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Non-commutative Gauss map

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In this paper we develop the theory of the Gauss map and supporting functions of hypersurfaces in a compact Lie group G. If M is such a hypersurface, then left and right Gauss maps from M to the unit sphere of the Lie algebra $\mathfrak{g}$ are defined as $\alpha_l(x) = x^{-1}n(x)$, $\alpha_r(x) = n(x)\cdot x^{-1}$, where $n(x)$ is the normal to M at x. Supporting map τ is defined by $\tau = \alpha_r \circ \alpha_l^{-1}$. We show that τ determines a family of symplectomorphisms on the orbits of adjoint representations, endowed by the Kyrillov–Kostant symplectic structure. This is true for "nondegenerate" M. We show that the maximal degree of degeneracy is such that $\alpha_l(M)$ intersects an orbit in $\mathfrak{g}$ by a coisotropic manifold which may be Lagrangian.

Conversely, we present a construction which prescribes, to a symplectomorphism of an adjoint orbit or to a generic pair of Lagrangian submanifolds, a foliation in G. This can be looked at as a generating object in the classical sense of Hamilton–Jacobi. This construction works in the case of noncompact G and even if G is infinite-dimensional (we shall pass to coadjoint orbits in these cases). The exposition for the infinite-dimensional case will appear later.

We derive from our approach the full description of flat surfaces in S^3, which were investigated earlier by Kitagawa and others ([Kit]). We show that Gauss images of such a surface are two smooth curves and some curvature inequalities are satisfied. Conversely, every two such curves determine a flat foliation in S^3 with an exceptional torus deleted, and we state the necessary and sufficient conditions for existence of a compact leaf.

1. Basic equations

Consider the standard euclidean sphere S^3, embedded in the quaternionical space $\mathbb{R}^4$, with the induced structure of the compact Lie group. We will identify the Lie algebra with the tangent space $\mathbb{R}^3$ at 1, consisting of imaginary quaternions. Let S^2 be the unit sphere in $\mathbb{R}^3$. We will freely identify the tangent vectors to S^3 with the elements of $\mathbb{R}^4$ and the left and right actions of S^3 in TS^3 with the usual quaternionical multiplication in $\mathbb{R}^4$.

Let M be a smooth oriented surface in S^3 and for $x \in M$ let $n(x)$ be the positive normal vector to M at x. We define left and right Gauss maps as $\alpha(x) = x^{-1}n(x)$, $\beta(x) = n(x)x^{-1}$ both maps from M to S^2.

DEFINITION. A point $x \in M$ will be called (left) regular if the Gauss map α is the local diffeomorphism at x.

DEFINITION. The support map of M at the regular point x is the locally defined smooth map $\tau = \beta \circ \alpha^{-1}$ from some neighbourhood of $\alpha(x)$ to S^2.

If all $x \in M$ are regular then τ is globally defined in S^2. Let $v \in S^2$, $v = \alpha(x)$, and x is regular, then evidently $\tau(v) = xvx^{-1} = (\mathrm{Ad}\, x)v$. From now on all computations will be made in some neighbourhoods of v and x. Let $X \in T_vS^2$ and let us write simply $x = x(v)$ instead of $x = \alpha^{-1}(v)$. Differentiating the equality $xv = \tau(v)x$ along X we obtain $x'_X v + xX = \tau_* X x + \tau(v)x'_X$ where $\tau_*\colon T_vS^2 \to T_{\tau(v)}S^2$ is the derivative of τ. Multiplying by x^{-1} from the left and taking into account that $x^{-1}\tau(v) = vx^{-1}$ we will have $x^{-1}x'_X v - vx^{-1}x'_X + X = x^{-1}\tau_* X x$ or $[x^{-1}x'_X, v] + X = (\mathrm{Ad}\, x^{-1})\tau_* X$. From now on denote by J_v or simply J the linear orthogonal operator in T_vS^2 defined by the formula $J_v(\cdot) = \frac{1}{2}[\cdot, v]$ (we use the Lie algebra brackets in $\mathbb{R}^3$). Further, since $x'_X \in T_xM$, $n(x)$ is orthogonal to T_xM and $x^{-1}n(x) = v$, we have $x^{-1}x'_X \in T_vS^2$. We will denote the linear operator $X \mapsto x^{-1}x'_X$ in T_vS^2 by Φ_v or Φ. Thus we obtain

$$2J_v\Phi_v + E_v = \mathrm{Ad}\, x^{-1} \circ \tau_* \tag{1}$$

where E_v is the identity map. Note that $J_v^2 = -E_v$.

Now we want to use the "integrability" of the distribution of the tangent planes to M to obtain additional equations containing Φ. For this purpose we will compute the second fundamental operator of M.

LEMMA 1. *Let G be a compact Lie group supplied with bi-invariant positive Riemannian metric and the corresponding Levi–Civitta connection ∇. Let $x(t)\colon [0, d] \to G$, $x(0) = e$, and $v(t)\colon [0, d] \to \mathfrak{g}$ be smooth curves and let $n(t) = x(t)v(t)$ be the left shift of $v(t)$ so $n(t)$ is a vector field along $x(t)$. Then*

$$\nabla_{x'(0)}n(t) = \tfrac{1}{2}[x'(0), v(0)] + v'(0). \tag{2}$$

Proof. We can decompose $v(t)$ as $v(0) + t\mu(t)$, $\mu(0) = v'(0)$. Since $x(t)v(0)$ is the restriction of the left-invariant vector field on G, and $\nabla_X Y = \frac{1}{2}[X, Y]$ for left-invariant fields ([Ar]), then $\nabla_{x'(0)}x(t)v(0) = \frac{1}{2}[x'(0), v(0)]$. It is easy to show that $\nabla_{x'(0)}(tx(t)\mu(t)) = \mu(0)$ which proves the lemma.

