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JOLANTA SŁLOMIŃSKA

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Smith theory and quasi-periodicity in Bredon cohomology

JOLANTA SŁOMIŃSKA

Instytut Matematyki Universytetu M. Kopernika, ul. Chopina 12-18, 87-100, Torun, Poland

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1. Introduction

Let G be a finite group. A CW-complex K with a given action of G is said to be a G-CW-complex, if for every subgroup H of G, the fixed point set K^H is a subcomplex of K. In this paper we study the equivariant cohomology theories defined on the category G-CW of G-CW-complexes by Bredon in [1]. In terms of Bredon cohomology, we generalize some well-known results belonging to the Smith theory (described, for instance in [2], III), and to the cohomology theory of groups. The following fact is a specialization of one of our results.

1.1. COROLLARY. *Let K be a G-CW-complex. Assume that A is an abelian group, and that q and m are natural numbers such that q is greater than m. Suppose that, for every subgroup H of G,*

$$H^n(K/H, A) = 0$$

whenever $n = q$, $q-1$, and that

$$H^n(K, A) = 0$$

whenever $m \leqslant n \leqslant q$. Then

$$H^n(K/G, A) = 0$$

whenever $m \leqslant n \leqslant q$. □

This fact yields Theorem 5.4 in Ch. III of [2].

In order to state our main results, we now recall briefly the definition of the Bredon cohomology. Coefficients of Bredon cohomology theories are contravariant functors $M\colon O_G^{\mathrm{op}} \to \mathrm{Ab}$, where Ab is the category of abelian groups and O_G is the category of canonical G-orbits. The objects of O_G are the G-sets of the form G/H, where H is a subgroup of G. The morphisms of O_G are the equivariant maps. Let $C_*(-)$ denote the cellular chain complex functor from the category

CW of CW-complexes to the category Ab_* of chain complexes in Ab. Assume that K is a G-CW-complex. Let us consider the contravariant functor

$$c_*(K)\colon O_G^{\mathrm{op}} \to \mathrm{Ab}_*,$$

which is equal to the functor $C_*(\mathrm{Map}_G(-, K))$. For every subgroup H of G,

$$c_*(K)(G/H) = C_*(K^H).$$

The nth Bredon cohomology group $H_G^n(K, M)$ can be defined as the nth cohomology group of the cochain complex

$$\mathrm{Hom}_{O_G}(c_*(K), M)$$

where $\mathrm{Hom}_{O_G}(-, -)$ denotes the abelian group of all natural transformations of contravariant functors from O_G to Ab.

We need also the following definitions from [12] and [13].

1.2. DEFINITION. (i) *For any abelian group A, let*

$$A[G/H]\colon O_G^{\mathrm{op}} \to \mathrm{Ab}$$

be the coefficient system such that

$$A[G/H](-) = A \otimes \mathrm{Hom}_{\mathbb{Z}(G)}(\mathbb{Z}(-), \mathbb{Z}(G/H))$$

where $\mathbb{Z}(G/H)$ is the $\mathbb{Z}(G)$-permutation module defined over the G-set G/H.

(ii) *For any coefficient system M, let $M[G/H]$ be the coefficient system defined by*

$$M[G/H](-) = M(-) \otimes \mathbb{Z}[G/H](-). \qquad \square$$

It is obvious that $M[G/G] = M$ and that

$$A[G/G][G/H] = A[G/H].$$

It follows from the "classical uniqueness" theorem in [1], IV, 5, that, for every subgroup H of G, there are isomorphisms

$$H_G^*(K, A[G/H]) = H^*(K/H, A)$$

and

$$H_G^*(K, M[G/H]) = H_G^*(K x_{K/G} K/H, M),$$

where $Kx_{K/G}K/H$ is the G-CW-complex obtained as the fibre product (pull-back) of the natural projections to the orbit space. The group G acts on $Kx_{K/G}K/H$ by the action on the first coordinate. If K' is a H-subcomplex of K then

$$H^*_G(Kx_{K/G}K'/H, A[G/G]) = H^*(K'/H, A).$$

We can now state our main results. By e we shall denote the neutral element of G.

1.3. PROPOSITION. *Let H be a subgroup of G and let m and q be natural numbers such that $m \leqslant q$. Suppose that*

$$H^n_G(K, M[G/e]) = 0$$

whenever $m \leqslant n \leqslant q$ and that, for every prime number p and every p-subgroup H' of H,

$$H^n_G(K, M[G/H']) = 0$$

for $n = q, q-1$. Then

$$H^n_G(K, M[G/H]) = 0$$

whenever $m \leqslant n \leqslant q$. □

It is obvious that Corollary 1.1 is an immediate consequence of 1.3. Proposition 1.3 will be proved in Section 4. In order to state the next result we need the following notation. Let $G(K)$ be the subgroup of G generated by all subgroups of the form G_k where $G_k = \{g \in G: gk = k\}$ is the isotropy group of the action of G at the point $k \in K$. For any prime p and any two subgroups G' and G'' of G, by $\mathscr{E}_p(G', G'')$ we shall denote the set of all p-subgroups H of G'' such that $H/H \cap G'$ is an elementary abelian p-group. Let P be the set of all prime numbers. We shall also use the notation

$$\mathscr{E}_0(K) = \bigcup_{p \in P} \bigcup_{H \in S(p)} \mathscr{E}_p(H(K), H),$$

where $S(p)$ is the set of all Sylow p-subgroups of G.

1.4. PROPOSITION. *Let m be a natural number. Assume that, for every $n \geqslant m$ and for every element H of $\mathscr{E}_0(K)$,*

$$H^n_G(K, M[G/H]) = 0.$$

Then, for every $n \geqslant m$,

$$H^n_G(K, M) = 0. \qquad \square$$

The above proposition will be proved in Section 3. If $K = EG$ is a universal free G-CW-complex, then for any coefficient system M, there is an isomorphism

$$H^*_G(EG, M[G/H]) = H^*(H, M(G)).$$

In this case, $H(K) = (e)$, and Proposition 1.4 specializes to the well-known result of the cohomology theory of groups described in [7] and [5].

