}_{n-1}^* \wedge e_K^* \right\},$$

where $N = (1, 2, \ldots, n-1)$.

Proof.

$$\begin{aligned} &i_q\phi(\zeta, \bar{e}_1, \bar{e}_2, \ldots \vee^i \ldots, \bar{e}_{n-1}, e_{k_1}, e_{k_2}, \ldots, e_{k_q}) \\ &= \omega(\zeta, \bar{e}_1, \bar{e}_2, \ldots \vee^i \ldots, \bar{e}_{n-1}, \phi(e_{k_1}, e_{k_2}, \ldots, e_{k_q})) \\ &= \omega\left(\zeta, \bar{e}_1, \bar{e}_2, \ldots \vee^i \ldots, \bar{e}_{n-1}, \sum_{j=1}^{q} \phi_K^j \bar{e}_j\right) = (-1)^{n-i-1} \omega \phi_K^i. \end{aligned}$$

$$\begin{aligned} &i_q\phi(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1}, e_{k_1}, e_{k_2}, \ldots, e_{k_q}) \\ &= \omega(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1}, \phi(e_{k_1}, e_{k_2}, \ldots, e_{k_q})) \\ &= \omega(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{n-1}, \phi_K \zeta) = (-1)^{n-1} \omega \phi_K. \end{aligned}$$

□

LEMMA 4.4. *If*

$$\alpha = \frac{1}{p!q!} \sum_{|I'|=p} \sum_{|K'|=q} \alpha_{I'K'} (C\zeta)^* \wedge \bar{e}_{I'}^* \wedge e_{K'}^* \in A^{p,q}(M)$$
$$= \Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*),$$

$$d'\alpha(p) = \frac{-1}{(p+1)!q!} \sum_{|K'|=q} \sum_{|I|=p+1} \sum_{(i,I')=I} \sigma\binom{I}{i,I'} \bar{e}_i(\phi_{I'K'})(p)(\mathbb{C}\zeta)_p^* \wedge (\bar{e}_I^*)_p \wedge (e_{K'}^*)_p,$$

$$d''\alpha(p) = \frac{(-1)^{p+1}}{p!(q+1)!} \sum_{|I'|=p} \sum_{|K|=q+1} \sum_{(k,K')=I} \sigma\binom{K}{k,K'} e_k(\phi_{I'K'})(p)(C\zeta)_p^* \wedge (\bar{e}_{I'}^*)_p \wedge (e_K^*)_p,$$

where $\sigma\binom{I}{i,I'}$ *(resp.* $\sigma\binom{K}{k,K'}$*) denotes the sign of permutation changing* (i, I') *into* I *(resp.* (k, K') *into* K*).*

Proof.

$$
\begin{aligned}
&d'\alpha(\zeta, \bar{e}_{i_1}, \bar{e}_{i_2}, \ldots, \bar{e}_{i_{p+1}}, e_{k_1}, e_{k_2}, \ldots, e_{k_q}) \\
&\quad = \sum_{j=1}^{p+1} (-1)^j \bar{e}_{i_j} \alpha(\zeta, \bar{e}_{i_1}, \ldots \vee^{i_j} \ldots, \bar{e}_{i_{p+1}}, e_{k_1}, e_{k_2}, \ldots, e_{k_q}) \\
&\qquad + \sum_{j<l} (-1)^{j+l} \alpha([\bar{e}_{i_j}, \bar{e}_{i_l}]_{^0\bar{T}''}, \zeta, \bar{e}_{i_1}, \ldots \vee^{i_j} \cdots \vee^{i_l} \ldots, \bar{e}_{i_{p+1}}, e_{k_1}, e_{k_2}, \ldots, e_{k_q}).
\end{aligned}
$$

By Lemma 4.1,

$$
\begin{aligned}
&d'\alpha(\zeta, \bar{e}_{i_1}, \bar{e}_{i_2}, \ldots, \bar{e}_{i_{p+1}}, e_{k_1}, e_{k_2}, \ldots, e_{k_q})(p) \\
&\quad = \sum_{j=1}^{p+1} (-1)^j \bar{e}_{i_j} (\alpha_{i_1 i_2 \cdots \vee^{i_j} \cdots i_{p+1} k_1 k_2 \cdots k_q})(p).
\end{aligned}
$$

Next,

$$
\begin{aligned}
&d''\alpha(\zeta, \bar{e}_{i_1}, \bar{e}_{i_2}, \ldots, \bar{e}_{i_p}, e_{k_1}, e_{k_2}, \ldots, e_{k_{q+1}}) \\
&\quad = \sum_{j=1}^{q+1} (-1)^{p+j} e_{k_j} \alpha(\zeta, \bar{e}_{i_1}, \ldots, \bar{e}_{i_p}, e_{k_1}, e_{k_2}, \ldots \vee^{k_j} \ldots, e_{k_{q+1}}) \\
&\qquad + \sum_{j<l} (-1)^{j+l} \alpha([e_{k_j}, e_{k_l}]_{^0\bar{T}''}, \zeta, \bar{e}_{i_1}, \ldots, \bar{e}_{i_p}, e_{k_1}, e_{k_2}, \ldots \vee^{k_j} \cdots \vee^{k_l} \ldots, e_{k_{q+1}}).
\end{aligned}
$$

By Lemma 4.1,

$$
\begin{aligned}
&d''\alpha(\zeta, \bar{e}_{i_1}, \bar{e}_{i_2}, \ldots, \bar{e}_{i_p}, e_{k_1}, e_{k_2}, \ldots, e_{k_{q+1}})(p) \\
&\quad = \sum_{j=1}^{q+1} (-1)^{p+j} e_{k_j} (\alpha_{i_1 i_2 \cdots i_p k_1 k_2 \cdots \vee^{k_j} \cdots k_{q+1}})(p). \qquad \square
\end{aligned}
$$

LEMMA 4.5. *For*

$$\phi = \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \phi_k^i \bar{e}_i \otimes e_k^*$$

and

$$\psi = \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \psi_k^i \bar{e}_i \otimes e_k^* \in \Gamma(M, {}^0\bar{T}'' \otimes ({}^0T'')^*),$$

$$
\begin{aligned}
[\phi, \psi](p) = \frac{1}{2} \sum_{i=1}^{n-1} \sum_{k,l=1}^{n-1} \sum_{j=1}^{n-1} \{ &\phi_k^j(p) \bar{e}_j(\psi_l^i)(p) - \psi_l^j(p) \bar{e}_j(\phi_k^i)(p) \\
&+ \psi_k^j(p) \bar{e}_j(\phi_l^i)(p) - \phi_l^j(p) \bar{e}_j(\psi_k^i)(p) \} (\bar{e}_i)_p \otimes (e_k^*)_p \wedge (e_l^*)_p.
\end{aligned}
$$

Proof.