Now let $x \in M$ be regular, $v = \alpha(x)\, X \in T_vS^2$ and $Z = x_*(X)$ (expressions x'_X and $x_*(X)$ means the same vector in T_xM, but we prefer the former expression when computations are made in $\mathbb{R}^4$). Let $v(t)$ be a smooth curve tangent to X, $v(0) = v$, and $x(t) = \alpha^{-1}(v(t))$. Since $n(x(t)) = x(t)v(t)$, the second fundamental symmetric operator in T_xM can be expressed as $x'(0)\mapsto\nabla_{x'(0)}x(t)v(t)$. Let $\tilde{x}(t)=x^{-1}(0)x(t)$, then $\tilde{x}(0)=1$ and by the previous lemma we will have

$$\nabla_{x'(0)}x(t)v(t)=x(0)\nabla_{\tilde{x}'(0)}\tilde{x}(t)v(t)=\tfrac{1}{2}x(0)[\tilde{x}'(0),v(0)]+x(0)v'(0)=\tfrac{1}{2}x(0)\\ \times[x^{-1}(0)x'(0),v(0)]+x(0)v'(0)=\tfrac{1}{2}x[x^{-1}Z,v]+xX.$$

So the second fundamental operator A_x has the form $A_x(Z)=\frac{1}{2}x[x^{-1}Z,v]+xX$. Since the left shift $X\mapsto xX$ orthogonally maps T_vS^2 onto T_xM, we can pull back the operator A_x to T_vS^2 and denote $A_vX=x^{-1}A_x(xX)$. As $Z=x^*X$ and $x^{-1}Z=x^{-1}x_*X=\Phi_vX$ by the definition of Φ_v, we obtain that $A_v\Phi_vX=\frac{1}{2}[\Phi_vX,v]+X=J_v\Phi_vX+X$, so

$$A_v\Phi_v=J_v\Phi_v+E_v \tag{3}$$

or

$$A_v=J_v+\Phi_v^{-1} \tag{4}$$

because Φ_v is invertible, and

$$\Phi_v=(A_v-J_v)^{-1}. \tag{5}$$

Recalling (1), we can write

$$(\mathrm{Ad}\ x^{-1})\circ\tau_*=2J_v\Phi_v+E_v=2J_v(A_v-J_v)^{-1}+(A_v-J_v)(A_v-J_v)^{-1}\\ =(A_v+J_v)(A_v-J_v)^{-1}.$$

THEOREM 1. *For any regular $x\in M$ the support map τ is an area-preserving map from a neighbourhood of $v=\alpha(x)$ to a neighbourhood of $\tau(v)=\beta(x)$.*

Proof. As we have just seen,

$$(\mathrm{Ad}\ x^{-1})\circ\tau_*=(A_v+J_v)(A_v-J_v)^{-1}. \tag{6}$$

As Ad x is the rotation of S^2 it is sufficient to show that $\det((\mathrm{Ad}(x^{-1})\circ\tau_*)=1$. But for a symmetric operator A in the euclidean oriented 2-space and the "multiplication by $\sqrt{-1}$ in J, having the matrix $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ in every oriented orthonormed base, $\det(A\pm J)=\det A+1$, which proves the theorem.

Let $K(x)$ be the sectional curvature of M at x, then by the Gauss formula, $K(x)=\det A_x+1$. We can replace A_x by A_v and write

$$K(\alpha^{-1}v)=\det A_v+1 \tag{7}$$

when $x \in M$ is regular and $v = \alpha(x)$. Let ds and dv be the area 2-forms on M and S^2 respectively. Since $\varphi_v = x^{-1}x_*$, we see that in some neighbourhoods of x, v, $(\alpha^{-1})^* \mathrm{d}s = (\det \Phi)\mathrm{d}v$, so $\alpha^* \mathrm{d}v = (\det \Phi^{-1})\mathrm{d}s$. Using (5) and (7) we obtain $\alpha^* \mathrm{d}v = K \mathrm{d}s$.

THEOREM 2. *For any M, the following "Gauss formula" is valid*:

$$\alpha^* \mathrm{d}v = \beta^* \mathrm{d}v = K \mathrm{d}s. \tag{8}$$

Proof. If $x \in M$ is regular, we have just obtained that $\alpha^* \mathrm{d}v = K \mathrm{d}s$ in some neighbourhoods of x and v. By Theorem 1, $\alpha^* \mathrm{d}v = \beta^* \mathrm{d}v$ because $\tau = \beta \circ \alpha^{-1}$ is area-preserving. Note that the regularity of x is equivalent to $(\alpha^* \mathrm{d}v)_x \neq 0$. So (8) is valid where the left side $\neq 0$. It is clear that we could start from β instead of α, so (8) is valid where $\beta^* \mathrm{d}v \neq 0$. Hence $\alpha^* \mathrm{d}v = \beta^* \mathrm{d}v$ everywhere. Approximating M by analytic surfaces we see (8) to be valid if $\alpha^* \mathrm{d}v$ or $\beta^* \mathrm{d}v$ is not identically equal to zero. So the only thing remaining is to show that if $\alpha^* \mathrm{d}v = \beta^* \mathrm{d}v = 0$ on M then $K = 0$. We will show it later in Section 5. Note that the implication $K = 0 \Rightarrow \alpha^* \mathrm{d}v = \beta^* \mathrm{d}v = 0$ is already shown.

COROLLARY. *A point $x \in M$ is regular if and only if $K(x) \neq 0$. If M is compact and $K \neq 0$ on M then $K > 0$, M is diffeomorphic to S^2 and τ is the globally defined area-preserving diffeomorphism of S^2.*

Proof. The only thing that needs to be proved is $K \neq 0 \Rightarrow K > 0$. But if $K < 0$ then the Euler number $\chi(M) < 0$ by the Gauss–Bonnet formula, which contradicts with $\alpha: M \to S^2$ being the diffeomorphism.

We will conclude this section with some curvature formulas. Let $H(x)$ be the mean curvature of M at x, so

$$H(x) = \frac{\lambda_x + \mu_x}{2}, \qquad K(x) = \lambda_x \mu_x + 1,$$

where λ_x, μ_x are the eigenvalues of A_x. If x is regular and $v = \alpha(x)$, then A_v can be represented by the matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ in some oriented orthonormed base, so by (1), $(\mathrm{Ad}\, x^{-1}) \circ \tau_*$ will be represented by the matrix

$$\frac{1}{\lambda\mu + 1} \begin{pmatrix} \lambda\mu - 1 & -2\lambda \\ 2\mu & \lambda\mu - 1 \end{pmatrix}. \tag{9}$$

It follows immediately that

$$K(x) = \frac{4}{2 - \mathrm{Tr}(\mathrm{Ad}(x^{-1})) \circ \tau_*}, \qquad H(x) = -\frac{K(x)}{4} \mathrm{Tr}((\mathrm{Ad}(x^{-1})) \circ \tau_* \circ J_v). \tag{10}$$

We will use these formulas in Section 4.

2. Some properties and examples

Let $\gamma(x, t)$ be the normal geodesic, orthogonal to M at the point $x = \gamma(x, 0)$. It is clear that $\gamma(x, t) = x \exp tv$, where $v = \alpha(x)$. Given $\varepsilon > 0$ we define the equidistant M_ε as the parameterized surface $x \mapsto \gamma(x, \varepsilon)$ (we do not use the usual metric definition to avoid the "boundary effect" when M is noncompact). To be sure that M_ε is the embedded surface, we always assume that ε is sufficiently small and M is a proper open set of some other embedded surface $\tilde{M}$. Let $\pi_\varepsilon\colon M_\varepsilon \to M$ be the natural projection.

