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Visibility cells and T-convexity spaces

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Abstract. The aim of this paper is to develop the basic tools to study the subjects of starshapedness and visibility in the axiomatic setting of convexity spaces. The analysis of geometrical notions such as convex components, visibility cells and kernel is accomplished in this context. The main result is a characterization of T-convexity spaces by means of the connections between visibility cells and convex components.

1. Introduction and basic definitions

A pair $(X, \mathscr{C})$ is a *convexity space* iff X is a nonempty set and $\mathscr{C}$ is a family of subsets of X such that $\emptyset$ and X belong to $\mathscr{C}$ and $\bigcap \mathscr{F} \in \mathscr{C}$ for each $\mathscr{F} \subset \mathscr{C}$. A set $S \subset X$ is called *$\mathscr{C}$-convex* iff $S \in \mathscr{C}$. The *$\mathscr{C}$-convex hull* of a subset S of X is defined in the usual way

$$\mathscr{C}(S) = \bigcap \{C \in \mathscr{C} \mid S \subset C\}.$$

If $\{x_1, x_2, \ldots, x_n\} \subset X$, $\mathscr{C}(\{x_1, x_2, \ldots, x_n\})$ is denoted simply $\mathscr{C}(x_1, x_2, \ldots, x_n)$. Also, if $p \in X$ and $S \subset X$, $\mathscr{C}(\{p\} \cup S)$ is written $\mathscr{C}(p \cup S)$. Let $S \subset X$ and $p \in S$. The *$\mathscr{C}$-star of x in S* is the set

$$\mathscr{C}\,\mathrm{st}(x, S) = \{y \in S \mid \mathscr{C}(x, y) \subset S\}.$$

The set S is said to be *$\mathscr{C}$-starshaped relative to* $x \in S$ if and only if $\mathscr{C}\,\mathrm{st}(x, S) = S$. The *$\mathscr{C}$-kernel of* S is the set

$$\mathscr{C}\ker(S) = \{x \in S \mid \mathscr{C}\,\mathrm{st}(x, S) = S\}.$$

S is *$\mathscr{C}$-starshaped* iff it is $\mathscr{C}$-starshaped relative to some of its points, that is iff $\mathscr{C}\ker(S) \neq \emptyset$.

We recall the following definitions from [4], [5] and [6]. A convexity space $(X, \mathscr{C})$ is said to be *domain finite* (DF) iff for each $S \subset X$ holds

$$\mathscr{C}(S) = \bigcup \{\mathscr{C}(F) \mid F \subset S \text{ and } \mathrm{card}(F) < \infty\}.$$

$(X, \mathscr{C})$ is said to be *join-hull commutative* (JHC) iff for each $p \in X$ and each nonempty subset S of X, $\mathscr{C}(p \cup S) = \bigcup\{\mathscr{C}(p, s) | s \in \mathscr{C}(S)\}$. $(X, \mathscr{C})$ is said to be a *JD-convexity space* iff it is a JHC-convexity space and a DF-convexity space. $(X, \mathscr{C})$ is said to be $\mathbb{T}_1$ iff $\forall x \in X$ $\{x\} \in \mathscr{C}$. $(X, \mathscr{C})$ is said to be a *B-convexity space* iff $\forall S \subset X$, $\mathscr{C}\ \mathrm{ker}(S) \in \mathscr{C}$. $(X, \mathscr{C})$ is said to be a *T-convexity space* iff $\forall S \subset X$,

$$\mathscr{C}\ \mathrm{ker}(S) = \bigcap \mathscr{M}(S),$$

where $\mathscr{M}(S)$ is the family of all the maximal $\mathscr{C}$-convex subsets of S, also called $\mathscr{C}$-convex components of S. B-convexity spaces and T-convexity spaces were studied by Kołodziejczyk [6].

In this paper we define the concept of visibility cell in a convexity space, and we study its connections with other ideas related with visibility and starshapedness.

2. Visibility cells

Let $(X, \mathscr{C})$ be a convexity space. For each nonempty subset $S \subset X$ and each point $x \in S$ the *$\mathscr{C}$-visibility cell of x relative to S* is

$$\mathscr{C}\ \mathrm{vis}(x, S) = \{y \in \mathscr{C}\ \mathrm{st}(x, S) \,|\, \mathscr{C}\ \mathrm{st}(x, S) \subset \mathscr{C}\ \mathrm{st}(y, S)\}.$$

The notion of visibility cell was introduced by Toranzos [8] for ordinary convexity in a real linear space. We intend to show that some of its properties can be generalized to convexity spaces. In the next result we collect some basic properties of visibility cells[1].

