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Non-existence of singular cusp forms

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1. Introduction

The notion of singular automorphic forms on the Siegel upper half space was introduced by Maass [10] in the holomorphic case, and by Howe [4] in the adelic setting. As the word "singular" indicates, such forms are very degenerate. For example, their matrix coefficients tend to zero very slowly at infinity, and in particular can not be tempered in the sense of Harish-Chandra. It has been expected for several years [12] that singular forms can not be cuspidal. The purpose of this paper is to confirm this expectation.

Let k be a global field of characteristic not equal to 2 and let (D, σ) be a k-algebra with involution of one of the following three types:

$$D = \begin{cases} k & \text{case 1} \\ \text{a quadratic extension } F/k & \text{case 2} \\ \text{a quaternion algebra with center } k & \text{case 3} \end{cases} \tag{1}$$

and

$$\sigma = \begin{cases} id & \text{case 1} \\ \text{the Galois involution of } F/k & \text{case 2} \\ \text{the standard involution} & \text{case 3} \end{cases} \tag{2}$$

Let V be a finite dimensional vector space over D endowed with a non-degenerate sesqui-linear form (,) which is either σ-hermitian or σ-skew hermitian. We let $\eta = 1$ or -1 to indicate the two possibilities. Let G be the isometry group of the form (,).

Let X and Y be maximal totally isotropic subspaces of V which are non-degenerately paired by (,). Let V_0 be the orthogonal complement of $X + Y$. Let $P = P_X$ be the maximal parabolic subgroup of G preserving X. The unipotent radical N of P is at most two-step nilpotent. Let $Z \subseteq N$ be the subgroup consisting of elements which act as identity on V_0. Then both Z and $Z\backslash N$ are

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abelian. The latter can be identified with $\mathrm{Hom}_D(V_0, X)$ and we have the short exact sequence

$$Z \to N \to \mathrm{Hom}_D(V_0, X)$$

Let $\eta' = -\eta$. The group Z is isomorphic to the vector space $B(Y, \eta')$ of all sesquilinear forms on Y having the opposite symmetry as (,) under interchange of the two variables. The dual space of $B(Y, \eta')$ is isomorphic to $B(X, \eta')$ in a natural way. Let $\mathbf{A}$ be the ring of adeles of k. We fix a non-trivial character ψ of $\mathbf{A}/k$. The Pontrjagin dual of $Z(\mathbf{A})/Z(k)$ is then identified with $B(X, \eta')$. For $T \in B(X, \eta')$ we let ψ_T be the corresponding character of $Z(\mathbf{A})/Z(k)$.

For a smooth function φ in $L^2(G(k)\backslash G(\mathbf{A}))$ and $T \in B(X, \eta')$ we let φ_T be the Fourier coefficient along Z defined by

$$\varphi_T(g) = \int_{Z(k)\backslash Z(\mathbf{A})} \varphi(zg)\overline{\psi_T(z)}\,dz \qquad (g \in G(\mathbf{A})) \tag{3}$$

MAIN THEOREM. *Let $\pi \subseteq L^2(G(k)\backslash G(\mathbf{A})$ be a cusp form. Then for any smooth function φ in the space of π there exists a $T \in B(X, \eta')$ of maximal rank, such that $\varphi_T \neq 0$.*

For an application of this result to the construction of automorphic L-functions see [11].

A comparison with the situation for $\mathrm{GL}(n)$ is interesting. It is a well known fact (due independently to Piatetski-Shapiro and Shalika) that cusp forms on $\mathrm{GL}(n)$ are generic. While this is certainly not valid for any groups other than $\mathrm{GL}(n)$, our result here provides the best possible analog of the result of Piatetski-Shapiro and Shalika for type I classical groups.

Jacquet and Shalika [5] have given the best possible estimate of matrix coefficients for generic unitary representations of $\mathrm{GL}(n)$. Because of the fact that cusp forms are generic, their estimate is valid for cusp forms on $\mathrm{GL}(n)$. Likewise, when coupled with the local results of Howe [3], the above result will lead to non-trivial estimates of matrix coefficients of cusp forms for other classical groups.

A proof of the above theorem in the symplectic group case was briefly indicated in [9, Lemma 1.1]. It was insufficient for the general case.

2. The theory of singular forms

For $T \in B(X, \eta')$ we define its rank, denoted $\mathrm{rank}(T)$, as follows. First suppose D is a quaternion algebra. We let $\mathrm{rank}(T)$ be *twice* the dimension over D of a maximal subspace of X which is non-degenerate with respect to T.