$$[\phi, \psi](e_k, e_l) = \left[\sum_{i=1}^{n-1} \phi_k^i \bar{e}_i, \sum_{j=1}^{n-1} \psi_l^j \bar{e}_j\right] + \left[\sum_{i=1}^{n-1} \psi_k^i \bar{e}_i, \sum_{j=1}^{n-1} \phi_l^j \bar{e}_j\right]$$
$$-\phi\left(\left[e_k, \sum_{j=1}^{n-1} \psi_l^j \bar{e}_j\right]_{^0T''} + \left[\sum_{i=1}^{n-1} \psi_k^i \bar{e}_i, e_l\right]_{^0T''}\right)$$
$$-\psi\left(\left[e_k, \sum_{j=1}^{n-1} \phi_l^j \bar{e}_j\right]_{^0T''} + \left[\sum_{i=1}^{n-1} \phi_k^i \bar{e}_i, e_l\right]_{^0T''}\right).$$

By Lemma 4.1, we have Lemma 4.5. □

Before proving a CR-analogue of Tian–Todorov's lemma, we recall the inner product between $T' \otimes \wedge^q({}^0T'')^*$ and $\wedge^p(T')^* \wedge \wedge^r({}^0T'')^*$: by

$$\phi \rfloor \alpha(X_1, X_2, \ldots, X_{p-1}, \bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_{q+r}) = \frac{1}{(q+r)!} \sum_{(K,L)=(1,2,\ldots,q+r)} \sigma\binom{1,2,\ldots,q+r}{K,L}$$
$$\times \alpha(X_1, \ldots, X_{p-1}, \phi(\bar{Y}_{k_1}, \bar{Y}_{k_2}, \ldots, \bar{Y}_{k_q}), \bar{Y}_{l_1}, \bar{Y}_{l_2}, \ldots, \bar{Y}_{l_r})$$

for $X_1, X_2, \ldots, X_{p-1} \in {}^0\bar{T}''$ and $\bar{Y}_1, \bar{Y}_2, \ldots, \bar{Y}_{q+r} \in {}^0T''$.

Note that $\phi \rfloor \alpha \in (\mathbb{C}\zeta)^* \wedge \wedge^{p-2}({}^0\bar{T}'')^* \wedge \wedge^{q+r}({}^0T'')^*$ if $\phi \in {}^0\bar{T}'' \otimes ({}^0T'')^*$ and $\alpha \in (\mathbb{C}\zeta)^* \wedge \wedge^{p-1}({}^0\bar{T}'')^* \wedge \wedge^r({}^0T'')^*$.

PROPOSITION 4.6 (A CR-analogue of Tian–Todorov's lemma). *If α, $\beta \in A^{n-2,1}(M)$ and satisfy $d'\alpha = d'\beta = 0$, then*

$$2i_2[i_1^{-1}\alpha, i_1^{-1}\beta] = d'(i_1^{-1}\alpha \rfloor \beta + i_1^{-1}\beta \rfloor \alpha).$$

Proof. Write

$$\alpha = \sum_{i=1}^{n-1}\sum_{k=1}^{n-1} \alpha_{i,k}(\mathbb{C}\zeta)^* \wedge \bar{e}_1^* \wedge \bar{e}_2^* \wedge \cdots \vee^i \cdots \wedge \bar{e}_{n-1}^* \wedge e_k^*$$

and

$$\beta = \sum_{j=1}^{n-1}\sum_{l=1}^{n-1} \beta_{j,l}(\mathbb{C}\zeta)^* \wedge \bar{e}_1^* \wedge \bar{e}_2^* \wedge \cdots \vee^j \cdots \wedge \bar{e}_{n-1}^* \wedge e_l^*.$$

Then

$$i_1^{-1}\alpha = \frac{(-1)^{n-1}}{\omega} \sum_{i=1}^{n-1}\sum_{k=1}^{n-1} (-1)^i \alpha_{i,k} \bar{e}_i \otimes e_k^*$$

and

$$i_1^{-1}\beta=\frac{(-1)^{n-1}}{\omega}\sum_{j=1}^{n-1}\sum_{l=1}^{n-1}(-1)^j\beta_{j,l}\bar{e}_j\otimes e_l^*.$$

By Lemma 4.5, at $p\in M$,

$$\begin{aligned}[i_1^{-1}\alpha, i_1^{-1}\beta]=&\frac{1}{2}\sum_{k,l=1}^{n-1}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}\left\{\frac{\alpha_{j,k}}{\omega}\bar{e}_j\left(\frac{\beta_{i,l}}{\omega}\right)-\frac{\beta_{j,l}}{\omega}\bar{e}_j\left(\frac{\alpha_{i,k}}{\omega}\right)\right.\\&\left.+\frac{\beta_{j,k}}{\omega}\bar{e}_j\left(\frac{\alpha_{i,l}}{\omega}\right)-\frac{\alpha_{j,l}}{\omega}\bar{e}_j\left(\frac{\beta_{i,k}}{\omega}\right)\right\}\bar{e}_i\otimes e_k^*\wedge e_l^*\\=&\frac{1}{2}\sum_{k,l=1}^{n-1}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}(-1)^{i+j}\left\{\frac{1}{\omega^2}(\alpha_{j,k}\bar{e}_j\beta_{i,l}-\beta_{j,l}\bar{e}_j\alpha_{i,k}+\beta_{j,k}\bar{e}_j\alpha_{i,l}-\alpha_{j,l}\bar{e}_j\beta_{i,k})\right.\\&\left.+\frac{1}{\omega}(\alpha_{j,k}\beta_{i,l}-\beta_{j,l}\alpha_{i,k}+\beta_{j,k}\alpha_{i,l}-\alpha_{j,l}\beta_{i,k})\bar{e}_j\left(\frac{1}{\omega}\right)\right\}\bar{e}_i\otimes e_k^*\wedge e_l^*.\end{aligned}$$

Hence, at $p\in M$,

$$\begin{aligned}i_2[i_1^{-1}\alpha, i_1^{-1}\beta]=&\frac{(-1)^{n-1}}{2}\sum_{k,l=1}^{n-1}\sum_{j=1}^{n-1}\sum_{i=1}^{n-1}(-1)^j\left\{\frac{1}{\omega}(\alpha_{j,k}\bar{e}_j\beta_{i,l}-\beta_{j,l}\bar{e}_j\alpha_{i,k}\right.\\&\left.+\beta_{j,k}\bar{e}_j\alpha_{i,l}-\alpha_{j,l}\bar{e}_j\beta_{i,k})+(\alpha_{j,k}\beta_{i,l}-\beta_{j,l}\alpha_{i,k}+\beta_{j,k}\alpha_{i,l}-\alpha_{j,l}\beta_{i,k})\bar{e}_j\left(\frac{1}{\omega}\right)\right\}\\&\times(\mathbb{C}\zeta)^*\wedge\bar{e}_1^*\wedge\bar{e}_2^*\wedge\cdots\vee^i\cdots\wedge\bar{e}_{n-1}^*\wedge e_k^*\wedge e_l^*.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(i_1^{-1}\alpha)\rfloor\beta+(i_1^{-1}\beta)\rfloor\alpha\\&=\sum_{i<j}\sum_{k,l=1}^{n-1}\frac{1}{\omega}\{\alpha_{i,k}\beta_{j,l}-\alpha_{j,k}\beta_{i,l}-\alpha_{i,l}\beta_{j,k}+\alpha_{j,l}\beta_{i,k}\}\\&\quad\times\zeta^*\wedge\bar{e}_1^*\wedge\bar{e}_2^*\wedge\cdots\vee^i\cdots\vee^j\cdots\wedge\bar{e}_{n-1}^*\wedge e_k^*\wedge e_l^*.\end{aligned}$$

