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$C^{-\infty}$-Whittaker vectors corresponding to a principal nilpotent orbit of a real reductive linear Lie group, and wave front sets

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0. Introduction

For an infinite-dimensional admissible irreducible representation V of $GL(2, \mathbb{R})$ (or more general groups), a realization of V in the space of (classical) Whittaker functions is called a Whittaker model ([19], etc.). Whittaker models and their various generalizations have been studied by various mathematicians. Considering a non-trivial Whittaker vector in the "dual" of V amounts to considering an embedding of V into "the space of Whittaker functions".

To be precise, we introduce the following notations and notions. Let G be a real reductive linear Lie group and let $\mathfrak{g}_0$ be its Lie algebra. (The precise definition of "a real reductive linear Lie group" is given in 1.2. This definition is found in [67].) We denote by $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. We fix a maximal compact subgroup K of G. Let ${}^sP$ be a minimal parabolic subgroup and let $\bar{N}$ be the nilradical of the opposite parabolic subgroup to ${}^sP$. We denote by $\bar{\mathfrak{n}}$ the complexified Lie algebra of $\bar{N}$. We fix an admissible (= non-degenerate) unitary character ψ on $\bar{N}$ and denote the corresponding complexified differential character of $\bar{\mathfrak{n}}$ by the same letter. Let M be a $\mathfrak{g}$-module and we call $v \in M$ a Whittaker vector if $Xv = \psi(X)v$ for all $X \in \bar{\mathfrak{n}}$ ([36]). For quasi-split groups, in [36], Kostant studied Whittaker vectors. In particular, Kostant proved the dimension of the space of Whittaker vectors in the (algebraic) dual of the Harish-Chandra module of a non-unitary principal series representation is equal to the cardinality of the little Weyl group (say w_G). (In his thesis at MIT ([40]), Lynch extended this result to the non-quasi-split case.) In order that a Whittaker vector v in V^* actually define a (continuous) homomorphism of V to the space of Whittaker function, we need to extend v to some continuous linear functional on a G-globalization of V. Let V be a Harish-Chandra $(\mathfrak{g}, K)$-module and we fix an admissible Hilbert G-representation H whose K-finite part is V. Let V_∞ (resp. V_ω) be the space of C^∞ (resp. real analytic) vectors in H. Casselman and Wallach (respectively, Schmid) established a remarkable result; V_∞ (resp. V_ω) does not

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depend on the choice of H. We call a Whittaker vector in the continuous dual of V_∞ (resp. V_ω) a $C^{-\infty}$-(resp. $C^{-\omega}$)-Whittaker vector. We denote by $\mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V)$ (resp. $\mathrm{Wh}^{\omega}_{\bar{\mathfrak{n}},\psi}(V)$) the space of $C^{-\infty}$ (resp. $C^{-\omega}$-Whittaker vectors). Goodman and Wallach ([12]) observed that every Whittaker vector in V^* really extends to a $C^{-\omega}$-Whittaker vector for quasi-split groups. (Actually, they proved more. Their result is described in terms of Gevrey vectors.) For $C^{-\infty}$-Whittaker vectors, the situation is quite different. As Kostant observed in [36], a Whittaker vector in the algebraic dual of the Harish-Chandra module of a principal series representation is not necessary to extend to a $C^{-\infty}$-Whittaker vector. Actually, the Whittaker vector which has such an extension should come from (the analytic continuation of) a Jacquet integral studied by [18], [57], [12], etc. As pointed out in [12] and [72], a $C^{-\infty}$-Whittaker vector corresponds to a homomorphism to the space of Whittaker functions satisfying certain growth condition.

A Harish-Chandra module V is called quasi-large if the Gelfand-Kirillov dimension $\mathrm{Dim}(V)$ of V is equal to $\dim \bar{N}$. For the definition of $\mathrm{Dim}(V)$, see [66]. (If G is quasi-split, V is quasi-large if and only if V is large in the sense of [66]). In [46, 47], we generalized some results in [36] and [12] to non-quasi-split case and obtained the following results.

THEOREM A′. *Let ψ be an admissible (= non-degenerate) (unitary) character on $\bar{\mathfrak{n}}$ and let V be a Harish-Chandra $(\mathfrak{g}, K)$-module. Then, we have $\mathrm{Wh}^{\omega}_{\bar{\mathfrak{n}},\psi}(V) \neq 0$ if and only if V is quasi-large.*

THEOREM B′. *Let ψ be an admissible (= non-degenerate) (unitary) character on $\bar{\mathfrak{n}}$ and let V be a quasi-large Harish-Chandra module. Then, $\dim \mathbf{Wh}^{\omega}_{\bar{\mathfrak{n}},\psi}(V)$ coincides with the multiplicity (or Bernstein degree) $c(V)$ of V.*

For the definition of $c(V)$, see [66].