PROPOSITION 1. $\alpha_\varepsilon = \alpha \circ \pi_\varepsilon$, $\beta_\varepsilon = \beta \circ \pi_\varepsilon$, $\tau_\varepsilon = \tau$.

Proof. It is clear that the normal vector to M_ε at the point $\gamma(x, \varepsilon)$ is $d/(d\varepsilon)\gamma(x, \varepsilon)$. As $\gamma(x, t) = x \exp tv$, we see that $n_\varepsilon(\gamma(x, \varepsilon)) = x \exp \varepsilon v \cdot v$ so $\alpha_\varepsilon(\gamma(x, \varepsilon)) = (x \exp \varepsilon v)^{-1} x \exp \varepsilon v \cdot v = v$. This proves the lemma for $v = \alpha(x)$ and $x = \pi_\varepsilon(\gamma(x, \varepsilon))$.

This proposition shows that given τ, we cannot expect the correspondent M to be unique, because τ determines the "equidistant foliation" rather than the single leaf M. This is exactly so, as we will see later in Section 5. The situation becomes different, however, if we put additional restrictions on M.

PROPOSITION 2. *If $K \neq 0$ on M then M is minimal if and only if* $(\mathrm{Ad}\, x^{-1}) \circ \tau_*$ *is symmetric for all $v \in \alpha(M)$.*

Proof. This follows immediately from (10) and the fact that a linear operator B in the euclidean 2-space is symmetric if and only if $\mathrm{Tr}\, BJ = 0$.

The two conditions: (1) $(\mathrm{Ad}\, x)v = \tau(v)$ and (2) $(\mathrm{Ad}\, x^{-1}) \circ \tau_*$ is symmetric determine $x = x(v)$. Namely, $\tau_*\colon T_v S^2 \to T_{\tau(v)} S^2$ admits the polar decomposition $\tau_* = U_v P_v$ where $P_v\colon T_v S^2 \to T_v S^2$ is symmetric and positive and $U_v\colon T_v S^2 \to T_{\tau(v)} S^2$ is orthogonal. It follows immediately that $\mathrm{Ad}\, x|_{T_v S^2} = U_v$ which determined $\mathrm{Ad}\, x$, and, consequently, determines x up to the (± 1) multiplier.

The condition, xv is normal to M at x, means that there are some equations the support function (map) τ of the minimal M must yield.

PROPOSITION 3. *M has the constant curvature if and only if* $\mathrm{Tr}(\mathrm{Ad}\, x^{-1}) \circ \tau_* = \mathrm{const}$.

Proof. This follows from (10). One can see that the condition $\mathrm{Tr}(\mathrm{Ad}\, x^{-1}) \circ \tau_* = C$ determines $x(v)$ by τ, so some additional equations on τ of the *sh*-Gordon type must exist.

Let us look at some examples. If M is the sphere $S(1, r)$ with center 1, then τ is the identical map. If M is the sphere $S(u, r)$ with center u then it can be parameterized as

$$v \xrightarrow{\alpha^{-1}} u \exp rv \quad \text{and} \quad \tau = \mathrm{Ad}\, u,$$

so τ is an isometry. Let M be the quadric $x_0^2 - x_2^2 - x_3^2 = 0$ with two singular points $\pm(0, 1, 0, 0)$. Then the direct computation shows that $\tau(v_1 i + v_2 j + v_3 k) = \beta_1 i + \beta_2 j + \beta_3 k$, where

$$\beta_1 = v_1, \qquad \begin{pmatrix}\beta_2\\ \beta_3\end{pmatrix} = \begin{pmatrix} a(v_1) & b(v_1)\\ -b(v_1) & a(v_1)\end{pmatrix}\begin{pmatrix}v_2\\ v_3\end{pmatrix} \tag{11}$$

for some $a(v_1)$, $b(v_1)$ satisfying $a^2(v_1)+b^2(v_1) = 1$ (namely, $a(v_1) = 1-3v_1^2/1-v_1^2$ and $b(v_1) = 2v_1\sqrt{1-2v_1^2}/1-v_1^2$).

DEFINITION. A Blaschke product is a map of the form $\tau = \psi_1^{-1}\rho_1\psi_1\psi_2^{-1}\rho_2\psi_2\cdots\psi_m^{-1}\rho_m\psi_m$ where ψ_k are area-preserving diffeomorphisms of S^2 and ρ_k have the form (11) with some C^∞-functions $a(v_1)$, $b(v_1)$.

CONJECTURE. *Every area-preserving diffeomorphism of S^2 is a C^0-limit of Blaschke products.*

3. The description of flat surfaces

LEMMA 2. *For any M and $x \in M$*

(1) rank $\alpha_*|_{T_xM} \geqslant 1$, rank $\beta_*|_{T_xM} \geqslant 1$,
(2) *if* $\alpha_* X = 0$ *then* $(A_x X, X) = 0$,
(3) ker $\alpha_* \cap$ ker $\beta_* = 0$ *in* T_xM.

We will prove the lemma in a more general context in Section 5. Assume that M is flat, so $K = 0$ and rank $\alpha_* < 2$, rank $\beta_* < 2$ by Theorem 2. Then we see that α_*, β_* have the constant rank one and that their kernels are asymptotic directions in T_xM. So the next proposition is valid.

PROPOSITION 3. *If M is flat, then $\alpha(M)$ and $\beta(M)$ are immersed curves in S^2 (maybe, with self-intersections). Both maps α, β foliate M onto foliations with asymptotical lines as their leaves. In particular, every asymptotic line is closed in M.*

We are now able to prove the main result of Kitagawa ([Kit]):

THEOREM 3 (Kitagawa). *If M is flat and compact, then all its asymptotic lines are periodic.*

Kitagawa proved this by using special coordinate systems in his profound investigation of flat surfaces. In this case both $\alpha(M)$, $\beta(M)$ are closed immersed curves in S^2.

PROPOSITION 4. *If M is flat, then for sufficiently small $|\varepsilon|$, all its equidistants M_ε are also flat.*

Proof. By Theorem 2, $K=0 \Leftrightarrow \text{rank}\ \alpha_* < 2$ on M. Since $\alpha_\varepsilon = \alpha \circ \pi_\varepsilon$ (see Proposition 1) we have $\text{rank}\, \alpha_\varepsilon < 2$, so $K_\varepsilon = 0$. Moreover, the curves $\alpha_\varepsilon(M_\varepsilon)$, $\beta_\varepsilon(M_\varepsilon)$ coincide with $\alpha(M)$, $\beta(M)$.

