COMPOSITIO MATHEMATICA

FRÉDÉRIC CAMPANA
THOMAS PETERNELL

On the second exterior power of tangent bundles of threefolds

Compositio Mathematica, tome 83, nº 3 (1992), p. 329-346
<http://www.numdam.org/item?id=CM_1992__83_3_329_0>

Compositio Mathematica **83**: 329–346, 1992.

On the second exterior power of tangent bundles of threefolds

FRÉDÉRIC CAMPANA[1] and THOMAS PETERNELL
Mathematisches Institut der Universität Bayreuth, Postfach 101251, 8580 Bayreuth

Received 31 October 1990; accepted February 2, 1992

Introduction

Projective manifolds X whose anti-canonical class $-K_X$ is numerically effective ("nef") constitute an interesting class of manifolds with Kodaira Dimension $\kappa(X) \leqslant 0$. For instance, they include Fano manifolds. But it seems to be very hard to obtain a "kind of classification" of those manifolds, even in low dimensions. Therefore we considered in [CP] a special subclass; namely those X, whose tangent bundles T_X are nef[(1)]. Besides from some results in all dimensions, we gave a complete classification in dimension 3. In this paper we go a step further and consider as another special subclass 3-folds X such that $\Lambda^2 T_X$ is nef. The main result can now be stated as follows.

THEOREM. *Let X be a projective 3-fold with $\Lambda^2 T_X$* nef. *Then either T_X is* nef or *X is one of the following.*

(a) *X is the blow-up of $\mathbb{P}_3$ in one point*
(b) *X is a Fano 3-fold of index 2 and $b_2(X) = 1$ except for those of degree 1, which are exactly those arising as certain double covers of the Veronese cone in $\mathbb{P}_6$.*

Recall from [CP] the classification of 3-folds with T_X nef:

(1) $\mathbb{P}_3$, Q_3 (3-dimensional quadric), $\mathbb{P}_1 \times \mathbb{P}_2$, $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$, $\mathbb{P}(T_{\mathbb{P}_2})$ and up to finite étale cover:
(2) $X = \mathbb{P}(E)$, E a flat 3-bundle over an elliptic curve
(3) $X = \mathbb{P}(E)$, E a flat 2-bundle over an abelian surface
(4) $X = \mathbb{P}(E) \times_C \mathbb{P}(F)$, E, F flat 2-bundles over the elliptic curve C.
(5) $X =$ abelian 3-fold.

Then we get putting things together a complete list of 3-fold with $\Lambda^2 T_X$ nef.

It is quite remarkable that there are both Fano 3-folds X of index 2 with $\Lambda^2 T_X$ nef and $\Lambda^2 T_X$ not nef.

---

[1]Also at Université de Nancy I, Faculté des Sciences, Département de Mathématiques, B.P. 239, 54505 Vandoevre les Nancy Cedex, France.

[(1)]i.e. the tautological line bundle $\mathcal{O}(1)$ is nef on $\mathbb{P}(T_X)$.

In the last section we define a new invariant for Fano manifolds X:

$$\lambda(X) = \inf\{\lambda \in \mathbb{Q} \mid T_X \otimes \mathcal{O}_X(-\lambda K_X) \text{ is nef}\}$$

$\lambda(X)$ somehow measures how positive T_X is. For example, $\lambda(X) \leqslant 0$ is equivalent to T_X being nef.

If $\dim X = n$ and $\Lambda^{n-1} T_X$ is nef, then $\lambda(X) \leqslant n - 2$. We prove some results on $\lambda(X)$ for Fano 3-folds and it turns out that already here it is in general very difficult to determine $\lambda(X)$ exactly. We conjecture that $\lambda(X)$ is already determined by the splitting behavior of T_X on lines.

We would like to thank J. Wiśniewski for very interesting discussions.

0. Preliminaries

(0.1) DEFINITION. Let X be a projective complex space, $\mathscr{E}$ a locally free sheaf (= vector bundle) on X. $\mathscr{E}$ is called numerically effective ("nef" for short) iff the "tautological" line bundle $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$ on $\mathbb{P}(\mathscr{E})$ is nef, i.e.

$$c_1(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) \mid C) \geqslant 0$$

for any irreducible curve $C \subset X$.

Here we take $\mathbb{P}(\mathscr{E})$ always in Grothendieck-Hartshorne's sense (hyperplanes).

For basic properties of nef bundles we refer to [CP]. For our purposes we need some additional information.

(0.2) LEMMA. *Let X be a projective manifold, E a nef vector bundle on X. Let p be a positive polynomial for ample vector bundles (meaning that for any X and any ample E on X, $p(c_1, c_2, \ldots,) > 0$, see* [Fu, p. 216]). *Then*:

$$p(c_1, \ldots, c_d) \geqslant 0,$$

$c_i = c_i(E)$.

In particular:

$$c_i \geqslant 0,$$

$$c_1^2 - c_2 \geqslant 0,$$

$$c_1 c_2 - c_3 \geqslant 0,$$

$$c_1^3 - 2c_1 c_2 + c_3 \geqslant 0.$$

Proof. The inequality for nef bundles is easily deduced from that one for ample bundles by using the fact that E is nef iff $E \otimes \mathcal{O}(D)$ is an ample "$\mathbb{Q}$-vector bundle" for any ample $\mathbb{Q}$-divisor D on X (see [CP]).

(0.3) LEMMA. *Let C be a smooth projective curve, $\mathcal{E}$, $\mathcal{F}$ locally free sheaves on C of the same rank. Let $\alpha: \mathcal{E} \hookrightarrow \mathcal{F}$ be a monomorphism (of sheaves). If $\mathcal{E}$ is* nef, *then $\mathcal{F}$ is* nef, *too.*

Proof. There is an exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to Q \to 0$$

with supp(Q) finite.

By [CP, 1.2] it is sufficient to prove that $\mathcal{F} \otimes \mathcal{O}(D)$ is ample for every ample $\mathbb{Q}$-divisor D on C.

Fix D. After passing to a (ramified) covering of C, D will be Cartier. Now we have an exact (!) sequence

$$0 \to \mathcal{E} \otimes \mathcal{O}(D) \to \mathcal{F} \otimes \mathcal{O}(D) \to Q \otimes \mathcal{O}(D) \to 0.$$

Now $\mathcal{F} \otimes \mathcal{O}(D)$ is ample by standard arguments [Ha] (for the come back from the ramified covering see [CP, 1.2]).