In any other case rank(T) is defined to be the maximal dimension over D of subspaces of X which are non-degenerate with respect to T.

Next we define an integer r_G as follows. (i). If $D = k$, $\eta = 1$ then r_G is the largest even integer not greater than $\dim_k X$. (ii). If D is a quaternion algebra $r_G = 2 \cdot \dim_D(X)$. (iii). In all other cases $r_G = \dim_D(X)$.

DEFINITION 2.1. Let π be an irreducible automorphic representation of $G(\mathbf{A})$ and r an integer.

(a) We say that π has rank r if (i) For any smooth $\varphi \in \pi$ and any $T \in B(X, \eta')$ with rank(T) $> r$ we have $\varphi_T = 0$. (ii) There exists $\varphi \in \pi$ and T with rank(T) $= r$ such that $\varphi_T \neq 0$.

(b) We say π is *singular* or *of low rank* if it has rank $< r_G$.

The notion of rank has its local analog [3], [8], [14]. Let v be a place of k. First assume D is a quaternion algebra which splits at v. Then $G(k_v)$ is isomorphic to an orthogonal or symplectic group. We describe this isomorphism (see [13]). There is a decomposition

$$V \otimes k_v = V_v \oplus V'_v \tag{4}$$

where V_v and V'_v are vector spaces over k_v of dimension $2 \cdot \dim_D V$. There will be a non-degenerate bilinear form $(\ ,\ )_v$ on V_v which is symmetric or skew-symmetric according as $\eta = -1$ or 1, and $G(k_v)$ is isomorphic to the isometry group of $(\ ,\ )_v$. Let X_v, Y_v, V_{0v} respectively be the projections of $X(k_v)(= X \otimes_e k_v)$, $Y(k_v)$ and $V_0(k_v)$ to V_v. Set $D_v = k_v$; $\eta'_v = -\eta' = \eta$ in this case. Then $Z(k_v) \simeq B(Y_v, \eta'_v)$, the vector space of bilinear forms on Y_v which are symmetric or skew-symmetric according as $\eta'_v = 1$ or -1. The linear dual of $B(Y_v, \eta'_v)$ is $B(X_v, \eta'_v)$. Note that under the inclusion $B(X, \eta') \subseteq B(X_v, \eta'_v)$, the rank of an element $T \in B(X, \eta')$ as a member of $B(X, \eta')$ is the same as the rank of T as a member of $B(X_v, \eta'_v)$.

Next assume $D = F$ is a quadratic extension of k and v is a place at which the extension splits. Then $G(k_v)$ is isomorphic to a general linear group. We have a decomposition as in (4) with V_v of dimension over k_v equal to the dimension of V over F. Define X_v, Y_v and V_{0v} as before. The group $Z(k_v)$ is now isomorphic to the vector space of all k_v-linear maps from Y_v to X_v. To have a consistent notation we denote this space by $B(Y_v, \eta'_v)$. Its linear dual is the vector space of all linear maps from X_v to Y_v and is denoted $B(X_v, \eta'_v)$. The rank of an element in it is just its rank as a linear transformation.

In the remaining cases we let $V_v = V(k_v)$, $X_v = X(k_v)$ etc. Let $(\ ,\ )_v$ be the obvious extension of $(\ ,\ )$ to V_v and $\eta'_v = \eta'$. Then $B(Y_v, \eta'_v)$ and $B(X_v, \eta'_v)$ have obvious meaning.

Now our basic character ψ is a product of local characters: $\psi = \Pi \psi_v$. Using the local character ψ_v we identify the Pontrjagin dual of $Z(k_v)$ with $B(X_v, \eta'_v)$.

DEFINITION 2.2. Let r be an integer. The representation π_v of $G(k_v)$ is of $Z(k_v)$-

rank $\leqslant r$ if whenever $f \in L^1(Z(k_v))$ is such that its Fourier transform $\hat{f}$, a continuous function on $B(X_v, \eta'_v)$, vanishes on the subvariety of forms of rank $\leqslant r$, we have $\pi_v(f) = 0$. It is *of rank* r if it has rank $\leqslant r$ but not strictly less than r.

Let now X'_v and Y'_v be maximal totally isotropic subspaces of V_v containing X_v and Y_v respectively, such that the form $(\ , \ )_v$ is non-degenerate on $X'_v + Y'_v$. (In the case $D = F$ is a quadratic extension of k splitting at v we just require X'_v and Y'_v to have trivial intersection. In the following similar remarks apply in many places but we will not mention it again). Let Z'_v be the (unipotent abelian) subgroup of $G(k_v)$ leaving X'_v and the orthogonal complement of $X'_v + Y'_v$ pointwise fixed. Then as with $Z(k_v)$ we have $Z'_v \simeq B(Y'_v, \eta'_v)$ and the Pontrjagin dual of Z'_v is identified with $B(X'_v, \eta'_v)$. Thus we can define the Z'_v-rank of representations of $G(k_v)$ as before.