In return, we will see in Sections 5 and 6 that any two curves in S^2 determine some foliation with flat leaves in an appropriate open set in S^3. Given some additional conditions, some leaves of this foliation turn out to be compact.

THEOREM 4. *In the conditions of Theorem* 3, *every two unknotted asymptotic lines belonging to the same* (*left or right*) *foliation are linked in* S^3.

Proof. Let $\delta(t)$ be an asymptotic line belonging to the left foliation, so $\alpha(\delta(t)) = v = \text{const}$. It follows that $\delta'(t) \perp \delta(t)v$ in $T_{\delta(t)}S^3$, because $n(t) = \delta(t)v$ by the definition of the map α. Consider the left-invariant unit vector field $v_v(x) = xv$. Let $V_v(x)$ be the plane distribution, orthogonal to $v_v(x)$. It is well-known that V_v determines the standard contact structure in S^3 (and also the canonical connection in the Hopf principal $SO(2)$-bundle over S^2). We see that $\delta(t)$ is a horizontal curve of this contact structure. By the Bennequin theorem ([Ben]) the linking number between $\delta(t)$ and its small shift $\delta_1(t)$ in the direction $n(\delta(t))$ is non-zero. Consider a unit vector field $m(t)$ along $\delta(t)$ defined by the following conditions: (1) $m(t) \in T_{\delta(t)}M$ and (2) $m(t) \perp \delta'(t)$. It is evident that every leaf of the left foliation which is sufficiently close to $\delta(t)$ can be isotopically deformed to the shift $\delta_2(t)$ of $\delta(t)$ in the direction $m(t)$, such that it will never intersect $\delta(t)$. Let $p_\sigma(t)$, $0 \leqslant \sigma \leqslant \pi/2$, be the vector field $\cos \sigma m(t) + \sin \sigma n(t)$ along $\delta(t)$. Since $n(t) \perp m(t)$, the shift $\delta_\sigma(t)$ in the direction $p\sigma(t)$ determines the isotopy between $\delta_1(t)$ and $\delta_2(t)$ which proves the theorem.

Using the methods of Section 5, one can show that every embedded horizontal curve of the standard contact structure in S^3, having the "good" (with only transversal self-intersections) front in S^2, lies on some flat surface.

4. Curvature of equidistants and the Weyl tube's volume formula in S^3

LEMMA 3. *In the notation of Proposition* 1, *let* K_ε *be the* (*sectional*) *curvature of* M_ε, *let* $x \in M$ *be regular and let* $\pi_\varepsilon(x_\varepsilon) = x$. *Then*

$$K_\varepsilon(x_\varepsilon) = \frac{K(x)}{K(x)\sin^2\varepsilon + \cos 2\varepsilon + H(x)\sin 2\varepsilon}. \tag{12}$$

Proof. Again denote $v = \alpha(x)$, so $x_\varepsilon = x \exp \varepsilon v$ by the proof of Proposition 3. We are going to use (10), so we write

$$K_\varepsilon(x_\varepsilon) = \frac{4}{2 - \text{Tr}((\text{Ad}\, x_\varepsilon^{-1}) \circ \tau_*)}.$$

Assume that $(\mathrm{Ad}\, x^{-1})\circ\tau_*$ is represented by a matrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ in some oriented orthnonormed base. Since $\mathrm{Ad}\ \exp(-\varepsilon v) = \exp \mathrm{ad}(-\varepsilon v) = \exp 2\varepsilon J_v$ and J_v is represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$, the matrix of the operator $\mathrm{Ad}\exp(-\varepsilon v)$ will be $\begin{pmatrix} \cos 2\varepsilon & -\sin 2\varepsilon \\ \sin 2\varepsilon & \cos 2\varepsilon\end{pmatrix}$. Hence

$$\mathrm{Tr}\ \mathrm{Ad}\ \exp(-\varepsilon v)\circ \mathrm{Ad}\, x^{-1}\circ\tau_* = \cos 2\varepsilon(a+d) + \sin 2\varepsilon(b-c).$$

From (10) we derive that

$$a + d = 2 - \frac{4}{K(x)} \quad \text{and} \quad (b-c) = -\frac{4H(x)}{K(x)}$$

so

$$K_\varepsilon(x_\varepsilon) = \frac{4}{2-\cos 2\varepsilon\left(2-\dfrac{4}{K(x)}\right) + \sin 2\varepsilon\,\dfrac{4H(x)}{K(x)}}$$

which is equivalent to (12).

Moreover, in the same way we obtain

$$H_\varepsilon(x_\varepsilon) = \frac{H(x)\cos 2\varepsilon + \frac{1}{2}K(x)\sin 2\varepsilon - \sin 2\varepsilon}{K(x)\sin^2\varepsilon + \cos 2\varepsilon + H(x)\sin 2\varepsilon}.$$

LEMMA 4. *If S_ε, S are respectively the areas of M_ε, M, then*

$$S_\varepsilon = \sin^2\varepsilon \int_M K\,\mathrm{d}s + \cos 2\varepsilon S + \sin 2\varepsilon \int_M H\,\mathrm{d}s. \tag{14}$$

Proof. Assume first that $K\neq 0$ on M, so all $x\in M$ are regular. From Proposition 1 and Theorem 2 it follows that

$$\mathrm{d}s_\varepsilon = \frac{K}{K_\varepsilon}\pi_\varepsilon^*\,\mathrm{d}s \quad \text{where} \quad \mathrm{d}s_\varepsilon$$

is the area 2-form on $M-\varepsilon$. Hence

$$S_\varepsilon = \int_{M_\varepsilon} \mathrm{d}s_\varepsilon = \int_M \frac{K}{K_\varepsilon}\,\mathrm{d}s,$$

which together with (12) implies (14). In the general case, we can divide M into small pieces N_k. Every such piece can be deformed in such a way that its curvature becomes non-zero, which enables us to apply (14). By the limit procedure, (14) remains valid for N_k, and, by additivity, for the whole of M.