By Lemma 4.4, at $p\in M$,

$$\begin{aligned}&d'(i_1^{-1}\alpha\rfloor\beta+i_1^{-1}\beta\rfloor\alpha)\\&=\sum_{k,l=1}^{n-1}\left\{\sum_{j<i}(-1)^j\bar{e}_j\left(\frac{1}{\omega}(\alpha_{j,k}\beta_{i,l}-\alpha_{i,k}\beta_{j,l}-\alpha_{j,l}\beta_{i,k}+\alpha_{i,l}\beta_{j,k})\right)\right.\end{aligned}$$

$$+\sum_{j>i}(-1)^{j-1}\bar{e}_j\left(\frac{1}{\omega}(\alpha_{i,k}\beta_{j,l}-\alpha_{j,k}\beta_{i,l}-\alpha_{i,l}\beta_{j,k}+\alpha_{j,l}\beta_{i,k})\right)\Bigg\}$$

$$\times(\mathbb{C}\zeta)^*\wedge\bar{e}_1^*\wedge\bar{e}_2^*\wedge\cdots\vee^i\cdots\wedge\bar{e}_{n-1}^*\wedge e_k^*\wedge e_l^*$$

$$=\sum_{k,l=1}^{n-1}\Bigg\{\frac{1}{\omega}\sum_{j\neq i}(-1)^j\bar{e}_j(\alpha_{j,k}\beta_{i,l}-\alpha_{i,k}\beta_{j,l}-\alpha_{j,l}\beta_{i,k}+\alpha_{i,l}\beta_{j,k})$$

$$+\sum_{j\neq i}(-1)^j(\alpha_{j,k}\beta_{i,l}-\alpha_{i,k}\beta_{j,l}-\alpha_{j,l}\beta_{i,k}+\alpha_{i,l}\beta_{j,k})\bar{e}_j\left(\frac{1}{\omega}\right)\Bigg\}$$

$$\times(\mathbb{C}\zeta)^*\wedge\bar{e}_1^*\wedge\bar{e}_2^*\wedge\cdots\vee^i\cdots\wedge\bar{e}_{n-1}^*\wedge e_k^*\wedge e_l^*$$

$$=\sum_{k,l=1}^{n-1}\Bigg\{\frac{1}{\omega}\sum_{j=1}^{n-1}(-1)^j\bar{e}_j(\alpha_{j,k}\beta_{i,l}-\alpha_{i,k}\beta_{j,l}-\alpha_{j,l}\beta_{i,k}+\alpha_{i,l}\beta_{j,k})$$

$$+\sum_{j=1}^{n-1}(-1)^j(\alpha_{j,k}\beta_{i,l}-\alpha_{i,k}\beta_{j,l}-\alpha_{j,l}\beta_{i,k}+\alpha_{i,l}\beta_{j,k})\bar{e}_j\left(\frac{1}{\omega}\right)\Bigg\}$$

$$\times(\mathbb{C}\zeta)^*\wedge\bar{e}_1^*\wedge\bar{e}_2^*\wedge\cdots\vee^i\cdots\wedge\bar{e}_{n-1}^*\wedge e_k^*\wedge e_l^*$$

Now, since

$$d'\alpha(p)=\sum_{k=1}^{n-1}\sum_{j=1}^{n-1}(-1)^j(\bar{e}_j\alpha_{j,k})(p)(\mathbb{C}\zeta)^*\wedge(\bar{e}_N^*)_p\wedge(e_k^*)_p,$$

$$d'\alpha=d'\beta=0\Rightarrow\sum_{j=1}^{n-1}(\bar{e}_j\alpha_{j,k})(p)=\sum_{j=1}^{n-1}(\bar{e}_j\beta_{j,k})(p)=0\quad(k=1,2,\ldots,n-1).$$

Therefore, at $p\in M$,

$$d'(i_1^{-1}\alpha\rfloor\beta+i_1^{-1}\beta\rfloor\alpha)$$

$$=\sum_{k,l=1}^{n-1}\Bigg\{\frac{1}{\omega}\sum_{j=1}^{n-1}(-1)^j(\alpha_{j,k}(\bar{e}_j\beta_{i,l})-(\bar{e}_j\alpha_{i,k})\beta_{j,l}-\alpha_{j,l}(\bar{e}_j\beta_{i,k})$$

$$+(\bar{e}_j\alpha_{i,l})\beta_{j,k})$$

$$+\sum_{j=1}^{n-1}(-1)^j(\alpha_{j,k}\beta_{i,l}-\alpha_{i,k}\beta_{j,l}-\alpha_{j,l}\beta_{i,k}+\alpha_{i,l}\beta_{j,k})\bar{e}_j\left(\frac{1}{\omega}\right)\Bigg\}$$

$$\times(\mathbb{C}\zeta)^*\wedge\bar{e}_1^*\wedge\bar{e}_2^*\wedge\cdots\vee^i\cdots\wedge\bar{e}_{n-1}^*\wedge e_k^*\wedge e_l^*.$$

Thus we have Proposition 4.6. □

Next, we will consider the compatibility of $\bar{\partial}_b$ and d'' via i_q.

LEMMA 4.7. *If*

$$\phi = \frac{1}{q!}\sum_{i=1}^{n-1}\sum_{|K'|=q}\phi^i_{K'}\bar{e}_i \otimes e^*_{K'} \in \Gamma(M, {}^0\bar{T}'' \otimes \wedge^q({}^0T'')^*),$$

then

$$\bar{\partial}_b\phi(p) = \frac{1}{(q+1)!}\sum_{|K|=q+1}\sum_{(k,K')=K}\sigma\binom{K}{k,K'}\left\{-\sqrt{-1}\,\phi^k_{K'}(p)\zeta_p \otimes (e^*_K)_p + \sum_{i=1}^{n-1} e_k(\phi^i_{K'})(p)(\bar{e}_i)_p \otimes (e^*_K)_p\right\},$$

where $\sigma\binom{K}{k,K'}$ *denotes the sign of permutation changing* (k, K') *into* K.