LEMMA 2.3. *Let π_v be a unitary representation of $G(k_v)$ of $Z(k_v)$-rank $r < r_G$. Then π_v also has Z'_v-rank r.*

The proof is exactly the same as that of [3, Lemma 2.9] so we omit it.

LEMMA 2.4. *Let $\pi = \otimes \pi_v$ be an irreducible automorphic representation of $G(\mathbf{A})$ occurring discretely in $L^2(G(k)\backslash G(\mathbf{A}))$. Suppose π has rank $r < r_G$. Let v be a place of k. Then π_v is a representation of Z'_v-rank r.*

Proof. Arguing exactly as in the proof of [4, Lemma 2.4] we see that π_v has $Z(k_v)$-rank equal to r. But then the previous lemma says π_v also has Z'_v-rank r since $r < r_G$. Q.E.D.

The restriction of π_v to $Z(k_v)$ is determined by a projection valued measure μ_v on the Pontrjagin dual of $Z(k_v)$ (see [3] or [8]), namely on $B(X_v, \eta'_v)$. Now $P(k_v)$ normalizes $Z(k_v)$ so it acts on its dual $B(X_v, \eta'_v)$. For $T \in B(X_v, \eta'_v)$ let $\mathcal{O}^v_T$ denote its orbit under $P(k_v)$. This orbit is of course just the set of sesqui-linear forms on X_v which are equivalent to T.

LEMMA 2.5. *Let π, v and r be as in Lemma 2.4. Choose $T \in B(X, \eta)$ so that* $\mathrm{rank}(T) = r$ *and there is a smooth function φ in the space of π such that $\varphi_T \neq 0$. Then the representation π_v has $Z(k_v)$-spectrum μ_v supported on the closure of $\mathcal{O}^v_T$.*

Proof. The representation π_v is irreducible and has rank $r < r_G$. So *a priori* its $Z(k_v)$-spectrum is supported on (the closure of) a single orbit by [3, Theorem 2.10] and [8, Theorem 3.1]. From the proof of [4, Lemma 2.4] we see that this must be the orbit of T. Q.E.D.

In fact, one can use the Hasse principle to further conclude that the set of rank r forms $T \in B(X, \eta')$ for which there exist a $\varphi \in \pi$ with $\varphi_T \neq 0$ must belong to the same $P(k)$-orbit. See [4, Theorem 2.3].

3. Proof of the main theorem

Let π be an irreducible automorphic representation of $G(\mathbf{A})$ occurring discretely in $L^2(G(k)\backslash G(\mathbf{A}))$. Assume that π is of rank $r < r_G$. Let H_π^∞ be the space of smooth functions belonging to π. Fix a $T \in B(X, \eta')$ of rank r such that there exists $\varphi \in H_\pi^\infty$ with $\varphi_T \neq 0$.

PROPOSITION 3.1. *Let* $\mathrm{Rad}(T) \subseteq X$ *be the radical of the form* T. *Let* Q *be the maximal parabolic subgroup of* G *preserving* $\mathrm{Rad}(T)$ *and let* R *denote the unipotent radical of* Q. *Then the linear functional* $\varphi \to \varphi_T(1)$ *on* H_π^∞ *is invariant under* $R(\mathbf{A})$.

The following result, which we state in the non-archimedean case only, is in fact valid for any place v. Let ψ_T be the character of $Z(k_v)$ corresponding to T.

LEMMA 3.2. *Let* v *be a finite place. Let* H_v^∞ *be the space of smooth vectors for* π_v. *If* λ *is any linear functional on* H_v^∞ *which satisfies the condition*

$$\lambda(\pi_v(z)\phi) = \psi_T(z) \cdot \lambda(\phi) \tag{5}$$

for all $z \in Z(k_v)$ *and* $\phi \in H_v^\infty$, *then* λ *is invariant under* $R(k_v)$.

The proof of this lemma will be given in the next section. We now use it to prove our proposition.