Next we will state some results of Mori and Mukai's classification of Fano 3-folds X with $b_2(X) \geqslant 2$ [MM].

(0.4) DEFINITION. A Fano 3-fold X is called imprimitive iff it is the blow-up of another Fano 3-fold along a smooth curve. X is called primitive if it is not imprimitive.

(0.5) THEOREM [MM]. (1) *A primitive Fano 3-fold satisfies $b_2(X) \leqslant 3$.*
(2) *If $b_2(X) = 3$ then X is a conic bundle over $\mathbb{P}_1 \times \mathbb{P}_1$.*
(3) *If $b_2(X) = 2$ and if both of the two Mori contractions $\phi: X \to Y$ are $\mathbb{P}_1$-bundles, $\mathbb{P}_2$-bundles, quadric bundles or blow up's of smooth points, then X is one of the following:*

$$\mathbb{P}(T_{\mathbb{P}_2}),\ \mathbb{P}_1 \times \mathbb{P}_2,\ \mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(1)).$$

1. General results

We begin by studying surfaces whose anticanonical bundles are nef.

1.1 PROPOSITION. *Let X be a projective surface with $-K_X$ nef. Then X belongs exactly to the following list*

(a) *X is minimal with $\kappa(X) = 0$.*

(b) $X = \mathbb{P}(E)$, *E a rank 2-vector bundle over an elliptic curve C with*
(1) $E = \mathcal{O} \oplus L$, $L \in \mathrm{Pic}^0(C)$.
(2) *E is given by a non-split extension*

$$0 \to \mathcal{O} \to E \to L \to 0$$

with $L = \mathcal{O}$ *or* $\deg L = 1$ *(in particular* T_X *is nef, cp.* [CP]).

(c) *X is the blow-up of* $\mathbb{P}_2$ *in at most* 8 *points in almost general position in the sense of Demazure* [D] *or such a surface blown up in another (general) point such that* $|-K_X|$ *is base point free or consists of an irreducible curve.*

(d) $X = \mathbb{P}_1 \times \mathbb{P}_1$ *or* $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ *over* $\mathbb{P}_1$.

Proof. Since $-K_X$ nef, we clearly have $\kappa(X) \leqslant 0$. Moreover, if $\kappa(X) = 0$, then X is minimal, $K_X \equiv 0$. These are exactly the minimal surfaces with $\kappa = 0$,. So assume $\kappa(X) = -\infty$. Suppose X non-rational. Since $K_X^2 \geqslant 0$, X must be minimal (observe that $K^2 = 8(1-g)$ for a ruled surface over a curve of genus g) and $X = \mathbb{P}(E)$, E a rank 2-bundle over an elliptic curve C. Now $-K_X$ is nef iff $-K_X \mid C$ is nef which in turn is equivalent to T_X being nef (see [CP, proof of Theorem 3.1]). Hence by applying this theorem we obtain (b). Now let X be rational. If X is minimal, $X = \mathbb{P}_2, \mathbb{P}_1 \times \mathbb{P}_1$ or $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2))$. If X is the blow-up of $\mathbb{P}_2$ in k points then by $K_X^2 \geqslant 0$ we get $k \leqslant 9$. Conversely it is proved in [D] that any blow-up X of $\mathbb{P}_2$ in at most 8 points in almost general position has $-K_X$ nef. The rest also follows from [D].

Now we turn to the higher dimensional case.

1.2 PROPOSITION. *Let X be a projective manifold of dimension at least* 3 *with* $\Lambda^2 T_X$ *nef. Then* $\kappa(X) \leqslant 0$ *and* $\kappa(X) = 0$ *iff X is covered (étale) by an abelian variety (which is in turn equivalent to* T_X *being nef).*

Proof. Since $\det(\Lambda^2 T_X) \simeq -2K_X$, $-K_X$ is nef, too. Let $\kappa(X) \geqslant 0$. Then for some $m \in \mathbb{N}$, mK_X is effective. We conclude $mK_X \simeq \mathcal{O}_X$, so $\kappa(X) = 0$. In order to show that X is covered by an abelian variety, it is by Yau [Y] sufficient to prove $c_2(X) = 0$ in $H^4(X, \mathbb{R})$. Since $c_2(\Lambda^2 T_X) = c_1^2(T_X) + c_2(T_X)$, we have $c_2(T_X) = c_2(\Lambda^2 T_X) \geqslant 0$. On the other hand, $c_1^2(\Lambda^2 T_X) \geqslant c_2(\Lambda^2 T_X)$ by (0.2), hence we conclude $c_2(X) = 0$.

1.3 LEMMA. *Let X be a projective manifold with* $\Lambda^2 T_X$ *nef. Let* $C \subset X$ *be a rational curve. Then* $(-K_X \cdot C) \geqslant 2$.

Proof. If $T_X \mid C$ is nef, this is clear. If $T_X \mid C$ is not nef, let $f\colon \mathbb{P}_1 \to C$ be the normalization. Write

$$f^*(T_X \mid C) = \oplus \mathcal{O}(a_i)$$

with $a_1 \geqslant \cdots \geqslant a_n$, $a_n \leqslant -1$. Since $\Lambda^2 T_X$ is nef, we conclude $a_i + a_n \geqslant 0$. Since $a_1 \geqslant 2$, we are done.

We now start to investigate projective 3-folds X with $\Lambda^2 T_X$ nef. By (1.2) we may assume K_X not nef, moreover $\kappa(X) = -\infty$. Hence there exists an extremal ray on X, we let $\phi: X \to Y$ be associated contraction.

1.4 PROPOSITION. (1) *If ϕ is a modification, it is the blow-up of a smooth point on Y.*

(2) *If* $\dim Y = 2$, *ϕ is a $\mathbb{P}_1$-bundle over Y.*

(3) *If* $\dim Y = 1$, *ϕ is a $\mathbb{P}_2$-bundle or a quadric bundle over Y.*

Proof. Combination of (0.5) and (1.3) and Mori's classification of extremal contractions in dimension 3 [Mo]: compute $(-K_X \cdot l)$ for l extremal rational curve.

1.5 PROPOSITION. *Assume $K_X^3 \neq 0$. Then X is Fano.*

Proof. $-K_X$ being nef, one has $(-K_X)^3 \geqslant 0$. Hence by our assumption $-K_X$ is big and nef. By the Kawamata-Shokurov base point free theorem, $|-mK_X|$ is base point free for $m \gg 0$.