Proof. By the definition of $\bar{\partial}_b$,

$$\begin{aligned}
&\bar{\partial}_b\phi(e_{k_1}, e_{k_2}, \ldots, e_{k_{q+1}}) \\
&= \sum_{j=1}^{q+1}(-1)^{j-1}[e_{k_j}, \phi(e_{k_1}, \ldots \vee^{k_j} \ldots, e_{k_{q+1}})] \\
&\quad + \sum_{j<l}(-1)^{j+l}\phi([e_{k_j}, e_{k_l}]_{{}^0T''}, e_{k_1}, \ldots \vee^{k_j} \cdots \vee^{k_l} \ldots, e_{k_{q+1}})
\end{aligned}$$

By Lemma 4.1,

$$\begin{aligned}
&\bar{\partial}_b\phi(e_{k_1}, e_{k_2}, \ldots, e_{k_{q+1}})(p) \\
&= \sum_{j=1}^{q+1}(-1)^{j-1}\left\{-\sqrt{-1}\,\phi^{k_j}_{k_1k_2\cdots\vee^{k_j}\cdots k_{q+1}}(p)\zeta_p + \sum_{i=1}^{n-1} e_{k_j}(\phi^i_{k_1k_2\cdots\vee^{k_j}\cdots k_{q+1}})(p)(\bar{e}_i)_p\right\}.
\end{aligned}$$

□

PROPOSITION 4.8. *For* $\phi \in \Gamma(M, {}^0\bar{T}'' \otimes ({}^0T'')^*)$,

$$d'' \circ i_q(\phi) = i_{q+1} \circ \bar{\partial}_b(\phi) + \sqrt{-1}\,d\theta \wedge (\zeta \,\lrcorner\, i_q\phi).$$

Proof. Write

$$\phi = \frac{1}{q!}\sum_{i=1}^{n-1}\sum_{|K|=q}\phi^i_K\bar{e}_i \otimes e^*_K.$$

Then, by Lemmas 4.2, 4.3 and 4.4, at $p \in M$,

$$d'' \circ i_q(\phi) = \frac{(-1)^{n-1}}{(q+1)!} \sum_{i=1}^{n-1} \sum_{|K|=q+1} \sum_{(k,K')=K} \sigma\binom{K}{k,K'}(-1)^i \omega(e_k \phi^i_{K'}) \times (\mathbb{C}\zeta)^* \wedge \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \cdots \vee^i \cdots \wedge \bar{e}^*_{n-1} \wedge e^*_K.$$

On the other hand, by Lemmas 4.7 and 4.3, at $p \in M$,

$$\begin{aligned} i_{q+1}\bar{\partial}_b \phi &= \frac{-\sqrt{-1}}{(q+1)!} \sum_{|K|=q+1} \sum_{(k,K')=K} \sigma\binom{K}{k,K'} \omega \phi^k_{K'} \bar{e}^*_N \wedge e^*_K \\ &+ \frac{(-1)^{n-1}}{(q+1)!} \sum_{|K|=q+1} \sum_{(k,K')=K} \sum_{i=1}^{n-1} \sigma\binom{K}{k,K'}(-1)^i \omega(e_k \phi^i_{K'}) \\ &\quad \times (\mathbb{C}\zeta)^* \wedge \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \cdots \vee^i \cdots \wedge \bar{e}^*_{n-1} \wedge e^*_K. \end{aligned}$$

Now, since $d\theta(p) = \Sigma_{i=1}^{n-1} (\bar{e}^*_i)_p \wedge (e^*_i)_p$ and by Lemma 4.3, at $p \in M$, $\omega \wedge (\zeta \rfloor i_q \phi) = \dfrac{1}{(q+1)!} \Sigma_{i=1}^{n-1} \Sigma_{|K|=q+1} \sigma\binom{K}{k,K'} \omega \phi^k_{K'} \bar{e}^*_N \wedge e^*_K$. We have Lemma 4.8. □

Thus we are led to a new subspace $\mathbf{F}^{p,q} = \{u \in A^{p,q}(M) \mid d\theta \wedge (\zeta \rfloor u) = 0\}$ (cf. Section 5 for the details). We will conclude this section by proving the following proposition indicating the naturality of considering the new subspace $\mathbf{F}^{p,q}$.

PROPOSITION 4.9. *If* $\alpha, \beta \in \mathbf{F}^{n-2,1}$, *then* $i_1^{-1}\alpha \rfloor \beta \in \mathbf{F}^{n-3,2}$.

Proof. Write

$$\alpha = \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \alpha_{i,k} (\mathbb{C}\zeta)^* \wedge \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \cdots \vee^i \cdots \wedge \bar{e}^*_{n-1} \wedge e^*_k$$

and

$$\beta = \sum_{j=1}^{n-1} \sum_{l=1}^{n-1} \beta_{j,l} (\mathbb{C}\zeta)^* \wedge \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \cdots \vee^j \cdots \wedge \bar{e}^*_{n-1} \wedge e^*_l.$$

Then $\alpha, \beta \in F^{n-2,1}$ implies $(-1)^i \alpha_{i,k}(p) - (-1)^k \alpha_{k,i}(p) = 0$ and $(-1)^i \beta_{i,k}(p) - (-1)^k \beta_{k,i}(p) = 0$ for $i, k = 1, 2, \ldots, n-1$.

As in the proof of Proposition 4.6,

$$(i_1^{-1}\alpha) \rfloor \beta = \frac{1}{2} \sum_{i<j} \sum_{k,l=1}^{n-1} \frac{1}{\omega} \{\alpha_{i,k}\beta_{j,l} - \alpha_{j,k}\beta_{i,l} - \alpha_{i,l}\beta_{j,k} + \alpha_{j,l}\beta_{i,k}\} \times (\mathbb{C}\zeta)^* \wedge \bar{e}^*_1 \wedge \bar{e}^*_2 \wedge \cdots \vee^i \cdots \vee^j \cdots \wedge \bar{e}^*_{n-1} \wedge e^*_k \wedge e^*_l.$$

Since

$$d\theta(p)=\sum_{i=1}^{n-1}(\bar{e}_i^*)_p\wedge(e_i^*)_p,\quad \text{at } p\in M,$$

$$\begin{aligned}d\theta\wedge(i_1^{-1}\alpha\,\rfloor\,\beta)=\frac{(-1)^{n-3}}{12\omega}\sum_{k,l=1}^{n-1}\Bigg[&\sum_{i<j}\{(-1)^{i-1}(\alpha_{i,k}\beta_{j,l}-\alpha_{i,l}\beta_{j,k}-\alpha_{j,k}\beta_{i,l}+\alpha_{j,l}\beta_{i,k})\\ &+(-1)^{k-1}(\alpha_{k,l}\beta_{j,i}-\alpha_{k,i}\beta_{j,l}-\alpha_{j,l}\beta_{k,i}+\alpha_{j,i}\beta_{k,l})\\ &+(-1)^{l-1}(\alpha_{l,i}\beta_{j,k}-\alpha_{l,k}\beta_{j,i}-\alpha_{j,i}\beta_{l,k}+\alpha_{j,k}\beta_{l,i})\}\\ &+\sum_{i>j}\{(-1)^{i}(\alpha_{j,k}\beta_{i,l}-\alpha_{j,l}\beta_{i,k}-\alpha_{i,k}\beta_{j,l}+\alpha_{i,l}\beta_{j,k})\\ &+(-1)^{k}(\alpha_{j,l}\beta_{k,i}-\alpha_{j,i}\beta_{k,l}-\alpha_{k,l}\beta_{j,i}+\alpha_{k,i}\beta_{j,l})\\ &+(-1)^{l}(\alpha_{j,i}\beta_{l,k}-\alpha_{j,k}\beta_{l,i}-\alpha_{l,i}\beta_{j,k}+\alpha_{l,k}\beta_{j,i})\}\Bigg]\\ &\times(\mathbb{C}\zeta)^*\wedge\bar{e}_1^*\wedge\bar{e}_2^*\wedge\cdots\vee^{j}\cdots\wedge\bar{e}_{n-1}^*\wedge e_i^*\wedge e_k^*\wedge e_l^*.\end{aligned}$$

Since α, $\beta\in\mathbf{F}^{n-2,1}$, we have $d\theta\wedge(i_1^{-1}\alpha\rfloor\beta)(p)=0$.