THEOREM 2.1. *Let $(X, \mathscr{C})$ be a convexity space, S a nonempty subset of X and $x \in S$. Then:*

(i) *The following four statements are equivalent:*
   1. $\mathscr{C}\ \mathrm{vis}(x, S) \neq \emptyset$.
   2. $\mathscr{C}\ \mathrm{st}(x, S) \neq \emptyset$.
   3. $x \in \mathscr{C}\ \mathrm{st}(x, S)$.
   4. $x \in \mathscr{C}\ \mathrm{vis}(x, S)$.

(ii) $\mathscr{C}\ \mathrm{ker}(\mathscr{C}\ \mathrm{st}(x, S)) \subset \mathscr{C}\ \mathrm{vis}(x, S) \subset \mathscr{C}\ \mathrm{st}(x, S)$.

(iii) $\mathscr{C}\ \mathrm{ker}(S) \subset \mathscr{C}\ \mathrm{vis}(x, S)$.

(iv) $x \in \mathscr{C}\ \mathrm{ker}(S)$ *if and only if* $\mathscr{C}\ \mathrm{vis}(x, S) = \mathscr{C}\ \mathrm{ker}(S) \neq \emptyset$.

Proof. (i) $\mathscr{C}\ \mathrm{vis}(x, S) \neq \emptyset$ implies trivially that $\mathscr{C}\ \mathrm{st}(x, S) \neq \emptyset$. Assume now that

[1]Similar results were presented by J.C. Bressan in a communication to the 1988 Annual Meeting of Unión Matemática Argentina.

$\mathscr{C}\,\mathrm{st}(x, S) \neq \varnothing$ and let $y \in \mathscr{C}\,\mathrm{st}(x, S)$. Then $\mathscr{C}(x) \subset \mathscr{C}(x, y) \subset S$ and consequently $x \in \mathscr{C}\,\mathrm{st}(x, S)$. From the definition of $\mathscr{C}\,\mathrm{vis}(x, S)$ it follows that $x \in \mathscr{C}\,\mathrm{st}(x, S)$ implies that $x \in \mathscr{C}\,\mathrm{vis}(x, S)$.

(ii) Let $y \in \mathscr{C}\,\mathrm{ker}(\mathscr{C}\,\mathrm{st}(x, S))$. Hence $y \in \mathscr{C}\,\mathrm{st}(x, S)$ and for each $z \in \mathscr{C}\,\mathrm{st}(x, S)$, $\mathscr{C}(y, z) \subset \mathscr{C}\,\mathrm{st}(x, S) \subset S$, and $z \in \mathscr{C}\,\mathrm{st}(y, S)$. Hence $y \in \mathscr{C}\,\mathrm{vis}(x, S)$. The remaining inclusion is trivial.

(iii) Take $y \in \mathscr{C}\,\mathrm{ker}(S)$. Then $\mathscr{C}\,\mathrm{st}(y, S) = S$ and $y \in \mathscr{C}\,\mathrm{st}(x, S) \subset S$. Hence $y \in \mathscr{C}\,\mathrm{vis}(x, S)$.

(iv) If $x \in \mathscr{C}\,\mathrm{ker}(S)$ take $y \in \mathscr{C}\,\mathrm{vis}(x, S)$. Then $S = \mathscr{C}\,\mathrm{st}(x, S) \subset \mathscr{C}\,\mathrm{st}(y, S)$. Hence $y \in \mathscr{C}\,\mathrm{ker}(S)$. By (iii) the reverse inclusion holds. Conversely, assume now that $\mathscr{C}\,\mathrm{vis}(x, S) = \mathscr{C}\,\mathrm{ker}(S) \neq \varnothing$. Part (i) implies that $x \in \mathscr{C}\,\mathrm{ker}(S)$. □

Example 4.1 will show that the condition $\mathscr{C}\,\mathrm{ker}(S) \neq \varnothing$ is indispensable in part (iv) of the previous theorem.