Let $\psi: X \to Z$ be the associated morphism to Z a normal projective variety. Assume X not to be Fano, then ψ has some positive-dimensional fibers. In such fibers we find rational curves C (cp. [P]), satisfying

$$(-K_X \cdot C) = 0.$$

Now we obtain a contradiction by (1.3).

REMARK. An analogous proof should generalize (1.5) to any dimension.

1.6 PROPOSITION. *Assume $K_X^3 = 0$. Then $\chi(X, \mathcal{O}_X) = 0$.*

Proof. We must show $c_1(X)c_2(X) = 0$ (Riemann-Roch). Using the formula

$$c_2(\Lambda^2 T_X) = c_1^2(X) + c_2(X)$$

we obtain: $c_1(X)c_2(X) = c_1(X)c_2(\Lambda^2 T) - c_1^3(X) = c_1(X)c_2(\Lambda^2 T) \geqslant 0$ by our nefness assumption.

On the other hand we use the inequality

$$c_1^2(\Lambda^2 T_X) \geqslant c_2(\Lambda^2 T_X) \tag{0.2}$$

to obtain (as in (1.2)):

$$c_2(X) \leqslant 3c_1^2(X),$$

hence

$$c_1(X)c_2(X) \leqslant 0.$$

1.7 COROLLARY. *Either X is Fano (hence $\chi(X, \mathcal{O}_X) = 1$) or $\chi(X, \mathcal{O}_X) = 0$.*
Proof. Combine (1.5) and (1.6).

We will now treat more or less separately the cases dim $Y = 0, 1, 2, 3$. In order to shorten the presentation, we have to treat in section 3 the case of Fano 3-folds also with $b_2(X) \geqslant 2$. So we proceed as follows:

Section 2, The Fano case
Section 3, Case dim $Y = 1$
Section 4, Case dim $Y = 2$
Section 5, Case dim $Y = 3$.

2. Fano 3-folds

(2.1) THEOREM. *Let X be a Fano 3-fold with $b_2(X) = 1$. Then $\Lambda^2 T_X$ is* nef *iff*

(1) *its index $r \geqslant 2$, and*
(2) *X is not a Fano 3-fold V_1 of degree 1, i.e. $H^3 = 1$ for the ample generator H of* Pic(X) *(these X can be realized as $2:1-$covers over the Veronese cone $W \subset \mathbb{P}_6$).*

Proof. Let l be a line on X ([Sh]). Then

$$(-K_X \cdot l) = r.$$

So (1.3) gives $r \geqslant 2$.

Now we have to prove that $\Lambda^2 T_X$ is nef for every Fano 3-fold X of index $\geqslant 2$ except for V_1. By Kobayashi-Ochiai [KO] $r \leqslant 4$ and $r = 4$ iff $X \simeq \mathbb{P}_3$, $r = 3$ iff $X \simeq Q_3$, the 3-dimensional smooth quadric. So for $r \geqslant 3$, $\Lambda^2 T_X$ is nef. Now let X be of index 2. According to Iskovskih's classification of Fano 3-folds with $b_2 = 1$ ([Is]) we treat 3 different cases:

(i) the ample generator L of Pic(X) (which satisfies $2L = -K_X$) is very ample
(ii) X is a double cover of $\mathbb{P}_3$ ramified along a smooth quartic Q
(iii) X is a double cover of the Veronese cone $W \subset \mathbb{P}_6$ ramified along a cubic hypersurface passing through the vertex v_0 of W, i.e. $X = V_1$.

We will treat these cases separately.

(2.2) PROPOSITION. *Let X be of type* (i). *Then $\Lambda^2 T_X$ is generated by global sections, hence nef.*

Proof. It is well known that $H^0(X, L)$ embeds $X \hookrightarrow \mathbb{P}_{d+1}$ of degree $d = L^3$ (apply Kodaira vanishing and Riemann-Roch). Now we have the obvious.

(2.3) LEMMA. *Let X be a submanifold of the complex manifold Y of codimension* *c. Let N be the normal bundle and $m \geqslant 0$ an integer. Then there exists an exact sequence*

$$\Lambda^{m+c} T_Y | X \xrightarrow{\alpha} \Lambda^m T_X \otimes \det N \to 0.$$

Moreover if $c = 1$, then $\operatorname{Ker} \alpha \simeq \Lambda^m T_X$.

Using (2.3) we finish the proof of (2.2) as follows with $Y = \mathbb{P}_{d+1}$. Then we have

$$\Lambda^d T_{\mathbb{P}_{d+1}} | X \to \Lambda^2 T_X \otimes \det N \to 0 \qquad (*)$$

By adjunction formula, $\det N \simeq dL$.
So (*) can be tensorized to read

$$\Lambda^d(T_{\mathbb{P}_{d+1}}(-1)) \to \Lambda^2 T_X \to 0.$$

Since $T_{\mathbb{P}_{d+1}}(-1)$ is generated by sections, so is $\Lambda^2 T_X$.

(2.4) PROPOSITION. *Let X be of type* (ii). *Then $\Lambda^2 T_X$ is nef.*

Proof. Let $p: L \to \mathbb{P}_3$ be the total space associated to $\mathcal{O}_{\mathbb{P}_3}(2)$. Let $s \in H^0(\mathcal{O}_{\mathbb{P}_3}(4))$ vanishing on Q. Let $\pi: L \to 2L$ be the natural map given by $u \to u \otimes u$. Then X is given by

$$X = \pi^{-1}(s(\mathbb{P}_3)).$$

Because of the exact sequence

$$0 \to \Lambda^3 T_X \to \Lambda^3 T_L | X \to \Lambda^2 T_X \otimes \underset{\substack{\wr\| \\ p^*(2L)}}{N^X{}_{|L}} \to 0$$

it is sufficient to show that $\Lambda^3 T_L|_X \otimes p^*(-2L)$ is nef. The sequence

$$0 \to p^*L \to T_L \to p^*(T_{\mathbb{P}_3}) \to 0$$

yields a sequence

$$0 \to p^*(\Lambda^2 T_{\mathbb{P}_3} \otimes L) \to \Lambda^3 T_L \to p^*(\Lambda^3 T_{\mathbb{P}_3}) \to 0.$$

Tensoring with $p^*(-2L)$ and using nefness of $(\Lambda^3 T_{\mathbb{P}_3})(-2L)$ and $(\Lambda^2 T_{\mathbb{P}_3})(-L)$, we conclude the proof of (2.4).