Using the results of this paper, by methods like those in [14], one can prove the following results.

THEOREM A. *Assume that for every subgroup* H *of* G *and for every* $n \geqslant m$,

$$H^n_G(K, M[G/H]) = 0.$$

Then, for every subgroup H *of* G *and every* $n \geqslant m$,

$$H^n_G(Kx_{K/G}K^H, M) = 0$$

and

$$H^n_G(Kx_{K/G}(K^H/NH), M) = 0,$$

where NH *is the normalizator of* H *in* G. $\square$

THEOREM B. *Let*

$$\mathscr{E}(K) = \bigcup_{k \in K} \bigcup_{p \in P} \mathscr{E}_p(G_k, G).$$

Assume that for every $n \geqslant m$ *and* $H \in \mathscr{E}(K)$

$$H^n_G(K, M[G/H]) = 0.$$

Then for every $n \geqslant m$,

$$H^n_G(K, M) = 0. \qquad \square$$

The above theorems will be proved in [15]. It is clear that $\mathscr{E}(K)$ is a subset of $\mathscr{E}_0(K)$. Hence Theorem B is more general than 1.4.

In our proofs of Propositions 1.3 and 1.4, we will use the following fact from homological algebra.

1.5. LEMMA. *Let*

$$\gamma\colon O \to M \to Q_q \to \cdots \to Q_0 \to N \to 0$$

be an exact sequence of G-coefficient systems. Then, for every $m \geqslant 0$, *there exists the homomorphism*

$$d^m\colon H^m_G(K, N) \to H^{m+q+1}_G(K, M),$$

which depends only on the class $[\gamma]$ *in* $\mathrm{Ext}^{q+1}_{O_G}(N, M)$ *and has the following properties.*

(i) *If for every* $i = 0, \dots, q$,

$$H^n_G(K, Q_i) = 0,$$

whenever $n = m + i,\ m + i + 1$, *then* d^m *is an isomorphism.*

(ii) *If, additionally,* $[\gamma] = 0$, *then*

$$H^{m+q+1}_G(K, M) = H^m_G(K, N) = 0. \qquad \square$$

The homomorphism d^m can be defined to be equal to the compositions of the boundary homomorphisms of the appropriate exact sequences. It is easy to prove the above lemma by methods like those in [6], IV, 9 and in [4], XIV, 1.

We shall consider exact sequences γ of coefficient systems such that for each $i = 0, \dots, q$, Q_i is a direct product of functors of the form $M[G/H]$. In particular, we describe exact sequences determined by homological spheres. In the case of representation spheres and appropriate G-CW-complex K, the vanishing of $H^n_G(K, N)$ implies the vanishing of $H^n_G(K, M)$. The exact sequence induced by a two dimensional free representation of a cyclic group is connected with the exact sequences which occur in the Smith theory and are described in [9]. In a proof of 1.4 we shall use an exact sequence γ such that $[\gamma] = 0$. The results of this paper, except 4.4 and 4.10, can be generalized to the case where M is a local G-coefficient system on K defined by Bredon in [1], Ch. I.5. The proofs remain the same.

2. Preliminaries

This section contains preliminary facts concerning Bredon cohomology. We recall here briefly some definitions and results of [13] about Bredon cohomology with coefficients of the form $M[G/H]$. In particular, we will use in our considerations the fact that there exists a subcategory $O_{G,H}$ of O_G such that $M[G/H]$ is the right Kan extension of the restriction of M to $O_{G,H}$.

Let G be a finite group and let H be a subgroup of G. By $O_{G,H}$ we denote the category whose objects are the canonical G-orbits and whose morphisms are the G-maps $f: G/H'' \to G/H'$ such that $f(H'') = hH'$, where $h \in H$ and $h^{-1}H''h \subseteq H'$.

Assume now that K is a G-CW-complex and that K' is a H-subcomplex of K. Then

$$c_*(K'): O^{\mathrm{op}}_{G,H} \to \mathrm{Ab}_*$$

is the functor such that, for every subgroup of H' of G,

$$c_*(K')(G/H') = C_*(K'^{H'})$$

and for every morphism f of $O_{G,H}$,

$$c_*(K')(f) = C_*([h]),$$

where

$$[h]: K'^{H'} \to K'^{H''}$$

is the map such that, for every point k of $K'^{H'}$, $[h](k) = hk$.

If M is a contravariant functor from $O_{G,H}$ to Ab, then we define $H^n_{\{H\}}(K', M)$ to be equal to the nth cohomology group of the cochain complex

$$\mathrm{Hom}_{O_{G,H}}(c_*(K'), M).$$

The groups $H^n_{\{H\}}(K', M)$ depend not only on the action of H on K', but also on the action of G on K. We introduce these groups because in some particular cases the groups $H^n_G(K, M[G/H])$ can be described using groups of the form $H^m_{\{H'\}}(K', M)$, where K' is a H'-subcomplex of K but not a G-subcomplex.

It is clear that if $H = G$, then $O_{G,G} = O_G$ and

$$H^*_{\{G\}}(K', M) = H^*_G(K', M).$$

If $H = e$, where e is the neutral element of G, then $O_{G,e}$ is the category associated to the poset of all subgroups of G.

We shall use the notation

$$C^*(K', M) = \mathrm{Hom}_{O_{G,e}}(c_*(K'), M).$$

There exists the natural action of H on $C^*(K', M)$ such that

$$C^*(K', M)^H = \mathrm{Hom}_{O_{G,H}}(c_*(K'), M).$$

Let F be a set of subgroups of G. By $O_{F,H}$ we denote the full subcategory of $O_{G,H}$ whose objects are the canonical G-orbits G/H' such that H' is an element of F. Let

$$F(K') = \{G_k : k \in K'\}.$$

Assume that, $F(K') \subseteq F$. Then

$$\mathrm{Hom}_{O_{G,H}}(c_*(K'), M) = \mathrm{Hom}_{O_{F,H}}(c_*(K')\imath_F, M\imath_F)$$

where $\imath_F : O_{F,H} \to O_{G,H}$ is the natural inclusion of categories.