First observe that $R \subseteq P$ and hence R normalizes Z. In fact it is not difficult to see that

$$\psi_T(rzr^{-1}) = \psi_T(z) \quad \text{for all } z \in Z(\mathbf{A}),\ r \in R(\mathbf{A}) \tag{6}$$

Let $\gamma \in R(k)$. We have for $\varphi \in H_\pi^\infty$

$$\begin{aligned} \varphi_T(\gamma) &= \int_{Z(k)\backslash Z(\mathbf{A})} \varphi(z\gamma)\overline{\psi_T(z)}\,dz \\ &= \int_{Z(k)\backslash Z(\mathbf{A})} \varphi(\gamma^{-1}z\gamma)\overline{\psi_T(z)}\,dz \\ &= \varphi_T(1) \end{aligned} \tag{7}$$

by an obvious change of variable and (6).

Choose a finite place v of k. By Lemma 3.2 the functional $\varphi \to \varphi_T(1)$ is invariant under $R(k_v)$. By (7) it is invariant under $R(k)$. Using strong approximation for unipotent groups [6] we get $R(\mathbf{A}) = R(k) \cdot R(k_v)$. Hence the functional is invariant under $R(\mathbf{A})$. This concludes the proof of Proposition 3.1.

Now let π be a cusp form of rank $r < r_G$ and choose T as in Proposition 3.1.

We normalize the Haar measure so that

$$\int_{R(k)\backslash R(\mathbf{A})} dr = 1$$

Then Proposition 3.1 implies

$$\varphi_T(1) = \int_{R(k)\backslash R(\mathbf{A})} \varphi_T(r)dr$$

Substituting (3) into the above formula gives

$$\varphi_T(1) = \int_{R(k)\backslash R(\mathbf{A})} \int_{Z(k)\backslash Z(\mathbf{A})} \overline{\psi_T(z)}\varphi(zr)dzdr$$

Making the change of variable $z \to rzr^{-1}$ in the inner integral and using (6) we obtain

$$\begin{aligned}\varphi_T(1) &= \int_{R(k)\backslash R(\mathbf{A})} \int_{Z(k)\backslash Z(\mathbf{A})} \overline{\psi_T(z)}\varphi(rz)dzdr \\ &= \int_{Z(k)\backslash Z(\mathbf{A})} \overline{\psi_T(z)} \int_{R(k)\backslash R(\mathbf{A})} \varphi(rz)drdz\end{aligned}$$

The inner integral in the last formula is zero since π is cuspidal. Thus $\varphi_T(1) = 0$ for all $\varphi \in H_\pi^\infty$ and hence $\varphi_T = 0$ for all φ. This contradicts the choice of T. The contradiction comes from our assumption of having a singular cusp form π. This proves the Main Theorem of Section 1.

4. A technical lemma

The purpose of this section is to prove Lemma 3.2. Everything in this section will be local so we drop the subscript v in our notations. Thus k is a non-archimedean local field of characteristic not equal to 2, $V = V_u$, $X = X_v$ and so on.

Let T be as in Lemma 3.2 but considered as a form on $X = X_v$ now. Let $\mathrm{Rad}(T)$ be the radical of the form T and set $V' = X/\mathrm{Rad}(T)$. Then T defines a nondegenerate form $(\ ,\ )'$ on V' which is of the opposite symmetry to the form $(\ ,\ )$ that we have on V. Let G' be the group of isometries of $(\ ,\ )'$. Then (G, G') form what is called a reductive dual pair [1]. The k-vector space $W = V \otimes_D V'$ is endowed with a symplectic form whose isometry group we denote by Sp. Then G and G' are mutual centralizers inside Sp.

Since $\pi = \pi_v$ is of rank $< r_G$ and its Z-spectrum is concentrated on the P-orbit $\mathcal{O}_T$ (Lemma 2.5), the main result of [8] implies that π must occur in Howe's duality correspondence with G'. Let $\widetilde{\mathrm{Sp}}$ be the metaplectic two fold cover of Sp and let ω be the Weil representation of $\widetilde{\mathrm{Sp}}$ corresponding to our chosen character ψ. Let $\mathcal{Y}^\infty$ be the space of smooth vectors for ω. Since π is a representation of G the covering $\widetilde{\mathrm{Sp}} \to \mathrm{Sp}$ essentially must be split over G. Thus ω restricts to a representation of G. Let H^∞ be the space of smooth vectors for π. Since π occurs in the duality correspondence H^∞ may be realized as a quotient of $\mathcal{Y}^\infty$:

$$H^\infty \simeq \mathcal{Y}^\infty / J \tag{8}$$

where J is a certain G-invariant subspace of $\mathcal{Y}^\infty$.