COROLLARY 1. *If M is compact and $\chi(M)$ is its Euler number, then*

$$S_\varepsilon = 2\pi \sin^2\varepsilon\chi(M) + \cos 2\varepsilon S + \sin 2\varepsilon \int_M H\, ds. \tag{15}$$

COROLLARY 2. *If M is flat, or M is compact and $\chi(M) = 0$, then $(d^2/d\varepsilon^2 S_\varepsilon)_{\varepsilon=0} > 0$. Hence no open subset U of S^3 can be fibrated over S^1 by flat equidistant fibers.*

5. Gauss map theory for hypersurfaces in a compact Lie group

Let G be a compact Lie group supplied with bi-invariant Riemannian metric (which is unique up to the constant multiplier if G is simple). Let S be the unit sphere in the Lie algebra $\mathfrak{g}$. The natural isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ enables us to pull back to $\mathfrak{g}$ the canonical Kyrillov–Kostant symplectic forms on the coadjoint orbits in $\mathfrak{g}^*$. If $v \in S$, $P(v)$ is its adjoint orbit in S, $V = T_vP \subset T_vS$, $J_v\colon T_vS \to T_vS$ is defined as $J_v = -\frac{1}{2}\operatorname{ad} v$, then we have the orthogonal decomposition $T_vS = T_vP \oplus \ker J_v$ and for $X, Y \in T_vP$ the value of the $K-K$ symplectic form Ω_v will be $\Omega_v(X, Y) = (J_v^{-1}X, Y)$, where $J_v^{-1}X$ means any vector Z such that $J_vZ = X$.

Let M be an oriented hypersurface in G. We define the Gauss maps $\alpha, \beta\colon M \to S$ and the support map $S \supset U \xrightarrow{\tau} S$ in the neighbourhood of $\alpha(x)$ where x is a regular point of M, exactly as in Section 1. Using any exact unitary representation of G, we can look at G as a subgroup of the group of invertible elements in some algebra R. This enables us to make computations which lead to (1), where Φ_v is defined in the same way. All formulas (2)–(6) remain valid, too. Since $\tau(v) = (\operatorname{Ad} x)v$ where $x = \alpha^{-1}(v)$, every adjoint orbit in S is invariant under the map τ.

THEOREM 1′. *If $x \in M$ is regular, $v = \alpha(x)$ then the restriction $\tau|_{P(v)}$ is the symplectomorphism from a neighbourhood of v to a neighbourhood of $\tau(v)$ in the symplectic manifold $P(v)$.*

Proof. If x is fixed then of course $\operatorname{Ad} x\colon P(v) \to P(v)$ is the symplectomorphism. Using (6) we reduce the statement of the theorem to the following lemma.

LEMMA 5. *Let W be an euclidean space, let J, A be respectively a skew-symmetric and symmetric operators in W, let $V = J(W)$, let $\Omega\colon V \wedge V \to \mathbb{R}$ be the* *symplectic form defined as* $\Omega(X, Y) = (J^{-1}X, Y)$. *Then if* $A - J$ *is invertible, then the operator* $(A+J)(A-J)^{-1}$ *has determinant* 1, *leaves* V *invariant and preserves the form* Ω.

Proof. Assume first that J is invertible, so $V = W$ (and dim W is even). For $\lambda = \pm 1$ and $Z, H \in W$ we have

$$\Omega((A+\lambda J)Z, (A+\lambda J)H) = (J^{-1}AZ + \lambda Z, AH + \lambda JH)$$
$$= (AJ^{-1}AZ, H) + \lambda^2(Z, JH) + \lambda(Z, AH) + \lambda(J^{-1}AZ, JH)$$

Since A is symmetric and J is skew-symmetric, the last two terms vanish, so the right side does not depend on λ, which proves the lemma. In the general case we see that V is invariant because $(A+J)(A-J)^{-1} = 2J(A-J)^{-1} + E$. Disturbing J to be invertible and expanding W to $W \oplus \mathbb{R}$ if dim W is odd we reduce this case to the previous one.

THEOREM 2′. *For any* M, $\alpha^* \mathrm{d}v = \beta^* \mathrm{d}v$.

Proof. This follows from Lemma 5 (see the proof of Theorem 2).

The full analogue of Proposition 1 is valid, too. Now we will formulate the analogue of Lemma 2.

LEMMA 2′. *Let* $x \in M$, $v = \alpha(x)$ *and let* $P(v)$ *be the adjoint orbit of* v *in* S. *Then*

(1) $\alpha_* T_x M \cap T_v P(v)$ *is coisotropic in the symplectic space* $T_v P(v)$, *hence* $\dim \alpha_* T_X M \geqslant \frac{1}{2} \dim P(v)$,
(2) *if* $\alpha_* X = 0$ *then* $(A_x X, X) = 0$,
(3) $\dim(\ker \alpha_* | T_x M \cap \ker \beta_* | T_x M) \leqslant \dim S - \dim P(v)$.

Proof. Let $X \in T_x M$ and $x(t)$ be tangent to X. Let $v(t) = \alpha(x(t))$, so $n(x(t)) = x(t)v(t)$, hence $\nabla_{x'(0)} n(x(t)) = \nabla_{x'(0)} x(t)v(t)$. The left side is equal to $A_x(X)$, while the right side is equal to $xJ_v(x^{-1}X) + xv'(0)$ by Lemma 1. It is clear that $v'(0) = \alpha_* X$, so denoting $Z = x^{-1}X$ we have $A_x(X) = x\alpha_*(X) + xJ_v Z$. Similarly, $A_x(X) = \beta_*(X)x - J_\mu W x$, where $\mu = \beta(x)$, $W = Xx^{-1}$, if we use an evident analogue of Lemma 1. To prove (3) we note that $\alpha_*(X) = \beta_*(X) = 0$ implies $(\mathrm{Ad}\, x)J_v Z = -J_\mu W$ or $(\mathrm{Ad}\, x)[Z, v] = -[W, \mu]$, which together with $(\mathrm{Ad}\, x)v = \mu$, $(\mathrm{Ad}\, x)Z = W$ and Ad x's being the automorphism of $\mathfrak{g}$ implies $J_v Z = J_\mu W = 0$ so $\dim(\ker \alpha_* \cap \ker \beta_*) \leqslant \dim \ker J_v = \dim S - \dim P(v)$. Further, if $\alpha_*(X) = 0$ then $A_x(X) = xJ_v Z$ hence $(A_x X, X) = (J_v Z, Z) = 0$, because J_v is skew-symmetric. At last, it is not hard to show (1) following the proof of Lemma 5.

COROLLARY 1. *Clean intersections of the Gauss map's images* $\alpha(M)$, $\beta(M)$ *with every adjoint orbit in* S *are either empty sets, or coisotropic varieties.*

So the "extremal" case will occur if these intersections are Lagrangian. This does happen, as we will see soon. We now remark, that if $L_1 = \alpha_* T_x M \cap T_v P(v)$

and $L_2 = \beta_* T_x M \cap T_\mu P(v)$ are Lagrangian, then $(\mathrm{Ad}\, x)L_1$ is transversal to L_2, as similar arguments show.