If H' is a subgroup of H, then

$$I_{H',H} : O_{G,H'} \to O_{G,H}$$

will denote the natural inclusion of categories. Assume that $M : O^{\mathrm{op}}_{G,H} \to \mathrm{Ab}$. We shall use the notation

$$H^*_{\{H'\}}(K', M) = H^*_{\{H\}}(K', MI_{H',H}).$$

We define the functor

$$M[H/H'] : O^{\mathrm{op}}_{G,H} \to \mathrm{Ab}$$

in such a way that, for every subgroup H'' of G,

$$M[H/H'](G/H'') = M(G/H'') \otimes \mathrm{Hom}_{\mathbb{Z}(H)}(\mathbb{Z}(H/H \cap H''), \mathbb{Z}(H/H'))$$

and for every morphism $f : G/H'' \to G/G''$ of $O_{G,H}$ determined by $h \in H$,

$$M[H/H'](f) = M(f) \otimes \mathrm{Hom}_{\mathbb{Z}(H)}(\mathbb{Z}(f'), \mathrm{id})$$

where $f' : H/H \cap H'' \to H/H \cap G''$ is the H map also determined by h.

The following facts are proved in [13].

2.1. PROPOSITION. (i) *There is an isomorphism*

$$H^*_{\{H'\}}(K', M) = H^*_{\{H\}}(K', M[H/H']).$$

(ii) *Assume that $H = G$. Then there is an isomorphism*

$$H^*_{\{H'\}}(K', M) = H^*_G(K x_{K/G} K'/H', M). \qquad \square$$

Assume now that $H'' \subseteq H' \subseteq H$. It follows from the definition that there exists natural transformations of functors

$$r: M[H/H'] \to M[H/H'']$$

and

$$i: M[H/H''] \to M[H/H']$$

such that the composition ir is equal to the multiplication by $|H'/H''|$. This fact implies the next lemma.

2.2. LEMMA. (i) *Assume that H' is a subgroup of H such that $|H/H'|$ is a power of p. If $H^n_{\{H'\}}(K', M)$ is a p-group, then $H^n_{\{H\}}(K', M)$ is also a p-group.*

(ii) *If*

$$H^n_{\{e\}}(K', M) = 0,$$

then

$$|H| H^n_{\{H\}}(K', M) = 0.$$

(iii) *Assume that*

$$H^n_{\{e\}}(K', M) = 0$$

and that, for every Sylow p-subgroup H_p of H,

$$H^n_{\{H_p\}}(K', M) = 0.$$

Then

$$H^n_{\{H\}}(K', M) = 0. \qquad \square$$

The above lemma may be proved using methods like those described in [3], Ch. III, 10.

We introduce now the notion of cohomological dimension in Bredon cohomology. Let H be a subgroup of G and let $M: O^{\mathrm{op}}_{G,H} \to \mathrm{Ab}$ be a coefficient system. Assume that K is a G-CW-complex and that K' is a H-subcomplex of K. Let $N^+ = N \cup \{\infty\}$, where N is the set of all natural numbers.

We define $\mathrm{cd}_H(K', M)$ to be the smallest element d of N^+ such that, for every $n \geqslant d+1$,

$$H^n_{\{H\}}(K', M) = 0.$$

If F is a set of subgroups of H, then

$$\mathrm{cd}_F(K', M) = \max_{H' \in F} \mathrm{cd}_{H'}(K', M).$$

The number

$$\mathrm{cd}(H, K', M) = \max_{H' \subseteq H} \mathrm{cd}_{H'}(K', M)$$

will be called the cohomological dimension of the H-subcomplex K' of K with coefficients in M.

2.3. COROLLARY. *Let P be the set of all prime numbers. Then*

$$\mathrm{cd}(H, K', M) = \max_{p \in P} \mathrm{cd}(H_p, K', M),$$

where H_p is a Sylow p-subgroup of H. □

Proof. This result is an immediate consequence of 2.2(iii). □

3. Quasi-periodicity in Bredon cohomology

Let H be a subgroup of G and let F be a set of subgroups of G. Assume that K is a G-CW-complex and that K' is a H-subcomplex of K. Let R be a commutative ring. Then Lemma 1.5 can be generalized in the following way.

3.1. LEMMA. *Let*

$$\gamma : O \to M \to M_q \to \cdots \to M_0 \to N \to 0$$

be a sequence of contravariant functors from $O_{G,H}$ to the category R-Mod of R-modules. Assume that this sequence is exact after the restriction to the category $O_{F,H}$ and that $F(K') \subseteq F$.

(i) *Let m be a natural number such that*

$$H^{m+i}_{\{H\}}(K', M_i) = O = H^{m+i+1}_{\{H\}}(K', M_i)$$

whenever $i = 0, \ldots, q$. Then there is an isomorphism

$$d^m : H^m_{\{H\}}(K', N) \to H^{m+q+1}_{\{H\}}(K', M).$$

(ii) *If, additionally, $[\gamma] = 0$ in $\mathrm{Ext}^{q+1}_{O_{G,H}}(N, M)$, then $d^m = 0$ and*

$$H^m_{\{H\}}(K', N) = 0.$$ □

A proof of this lemma is standard and will be left to the reader.

In order to prove Proposition 1.4 we shall consider the case where $N=M$ and $[\gamma]=0$. Let $H(K')$ be the subgroup of H generated by all subgroups of the form $H \cap G_k$ where $k \in K'$. Lemma 3.1 implies the following fact.