(2.5) PROPOSITION. *Let X be of type* (iii). *Then $\Lambda^2 T_X$ is not nef.*

Proof. Assume $\Lambda^2 T_X$ to be nef, then the Chern classes $\tilde{c}_i = c_i(\Lambda^2 T_X)$ satisfy the inequality

$$\tilde{c}_1^3 - 2\tilde{c}_1\tilde{c}_2 + \tilde{c}_3 \geqslant 0 \tag{0.2}$$

Let $c_i = c_i(X)$. Then: $\tilde{c}_1 = 2c_1$, $\tilde{c}_2 = c_1^2 + c_2$, $\tilde{c}_3 = c_1c_2 - c_3$. Now $c_1^3 = 8$, $c_3 = -38$ (cp. [ISh]), $c_1c_2 = 24\chi(X, \mathcal{O}_X) = 24$, hence $\tilde{c}_1^3 - 2\tilde{c}_1\tilde{c}_2 + \tilde{c}_3 = -2$, a contradiction.

The proof of (2.1) is now complete.

2.6 PROPOSITION. *Let X be a Fano 3-fold with $b_2(X) \geqslant 2$ and $\Lambda^2 T_X$ nef. Then X is one of the following*: $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$, $\mathbb{P}_1 \times \mathbb{P}_2$, $\mathbb{P}(T_{\mathbb{P}_2})$, $\mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(1))$, the latter being nothing else than the blow up of $\mathbb{P}_3$ in a point.

Proof. By (0.5) and (1.3) we have $b_2(X) = 2$ or 3. Going into classification of primitive Fano 3-folds with $b_2(X) = 2$ (0.5) *and using* (1.4) X must be of the last 3 types as stated in the proposition.

If $b_2(X) = 3$, then by (0.5) X is a conic bundle over $\mathbb{P}_1 \times \mathbb{P}_1$, hence by (1.4) even a $\mathbb{P}_1$-bundle over $\mathbb{P}_1 \times \mathbb{P}_1$. So $X = \mathbb{P}(E)$ with a 2-bundle E on $\mathbb{P}_1 \times \mathbb{P}_1$.

Now let $C = \mathbb{P}_1 \times \{q\}$ for some q and consider the ruled surface $\mathbb{P}(E \mid C) \simeq \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(a))$, $a \geqslant 0$. If $a \neq 0$, we find a rational curve $C_0 \subset \mathbb{P}(E \mid C)$ with $C_0^2 < 0$.

From the normal bundle sequence

$$\begin{array}{c} 0 \to N_{C_0|\mathbb{P}(E|C)} \to N_{C_0|X} \to N_{\mathbb{P}(E|C)|\mathbb{P}(E)} \to 0 \\ \qquad\qquad\qquad\qquad \wr\| \\ \qquad\qquad\qquad\qquad \mathcal{O} \end{array}$$

we obtain $c_1(N_{C_0|X}) < 0$, contradicting $(-K_X \cdot C_0) \geqslant 2$.

Hence $a = 0$ and $E \mid C \simeq \mathcal{O}(a) \oplus \mathcal{O}(\alpha)$ for some $\alpha \in \mathbb{Z}$. Of course α does not depend on q, so we may assume $E \mid \mathbb{P}_1 \times \{q\} \simeq \mathcal{O} \oplus \mathcal{O}$ for all q.

With analogous arguments we can simultaneously (!) achieve

$$E \mid \{p\} \times \mathbb{P}_1 \simeq \mathcal{O} \oplus \mathcal{O} \quad \text{for all } p,$$

hence $E \simeq \mathcal{O} \oplus \mathcal{O}$, so $X = \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$.

2.7 REMARK. One can easily avoid to use the Mori-Mukai classification in the proof of (2.6). Namely, we take our contraction $\phi: X \to Y$ and know by (1.3) that ϕ is a $\mathbb{P}_r$-bundle ($r = 1, 2$), a quadric bundle (over $\mathbb{P}_1$) or the blow-up of a smooth point. If $\dim Y = 1$ or 2 then $Y \simeq \mathbb{P}_1$ or Y is a rational surface. In this last case we have $X = \mathbb{P}(E)$ for some 2-bundle E on Y and proceed by restricting E to lines, (-1)-curves C etc. and then by considering curves in $\mathbb{P}(E \mid C)$ and computing normal bundles. Details and the case dim $Y = 1$ are left to the reader. If dim $Y = 3$ just proceed by induction: $\Lambda^2 T_Y$ is nef again, Y is Fano (see sect. 5).

3. Case: dim $Y = 1$

Here we treat the case where the Mori contraction ϕ goes to a (smooth) curve Y.

(3.1) PROPOSITION. *$\Lambda^2 T_X$ is nef iff $\phi: X \to Y$ is one of the following*

(a) $\phi: \mathbb{P}_1 \times \mathbb{P}_2 \to \mathbb{P}_1$ *the projection*
(b) *Y is an elliptic curve, and after possibly an unramified base change of degree* 2, $X \simeq \mathbb{P}(E) \to Y$ *with a flat rank* 3*-bundle E on Y*
(c) *Y is elliptic and – again after base change –* $X \simeq \mathbb{P}(E) \times_Y \mathbb{P}(F)$ *for flat rank* 2*-bundles E, F over Y.*

In particular: *$\Lambda^2 T_X$ nef iff T_X is nef* (see [CP]).

Proof. By [Mo] the structure of ϕ is as follows.

(α) ϕ is a $\mathbb{P}_2$-bundle
(β) ϕ is a quadric bundle: every fiber of ϕ is isomorphic to a normal quadric in $\mathbb{P}_3$
(γ) the general fiber F of ϕ is a del Pezzo surface with $1 \leqslant K_F^2 \leqslant 6$.

We will treat all these cases in the next lemmas.

(3.2) LEMMA. *ϕ cannot be a del Pezzo fibration, if $\Lambda^2 T_X$ is nef.*

Proof. Take a smooth fiber F of ϕ and let $C \subset F$ *be a* (-1)-curve. Then $(-K_X \cdot C) = 1$, contradicting (1.3).