THEOREM 2.2. *Let $(X, \mathscr{C})$ be a convexity space and S a nonempty subset of X. Then $\mathscr{C}\,\mathrm{ker}(S) = \bigcap \{\mathscr{C}\,\mathrm{vis}(x, S) \mid x \in S\}$.*

Proof. From part (iii) of 2.1 it follows that $\mathscr{C}\,\mathrm{ker}(S)$ is included in the intersection of the visibility cells of the points of S. The converse inclusion follows from the conjunction of Property 2.2 of [6] and part (ii) of 2.1. □

COROLLARY 2.3. *Let $(X, \mathscr{C})$ be a convexity space such that for each nonvoid subset $S \subset X$ and $\forall x \in S$, $\mathscr{C}\,\mathrm{vis}(x, S) \in \mathscr{C}$. Then $(X, \mathscr{C})$ is a B-convexity space.*

It will be shown below, by means of an example (cf. Example 4.2) that the converse statement of Corollary 2.3 is false. Part (i) of the following result has been proved by R. Wachenchauzer [9] in a linear space. It is a useful description of the visibility cell in constructive terms, that may be implemented algorithmically.

THEOREM 2.4. *Let $(X, \mathscr{C})$ be a JHC-convexity space, and S a nonempty subset of X. Then*

(i) $\forall x \in S$, $\mathscr{C}\,\mathrm{vis}(x, S) = \mathscr{C}\,\mathrm{ker}(\mathscr{C}\,\mathrm{st}(x, S))$.
(ii) $\mathscr{C}\,\mathrm{ker}(S) = \bigcap \{\mathscr{C}\,\mathrm{ker}(\mathscr{C}\,\mathrm{st}(x, S)) \mid x \in S\}$.

Proof. (i) By part (ii) of Theorem 2.1, one of the inclusions holds. The remaining inclusion is trivial if $\mathscr{C}\,\mathrm{vis}(x, S) = \varnothing$. Otherwise suppose that $\mathscr{C}\,\mathrm{vis}(x, S) \neq \varnothing$, and let $y \in \mathscr{C}\,\mathrm{vis}(x, S)$. Then $y \in \mathscr{C}\,\mathrm{st}(x, S) \subset \mathscr{C}\,\mathrm{st}(y, S)$. Let $z \in \mathscr{C}\,\mathrm{st}(x, S)$ and w be a generic point of $\mathscr{C}(y, z)$. An easy consequence of JHC property (that was observed in [1], Section 6) implies that for each $v \in \mathscr{C}(x, w)$ there exists $u \in \mathscr{C}(x, z)$ such that $v \in \mathscr{C}(y, u)$. Since $u \in \mathscr{C}\,\mathrm{st}(y, S)$ this means that $v \in S$. Thus $\mathscr{C}(x, w) \subset S$ and $\mathscr{C}(y, z) \subset \mathscr{C}\,\mathrm{st}(x, S)$. Consequently, $y \in \mathscr{C}\,\mathrm{ker}(\mathscr{C}\,\mathrm{st}(x, S))$.

(ii) It follows directly from part (i) and Theorem 2.2. □

Example 4.2 shows that JHC conditions cannot be omitted in Theorem 2.4. Example 4.3 below will show that there exist convexity spaces that satisfy (i) and (ii) but are not JHC.

3. Visibility cells in T-convexity spaces

The convex kernel of a set S in a linear space was characterized by Toranzos [7] as the intersection of the family $\mathscr{M}(S)$ of all the maximal convex subsets (convex components) of S, while Bressan ([2], [3]) proved the same characterization in the context of an axiomatic system for convex hull operators. This axiomatic system is logically equivalent to that of a $\mathbb{T}_1$ JD-convexity space considered by Kołodziejczyk [6].

Let $(X, \mathscr{C})$ be a convexity space and $S \subset X$. A set M is said to be a *$\mathscr{C}$-convex component* of S if M is a maximal $\mathscr{C}$-convex subset of S. In the sequel we will denote by $\mathscr{M}(S)$ the family of all $\mathscr{C}$-convex components of S. A nonvoid family $\mathscr{N}(S)$ of subsets of S is a *covering of S* iff $S = \bigcup \mathscr{N}(S)$. The following proposition is a direct consequence of (3.6) and (3.7) of [3].