COROLLARY 2. *If M is compact and its second fundamental form is positive then M is diffeomorphic to the sphere $S^{\dim G - 1}$* by any of Gauss maps.

Proposition 1 holds without any alteration. It follows that we must expect that the support map τ determines an equidistant codimension 1 foliation in G rather than a single hypersurface. As we saw in Theorem 1′, the support map of a surface M in the neighbourhood of a regular $x \in M$ can be looked at as a family of adjoint orbit's symplectomorphisms. It seems to be a complicated problem to reconstruct M from these data. However, if we have a symplectomorphism of a *single* orbit P, it does determine some foliation which we will call to be of P-type, because the image of the Gauss maps α, β in S coincide with P. In the case $G = S^3$ it does not put any restrictions, because there is only one orbit in S^2.

THEOREM 5. *Let U_1, U_2 be open sets in P, let $\tau: U_1 \to U_2$ be a symplectomorphism and let $\psi(\cdot)$ be the following multivalued function: $\psi(x)$ = set of the fixed points of $(\mathrm{Ad}\, x^{-1}) \circ \tau$. If $U \subset G$ is an open set and $v(x)$ is a smooth branch of $\psi(x)$, then the hyperplane distribution $V(x)$ orthogonal to $x \cdot v(x)$ is integrable in U and the support map of its leaves coincide with τ where both maps are defined.*

Proof. For any Riemannian manifold N and a unit vector field $n(x)$ the second fundamental operator $A_x: V(x) \to V(x)$ in the orthogonal hyperplane can be defined by the formula $X \mapsto \nabla_X n$. It is well-known that the distribution $V(x)$ is integrable if and only if A_x is symmetric and the correspondent foliation is equidistant if and only if $\nabla_n n = 0$. In our case the verification of the conditions can be easily made if we follow the proof of Theorem 1′ in the opposite direction.

EXAMPLE. Let $G = S^3 \times S^3$, so $\mathfrak{g} = \mathbb{R}^3 \oplus \mathbb{R}^3$. Take $P = S^2 \times S^2 \subset S = S^5$ and $\tau: (p, q) \mapsto (q, p)$. Then the leaves of the correspondent foliation will be $\{(x, y) \mid \mathrm{Tr}\, \mathrm{Ad}\, x \cdot \mathrm{Ad}\, y = \mathrm{const}\}$.

We will say briefly about the non-compact case a bit later and now we describe a "flat" situation.

THEOREM 6. *Let L_1, L_2 be Lagrangian submanifolds in P and let $\psi(\cdot)$ be the following multivalued function: $\psi(x) = (\mathrm{Ad}\, x)L_1 \cap L_2$. If $U \subset G$ is an open set and $v(x)$ is a smooth branch of $\psi(x)$ such that $(\mathrm{Ad}\, x)L_1$ intersects L_2 transversally at $(\mathrm{Ad}\, x)v(x)$ then the hyperplane distribution $V(x)$ is integrable and the images of α, β lie in L_1, L_2.*

Proof. Let $x \in U$, $v = v(x)$, $Z \in T_v S$, so $xZ = X \in V(x)$. Let $x(t) = x \exp tZ$. Then by Lemma 1 $A_x(xZ) = xv'_{xZ} + xJ_v Z$. As $y(t) = (\mathrm{Ad}(x \exp tZ))v(x \exp tZ) \in L_2$, we see that $\mathrm{d}/\mathrm{d}t_{t=0}\, y(t) \in T_{(\mathrm{Ad}\, x)v} L_2$. The left side is equal to $\mathrm{d}/\mathrm{d}t_{t=0}(\mathrm{Ad}\, x \cdot \exp \mathrm{ad}(tZ))v(x \exp tZ) = \mathrm{Ad}\, x(v'_{xZ} + \mathrm{ad}\, Zv) = \mathrm{Ad}\, x(v'_{xZ} + 2J_v Z)$. Let

$l_1 = T_v L_1$, $l_2 = (\mathrm{Ad}\, x^{-1}) T_{(\mathrm{Ad}\, x)v} L_2$, $l = T_v P$, then $l = l_1 \oplus l_2$ by transversality and we see that $v'_{xZ} + 2J_v Z \in l_2$. Also $v'_{xZ} \in l_1$ because $v(x) \in L_1$ for all x. Let $p_i: l \to l_i$, $i = 1, 2$ be natural projections, then we see that $p_1(J_v Z) = -\frac{1}{2} v'_{xZ}$, $p_2(J_v Z) = J_v Z + \frac{1}{2} v'_{xZ}$. Hence $x^{-1} A_x(xZ) = v'_{xZ} + J_v Z = (-2p_1 + E_v) J_v Z$. So we must show that the operator $Z \mapsto (-2p_1 + E_v) J_v Z$ is symmetric, or $((-2p_1 + E_v) J_v Z, H) = ((-2p_1 + E_v) J_v H, Z)$. Let $Z_1 = J_v Z$, $H_1 = J_v H$. By the formula $\Omega(X, Y) = (J^{-1} X, Y)$, $((-2p_1 + E_v) J_v Z, H) = -\Omega((-2p_1 + E_v) Z_1, H_1)$. So we have reduced the statement of the theorem to the following: given a Lagrangian decomposition $l = l_1 \oplus l_2$ of a symplectic space (l, Ω) to show that $\Omega(AX, Y) = \Omega(AY, X)$ where $A = -2p_1 + E = p_2 - p_1$, which is obvious.

In the case $G = S^3$ we have $\det J_v = 1$ and $\det(p_2 - p_1) = -1$, so $\det A_x = -1$. This enables us to finish the proof of Theorem 3 in Section 1. Indeed, if $\operatorname{rank} \alpha_* = \operatorname{rank} \beta_* = 1$, then M is a leaf of the corresponding foliation constructed by $\alpha(M)$, $\beta(M)$, and $K(x) = \det A_x + 1 = 0$, hence M is flat.

We will say some words about the non-compact case. If G is an arbitrary real Lie group, then the torsion-free connection ∇ can be defined by the formula $\nabla_X Y = \frac{1}{2}[X, Y]$ where X, Y are left-invariant vector fields. If $U \subset G$ is an open set and $\omega(x)$ is a 1-form in U which is nowhere zero, then the hyperplane distribution $V(x) = \ker \omega(x)$ is integrable if and only if the second fundamental form $A_x(X, Y) = (\nabla_X \omega)(Y)$ is symmetric on $V(x)$. Using this tool one can show that Theorems 5, 6 still hold if we replace the words "adjoing orbit P" to "coadjoint orbit $\mathfrak{g}^*$. However, the path from a hypersurface in G to the orbits symplectomorphisms seems to be lost for there is no rasonable way to define the Gauss maps.