(3.3) LEMMA. *Let ϕ be a $\mathbb{P}_2$-bundle. Then $\Lambda^2 T_X$ is nef iff X is of type* (a) or (b) in (3.1).

Proof. By (1.6): $Y \simeq \mathbb{P}_1$ iff $K_X^3 < 0$ and Y is elliptic iff $K_X^3 = 0$. Write $X = \mathbb{P}(E)$. If Y is elliptic,

$$-K_X \simeq \mathcal{O}_{\mathbb{P}(E)}(3) \otimes \phi^*(\det E^*).$$

Hence $E \otimes (\det E^*/3)$ is nef. Conversely if $E \otimes (\det E^*/3)$ is nef, $-K_X$ and also T_X are nef (cf [CP, 7.4]).

Hence by [CP 7.2, 7.3], X is of the form (a) or (b) in (3.1). If Y is rational, X is Fano, hence $X \simeq \mathbb{P}_1 \times \mathbb{P}_2$ by (2.6).

(3.4) LEMMA. *Let ϕ be a quadric bundle. Then $\Lambda^2 T_X$ is nef iff every fiber of ϕ is smooth and X is of type* (c) in (3.1), *in particular*, Y is elliptic.

Proof. Assume $\Lambda^2 T_X$ nef, so does $-K_X$. If $K_X^3 < 0$, X is Fano (1.5), hence by (2.6) $X \simeq \mathbb{P}_1 \times \mathbb{P}_2$, $\mathbb{P}(T_{\mathbb{P}_2})$, $\mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(1))$, none of them having a contraction of quadric bundle type. Hence $K_X^3 = 0$ and Y is elliptic. $\Lambda^2 T_X$ being nef, the inequality

$$c_1(\Lambda^2 T_X)c_2(\Lambda^2 T_X) - c_3(\Lambda^2 T_X) \geqslant 0$$

holds (0.2).

Putting in $c_1(\Lambda^2 T_X) = 2c_1(X)$, $c_2(\Lambda^2 T_X) = c_1^2(X) + c_2(X)$, $c_3(\Lambda^2 T_X) = (c_1 c_2 - c_3)(X)$ gives:

$$2c_1^3 + c_1 c_2 + c_3 \geqslant 0$$

for $c_i = c_i(X)$.

Now $c_1^3 = 0$, $c_1 c_2 = 24\chi(\mathcal{O}_X) = 24\chi(\mathcal{O}_Y) = 0$, hence $c_3(X) \geqslant 0$. On the other hand, take a non-zero holomorphic 1-form ω on Y and let $\eta = \phi^*(\omega) \in H^0(\Omega_X^1)$. Then η vanishes exactly at the singular points of the singular fibers, which is a finite set. Hence $c_3(\Omega_X^1) \geqslant 0$, so $c_3(X) \leqslant 0$, and:

$$c_3(X) = 0 \text{ iff } \phi \text{ is a } \mathbb{P}_1 \times \mathbb{P}_1\text{-bundle.}$$

Since $c_3(X) \geqslant 0$, too, ϕ is a $\mathbb{P}_1 \times \mathbb{P}_1$-bundle. After pässing to an étale 2-sheeted cover of C, X is of the form $\mathbb{P}(E) \times_C \mathbb{P}(F)$ with 2-bundles E, F on C. Since $-K_X$ is nef, $-K_{\mathbb{P}(E)}$, $-K_{\mathbb{P}(F)}$ are nef. Hence X is of type (c) in (3.1) (see (1.1)).

With some more work we can even show:

(3.5) PROPOSITION. *Let $\phi: X \to Y$ be a quadric bundle over an elliptic curve Y. Assume $c_1^3 = c_1(X)^3 = 0$. Then ϕ is a $\mathbb{P}_1 \times \mathbb{P}_1$-bundle.*

Proof. Let $X \hookrightarrow Z$ be embedded as a quadric bundle in the $\mathbb{P}_3$-bundle $\phi: Z \to C$, where $\pi: Z = \mathbb{P}(E) \to X$ is a rank 4 bundle over C.

We can assume (computing in the cohomology algebra of Z over $\mathbb{Q}$) that $c_1(X) = \mathcal{O}_Z(2) = \mathcal{O}_{\mathbb{P}(E)}(2)$.

We now compute the Chern classes of X, using the following two exact sequences of vector bundles:

$$0 \to T_X \to T_Z \to N_{X|Z} \to 0$$

$$0 \to \mathcal{O}_Z \to \phi^*(E^*) \otimes \mathcal{O}_Z(1) \to T_Z \to \pi^* T_C \simeq \mathcal{O}_Z \to 0$$

from which we get:

(1) $(1 + t_1 + t_2 + t_3 + t_4) = (1 + c_1' + c_2' + c_3' + c_4')$, where $t_i := c_i(Z)$ and $c_i' := c_i(\phi^* E^* \otimes \mathcal{O}_Z(1))$ and

(2) $(1 + \bar{t}_1 + \bar{t}_2 + \bar{t}_3 + \bar{t}_4) = (1 + c_1 + c_2 + c_3) \cdot (1 + 2\gamma)$ where:

$$\gamma = c_1(\mathcal{O}_Z(1)), \bar{t}_i := t_{i|X};\ c_i := c_i(X).$$

An easy computation now shows that:

$$t_1 = 4\gamma - \tilde{c},\ t_2 = 6\gamma^2 - 3\gamma\cdot\tilde{c}$$

and

$$t_3 = 4\gamma^3 - 3\gamma^2\cdot\tilde{c},$$

if $\tilde{c} := \phi^*(c_1(E))$.

Moreover, from (3.8.1) and (3.8.2) we get:

$$c_1 = (2\gamma - \tilde{c})_{|X}$$

$$c_2 = (2\gamma - \tilde{c})\cdot\gamma_{|X}$$

$$c_3 = (-\gamma^2\cdot\tilde{c})_{|X}$$

Now the relations $c_1c_2 = 0$ and $c_1^3 = 0$ give:

$$0 = c_1\cdot c_2 = 2(2\gamma - \tilde{c})^2\cdot\gamma^2 = 8(\gamma^4 - \gamma^3\cdot\tilde{c})$$

since $\tilde{c}^2 = 0$

$$0 = c_1^3 = 2\cdot(2\gamma - \tilde{c})^3\cdot\gamma = 2(8\gamma^4 - 6\gamma^3\cdot c).$$

Thus $\gamma^4 = \gamma^3\cdot\tilde{c} = 0$. Hence $c_3 = -\gamma^3\cdot\tilde{c} = 0$ and as in the proof of (3.4) X is a $\mathbb{P}_1\times\mathbb{P}_1$-bundle.