PROPOSITION 3.1. *Let $(X, \mathscr{C})$ be a convexity space and $S \subset X$. Then:*

(i) *If $(X, \mathscr{C})$ is a $\mathbb{T}_1$ DF-convexity space, then $\mathscr{M}(S)$ is a covering of S.*
(ii) *If $(X, \mathscr{C})$ is a JHC-convexity space and $\mathscr{N}(S)$ is a covering of S such that $\mathscr{N}(S) \subset \mathscr{M}(S)$, then $\mathscr{C}\ker(S) = \bigcap \mathscr{N}(S)$.*
(iii) *If $(X, \mathscr{C})$ is a $\mathbb{T}_1$ JD-convexity space then $\mathscr{M}(S)$ is a covering of S and $\mathscr{C}\ker(S) = \bigcap \mathscr{M}(S)$.*

The following equivalence was proved by Kołodziejczyk (see Theorem 4.2 of [6]).

THEOREM 3.2. *Let $(X, \mathscr{C})$ be a $\mathbb{T}_1$ convexity space. Then $(X, \mathscr{C})$ is a T-convexity space if and only if it is a JD-convexity space.*

It is noteworthy that in the previous theorem, the $\mathbb{T}_1$ condition is used only to prove that a JD-convexity space is a T-convexity space. Furthermore, a T-convexity space is not necessarily $\mathbb{T}_1$, as Example 4.1 shows.

We have seen above (cf. Theorem 2.2) that the $\mathscr{C}$-kernel of a set can be described as intersection of the $\mathscr{C}$-visibility cells. In a T-convexity space, an analogous description holds using $\mathscr{C}$-convex components. It is natural to search for connections between $\mathscr{C}$-visibility cells and $\mathscr{C}$-convex components. It has been proved in the context of linear spaces ([8], Lemma 3.1) that the visibility cell of a point x in a set S is the intersection of all the convex components of S that include x. The next theorem synthesizes the results proved in [3] and [6] about T-convexity spaces, and its connection with the ideas just discussed[2].

THEOREM 3.3. *Let $(X, \mathscr{C})$ be a convexity space. The following statements are equivalent*:

(i) *$(X, \mathscr{C})$ is a* T-*convexity space such that $\forall S \subset X$ the family $\mathscr{M}(S)$ is a covering of S.*
(ii) *$(X, \mathscr{C})$ is a $\mathbb{T}_1$ JD-convexity space.*
(iii) $\forall S \subset X$, $\forall x \in S$
$\mathscr{C}\ \mathrm{vis}(x, S) = \bigcap \mathscr{M}(\mathscr{C}\ \mathrm{st}(x, S)) = \bigcap \{C \in \mathscr{M}(S) \mid x \in C\}$.

Proof. (i)⇒(ii). The last part of the proof of Theorem 4.2 of [6] shows that a T-convexity space enjoys the JD property. Let $x \in X$, then $\{x\} = \bigcup \mathscr{M}(\{x\})$. Hence $\{x\} \in \mathscr{M}(\{x\})$ and $(X, \mathscr{C})$ is $\mathbb{T}_1$.

(ii) ⇒ (iii). Let S be a nonempty subset of X and $x \in S$. By part (i) of Theorem 2.4 and part (iii) of Proposition 3.1 it holds that

$$\mathscr{C}\ \mathrm{vis}(x, S) = \mathscr{C}\ \ker(\mathscr{C}\ \mathrm{st}(x, S)) = \bigcap \mathscr{M}(\mathscr{C}\ \mathrm{st}(x, S)).$$

On the other hand, it is easy to see that

$$\mathscr{M}(\mathscr{C}\ \mathrm{st}(x, S)) = \{C \in \mathscr{M}(S) \mid x \in C\}.$$

Thus

$$\mathscr{C}\ \mathrm{vis}(x, S) = \bigcap \mathscr{M}(\mathscr{C}\ \mathrm{st}(x, S)) = \bigcap \{C \in \mathscr{M}(S) \mid x \in C\}.$$