3.2. PROPOSITION. *Let H be a p-subgroup of G and let H' be a subgroup of H such that $H(K') \subseteq H'$. Then*

$$\mathrm{cd}(H, K', M) \leqslant \mathrm{cd}_{\mathscr{E}'_p(H',H)}(K', M),$$

where

$$\mathscr{E}'_p(H', H) = \{H'' \in \mathscr{E}_p(H', H) \colon H' \subseteq H'' \subseteq H\}. \qquad \square$$

Proof. Assume that H does not belong to $\mathscr{E}_p(H', H)$. It follows from the results of [5] and [11] that there exists an exact sequence of $\mathbb{Z}(H/H')$-modules

$$\gamma\colon 0 \to \mathbb{Z} \to C_q \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

such that

$$[\gamma] = 0 \quad \text{in } H^{q+1}(H/H', \mathbb{Z})$$

and for every $i = 0, \ldots, q$, $C_i = \mathbb{Z}(H/H_i)$, where H_i is a maximal subgroup of H containing H'. The sequence γ induces the sequence of functors

$$\gamma[M]\colon 0 \to M \to M[H/H_q] \to \cdots \to M[H/H_0] \to M \to 0$$

which is exact on the category $O_{F(K'),H}$. It is clear that

$$[\gamma[M]] = 0 \quad \text{in } \mathrm{Ext}^{q+1}_{O_{F(K'),H}}(M, M).$$

Lemma 3.1 implies now that if, for every proper subgroup H'' of H, $\mathrm{cd}(H'', K', M) \leqslant d$, then $\mathrm{cd}(H, K', M) \leqslant d$. Hence the proposition can be proved by induction on $|H/H'|$. $\square$

PROOF OF PROPOSITION 1.4. Lemma 2.3 implies that

$$\mathrm{cd}(G, K, M) = \mathrm{cd}_{F(P)}(K, M),$$

where

$$F(P) = \bigcup_{p \in P} F_p,$$

and F_p is the set all p-subgroups of G. Proposition 3.2 yields that

$$\mathrm{cd}_{F(P)}(K, M) = \mathrm{cd}_{\mathscr{E}_0(K)}(K, M)$$

and this fact ends the proof of 1.4. □

We shall next study exact sequences of coefficient systems determined by homological spheres. Let R be a commutative ring. By a q-dimensional homological R-sphere we mean a CW-complex S' such that, for $n = 0$ or q, $H_n(S', R) = R$, and $H_n(S', R) = 0$ otherwise. Let S be a H-CW-complex which is a q-dimensional homological R-sphere. By $F_q(S, R)$ we denote the set of all subgroups H' of G such that $S/H \cap H'$ is a q-dimensional homological R-sphere.

3.3. PROPOSITION. *Assume that S is a H-CW-complex of dimension q which is also a q-dimensional homological R-sphere. Then, for any functor $M: O_{G,H}^{\mathrm{op}} \to R$-Mod, there exists a sequence*

$$\gamma(M, S)\colon 0 \to M \to M_q \to \cdots \to M_0 \to M(S) \to 0$$

of contravariant functors from $O_{G,H}$ to R-Mod satisfying the following conditions.

(i) *The sequence $\gamma(M, S)$ is exact after the restriction to the category $O_{F,H}$, where $F = F_q(S, R)$.*

(ii) *For $i = 0, \ldots, q$, the functor M_i is a direct product of functors of the form $M[H/H_x]$, where $x \in S$.*

(iii) *For every element H' of $F_q(S, R)$,*

$$\begin{aligned} M(S)(G/H') &= \mathrm{Hom}_R(H_q((H/H \cap H' \times S)/H, R), M(G/H')) \\ &= \mathrm{Hom}_R(H_q(S/H \cap H', R), M(G/H')), \end{aligned}$$

and for every morphism f of $O_{G,H}$,

$$M(S)(f) = \mathrm{Hom}(H_q((f/x\,\mathrm{id})/H, R), M(f)).$$ □

Proof. We construct the sequence $\gamma(M, S)$ in such a way that

$$M_i(G/H') = \mathrm{Hom}_R(C_{q-i}(S/H \cap H', R), M(G/H')).$$

The homomorphisms of this sequence are induced by the boundary homomorphisms of $C_*(S/H \cap H')$. It is obvious, that $\gamma(M, S)$ after the restriction to $O_{F,H}$ is exact. If S_i is the H-set of i-cells of S, then

$$M_{q-i} = \prod_{[s] \in S_i/H} M[H/H_s].$$

This follows from the fact that

$$M_{q-i}(G/H') = \mathrm{Hom}_R\left(\bigoplus_{[s]\in S_i/H} (R \otimes_{R(H\cap H')} R(H/H_s)),\ M(G/H')\right)$$
$$= \prod_{[s]\in S_i/H} M(G/H') \otimes \mathrm{Hom}_{\mathbb{Z}(H)}(\mathbb{Z}(H/H\cap H'),\ \mathbb{Z}(H/H_s))$$

and that the above isomorphisms are natural in G/H'. □

The following result is an immediate consequence of 3.1 and 3.3.

3.4. COROLLARY. *Let S and M be the same as in 3.3 and let m be a natural number. Assume that K is a G-CW-complex and that K' is a H-subcomplex of K such that $F(K') \subseteq F_q(S, R)$. Suppose that*

$$H^n_{\{H_x\}}(K',\ M) = 0$$

whenever $m \leqslant n \leqslant m+q+1$ and $x \in S$. Then there is an isomorphism

$$H^m_{\{H\}}(K',\ M(S)) = H^{m+q+1}_{\{H\}}(K',\ M). \qquad \square$$

Let us consider the following specialization of 3.3. Assume that V is an orthogonal, orientation preserving representation of H. We shall use the notation

$$SV = \{x \in V\colon |x| = 1\} \quad \text{and} \quad H_V = \bigcap_{x\in SV} H_x.$$

3.5. PROPOSITION. *Assume that V is a $(q+1)$-dimensional, orthogonal and orientation preserving representation of H. Then, for every element H' of $F_q(SV, R)$,*

$$M(SV)(G/H') = M(G/H'),$$

and for every morphism $f\colon G/H' \to G/H''$ of $O_{F_q(SV,R),H}$,

$$M(SV)(f) = (|H_V(H''\cap H)|/|H_V(H'\cap H)|)M(f). \qquad \square$$

Proof. It follows from 8.2 of [10], that there exists a point x of SV such that $H_x = H_V$. Thus H_V is the main orbit type on SV. (See [8], I.2.1.) The result can now be obtained from the description of $M(SV)$ in 3.3(iii) because $C_q(SV)$ is a free $\mathbb{Z}(H/H_V)$-module. □

As immediate consequence of 3.4 and 3.5, we obtain the following fact.