5. The new double complex $(\mathbf{F}^{p,q}, d', d'')$

In this section, we introduce a new double complex $(\mathbf{F}^{p,q}, d', d'')$. Namely, we set

$$\mathbf{F}^{p,q}=\{u\colon u\in\Gamma(M,(\mathbb{C}\zeta)^*\wedge\wedge^{p}({}^0\bar{T}'')^*\wedge\wedge^{q}({}^0T'')^*),\ d\theta\wedge(\zeta\,\rfloor\,u)=0\}.$$

Then by (2.4) and (2.5), for u in $\mathbf{F}^{p,q}$,

$$d''u\in\mathbf{F}^{p,q+1}\quad\text{and}\quad d'u\in\mathbf{F}^{p+1,q}.$$

So our $(F^{p,q}, d', d'')$ is a double complex. We study this complex. First, we have

$$d'd''+d''d'=0\quad\text{on } \mathbf{F}^{p,q}.$$

In fact, from (2.2), this follows. We see cohomology groups.

LEMMA 5.1. *The map from* $(\mathbb{C}\zeta)^*\wedge\wedge^{p-1}({}^0\bar{T}'')^*\wedge\wedge^{q-1}({}^0T'')^*$ *to* $\wedge^{p}({}^0\bar{T}'')^*\wedge\wedge^{q}({}^0T'')^*$, *defined by: for* u *in* $(\mathbb{C}\zeta)^*\wedge\wedge^{p-1}({}^0\bar{T}'')^*\wedge\wedge^{q-1}({}^0T'')^*$, $d\theta\wedge(\zeta\rfloor u)$ *in* $\wedge^{p}({}^0\bar{T}'')^*\wedge\wedge^{q}({}^0T'')^*$ *is surjective if* $p+q\geqslant n$, *and* p *or* $q\geqslant n-2$.

Proof. For any ordered set $I=(i_1,\ldots,i_p)$, $J=(j_1,\ldots,j_q)$, there are k and k' satisfying: $i_k=j_{k'}$, because of $p+q\geqslant n$ and p or $q\geqslant n-2$. Without loss of generality, we can assume $i_1 = j_1$. Namely, $I = (i, i_2, \ldots, i_p)$, $I \leqslant i < i_2 < \cdots < i_p \leqslant p$ and $J = (i, j_2, \ldots, j_p)$, $I \leqslant i < j_2 \cdots < j_q \leqslant q$. So if we set

$$u = \zeta^* \wedge \bar{e}^*_{i_2} \wedge \cdots \wedge \bar{e}^*_{i_p} \wedge e^*_{j_2} \wedge \cdots \wedge e^*_{j_q},$$

we have

$$-d\theta \wedge (\zeta \rfloor u) = \bar{e}^*_i \wedge \bar{e}^*_{i_2} \wedge \cdots \wedge \bar{e}^*_{i_p} \wedge e^*_i \wedge e^*_{j_2} \wedge \cdots \wedge e^*_{j_q}.$$

Hence we have our theorem. □

We set a vector bundle $F^{p,q}$ by:

$$F^{p,q} = \{u : u \in (\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*,\ d\theta \wedge (\zeta \rfloor u) = 0\}.$$

We note that the condition $d\theta \wedge (\zeta \rfloor u) = 0$ is equivalent to $d\theta \wedge u = 0$ because $d\theta$ is an element of $\Gamma(M, ({}^0\bar{T}'')^* \wedge ({}^0T'')^*)$ ($(M, {}^0T'')$ is normal). Then obviously, $\mathbf{F}^{p,q} = \Gamma(M, F^{p,q})$. We note that our $F^{n-2,q} = i_q(E_q)$, where i_q is defined in Section 2 in this paper, and E_q is introduced in [A3]. So if $p \geqslant n-2$ and $q \geqslant 2$, or $p \geqslant 2$ and $q \geqslant n-2$, then

$$\operatorname{Ker} d'' \cap \mathbf{F}^{p,q}/d''\mathbf{F}^{p,q-1} \cong H^q_{d''}(M, \wedge^{p+1}(T')^*),$$

and

$$\operatorname{Ker} d'' \cap \mathbf{F}^{n-2,1} \to H^1_{d''}(M, \wedge^{n-1}(T')^*) \to 0.$$

In fact, if $p \geqslant n-2$ and $q \geqslant 1$, or $p \geqslant 1$ and $q \geqslant n-2$, we first show

$$\operatorname{Ker} d'' \cap \mathbf{F}^{p,q} \to H^q_{d''}(M, \wedge^{p+1}(T')^*) \to 0.$$

For u in $\Gamma(M, \wedge^{p+1}(T')^* \wedge \wedge^q({}^0T'')^*)$ satisfying $d''u = 0$,

$$u = u_1 + u_2,\ u_1 \in \Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*),$$
$$u_2 \in \Gamma(M, \wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*).$$

While by Lemma 5.1, there is a v in $\Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^{q-1}({}^0T'')^*)$ satisfying

$$(d''v)_{\wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*} = u_2.$$

We set $u-d''v$. Then, obviously

$$u-d''v \quad \text{in } \Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^q({}^0T'')^*),$$

and

$$u-d''v \quad \text{in } \mathbf{F}^{p,q} \text{ (by } d''u=0).$$

So we have injectivity. Second we show

$$0 \to \operatorname{Ker} d'' \cap \mathbf{F}^{p,q}/d''\mathbf{F}^{p,q-1} \to H^q_{d''}(M, \wedge^{p+1}(T')^*).$$

For $u \in \mathbf{F}^{p,q}$, we assume

$$u = d''v, \qquad v = v_1 + v_2,$$
$$v_1 \in \Gamma(M, (\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^{q-1}({}^0T'')^*),$$
$$v_2 \in \Gamma(M, \wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^{q-1}({}^0T'')^*).$$

By the same way as in the proof of injectivity, by Lemma 5.1, there is a w in $\Gamma(M,(\mathbb{C}\zeta)^* \wedge \wedge^p({}^0\bar{T}'')^* \wedge \wedge^{q-2}({}^0T'')^*)$ satisfying

$$(d''w)_{\wedge^{p+1}({}^0\bar{T}'')^* \wedge \wedge^{q-1}({}^0T'')^*} = v_2.$$

So $u=d''(v-d''w)$. Obviously $v-d''w$ is in $\mathbf{F}^{p,q-1}$. Hence we have injectivity.