(3.6) COROLLARY. Let $\phi: X\to Y$ be a quadric bundle with $-K_X$ nef. Then Y is elliptic and ϕ *a* $\mathbb{P}_1\times\mathbb{P}_1$-bundle or $Y\simeq\mathbb{P}_1$ and X is Fano (described in [MM]).

4. Case: dim $Y = 2$

We next look at 3-folds X with dim $Y = 2$ and $\Lambda^2 T_X$ is nef where $\phi: X\to Y$ is our fixed contraction.

(4.1) PROPOSITION. Assume dim $Y = 2$ for the 3-fold X with $\Lambda^2 T_X$ nef. Then X belongs to the following list.

(1) $X = \mathbb{P}_1\times\mathbb{P}_2$, $\mathbb{P}(T_{\mathbb{P}_2})$, $\mathbb{P}(\mathcal{O}_{\mathbb{P}_2}\oplus\mathcal{O}_{\mathbb{P}_2}(1))$ $(Y\simeq\mathbb{P}_2)$
$X = \mathbb{P}_1\times\mathbb{P}_1\times\mathbb{P}_1 (Y\simeq\mathbb{P}_1\times\mathbb{P}_1)$

(2) Y is a ruled surface over an elliptic curve C with T_Y nef (equivalently $-K_Y$

nef, cp. (1.1)) and up to finite étale cover X is of the form $X = Y \times_C Y'$, where $Y' \to C$ is another ruled surface with the same properties as Y. In particular $\Lambda^2 T_X$ is nef iff T_X is nef.

(3) Y is abelian or hyperelliptic and up to finite étale cover X is of the form $\mathbb{P}(E) \to Y$ with E a flat bundle of rank 2 (in particular T_X is nef).

Proof. If X is Fano, we apply (2.6) and obtain (1). So assume now X not to be Fano. Then by (1.7):

$$\chi(X, \mathcal{O}_X) = 0,$$

hence (1) $\chi(Y, \mathcal{O}_Y) = 0$.

Because of the vector bundle epimorphism

$$\Lambda^2 T_X \to \phi^*(\Lambda^2 T_Y),$$

$-K_Y$ is nef. By applying (1), (1.1) and the Enriques-Kodaira classification of surfaces, we deduce that Y is abelian, hyperelliptic (i.e. an elliptic fiber bundle over an elliptic curve, see e.g. [BPV]) or a ruled surface over an elliptic curve as described in (1.1.b).

If Y is abelian or hyperelliptic, we may assume by performing a finite étale cover that Y is abelian and moreover that $X = \mathbb{P}(E)$ with a 2-bundle E on Y (see [CP, 7.4]), even with $\det E \simeq \mathcal{O}_Y$.

By the exact sequence

$$0 \to T_{X|Y} \otimes \phi^*(T_Y) \to \Lambda^2 T_X \to \phi^*(\Lambda^2 T_Y) \to 0$$

and triviality of T_Y, we see that $\Lambda^2 T_X$ is nef iff $T_{X|Y}$ is nef ([CP, 1.2]) which in turn is equivalent to T_X being nef. Now apply [CP, 10.1] to obtain (3).

It remains to treat the case of Y being ruled over an elliptic curve. In this case even T_Y is nef (see (1.1)). Now apply (9.3), (9.4) of [CP] to see that up to finite étale cover X is of the form

$$X = Y \times_C Y'$$

with $Y' \to C$ being ruled with $T_{Y'}$ also nef.

Observe that we may apply (9.3): the proof works still if $\Lambda^2 T_X$ is nef, since a smooth rational curve $C \subset X$ cannot have $c_1(N_{C|X}) < 0$.

Now let $p: Y \to C$, $p': Y' \to C$ be the projections, $\phi': X \to Y'$ the projection. Then

$$T_{X|C} \simeq \phi^*(T_{Y|C}) \oplus \phi^*(T_{Y'|C}),$$

moreover $T_{Y|C}$ is nef by the exact sequence

$$0 \to T_{Y|C} \to T_Y \to p^*(T_C) \to 0$$
$$p^*(T_C) \cong \mathcal{O}_C$$

analogously $T_{Y'|C}$, thus $T_{X|Y}$ is nef.

By using

$$0 \to T_{X|C} \to T_X \to (p \circ \phi)^*(T_C) \to 0,$$

T_X is nef.

Conversely any $X = Y \times_C Y'$ with T_Y, $T_{Y'}$, nef, has T_X nef, in particular $\Lambda^2 T_X$ nef.

4.2 REMARK. Observe that in the list of (4.1) there is only one 3-fold X whose tangent bundle is not nef, namely $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(1))$, the blow-up of $\mathbb{P}_3$ in one point.

5. Case: dim $Y = 3$

We are now dealing with the case of a modification ϕ. We already know by (1.4) that ϕ is nothing than the blow-up of a smooth point of Y; in particular Y is smooth again.

5.1 PROPOSITION. $\Lambda^2 T_Y$ is nef.

Proof. There is an exact sequence of sheaves (see [Fu, p. 299])

$$0 \to T_X \to \phi^*(T_Y) \to T_{\mathbb{P}_2}(-1) \to 0,$$

identifying the exceptional divisor E with $\mathbb{P}_2$.

Let $C \subset Y$ be an irreducible curve, $\hat{C}$ its strict transform in X. Then we have – maybe after normalization – the exact sequence

$$\Lambda^2 T_X \mid \hat{C} \xrightarrow{\alpha} \phi^*(\Lambda^2 T_Y \mid C) \to Q \to 0$$

where Q is a sheaf supported on $\hat{C} \cap E$.

The map α is generically injective, hence injective. Thus $\phi^*(\Lambda^2 T_Y \mid C)$ and hence $\Lambda^2 T_Y \mid C$ is nef (0.3).

5.2 PROPOSITION. *Y is Fano.*

Proof. $-K_X$ being nef, we have $(-K_X)^3 \geqslant 0$. From

$$K_X = \phi^*(K_Y) + 2E$$

we obtain

$$K_X^3 = K_Y^3 + (2E)^3,$$

so

$$K_X = K_Y^3 + 8,$$

hence $(-K_Y)^3 > 0$. By (1.5) and (4.1) Y is Fano.