3.6. COROLLARY. *Assume that H is a p-subgroup of G and that $M\colon O^{\mathrm{op}}_{G,H} \to \mathbb{Z}/p$-Mod. Suppose that the assumptions of* 3.4 *hold for $S = SV$, where V is the same as in* 3.5. *Then there is an isomorphism*

$$\bigoplus_{i=0}^{r} H^m_{\{H\}}(K_i, K_{i+1}; M) = H^{m+1+q}_{\{H\}}(K, M),$$

where $r = \log_p|H/H_V|$ *and*

$$K_i = \{k \in K : |(H_k H_V)/H_V| \geqslant p^i\}. \qquad \square$$

The next result is also a consequence of 3.4 and 3.5.

If R is a commutative ring, then the set of all natural numbers, which are invertible in R, will be denoted by $\mathscr{J}(R)$. By $F_R(H)$ we shall denote the set of all subgroups H' of H such that $|H'| \in \mathscr{J}(R)$. A H-CW-complex K' will be called $H-R$-free if, for every $k \in K'$, $H_k \in F_R(H)$.

3.7. COROLLARY. *Let m be a natural number. Assume that K is a G-CW-complex and that $M\colon O^{\mathrm{op}}_{G,H} \to R$-Mod is a coefficient system such that, for every $n \geqslant m$,*

$$H^n_{\{e\}}(K, M) = 0.$$

Suppose that, K is $H-R$-free. If there exists an orthogonal, orientation preserving, representation V of H such that SV is $H-R$-free, then for every $n \geqslant m$,

$$H^n_{\{H\}}(K, M) = H^{n+v}_{\{H\}}(K, M),$$

where v is equal to the dimension of V. $\square$

Proof. Lemma 2.2(ii) implies that

$$H^n_{\{H'\}}(K, M) = 0$$

whenever $n \geqslant m$ and $H' \in F_R(H)$. For every $H' \in F_R(H)$, the projection $\pi\colon SV \to (SV)/H'$ to the orbit space induces an isomorphism

$$H^*(SV/H', R) \to H^*(SV, R).$$

Let $\omega\colon M(S) \to M$ be the natural transformation of functors such that $\omega(G/H'')$ is the multiplication by $|H_V(H \cap H'')|$. The result is now a consequence of 3.4 and 3.5 because the restriction of ω to the category $O_{F(SV,R),H}$ is a natural equivalence of functors. $\square$

In the rest of this section we shall prove that the vanishing of the cohomology groups with coefficients in $M(SV)$ implies the vanishing of the cohomology groups with coefficients in M.

3.8. PROPOSITION. *Let n be a natural number. Suppose that V is the same as in* 3.5 *and that K′ is a H-subcomplex of a G-CW-complex K such that*

$$F(K') = \{G_k : k \in K'\} \subseteq F_q(SV, R)$$

Assume that $M: O^{\mathrm{op}}_{G,H} \to R\text{-}Mod$ *is a coefficient system such that*

$$H^n_{\{H\}}(K', M(SV)) = 0.$$

Then

$$H^n_{\{H\}}(K', M) = 0. \qquad \square$$

We shall prove the above fact in the end of this section using Lemma 3.10. It is clear that 3.8 and 3.4 yield immediately the next result.

3.9. COROLLARY. *Assume that V is the same as in* 3.5.

(i) *Suppose that the assumptions of* 3.4 *are fulfilled for* $S = SV$. *If*

$$H^{m+1+q}_{\{H\}}(K', M) = 0.$$

then

$$H^m_{\{H\}}(K', M) = 0.$$

(ii) *Assume that* $F(K') \subseteq F_q(SV, R)$. *Then*

$$\mathrm{cd}_H(K', M) \leqslant \max_{x \in S} \mathrm{cd}_{H_x}(K', M). \qquad \square$$

Proposition 3.8 can be easily obtained from the following fact. Let F be a family of subgroups of G, i.e. a set of subgroups closed under conjugation by elements of G and under taking subgroups. Assume that p is a prime number and that $M: O^{\mathrm{op}}_{G,H} \to \mathrm{Ab}$. Then

$$M_F(p): O^{\mathrm{op}}_{G,H} \to \mathrm{Ab}$$

is the functor such that, for every subgroup H' of G,

$$M_F(p)(G/H') = M(G/H'),$$

and for every morphism $f: G/H' \to G/H''$ of $O_{G,H'}$

$$M_F(p)(f) = pM(f)$$

whenever H' belongs to F and H'' does not belong to F, and

$$M_F(p)(f) = M(f)$$

otherwise.

3.10. LEMMA. *Let n be a natural number. Assume that*

$$H^n_{\{H\}}(K', M_F(p)) = 0.$$

Then

$$H^n_{\{H\}}(K', M) = 0. \qquad \square$$

Proof. There exists a natural transformation of functors

$$\tau: M_F(p) \to M$$

given by

$$\tau(G/H'') = \mathrm{id}_{M(G/H'')}$$

whenever $H'' \in F$, and

$$\tau(G/H') = p \cdot \mathrm{id}_{M(G/H')}$$

whenever H' does not belong to F.

Let $M': O^{\mathrm{op}}_{G,H} \to \mathrm{Ab}$ be the coefficient system such that, for every $H'' \in F$,

$$M'(G/H'') = M(G/H''),$$

and for every H' which does not belong to F,

$$M'(G/H') = 0.$$

If $f: G/H' \to G/H''$ is a morphism of $O_{G,H}$, then $M'(f) = M(f)$ whenever $H'' \in F$.