By the same way as in [A3], we have an a priori estimate for $(\mathbf{F}^{n-2,q}, d'')$ complex (at $q=2$). So we have the Kodaira–Hodge type decomposition theorem. Namely, we have the operators $N_{d''}$, $H_{d''}$ from $L_2(M, \mathbf{F}^{n-2,2})$ to $L_2(\mathbf{M}, F^{n-2,2})$, satisfying:

(1) $u=H_{d''}u+(d''^*d''+d''d''^*)N_{d''}u$ for u in $F^{n-2,2}$,
(2) $\Box_{d''}N_{d''} = N_{d''}\Box_{d''}$, $\Box_{d''} = d''^*d''+d''d''^*$,
(3) $N_{d''}H_{d''} = H_{d''}N_{d''} = 0$.

Finally, in this section, we set the projection from $A^{n-2,q}(M)$ to $\mathbf{F}^{n-2,q}$. Namely we set that for u in $A^{n-2,q}(M)$, we put

$$u-(1/(q+1))\Lambda Lu,$$

where L, Λ are introduced in Section 3. In fact, by (3.4),

$$L\{u-(1/(q+1))\Lambda Lu\}=0 \quad \text{because of } LLu=0.$$

And ΛLu is orthogonal to $\mathbf{F}^{n-2,q}$ with respect to the inner product $\langle\,,\rangle$, defined by the Levi metric.

6. The key equality

For v in $\mathbf{F}^{n-2,2}$, we show the key equality. Namely, we have

Key equality: For v in $\mathbf{F}^{n-2,2}$,

$$d'\square_{d''}v = (2/3)d''^*d''(d'v)+(1/2)d''d''^*(d'v).$$

Proof. By the definition of $\square_{d''}$,

$$d'\{d''^*d''v+d''d''^*v\} = d'\{(\delta''d''v)_{F^{n-2,2}}+d''\{(\delta''v)_{F^{n-2,1}}\}\},$$

where $(\delta''d''v)_{\mathbf{F}^{n-2,2}}$ means the projection of $\delta''d''v$ to $\mathbf{F}^{n-2,2}$ and $(\delta''v)_{\mathbf{F}^{n-2,1}}$ means the projection of $\delta''v$ to $\mathbf{F}^{n-2,1}$. And by the result in the last part of Section 3 in this paper,

$$(\delta''v)_{\mathbf{F}^{n-2,1}} = \delta''v-(1/2)\Lambda L\delta''v$$

and

$$(\delta''d''v)_{\mathbf{F}^{n-2,2}} = \delta''d''v-(1/3)\Lambda L\delta''d''v.$$

Hence we must compute

$$d'd''\{\delta''v-(1/2)\Lambda L\delta''v\}$$

and

$$d'\{\delta''d''v-(1/3)\Lambda L\delta''d''v\}.$$

The computation of $d'd''\{\delta''v-(1/2)\Lambda L\delta''v\}$

$$d'd''\{\delta''v-(1/2)\Lambda L\delta''v\} = -d''d'\{\delta''v-(1/2)\Lambda L\delta''v\}$$

(because on $\mathbf{F}^{p,q}$)

$$= -d''\{d'\delta''v-(1/2)d'\Lambda L\delta''v\}$$
$$= d''\delta''(d'v)+(1/2)d''d'\Lambda L\delta''v$$

(because of $d'\delta''+\delta''d'=0$). We see $(1/2)d''d'\Lambda L\delta''v$. By

$$\delta''L - L\delta'' = -\sqrt{-1}\,d',$$

$$(1/2)d''d'\Lambda L\delta''v = (1/2)d''d'\Lambda(\delta''L+\sqrt{-1}\,d')v$$
$$= (1/2)d''d'\Lambda\sqrt{-1}\,d'v.$$

By

$$d'\Lambda-\Lambda d' = \sqrt{-1}\,\delta'',$$
$$= (1/2)d''(\sqrt{-1}\,\delta''+\Lambda d')\sqrt{-1}\,d'v$$
$$= -(1/2)d''\delta''d'v.$$

Hence

$$d'd''\{\delta''v-(1/2)\Lambda L\delta''v\} = (1/2)d''\delta''d'v$$
$$= (1/2)d''\{\delta''d'v-(1/3)\Lambda L\delta''d'v\}$$

(because $d''\Lambda L\delta''d'v=d''\Lambda\{\delta''L+\sqrt{-1}\,d'\}d'v=0$)

$$= (1/2)d''d''^{*}d'v.$$

The computation of $d'\{\delta''d''v-(1/3)\Lambda L\delta''d''v\}$

$$d'\{\delta''d''v-(1/3)\Lambda L\delta''d''v\} = d'\delta''d''v-(1/3)d'\Lambda L\delta''d''v$$
$$= -\delta''d'd''v-(1/3)d'\Lambda L\delta''d''v$$

(because of $d'\delta''+\delta''d'=0$)

$$= \delta''d''d'v-(1/3)d'\Lambda L\delta''d''v$$

(because on $\mathbf{F}^{p,q}$). We see $(1/3)d'\Lambda L\delta''d''v$. By

$$d'\Lambda-\Lambda d' = \sqrt{-1}\,\delta'',$$

$$(1/3)d'\Lambda L\delta''v = (1/3)\{\Lambda d'+\sqrt{-1}\,\delta''\}L\delta''d''v$$
$$= (1/3)\Lambda d'L\delta''d''v+(1/3)\sqrt{-1}\,\delta''L\delta''d''v.$$ By

$$\delta''L - L\delta'' = -\sqrt{-1}\,d',$$

$$\begin{aligned}(1/3)\sqrt{-1}\,\delta''L\delta''d''v &= (1/3)\sqrt{-1}\,\delta''\{\delta''L + \sqrt{-1}\,d'\}d''v \\ &= -(1/3)\delta''d'd''v \\ &= (1/3)\delta''d''d'v.\end{aligned}$$

Hence

$$\begin{aligned}(1/3)d'\Lambda L\delta''d''v &= (1/3)\Lambda d'L\delta''d''v + (1/3)\delta''d''d'v \\ &= (1/3)\delta''d''d'v.\end{aligned}$$

Because

$$\begin{aligned}\Lambda d'L\delta''d''v &= \Lambda d'\{\delta''L + \sqrt{-1}\,d'\}d''v \\ &= \Lambda d'\delta''Ld''v + \sqrt{-1}\,\Lambda d'd'd''v \\ &= 0.\end{aligned}$$

Namely,

$$\begin{aligned}d'\{\delta''d''v - (1/3)\Lambda L\delta''d''v\} &= (2/3)\delta''d''d'v \\ &= (2/3)d''^{*}d''d'v.\end{aligned}$$