5.3 PROPOSITION. $Y \simeq \mathbb{P}_3$.

Proof. Let $p \in Y$ the point blown up.

(a) If Y is a 3-dimensional quadric Q_3, choose a line $l \subset Q_3$ through p. Let $\hat{l}$ be the strict transform of l in X.

Since $(-K_{Q_3} \cdot l) = 3$, we obtain:

$$(-K_X \cdot \hat{l}) = 1,$$

a contradiction.

If Y has $b_2(Y) = 1$ and index 2, we proceed in the same way (using the existence of lines, moreover there is a line through every point by Iskovskih [Is]).

Since (by nefness of $\Lambda^2 T_Y$) Y cannot have index 1, it is now sufficient to show

(b) Y cannot have $b_2 \geqslant 2$.

Now Y is a $\mathbb{P}_1$-bundle over $\mathbb{P}_2$ or $\mathbb{P}_1 \times \mathbb{P}_1$. Now let l be the fiber of the projection passing through p and $\hat{l}$ as in (a). Then we easily get

$$(-K_X \cdot \hat{l}) = 0,$$

which is impossible.

We already observed that in case X being $\mathbb{P}_3$ blown up in one point, $\Lambda^2 T_X$ is indeed nef.

6. Conclusion and Problems

Taking into account all our previous results we obtain

(6.1) THEOREM. *Let X be a projective smooth 3-fold. Then* $\Lambda^2 T_X$ is nef iff X

belong to the following list.

(1)(a) $X = \mathbb{P}_3$
(b) $X = Q_3$
(c) X a Fano 3-fold of index 2 with $b_2 = 1$ but not a double cover of the Veronese cone
(d) $X = \mathbb{P}_1 \times \mathbb{P}_2$ or $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$
(e) $X = \mathbb{P}(T_{\mathbb{P}_2})$
(f) $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(1))$

(2) up to finite étale cover $X = \mathbb{P}(E)$, E a flat rank 3-bundle over an elliptic curve
(3) up to finite étale cover $X = \mathbb{P}(E)$, E a flat 2-bundle over an abelian surface
(4) up to finite étale cover $X = \mathbb{P}(E) \times_C \mathbb{P}(F)$, E, F flat 2-bundles over the elliptic curve C
(5) some finite étale cover of X is an abelian 3-fold.

Comparing (6.1) with the classification of 3-folds with T_X nef gives

(6.2) COROLLARY. *Let X be a projective 3-fold with $\Lambda^2 T_X$ nef and T_X not nef. Then X is just one of the following.*

(a) $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}_2} \oplus \mathcal{O}_{\mathbb{P}_2}(1))$ (= blow-up of $\mathbb{P}_3$ in one point)
(b) X is Fano of index 2 with $b_2 = 1$ but not a double cover of the Veronese cone.

We conclude by proposing two problems in the higher-dimensional case.

(6.3) PROBLEM. Let X be a Fano manifold, say $b_2(X) = 1$. Assume $\Lambda^q T_X$ to be nef for some $q < \dim X$. Then find some estimate for the index r of X.

We know that $r \geqslant 2$ if $\dim X = 3$. But this is proved using the existence of lines, hence there is no apparent generalization to $\dim X > 3$.

(6.4) PROBLEM. In general it is very difficult to decide whether a vector bundle E is nef or not. In case $E = T_X$ or $\Lambda^q T_X$ and X is Fano we would ask the following.

Assume that $\Lambda^q T_X$ is nef on every (or some) extremal rational curve. Is then $\Lambda^q T_X$ already nef?

For 3-folds this is true.

What about twists $T_X \otimes \mathcal{O}(-\lambda K_X)$? Compare (7.5).

7. An invariant for Fano manifolds

A related problem to numerical effectiveness of T_X or $\Lambda^q T_X$ is to find λ as small as possible with $T_X \otimes \mathcal{O}(-\lambda K_X)$ nef. Observe that e.g. for 3-folds

$$\Lambda^2 \Lambda^2 T_X \simeq T_X \otimes \mathcal{O}_X(-K_X),$$

so if say $\Lambda^2 T_X$ is nef, then $T_X \otimes \mathcal{O}_X(-K_X)$ is nef.

(7.1) DEFINITION. Let X be a Fano manifold. Then we define

$$\lambda(X) = \inf\{\mu \in \mathbb{Q} \mid T_X \otimes \mathcal{O}(-\mu K_X) \text{ is nef}\}.$$

Remark that $T_X \otimes \mathcal{O}(-\mu K_X)$ is a $\mathbb{Q}$-vector bundle ($\mu \in \mathbb{Q}$) and it is nef by definition iff the $\mathbb{Q}$-Cartier divisor

$$\mathcal{O}_{\mathbb{P}(T_X)}(1) \otimes \pi^*(\mathcal{O}(-\mu K_X))$$

is nef.

The following is rather obvious.

(7.2) PROPOSITION

(1) $\lambda(\mathbb{P}_n) = -\left(\frac{1}{n+1}\right)$; $\lambda(X) < 0$ implies $X \simeq \mathbb{P}_n$
(2) $\lambda(Q_n) = 0$
(3) $\lambda(X) \leqslant 0$ iff T_X nef (X Fano)
(4) If X is Fano of dimension n and $\Lambda^{n-1} T_X$ is nef, then $\lambda(X) \leqslant n-2$.

Proof. Only the second part of (1) is non-trivial. So let X be Fano with $\lambda(X) < 0$. Then it follows immediately that T_X is ample.

Hence by Mori's theorem, $X \simeq \mathbb{P}_n$.

(4) follows from $\Lambda^{n-1}\Lambda^{n-1} T_X \simeq T_X \otimes \mathcal{O}(-(n-2)K_X)$.

We next want to compute $\lambda(X)$ for Fano 3-folds with $b_2(X) = 1$.