If $M'' = M/M'$, then we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "\mathrm{id}"'] & M_F(p) \arrow[r] \arrow[d, "\tau"'] & M'' \arrow[r] \arrow[d, "p\cdot\mathrm{id}"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Let

$$\delta^n, \mathscr{d}^n \colon H^n_{\{H\}}(K', M'') \to H^n_{\{H\}}(K', M')$$

be the boundary homomorphisms of the long exact sequences induced by the exact sequences described above. Then $\delta^n = p \cdot \mathscr{d}^n$. This implies that if δ^n is a monomorphism (an epimorphism), then the same is true for $\mathscr{d}^n$ in place of δ^n. The lemma is now a consequence of the fact that

$$H^n_{\{H\}}(K', M_F(p)) = 0, \qquad (H^n_{\{H\}}(K', M) = 0),$$

if and only if $\delta^n(\mathscr{d}^n)$ is a monomorphism and $\delta^{n-1}(\mathscr{d}^{n-1})$ is an epimorphism. □

PROOF OF PROPOSITION 3.8. Let H_0 be a normal subgroup of H. Assume that $|H/H_0| = p_1^{\alpha}1 \cdots p_r^{\alpha}r$, where $p_1, \ldots, p_r$ are different prime numbers. For any subgroup H' of G and $i \in \{1, \ldots, r\}$, let $\beta_i(H')$ be the natural number such that

$$|H/(H \cap H')H_0| = p_1^{\beta_1(H')} \cdots p_r^{\beta_r(H')}.$$

Assume that $M_{H_0}[p_i] \colon O^{\mathrm{op}}_{G,H} \to R\text{-Mod}$ is the functor such that, for every subgroup H' of G,

$$M_{H_0}[p_i](G/H') = M(G/H'),$$

and for every morphism $f \colon G/H' \to G/H''$ of $O_{G,H}$,

$$M_{H_0}[p_i](f) = p_i^{\beta_i(H') - \beta_i(H'')} M(f).$$

If $H_0 = H_V$, then

$$M(SV) = M_{H_V}[p_1]_{H_V}[p_2] \cdots {}_{H_V}[p_r].$$

Let

$$F_{i,j} = \{H' \in F \colon \beta_i(H') \leqslant j\}.$$

It is easy to check that for any coefficient system N,

$$N_{H_V} = N_{F_{i,1}(p_i)\cdots F_{i,\alpha_i}(p_i)}.$$

Now, it is sufficient to apply 3.10. □

4. Smith theory in Bredon cohomology

Assume that H is a subgroup of G and that H' is a normal subgroup of H such that H/H' is a cyclic p-group. There exists a 2-dimensional representation V of H such that $C_*(SV)$ is equal to the $\mathbb{Z}(H)$-module chain complex

$$O \to \mathbb{Z} \to \mathbb{Z}(H/H') \xrightarrow{\alpha} \mathbb{Z}(H/H') \to \mathbb{Z} \to 0$$

where $\alpha = \mathrm{id} - [h]\cdot\mathrm{id}$, and $[h]\cdot\mathrm{id}$ is the multiplication by a generator $[h]$ of H/H'. It is obvious that in this case all subgroups of G belong to $F_1(SV, \mathbb{Z})$ and that

$$M[H/H'](G/H'') = M(G/H'') \otimes \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(H/(H'' \cap H)H'), \mathbb{Z}).$$

The results of the previous section imply the following facts.

4.1. COROLLARY. *Let*

$$M: O^{\mathrm{op}}_{G,H} \to \mathrm{Ab}$$

be a coefficient system. Assume that H/H' is a cyclic p-group. Then there exists an exact sequence

$$0 \longrightarrow M \longrightarrow M[H/H'] \xrightarrow{\alpha_M} M[H/H'] \longrightarrow M' \longrightarrow 0$$

where, for every subgroup H'' of G,

$$M'(G/H'') = M(G/H''),$$

for every morphism $f: G/H'' \to G/G''$ of $O_{G,H}$,

$$M'(f) = (|(G'' \cap H)H'|/|(H'' \cap H)H'|)M(f),$$

and α_M is induced by α. □

Proof. This result is a consequence of 3.5. □

4.2. COROLLARY. *Assume that K is a G-CW-complex and that K' is a H-subcomplex of K.*

(i) *If*

$$H^n_{\{H\}}(K', M) = 0,$$

then

$$H^n_{\{H\}}(K', M') = 0.$$

(ii) *Let m and q be natural numbers such that $m \leqslant q$. If for $n = q, q-1$*

$$H^n_{\{H\}}(K', M) = 0$$

and

$$H^n_{\{H'\}}(K', M) = 0 \quad \textit{whenever } m \leqslant n \leqslant q,$$

then

$$H^n_{\{H\}}(K', M) = 0 \quad \textit{whenever } m \leqslant n \leqslant q.$$

Proof. The statement (i) follows from 3.8. If

$$H^n_{\{H\}}(K', M) = 0$$

whenever $m \leqslant n \leqslant q$, then, by 3.4,

$$H^n_{\{H\}}(K', M') = H^{n+2}_{\{H\}}(K', M)$$

whenever $m \leqslant n \leqslant q - 2$. This fact implies the assertion (ii). □

We can now prove one of the results stated in Section 1.

PROOF OF PROPOSITION 1.3. By Lemma 2.2(iii), it is sufficient to consider the case where H is a p-group. There exists a sequence of subgroups of H:

$$(e) = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_r = H,$$

such that, for every $i = 0, \ldots, r-1$, H_i is a normal subgroup of H_{i+1} and H_{i+1}/H_i is a cyclic p-group. Proposition 1.3 is now a consequence of 4.2(ii). □

Proposition 1.3 implies the following fact.

4.3. COROLLARY. *Assume that*

$$\mathrm{cd}(H, K', M) < \infty.$$

Then

$$\mathrm{cd}_{(e)}(K', M) = \mathrm{cd}(H, K', M). \qquad \square$$

In particular, if K is a finite dimensional G-CW-complex, $H = G$ and, for $n \geqslant m$,

$$H^n_G(K, M[G/e]) = 0,$$

then, for $n \geqslant m$,

$$H^n_G(K, M) = 0.$$

For a G-CW-complex K, let $d_G(K)$ be equal to the greatest natural number r with the property that there exists a sequence

$$G_0 \subset G_1 \subset \cdots \subset G_r$$

of different elements of the subgroup set $F(K) = \{G_k : k \in K\}$.