Because

$$\begin{aligned}\Lambda L\delta''d''d'v &= \Lambda\{\delta''L + \sqrt{-1}\,d'\}d''d'v \\ &= 0.\end{aligned}$$

Therefore we have our equality. □

Next by using this equality, we have

THEOREM 6.1. *If for u in $\mathbf{F}^{n-2,2}$, $d'u = 0$ holds, then*

$$d'd''^{*}N_{d''}u = 0.$$

Proof. In fact, for u in $\mathbf{F}^{n-2,2}$,

$$d'\Box_{d''}N_{d''}u = (2/3)d''^{*}d''(d'N_{d''}u) + (1/2)d''d''^{*}(d'N_{d''}u).$$

Hence

$$d'u - d'H_{d''}u = (2/3)d''^*d''(d'N_{d''}u) + (1/2)d''d''^*(d'N_{d''}u).$$

Namely,

$$-d'H_{d''}u = (2/3)d''^*d''(d'N_{d''}u) + (1/2)d''d''^*(d'N_{d''}u).$$

We show $d'H_{d''}u = 0$, $d''d'N_{d''}u = 0$ and $d''^*d'N_{d''}u = 0$. For this, it is sufficient to see

$$(d''^*d''(d'N_{d''}u),\ d'\alpha) = 0 \quad \text{for any } \alpha \text{ in } H_{d''}, \tag{6.1}$$

and

$$(d''d''^*(d'N_{d''}u),\ d'\alpha) = 0 \quad \text{for any } \alpha \text{ in } H_{d''}. \tag{6.2}$$

First, we see (6.1). For any α in $H_{d''}$,

$$\begin{aligned}(d''^*d''(d'N_{d''}u),\ d'\alpha) &= (d''(d'N_{d''}u),\ d''d'\alpha)\\ &= (d''(d'N_{d''}u),\ -d'd''\alpha)\\ &= 0.\end{aligned}$$

Second, we see (6.2). For any α in $H_{d''}$,

$$(d''^*d''(d'N_{d''}u),\ d'\alpha) = (d''(d'N_{d''}u),\ d''^*d'\alpha).$$

While we have

LEMMA 6.2. *For α in $H_{d''}$ and for v in $\mathbf{F}^{n-1,1}$,*

$$(v,\ d''^*d'\alpha) = 0.$$

Proof.

$$\begin{aligned}(v,\ d''^*d'\alpha) &= (v,\ \delta''d'\alpha - (1/3)\Lambda L\delta''d'\alpha)\\ &= (v,\ \delta''d'\alpha)\\ &= -(v,\ d'\delta''\alpha).\end{aligned}$$

We show $(v, d'\delta''\alpha)=0$ for v in $\mathbf{F}^{n-1,1}$ and α in $H_{d''}$. As α is an element of $H_{d''}$,

$$d''^*\alpha = 0.$$

Namely,

$$\delta''\alpha - (1/2)\Lambda L\delta''\alpha = 0.$$

So

$$d'\delta''\alpha - (1/2)d'\Lambda L\delta''\alpha = 0.$$

Hence

$$(v,\ d'\delta''\alpha)-(1/2)(v,\ d'\Lambda L\delta''\alpha) = 0.$$

By

$$d'\Lambda - \Lambda d' = \sqrt{-1}\,\delta'',$$

$$(v,\ d'\delta''\alpha)-(1/2)(v,\ (\Lambda d' + \sqrt{-1}\,\delta'')L\delta''\alpha) = 0.$$

So

$$(v,\ d'\delta''\alpha)-(1/2)(v,\ \sqrt{-1}\,\delta'' L\delta''\alpha) = 0 \quad (\text{by } Lv = 0).$$

By

$$L\delta'' - \delta'' L = \sqrt{-1}\,d',$$

$$(v,\ d'\delta''\alpha)-(1/2)(v,\ \sqrt{-1}\,\delta''(\delta'' L + \sqrt{-1}\,d')\alpha) = 0.$$

Namely,

$$(v,\ d'\delta''\alpha)-(1/2)(v,\ \sqrt{-1}\,\delta''(\sqrt{-1}\,d'\alpha)) = 0.$$

Hence

$$(v,\ d'\delta''\alpha) + (1/2)(v,\ \delta'' d'\alpha) = 0.$$

So

$$(v,\ d'\delta''\alpha) - (1/2)(v,\ d'\delta''\alpha) = 0.$$

Hence

$$(v, d'\delta''\alpha) = 0.$$

Hence we have our lemma. □

Hence at the same time, we have

$$d'(H_{d''}u) = 0, \tag{6.3}$$

$$d''d'N_{d''}u = 0, \tag{6.4}$$

$$d''^{*}d'N_{d''}u = 0. \tag{6.5}$$

As $F^{n-1,1} = A^{n-1,1}(M)$, $d''^{*}d'N_{d''}u = \delta''d'N_{d''}u$. So (6.5) means (6.6) $\delta''d'N_{d''}u = 0$. With (6.6) in mind, we show

COROLLARY 6.3. *If u in $\mathbf{F}^{n-2,2}$ and $d'u = 0$, then*

$$d'd''^{*}N_{d''}u = 0.$$

Proof. We compute $d'd''^{*}N_{d''}u$. Namely,

$$\begin{aligned} d'd''^{*}N_{d''}u &= d'(\delta'' - (1/3)\Lambda L\delta'')N_{d''}u \\ &= d'\delta''N_{d''}u - (1/3)d'\Lambda L\delta''N_{d''}u \\ &= -\delta''d'N_{d''}u - (1/3)(\Lambda d' + \sqrt{-1}\,\delta'')L\delta''N_{d''}u \\ &= -(1/3)(\Lambda d'L\delta''N_{d''}u + \sqrt{-1}\,\delta''L\delta''N_{d''}u) \\ &= (1/3)\Lambda L\delta''d'N_{d''}u - \sqrt{-1}\,(1/3)\delta''L\delta''N_{d''}u \\ &= -\sqrt{-1}\,(1/3)\delta''(\delta''L + \sqrt{-1}\,d')N_{d''}u \\ &= (1/3)\delta''d'N_{d''}u \\ &= 0. \end{aligned}$$

□

Hence we have our corollary.

7. A subcomplex

Let Z^q be a subspace of $\mathbf{F}^{n-2,q}$ given by $Z^q = \{\alpha \in \mathbf{F}^{n-2,q} \mid d'\alpha = 0\}$. Then by (5.1) $d''Z^q \subset Z^{q+1}$. The following proposition indicates the possibility of considering deformations of CR structures relying on this subcomplex $(Z^{\cdot}, d'')$ of $(\mathbf{F}^{n-2,\cdot}, d'')$.