EXAMPLE. Let (W, Ω) be a symplectic space, let $n = W \oplus \mathbb{R}E$ be the Geisenberg algebra with the Lie brackets $[x, y] = \Omega(x, y)E$, and let N be the correspondent Lie group. Let t be the second coordinate in $\mathfrak{n}^* \approx W \oplus \mathbb{R}$. Then each hyperplane $t = \mathrm{const} \neq 0$ is an orbit in $\mathfrak{n}^*$ and each pair L_1, L_2 of transversal Lagrangian affine subspaces in W defines a codimension 1 foliation in the whole N.

6. Existing of compact leaves

In this section we deal only with $G = S^3$ or $G = \mathrm{SO}(3) = S^3/\mathbb{Z}_2$. Let us start with the flat case. If M is a flat surface in S^3 then by Corollary 1 of Lemma 2′, $(\mathrm{Ad}\, x)\alpha(M)$ and $\beta(M)$ are transversal at $(\mathrm{Ad}\, x)v(x)$, where $v(x) = \alpha(x)$. So each flat M can be obtained by the construction of Theorem 6. Denote $\alpha(M) = L_1$, $\beta(M) = L_2$ and let σ_1, σ_2, be the length parameters on L_1, L_2. Then evidently $\sigma_1(\alpha(x))$, $\sigma_2(\beta(x))$ can serve as local coordinates in M. In other words, $x \in M$ is determined locally by the condition $(\mathrm{Ad}\, x)v = \mu$, $v \in L_1$, $\mu \in L_2$. Let $\varphi(x)$ be the angle between $(\mathrm{Ad}\, x)L_1$ and L_2 at $(\mathrm{Ad}\, x)\alpha(x)$ so $\varphi(x) \in \mathbb{R}/2\pi\mathbb{Z}$, $\varphi(x) \notin \pi\mathbb{Z}$.

LEMMA 6. *$\partial\varphi/\partial\sigma_i = k_i(\sigma_i)$, where k_i is the curvature of L_i.*

Proof. Let $\alpha(x)=v$, $\beta(x)=\mu$, so $(\operatorname{Ad} x)v=\mu$. As $n(x)=\beta(x)x=\mu x$, for any $Z\perp\mu$ we have $Zx\in T_xM$ so $\exp tZx$ is tangent to M. We will compute $\mathrm{d}/\mathrm{d}t_{t=0}\varphi(\exp tZ\cdot x)$ when Z is the unit tangent vector to L_2 at μ. Let us look at Z as vertical axis in $\mathbb{R}^3$, so $\exp tZ$ is the ordinary rotation group and the point μ lies on the equator. Denote $(\operatorname{Ad} x)L_1=\tilde{L}_1$, so we face the following problem: given two curves $\tilde{L}_1, L_2$ intersecting at the equator point μ, to find $\mathrm{d}/\mathrm{d}t\varphi(\exp tZ\tilde{L}_1, L_2)$. It is very convenient to use the stereographic projection from S^2 to $T_\mu S^2$ with center $-\mu$. Then the equator will be replaced by the axis Ox, the rotation group will be replaced by the hyperbolic rotation group g_t with some center a (which is the image of the northern pole) and the curves $\tilde{L}_1, L_2$ will be replaced by some M_1, M_2 intersecting at $\mu\in Ox$. It is more convenient to move M_2 (instead of M_1) under g_t^{-1}. Note that Ox is invariant under g_t^{-1} and actually serves as the hyperbolic absolute and $g_t^{-1}M_2$ remains orthogonal to Ox. All the angles remain the same by the conformity and it is easy to show that the curvatures of the curves at the point μ remain the same. Approximating M_1, M_2 by the corresponding circles and using the plane trigonometry we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t_{t=0}}\cos\varphi(\exp tZ\tilde{L}_1, L_2)=\tilde{k}_1(\mu)+k_2(\mu)\cos\varphi=k_1(v)+k_2(\mu)\cos\varphi.$$

Further the same arguments show that the intersection point moves as

$$\frac{\mathrm{d}\sigma_1}{\mathrm{d}t}=\frac{1}{\sin\varphi},\qquad \frac{\mathrm{d}\sigma_2}{\mathrm{d}t}=\frac{\cos\varphi}{\sin\varphi},$$

so

$$Zx=\frac{1}{\sin\varphi}\frac{\partial}{\partial\sigma_1}+\frac{\cos\varphi}{\sin\varphi}\frac{\partial}{\partial\sigma_2},$$

hence

$$k_1(v)+k_2(\mu)\cos\varphi=\frac{1}{\sin\varphi}\frac{\partial}{\partial\sigma_1}(\cos\varphi)+\frac{\cos\varphi}{\sin\varphi}\frac{\partial}{\partial\sigma_2}(\cos\varphi)$$
$$=-\left(\frac{\partial\varphi}{\partial\sigma_1}+\cos\varphi\frac{\partial\varphi}{\partial\sigma_2}\right).$$

Choosing Z to be tangent to $(\operatorname{Ad} x)L_1$ we find similarly

$$k_1(v)\cos\varphi+k_2(\mu)=-\left(\frac{\partial\varphi}{\partial\sigma_1}\cos\varphi+\frac{\partial\varphi}{\partial\sigma_2}\right)$$

which proves the lemma up to the change of orientations.

We are ready now to prove the main result of this section.

DEFINITION. A closed immersed curve $L \subset S^2$ is called pseudo-geodesic (or *pg*-curve) if it yields the two following conditions:

(1) $\int_L k\,\mathrm{d}\sigma = 0$, where σ is the length parameter,
(2) there exists $p \in L$ such that for all $q \in L$, $|\int_p^q k\,\mathrm{d}\sigma| < \pi/2$.

DEFINITION. A pair of two *pg*-curves L_1, L_2 is called compatible if there exists $p_i \in L_i$ such that $|\int_{p_1}^{q_1} k_1\,\mathrm{d}\sigma_1 + \int_{p_2}^{q_2} k_2\,\mathrm{d}\sigma_2| < \pi/2$ for all $q_i \in L_i$, $i = 1, 2$.