4.4. COROLLARY. *Let m and q be natural numbers. Assume that for every subgroup H of G,*

$$H_n(K^H, \mathbb{Z}) = 0$$

whenever $n \geqslant m$, and

$$H^n_G(K, M[G/H]) = 0$$

whenever $n \geqslant q$. Then, for $n \geqslant m + 1 + d_G(K)$,

$$H^n_G(K, M) = 0.$$

Proof. Let

$$K^{>H} = \{k \in K : H \subseteq G_k,\ H \neq G_k\}.$$

Then

$$H^n_{\{e\}}(K^H, K^{>H}; M) = H^n(K^H, K^{>H}; M(G/H)).$$

Using induction one can easily prove that, for $n \geqslant m + 1 + d_G(K)$ and for every subgroup H of G,

$$H^n(K^H, K^{>H}; M(G/H)) = 0.$$

This fact implies, also by induction, that

$$H^n_{\{e\}}(K, M) = 0.$$

The result now follows from 4.3. □

We shall now consider coefficient systems defined over $\mathbb{Z}/p$.

4.5. COROLLARY. *Let H be a subgroup of G and let H' be a normal subgroup of H such that H/H' is a cyclic p-group. Assume that $M: O_G^{\mathrm{op}} \to \mathbb{Z}/p$-Mod. If*

$$H^m_{\{H'\}}(K, M) = 0 = H^{m+1}_{\{H'\}}(K, M),$$

then there is an isomorphism

$$H^{m+2}_{\{H\}}(K, M) = \bigoplus_{i=0}^{d} H^m_{\{H\}}(K_i, K_{i+1}; M),$$

where $d = \log_p|H/H'|$,

$$K_i = \bigcup_{G' \in F(i)} K^{G'},$$

and

$$F(i) = \{G' \subseteq G: |((G' \cap H)H')/H'| = p^i\}.$$

Proof. This result is a specialization of 3.6. It can also be obtained immediately from 4.1. □

The following fact is a specialization of 4.5.

4.6. COROLLARY. *Assume that G is a cyclic p-group and that for $n = m$, $m+1$, $m+2$,*

$$H^n(K, \mathbb{Z}/p) = 0.$$

Then

$$H^{m+2}(K/G, \mathbb{Z}/p) = H^m(K^G, \mathbb{Z}/p) \oplus \bigoplus_{i=0}^{d-1} H^m(K^{H(i)}/G, K^{H(i+1)}/G; \mathbb{Z}/p),$$

where $H(i)$ is the subgroup of G isomorphic to $\mathbb{Z}/p^i$ and d is the natural number such that $H(d) = G$. □

If $H/H' = \mathbb{Z}/p$, then $K_0 = K$ and

$$K_1 = \bigcup_{(G' \cap H)H' = H} K^{G'}.$$

In this case, we may prove the following result.

4.7. PROPOSITION. *Let H' be a normal subgroup of G such that $H/H' = \mathbb{Z}/p$. Suppose that $M: O_G^{\mathrm{op}} \to \mathbb{Z}/p$ is a coefficient system such that*

$$H^n_{\{H'\}}(K, M) = 0$$

whenever $m \leqslant n \leqslant r$. Let

$$A^n = H^n_{\{H\}}(K, M), \qquad B^n = H^n_{\{H\}}(K_1, M), \qquad D^n = H^n_{\{H\}}(K, K_1; M).$$

Then there are isomorphisms

$$A^{n+2q} = D^n \oplus \bigoplus_{i=0}^{2q-2} B^{n+i}$$

whenever $m \leqslant n < n + 2q \leqslant r$,

$$A^{n+2q} = A^n \oplus \bigoplus_{i=-1}^{2q-2} B^{n+i}$$

whenever $m + 1 \leqslant n < n + 2q \leqslant r$,

$$D^{n+2q} = D^n \oplus \bigoplus_{i=0}^{2q-1} B^{n+i}$$

whenever $m \leqslant n < n + 2q \leqslant r$, and

$$D^{n+2q} = A^n \oplus \bigoplus_{i=-1}^{2q-1} B^{n+i}$$

whenever $m + 1 \leqslant n \leqslant n + 2q \leqslant r$.

Proof. Let

$$M_1 : O_G^{\mathrm{op}} \to \mathbb{Z}/p\text{-Mod}$$

be the coefficient system such that

$$M_1(G/H'') = \begin{cases} 0, & \text{if } H \cap H'' = H' \cap H'' \\ M(G/H''), & (H \cap H'')H' = H \end{cases}$$

and $M_1(f) = M(f)$ in the case where $f: G/H'' \to G/G''$ and $(H \cap H'')H' = H$. Then

$$H^*_{\{H\}}(K, M_1) = H^*_{\{H\}}(K_1, M) = H^*_{\{H'\}}(K_1, M).$$

There is a natural epimorphism $M \to M_1$. We shall denote its kernel by M_0. It is obvious that

$$H^*_{\{H\}}(K, M_0) = H^*_{\{H\}}(K, K_1; M).$$

and that

$$M' = M_0 \oplus M_1.$$

Let

$$\alpha_{M_0} : M_0[H/H'] \to M_0[H/H']$$

be the map described in 4.1. The image of α_{M_0} will be denoted by I. It is clear that I is equal to the image of α_M. We shall consider the following three exact sequences:

$$0 \to M \to M[H/H'] \to I \to 0,$$

$$0 \to I \to M[H/H'] \to M_1 \oplus M_0 \to 0,$$

$$0 \to M_0 \to M[H/H'] \to I \oplus M_1 \to 0.$$

The first and second of this sequence gives us the exact sequence described in 4.1. The second and third sequence occur in the Smith theory ([9]). The assumptions imply that there are isomorphisms

$$H^n_{\{H\}}(K, I) = A^{n+1},$$

$$H^{n+1}_{\{H\}}(K, I) = B^n \oplus D^n,$$

$$H^n_{\{H\}}(K, I) \oplus B^n = D^{n+1},$$

whenever $m \leqslant n \leqslant r-1$. Thus

$$A^{n+2} = B^n \oplus D^n, \quad \text{if } m \leqslant n \leqslant r-2,$$

$$D^{n+1} = B^n \oplus A^{n+1}, \quad \text{if } m \leqslant n \leqslant r-1,$$

$$A^{n+2} = B^{n-1} \oplus B^n \oplus A^n, \quad \text{if } m+1 \leqslant n \leqslant r-2,$$

$$D^{n+2} = B^n \oplus B^{n+1} \oplus D^n, \quad \text{if } m \leqslant n \leqslant r-2.$$

Using induction, we can now obtain the assertion of the proposition. □

The following fact is an immediate consequence of 4.7.