(iii) ⇒ (i). Let $S \subset X$. If $S = \varnothing$ both the $\mathscr{C}$-kernel and the only $\mathscr{C}$-convex component of S are empty. Assume then that $S \neq \varnothing$. By (iii) and Theorem 2.2 $\mathscr{C}\ker(S)$ is the intersection of the $\mathscr{C}$-convex components of S. Hence $(X, \mathscr{C})$ is a T-convexity space. It remains to be proved that $\forall S \subset X$ the family $\mathscr{M}(S)$ is a covering of S. If $\mathrm{card}(X) = 1$ or if $S = X$ the property is trivial. Assume now that $\mathrm{card}(X) \geqslant 2$ and S is a proper subset of X. If $\mathscr{M}(S)$ would not be a covering of S, there would exist $x \in S$ such that $\forall M \in \mathscr{M}(S)$, $x \notin M$. Hence, according to (iii), $\mathscr{C}\ \mathrm{vis}(x, S)$ would be the intersection of an empty family, i.e. the whole space X. This contradicts Theorem 2.1(ii). □

[2]A previous version of this theorem was presented by J.C. Bressan in a communication to the 1987 Annual Meeting of Unión Matemática Argentina.

4. Counterexamples

We collect here three explicit examples of convexity spaces with certain properties that help to fix the possibilities of generalization of the previous results.

EXAMPLE 4.1. Let X be a set of cardinality greater than or equal to 2, and $\mathscr{C}=\{\varnothing; X\}$. Thus $(X, \mathscr{C})$ is a convexity space. Let S be a proper nonvoid subset of X, and $x \in S$. Then $\mathscr{C}\operatorname{vis}(x, S)=\mathscr{C}\ker(S)=\varnothing$, and $x \notin \mathscr{C}\ker(S)$. This shows that in Theorem 2.1. (iv), the $\mathscr{C}$-starshapedness of S is necessary. Furthermore, using the notation introduced in Section 3, $\mathscr{M}(S)=\{\varnothing\}$, $\mathscr{M}(X)=\{X\}$, and $\mathscr{C}\ker(S)=\varnothing=\bigcap\mathscr{M}(S)$, $\mathscr{C}\ker(X)=X=\bigcap\mathscr{M}(X)$. Hence $(X, \mathscr{C})$ is a T-convexity space that is not $\mathbb{T}_1$, as was announced after Theorem 3.2. Besides, it is clear that $\mathscr{M}(S)$ is not a covering of S. Hence, in Proposition 3.1(iii), the hypothesis of the space being $\mathbb{T}_1$ is not superfluous.

EXAMPLE 4.2. Let $\{a, b, c\}$ be a set of three non-collinear points of $\mathbb{R}^2$. Denote $\Delta=\operatorname{conv}\{a, b, c\}$, where 'conv' stands for the usual convex hull in the plane. The open linear segment of extremes a and b is denoted by $(a\ b)$. Let $X=\Delta \sim (a\ b)$. Define $\mathscr{C}$ as the 'relative convexity' of X, that is $\mathscr{C}=\{A \cap X \mid A$ convex in $\mathbb{R}^2\}$. Then $(X, \mathscr{C})$ is a B-convexity space, but if we define $S=\operatorname{conv}\{a, c\} \cup \operatorname{conv}\{c, b\}$ then $\mathscr{C}\operatorname{vis}(a, S)=\{a, c\} \neq \mathscr{C}$. This disproves the converse statement of Corollary 2.3. Furthermore $\mathscr{C}\ker(\mathscr{C}\operatorname{st}(a, S))=\{a\}$. Hence $\mathscr{C}\operatorname{vis}(a, S) \neq \mathscr{C}\ker(\mathscr{C}\operatorname{st}(a, S))$. We see that $(X, \mathscr{C})$ is not a JHC-convexity space. This shows that JHC conditions cannot be omitted in Theorem 2.4.

EXAMPLE 4.3. Let X be a set such that card $X \geqslant 4$ and $\mathscr{C}$ the family formed by X and its subsets of cardinality less than or equal to 2. $(X, \mathscr{C})$ is a convexity space that is not JHC. But if S is a nonempty subset of X and $x \in S$ then $\mathscr{C}\operatorname{st}(x, S)=S$ and $\mathscr{C}\operatorname{vis}(x, S)=S=\mathscr{C}\ker(\mathscr{C}\operatorname{st}(x, S))$. Thus, Theorem 2.4 is not a characterization of a JHC-convexity space.

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