THEOREM 7. *If M is a compact flat surface in S^3 then its Gauss images $L_1 = \alpha(M)$ and $L_2 = \beta(M)$ are compatible pg-curves. In return, given a compatible pg-pair L_1, L_2 one can find a compact flat M such that $\alpha(M) = L_1$, $\beta(M) = L_2$.*

Proof. The first part of the theorem follows immediately from Lemma 6 and the transversality condition $\varphi(x) \notin \overline{\pi\mathbb{Z}}$. In return, given two compatible pg-curves L_1, L_2, we can find a smooth function $\varphi(\sigma_1, \sigma_2)$, $\varphi \notin \pi\mathbb{Z}$, satisfying $\partial\varphi/\partial\sigma_i = k_i(\sigma_i)$. The element $x \in S^3$ satisfying $(\mathrm{Ad}\,x)\sigma_1 = \sigma_2$ and $\varphi((\mathrm{Ad}\,x)L_1, L_2) = \varphi(\sigma_1, \sigma_2)$ is unique up to the (-1) multiplier, so we have a torus $M \subset \mathrm{SO}(3) = S^3/\mathbb{Z}_2$ covering $L_1 \times L_2$ and, consequently, one or two tori M_i in S^3 covering M. Let $x \in M_1$ covers (σ_1, σ_2) so $(\mathrm{Ad}\,x)\sigma_1 = \sigma_2$ and $\varphi((\mathrm{Ad}\,x)L_1, L_2) = \varphi(\sigma_1, \sigma_2)$. Let us construct a flat foliation corresponding to L_1, L_2 which exists by Theorem 6 and let $\tilde{M}(x)$ be the leaf containing x. From Lemma 6 we see that M_1 and $\tilde{M}$ are tangent at x, hence M_1 itself must be the leaf, i.e. M_1 is flat.

Let us remark that if L_1, L_2 are embedded pg-curves then by the Gauss–Bonnet formula, each component of $S^2 \backslash L_i$ has the area 2π, so for all x, $(\mathrm{Ad}\,x)L_1 \cap L_2 \neq \emptyset$. Given two embedded compatible pg-curves, say L_1, L_2, the whole picture looks as follows. There is the exceptional torus $T \subset S^3$ consisting of such x that $(\mathrm{Ad}\,x)L_1$ and L_2 are tangent at some point. In every component C_i of $S^3 \backslash T$ the number $b(x) = \#((\mathrm{Ad}\,x)L_1 \cap L_2) = \text{const}$, so C_i is filled with $b(C_i)$ flat foliations R_i^j, $j = 1, \ldots, b(C_i)$. For each j the union of compact leaves of R_i^j is an open set B_i^j and the closure of each noncompact leaf intersects T. The foliation R_i^j in B_i^j is actually the fibration π over some interval $I \subset \mathbb{R}$ and the function $S_i\colon t \mapsto$ the area of $\pi^{-1}(t)$ is concave by Corollary 2 of Lemma 4.

It will be fruitful work to compare accurately our analysis with that of Kitagawa ([Kit]).

Let us turn our attention to the general case.

LEMMA 7. *Let M be compact with $K \neq 0$, let $\alpha, \beta\colon M \to S^2$ be its Gauss maps and let $\tau\colon S^2 \to S^2$ be its support map. Consider a smooth function $\lambda\colon S^2 \to \mathbb{R}$ and the perturbation $M_\varepsilon\colon x \mapsto \exp(\varepsilon\lambda(\beta(x))\beta(x))x$ satisfying $\mathrm{d}/\mathrm{d}\varepsilon_{\varepsilon=0}\, M_\varepsilon(x) = \lambda(\beta(x))n(x)$. Then for all $v \in S^2$*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon_{\varepsilon=0}} \tau_\varepsilon(v) = 2J_{\tau(v)}(\mathrm{grad}\,\lambda)_{\tau(v)}. \tag{16}$$

We omit the proof which is based on direct computations. This statement means that normal perturbations of the surface correspond to Hamiltonian perturbations of its support map. Given a symplectomorphism τ sufficiently C^∞-close to the identity map, consider a symplectic isotopy τ_ε, $0 \leqslant \varepsilon \leqslant 1$, from $\tau_0 = \mathrm{id}$ to $\tau_1 = \tau$. It is well-known that we can find a smooth $\lambda(\varepsilon, \mu)$ satisfying (16). So we will be able to find a smooth M near the equator sphere $S(1, \pi/2)$ with the prescribed support map τ, if we solve the following problem, which seems to be non-trivial in the non-analytic case.

PROBLEM. Given two compact Riemannian manifolds Σ, N with $\dim N = \dim \Sigma + 1$, an embedding $\beta_0: \Sigma \to M$, and a smooth C^∞-function $\lambda: \Sigma \times [0, 1] \to \mathbb{R}$ with sufficiently small C^∞-norm, to find a smooth family of embeddings β_ε, $0 \leqslant \varepsilon \leqslant 1$, satisfying

$$\left(\frac{\partial}{\partial \varepsilon}\beta_\varepsilon(x),\ n_\varepsilon(x)\right) = \lambda(x, \varepsilon) \tag{17}$$

where $n_\varepsilon(x)$ is the unit normal to $\beta_\varepsilon(\Sigma)$ at $\beta_\varepsilon(x)$.

In conclusion we will explain the origin of "symplectic matter" in the case $G = S^3$. Consider the manifold CS^3 of the oriented geodesics (great circles) in S^3. It carries the natural symplectic structure, which is the Wenstein–Marsden reduction of the canonical symplectic structure in T^*S^3. Given a surface M, its conormal bundle is the Lagrangian submanifold in T^*S^3, so the reduction of this bundle is the Lagrangian submanifold in CS^3, consisting of all geodesics orthogonal to M at some point. So we have the Lagrangian immersion $j: M \to CS^3(p \mapsto$ the geodesic, orthogonal to M at p). Further, as a symplectic manifold, CS^2 is isomorphic to the product $S^2 \times S^2$ ([Be]). Let $\pi_i: CS^3 \to S^2$, $i = 1, 2$, be the natural projections. If a Lagrangian submanifold Q in the symplectic product $W \times W$ is locally a graph of a smooth map τ: $W \to W$ then this map τ is a local symplectomorphism. So we need to investigate whether $\pi_i \circ j$ or $\pi_2 \circ j$ are local diffeomorphisms. These maps are actually our α, β ([Re1], [Re2]). In return, let $\tau: S^2 \to S^2$ be a (local) symplectomorphism, then obtain a Lagrangian submanifold *graph* τ in CS^3. The distribution of the tangent planes, orthogonal to the geodesics from *graph* τ, is integrable where these geodesics foliate S^3. The last statement belongs to E. Cartan.

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