4.8. COROLLARY. *Assume that the hypothesis of* 4.7 *are fulfilled and that, for* $n = r,\ r-1$,

$$H^n_{\{H\}}(K, K_1; M) = 0.$$

Then

$$H^n_{\{H\}}(K, M) = 0$$

whenever $m \leqslant n \leqslant r$, *and*

$$H^n_{\{H\}}(K_1, M) = 0$$

whenever $m \leqslant n \leqslant r-1$. □

We shall now consider the case where G is a p-group and M is a G-coefficient system defined over $\mathbb{Z}/p$.

4.9. PROPOSITION. *Let G be a p-group and let*

$$M : O_G^{\mathrm{op}} \to \mathbb{Z}/p\text{-}Mod$$

be a G-coefficient system. Assume that K is a G-CW-complex and that q is a natural number such that, for every subgroup H of G,

$$H^q_{\{NH\}}(K^H, K^{>H}; M) = 0.$$

If

$$H^n_{\{e\}}(K, M) = 0 \quad \textit{whenever } m \leqslant n \leqslant q,$$

then, for every subgroup H of G,

$$H^n_G(K, M) = 0 = H^n_{\{NH\}}(K^H, K^{>H}; M) = H^n_{\{e\}}(K^H, K^{>H}; M)$$

whenever $m \leqslant n \leqslant q$. $\square$

Proof. Let

$$J: O_G^{\mathrm{op}} \to \mathbb{Z}/p\text{-Mod}$$

be the coefficient system such that, for every subgroup H of G, $J(G/H) = \mathbb{Z}/p$, and for every morphism f of O_G, $J(f) = 0 \cdot \mathrm{id}$. It is obvious that

$$J = \oplus J_{G/H},$$

where the sum is taken over all conjugacy classes of subgroups of G and $J_{G/H}$ satisfies the condition

$$H^*_G(K, J_{G/H}) = H^*((K^H, K^{>H})/NH; \mathbb{Z}/p).$$

For any coefficient system $T: O_G^{\mathrm{op}} \to \mathbb{Z}/p\text{-Mod}$,

$$H^n_G(K, T \otimes_{\mathbb{Z}/p} J_{G/H}) = H^n_G(GK^H, GK^{>H}; T) = H^n_{\{NH\}}(K^H, K^{>H}; T).$$

From Lemma 3.10 it follows that if

$$H^n_G(K, T \otimes_{\mathbb{Z}/p} J) = 0,$$

then

$$H^n_G(K, T) = 0.$$

(This well known fact can be also proved by induction on suitable subcomplexes of K.) There exists the natural epimorphism $\mathbb{Z}/p[G/e] \to J$. By I_p we shall denote the kernel of this map. The condition that

$$H^n_G(K, M \otimes_{\mathbb{Z}/p} J) = 0$$

implies that

$$H^n_G(K, M \otimes_{\mathbb{Z}/p} I_p \otimes_{\mathbb{Z}/p} J) = 0 = H^n_G(K, M \otimes_{\mathbb{Z}/p} I_p).$$

This is a consequence of the fact that, for any $\mathbb{Z}/p(NH/H)$-module W, there exists a composition series of $\mathbb{Z}/p(NH/H)$-modules

$$0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_s = W$$

such that, for each $i = 1, \ldots, s$, $W_i/W_{i-1} = \mathbb{Z}/p$ with the trivial NH/H-action. For any subgroup H of G, there is an exact sequence

$$0 \to M \otimes_{\mathbb{Z}/p} I_p \otimes_{\mathbb{Z}/p} J_{G/H} \to M[G/e] \otimes_{\mathbb{Z}/p} J_{G/H} \to M \otimes_{\mathbb{Z}/p} J_{G/H} \to 0$$

and

$$H^*_G(K, M[G/e] \otimes_{\mathbb{Z}/p} J_{G/H}) = H^*_{\{e\}}(K^H, K^{>H}; M).$$

Let us now consider the exact sequence

$$0 \to M \otimes_{\mathbb{Z}/p} I_p \to M[G/e] \to M \otimes_{\mathbb{Z}/p} J \to 0.$$

The result can be obtained using the long exact sequence of cohomology groups induced by this short exact sequence. □

4.10. COROLLARY. *Let G be a p-group and let K be a finite dimensional G-CW-complex. Assume that for each $n \geqslant m$,*

$$H^n(K, \mathbb{Z}/p) = 0.$$

Then, for any functor $T\colon O_G^{\mathrm{op}} \to \mathbb{Z}/p$-Mod and $n \geqslant m$,

$$H^n_G(K, T) = 0.$$ □

Proof. Proposition 4.9 implies that, for $n \geqslant m$ and $H \subseteq G$,

$$H^n(K^H, K^{>H}; \mathbb{Z}/p) = 0,$$

if we take $M = \mathbb{Z}/p[G/G]$. Hence, for $n \geqslant m$,

$$H^n_{\{e\}}(K^H, K^{>H}; T) = H^n(\mathrm{Hom}_{\mathbb{Z}/p}(C_*(K^H, K^{>H}; \mathbb{Z}/p), T(G/H))) = 0.$$

Thus, for $n \geqslant m$,

$$H^n_{\{e\}}(K, T) = 0$$

and we can apply 4.4. □

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