PROPOSITION 7.1. *If $\alpha \in Z^1$ then $i_2 P(i_1^{-1}\alpha) \in Z^2$.*

Proof. Recall that $P(\phi) = \bar{\partial}_b \phi + (1/2)[\phi, \phi]$ for $\phi \in \Gamma(M, {}^0\bar{T}'' \otimes ({}^0T'')^*)$. Then by Propositions 4.6 and 4.8, we have

$$i_2 P(i_1^{-1}\alpha) = d''\alpha + (1/2)d'(i_1^{-1}\alpha \lrcorner \alpha).$$

Hence Proposition 7.1 follows from Proposition 4.9 with

$$d'd'' + d''d' = 0 \quad \text{on } \mathbf{F}^{p,q}. \qquad \square$$

Let

$$J^{n-2,q} = ((\text{Ker } d'') \cap d'\mathbf{F}^{n-3,q})/(d''\mathbf{F}^{n-2,q-1} \cap d'\mathbf{F}^{n-3,q})(2 \leqslant q \leqslant n-1).$$

In Tian–Todorov's approach, $\partial\bar{\partial}$-lemma for a compact Kähler manifold plays an essential role. We call a $(\mathbf{F}^{p,q}, d', d'')$-version of $\partial\bar{\partial}$-lemma the $d'd''$-lemma. That is

$d'd''$-LEMMA. *If $\phi \in \mathbf{F}^{p,q}$ is d''-closed and d'exact, or d'-closed and d''-exact, then it is $d'd''$-exact.*

PROPOSITION 7.2. *If $d'd''$-lemma holds, then*

(1) *the natural homomorphism* $(\text{Ker } d'') \cap Z^1 \to H^1_{d''}(M, \wedge^{n-1}(T')^*)$ *is surjective,*
(2) $J^{n-2,q} = 0$ $(2 \leqslant q \leqslant n-1)$.

Proof. (1) Since $(\text{Ker } d'') \cap \mathbf{F}^{n-2,1} \to H^1_{d''}(M, \wedge^{n-1}(T')^*)$ is surjective (cf. Section 5), it is enough to show that $(\text{Ker } d'') \cap Z^1 = (\text{Ker } d'') \cap \mathbf{F}^{n-2,1}$. Let $\phi \in (\text{Ker } d'') \cap \mathbf{F}^{n-2,1}$. Then $d'\phi \in (\text{Ker } d'') \cap d'\mathbf{F}^{n-2,1}$. Since $d'd''$-lemma holds, $d'\phi \in d'd''\mathbf{F}^{n-2,0} = \{0\}$. (Note that $F^{p,0} = \{0\}$.) Therefore $\phi \in (\text{Ker } d'') \cap Z^1$.

(2) Is clear from the definition of $J^{n-2,q}$. $\square$

8. Main theorem

Our main theorem is as follows:

MAIN THEOREM. *Let $(M, {}^0T'')$ be a normal s.p.c. manifold with* $\dim_{\mathbb{R}} M = 2n-1 \geqslant 7$. *And we assume that its canonical line bundle $K_M = \Lambda^n(T')^*$ is trivial in CR-sense. Then the obstructions of deformations in $i_1^{-1}(Z^1)$ appear in $J^{n-2,2}$. That is, if $J^{n-2,2} = 0$, then any deformation of CR structures in $i_1^{-1}(Z^1)$ is unobstructed.*

Proof. Suppose that a Z^1-valued polynomial $\phi^{(k)}(t)$ in $t = (t_1, t_2, \ldots, t_r)$ satisfying $P(i_1^{-1}\phi^{(k)}(t)) \equiv 0 \bmod \mathfrak{m}^{k+1}$ is given, where $\mathfrak{m}$ denotes the maximal ideal of

$\mathbb{C}\{t_1, t_2, \ldots, t_r\}$. Then, by Proposition 4.8 and [A1] Theorem 4.10,

$$d'' i_2 P(i_1^{-1}\phi^{(k)}(t)) \equiv 0 \pmod{\mathfrak{m}^{k+2}}.$$

By Proposition 4.6,

$$i_2 P(i_1^{-1}\phi^{(k)}(t)) = d'(i_1^{-1}\phi^{(k)}(t) \lrcorner \phi^{(k)}(t)).$$

Hence the $(k+1)$th order homogeneous term of $i_2 P(i_1^{-1}\phi^{(k)}(t))$ is in $(\operatorname{Ker} d'') \cap d'\mathbf{F}^{n-3,2}$. Since $J^{n-2,2}=0$ by the assumption, we have

$$i_2 P(i_1^{-1}\phi^{(k)}(t)) \equiv d'' d''^* N_{d''} i_2 P(i_1^{-1}\phi^{(k)}(t)) \pmod{\mathfrak{m}^{k+2}}.$$

Therefore, if we set $\phi_{k+1}(t)$ to be the $(k+1)$th order homogeneous term of $-d''^* N_{d''} i_2 P(i_1^{-1}\phi^{(k)}(t))$ and set $\phi^{(k+1)}(t) = \phi^{(k)}(t) + \phi_{k+1}(t)$, then $\phi^{(k+1)}(t)$ is Z^1-valued by Theorem 6.1 and we have

$$P(i_1^{-1}\phi^{(k+1)})(t)) \equiv 0 \pmod{\mathfrak{m}^{k+2}}.$$

By the same argument as in [A3], we can prove the convergence of $\phi(t) = \lim_{k \to +\infty} \phi^{(k)}(t)$ with respect to the Folland–Stein norm $\| \ \|'_m$ (cf. [A2]), and we omit it. □

9. Smoothness of the versal family

If K_M is trivial in CR-sense and if $d'd''$-lemma holds in $(\mathbf{F}^{p,q}, d', d'')$, then from the Main Theorem together with Proposition 7.2, we have a holomorphic map

$$\psi = i_1^{-1} \circ \phi : \mathbb{C}^r \supset D \to (\Gamma(M, T' \otimes ({}^0T'')^*, \| \ \|'_k) \quad (r = \dim_{\mathbb{C}} H^1_{\bar\partial_b}(M, T')),$$

such that

$$P(\psi(t)) = 0 \quad \text{for all } t \in D,$$

and the infinitesimal deformation map

$$T_0 D \to H^1_{\bar\partial_b}(M, T')$$

is an isomorphism.

Also in this case, the argument of Section 3 in [Ak-Mi] works well and then the family $\psi(t)$ is versal in the sense of Kuranishi.

On the other hand, from the Main Theorem of [Mi2], we infer that all versal families of strongly pseudo convex CR structures (in the sense of Kuranishi) of $\dim_{\mathbb{R}} \geqslant 7$ are realized as families of real hypersurfaces of a canonical family of tubular neighbourhoods of M. In particular, their parameter spaces coincide with each other.

Hence we have

COROLLARY 9.1. *Suppose that* $\dim_{\mathbb{R}} M \geqslant 7$. *If* K_M *is trivial in CR-sense and if* $d'd''$*-lemma holds in* $(\mathbf{F}^{p,q}, d', d'')$, *then all versal families (in the sense of Kuranishi) of strongly pseudo-convex CR-structures are unobstructed.*

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