(7.3) PROPOSITION

(1) *If X is a Fano 3-fold of index 1, $b_2(X) = 1$, then $\lambda(X) \geqslant 1$*
(2) *If X is a Fano 3-fold of index 2, $b_2(X) = 1$, and not of type V_1 or V_2 then $\lambda(X) = \frac{1}{2}$.*

Proof. For (1), observe that a line $l \subset X$ has normal bundle

$$N_{l|X} = \mathcal{O}(a) \oplus \mathcal{O}(-1-a),$$

with some $a \geqslant 0$, the existence of such a line guaranteed by [Sh]. Since $(-K_X \cdot l) = 1$, $T_X \otimes \mathcal{O}((-1+\varepsilon)K_X)$ cannot be nef for positive rational ε. Hence $\lambda(X) \geqslant 1$.

(2) If X has index 2 and is not V_1 or V_2 then by the arguments of [CP, p. 180], we find lines $l \subset X$ with

$$N_{l|X} \simeq \mathcal{O}(a) \oplus \mathcal{O}(-a) \text{ for some } a > 0.$$

Hence $\lambda(X) \geqslant \frac{1}{2}$. The nefness of $T_X \otimes \mathcal{O}_X(-K_X/2)$ follows from (2.3) applied with $m = 1$ $\left(\text{to the embedding } X \subset \mathbb{P}_{d+1} \text{ given by } \left|-\frac{K_X}{2}\right|\right)$.

Recall that X is of type (V_1) iff $c_1(\mathcal{O}_X(1))^3 = 1$ for the ample generator $\mathcal{O}_X(1)$ of $\mathrm{Pic}(X) \simeq \mathbb{Z}$; in this case $|-K_X|$ realizes X as a double cover over the Veronese cone $W \subset \mathbb{P}_6$.

X is of type (V_2) iff $c_1(\mathcal{O}_X(1))^3 = 2$, here $-K_X|$ describes X as double cover over $\mathbb{P}_3$ ramified along a smooth quartic Q in $\mathbb{P}_3$.

Sometimes a Fano 3-fold of index 1 (of type "A_{22}") has a line with $N_l = \mathcal{O}(1) \oplus \mathcal{O}(-2)$. In these cases $\lambda(X) \geqslant 2$.

(7.4) PROPOSITION. *Suppose X is of type (V_2).*

(1) $\lambda(X) \leqslant 1$.
(2) *If the ramification locus $Q \subset \mathbb{P}_3$ contains a line l, then $\lambda(X) = 1$.*

Proof. (1) is just the nefness of $\Lambda^2(\Lambda^2 T_X)$.

(2) Since $f|f^{-1}(Q)$ is biholomorphic to Q, we consider Q also as subvariety of X.

By observing that the image of $T_X|Q \to p^*T_{\mathbb{P}_3}|Q$ is just p^*T_Q one sees easily

$$T_X|l = \mathcal{O}(2) \otimes \mathcal{O}(2) \otimes \mathcal{O}(-2).$$

Hence $\lambda(X) \geqslant 1$.

(7.5) CONJECTURE. Assume in (7.4) that Q does not contain a line. Then we conjecture that $\lambda(X) = \frac{1}{2}$. We have the following evidence for this conjecture.

For any line $l \subset X$:

$$T_X|l \simeq \begin{cases} \mathcal{O}(2) \oplus \mathcal{O} \oplus \mathcal{O} \\ \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O}(-1) \end{cases}.$$

Hence $T_X \otimes \mathcal{O}(-\frac{1}{2}K_X)|l$ is always nef. This should imply nefness of $T_X \otimes \mathcal{O}(-\frac{1}{2}K_X)$; cp. (6.4).

(7.6) PROPOSITION. *Assume X to be of type (V_1). Then $\lambda(X) > \frac{61}{63}$.*

Proof. Let $T_X \otimes \mathcal{O}(-\lambda K_X)$ be nef.

Then:

$$0 \leqslant c_3(T_X \otimes \mathcal{O}(-\lambda K_X)) = c_3 + \lambda c_1 c_2 + (\lambda^2 + \lambda^3)c_1^3$$
$$= -38 + 24\lambda + (\lambda^2 + \lambda^3)\cdot 8 \ (c_i = c_i(X)).$$

The last term is negative for $\lambda = \frac{61}{63}$, hence our claim.

(7.7) PROBLEM. Is $\lambda(X) = 1$ for X of type (V_1)?

We would like to finish with the following

(7.8) PROBLEM. Let X be a Fano manifold. Is $\lambda(X)$ a rational number?

References

[BPV] Barth, W., Peters, C., and van de Ven, A., *Compact complex surfaces*. Erd. d. Math. Bd. 3, Springer 1984.

[CP] Campana, F., and Peternell, Th., Manifolds whose tangent bundles are numerically effective. *Math. Ann.* 289, 169–187 (1991).

[Fu] Fulton, W., *Intersection Theory*. Erg. d. Math. 2. Springer 1984.

[Ha] Hartshorne, R., Ample vector bundles. *Publ. Math.* IHES 29, 319–350 (1966).

[Is] Iskovskih, V. A., *Fano 3-folds* I, II. Math. USSR Isv. 11, 485–527 (1977), ibid. 12, 469–506 (1978).

[ISh] Iskovskih, V. A. and Shokurov, V. V., Biregular theory of Fano 3-folds. *Lecture Notes in Math.* 732, 171–182 Springer 1978.

[Ko] Kobayashi, Ochiai, T., Characterizations of complex projective spaces and hyperquadrics. *J. Math.* Kyoto Univ. 13, 31–47 (1973).

[MM] Mori, S. and Mukai, S., On Fano 3-folds with $b_2 \geqslant 2$. *Adv. Studies in Math.* 1, 101–129 (1983).

[Mo] Mori, S., Threefolds whose canonical bundles are not numerically effective. *Ann. Math.* 116, 133–176 (1982).

[P] Peternell, Th., Calabi-Yau manifolds and a conjecture of Kobayashi. *Math. Z.* 207, 305–318 (1991).

[Sh] Shokurov, V. V., The existence of a straight line on Fano 3-folds. *Math. USSR Isv.* 15, 173–209 (1980).

[D] Demazure, M., Surface de del Pezzo I–V, *Lecture Notes in Math.* 777, 23–70. Springer 1980.

[Y] Yau, S. T., Calabi's conjecture and some new results in algebraic geometry. *Proc. Math. Acad. Sci.* USA 74, 1789 (1977).

Note added in proof

New results on manifolds whose tangent bundles are nef appear in the preprints:

Demailly, J. P., Peternell, Th., Schneider, N., Compact complex manifolds with numerically effective tangent bundles.

and:

Compact Kahler manifolds with semi-positive Ricci curvature.