Let λ and the group R be as in Lemma 3.2. By (8) we may lift λ to a linear functional on $\mathcal{Y}^\infty$. Thus we are finally reduced to the following:

LEMMA 4.1. *Let λ be any linear functional on $\mathcal{Y}^\infty$ satisfying the condition*

$$\lambda(\omega(z)\phi) = \psi_T(z) \cdot \lambda(\phi) \tag{9}$$

for all $z \in Z$, $\phi \in \mathcal{Y}^\infty$. Then λ is invariant under R.

Proof. We review a mixed model realization of ω (see for example [7, pp. 246–247]). Using the form (.) we may identify V with its own linear dual. This leads to the isomorphism $W \simeq \mathrm{Hom}_D(V, V')$. Recall the decomposition $V = X + V_0 + Y$. The subspace $W_0 = \mathrm{Hom}_D(V_0, V')$ (consisting of linear maps which vanishes on $X + Y$) is non-degenerate with respect to the symplectic form on W. So we may define $H(W_0)$, the Heisenberg group attached to the symplectic space W_0, as in [3]. Let ρ_0 be the unique irreducible unitary representation of $H(W_0)$ with central character ψ. Let $\mathcal{F}$ be the space of smooth vectors for ρ_0. Set $\mathbf{X} = \mathrm{Hom}_D(X, V')$. Then ω may be realized on

$$\mathcal{Y}^\infty = \mathbf{C}_c^\infty(\mathbf{X}, \mathcal{F}),$$

the space of locally constant compactly supported functions on $\mathbf{X}$ with values in $\mathcal{F}$. We describe the action of (part of) G in this model. Let M_0 be the subgroup of P preserving the decomposition $V = X + V_0 + Y$ and leaving V_0 point-wise fixed. Then $M_0 \simeq \mathrm{GL}_D(X)$. If $g \in M_0$ is of determinant (or reduced norm) one then for $\phi \in \mathcal{Y}^\infty$

$$\omega(g)(\phi)(u) = \phi(u \cdot g) \tag{10}$$

Next define the so called orbit parameter map

$$\mathbf{X} \to B(X, \eta'), \qquad u \mapsto T(u)$$

by the identity

$$T(u)(x_1, x_2) = (u(x_1), u(x_2))' \quad (x_1, x_2 \in X)$$

We have

$$\omega(z)(\phi)(u) = \psi_{T(u)}(z) \cdot \phi(u) \quad (z \in Z) \tag{11}$$

Finally, one may construct a cross section

$$\mathrm{Hom}_D(V_0, X) \to N$$

in analogy with ([2], formula (16)). This enables us to write

$$N \simeq \mathrm{Hom}_D(V_0, X) \oplus Z$$

as set. Then if $T \in \mathrm{Hom}_D(V_0, X)$ we have

$$\omega(T)(\phi)(u) = \rho_0(u \circ T)(\phi(u)) \tag{12}$$

We can now come to the proof of our lemma. We first claim that (9) and (11) implies λ must factor through evaluation on the subset

$$\mathbf{X}_1 = \mathrm{Hom}_D(X/\mathrm{Rad}(T), V') \subseteq \mathbf{X}$$

For any open compact subgroup $L \subseteq Z$ let

$$\phi^L = \frac{1}{\mathrm{Vol}(L)} \cdot \int_L \overline{\psi_T(z)}\, \omega(z)(\phi) dz$$

Then (9) implies $\lambda(\phi^L) = \lambda(\phi)$. By (11) we have

$$\phi^L(u) = \frac{1}{\mathrm{Vol}(L)} \cdot \int_L \overline{\psi_T(z)} \cdot \psi_{T(u)}(z) dz \cdot \phi(u)$$

Hence for L large ϕ^L has support in a small neighborhood of $\mathbf{X}_1$; and if $\phi^L(u) \neq 0$ then $\phi^L(u) = \phi(u)$. But ϕ is locally constant and compactly supported. Hence its value in a small enough neighborhood of $\mathbf{X}_1$ is determined by its restriction to

$\mathbf{X}_1$. Hence for L large enough ϕ^L is determined by the restriction of ϕ to $\mathbf{X}_1$. This proves our claim.

Now we have $R \subseteq M_0 \cdot N$. Using (10), (11) and (12) above one immediately verifies the following relation: we have

$$\omega(r)(\phi)|_{\mathbf{X}_1} = \phi|_{\mathbf{X}_1}$$

for all $\phi \in \mathscr{Y}^\infty$, $r \in R$. Hence $\lambda(\omega(r)(\phi)) = \lambda(\phi)$ for all ϕ and $r \in R$. This proves our lemma and concludes the paper.

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