The purpose of this article is to get $C^{-\infty}$-counterpart of this result. In order to describe our result, we introduce several conventions and notations. First, using the Killing form of $[\mathfrak{g}_0, \mathfrak{g}_0]$, we identify as follows

$$\bar{\mathfrak{n}}_0' \cong \mathfrak{n}_0 \subseteq [\mathfrak{g}_0, \mathfrak{g}_0] \cong [\mathfrak{g}_0, \mathfrak{g}_0]' \subseteq \mathfrak{g}_0'.$$

Here, "′" denotes the real dual, $\bar{\mathfrak{n}}_0$ is the Lie algebra of $\bar{N}$, and $\mathfrak{n}_0$ is the opposite subalgebra of $\mathfrak{g}_0$. The last inclusion is defined by the direct sum decomposition $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{c}_0$, where $\mathfrak{c}_0$ is the center of $\mathfrak{g}_0$. The above identification induces inclusions $i\bar{\mathfrak{n}}_0' \subseteq i\mathfrak{g}_0' \subseteq \mathfrak{g}^*$. Here, i is the imaginary unit. Using this inclusion, we regard an admissible unitary (differential) character ψ on $\bar{\mathfrak{n}}$ as a element of $i\mathfrak{g}_0'$ or $\mathfrak{g}^*$. A nilpotent G-orbit in $\mathfrak{g}_0$ of the maximal dimension ($= 2\dim \bar{N}$) is called a principal nilpotent G-orbit. We denote by $\mathscr{P}r_0(G)$ the set of principal nilpotent G-orbits. A unitary character ψ on $\bar{\mathfrak{n}}$ is admissible if and only if $i^{-1}\,\mathrm{Ad}(G)\psi \subseteq \mathfrak{g}_0^* \cong \mathfrak{g}_0$ is a principal nilpotent G-orbit.

Fix an irreducible Harish-Chandra module V and denote by I the annihilator of V in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. The associated variety $\mathrm{Ass}(U(\mathfrak{g})/I)$ is a closed conic subvariety of $\mathfrak{g}^*$ (for example, see [68]). The condition $\mathrm{Dim}(V) = \dim \bar{N}$ (i.e. V is quasi-large) is equivalent to the condition that $\psi \in \mathrm{Ass}(U(\mathfrak{g})/I)$. For $C^{-\infty}$ situation, the condition for $\mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) \neq 0$ is described in terms of wave front set $\mathrm{WF}(V)$ of V, which are studied by [17], [1, 2, 3] (also see [26]). As is explained in [17] or [1], the associated variety Ass $(U(\mathfrak{g})/I)$ of the annihilator I can be regarded as the complexification of $\mathrm{WF}(V)$ and the wave front set $\mathrm{WF}(V)$ contains more precise information of V than $\mathrm{Ass}(U(\mathfrak{g})/I)$. (For example, the holomorphic and the anti-holomorphic discrete series of $SL(2, \mathbb{R})$ are distinguished by the wave front sets but they have the same associated varieties of the annihilators.) The precise definition of the wave front set of a Harish-Chandra module in this article is discussed in 3.1. Here, we define the wave front set as a closed conic subset in $i\mathfrak{g}_0'$.

The main theorem of this article is:

THEOREM A (Theorem 3.3.3). *Let G be a real reductive linear Lie group. Let ψ be an admissible unitary character on $\bar{\mathfrak{n}}$ and let V be a Harish-Chandra* $(\mathfrak{g}, K)$-*module. Then, we have $\mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) \neq 0$ if and only if $\psi \in \mathrm{WF}(V)$.*

For quasi-split groups, this result gives a refinement of [36] Theorem L. We remark that the above Theorem A for a quasi-split group with only one principal nilpotent G-orbit is immediately deduced from this result of Kostant.

About the dimension of $\mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V)$, for quasi-split groups, the famous multiplicity one theorem tells us the dimension of $\mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V)$ is one or zero. (This multiplicity one theorem has a long history and studied by many mathematicians [19], [63], [36], [76], etc. The version, for general real reductive linear Lie groups, described here is due to Wallach ([72]).

Contrary to quasi-split groups, it is known that the multiplicity one theorem fails for non-quasi-split groups. The counterpart of Theorem B′ is:

THEOREM B (Theorem 5.5.2). *Let G be a connected real reductive linear Lie group and let V be an irreducible Harish-Chandra* $(\mathfrak{g}, K)$-*module. Let ψ be an admissible unitary character on $\bar{\mathfrak{n}}$ such that $\psi \in \mathrm{WF}(V)$. Then, there exists an integer s such that $1 \leqslant 2^s \leqslant \mathrm{card}\, \mathscr{P}r_0(G)$ which only depends on V such that*

$$\dim \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) = 2^s w_G^{-1} c(V).$$

Moreover, if V is the Harish-Chandra module of a discrete series representation, then 2^s equals $\mathrm{card}\, \mathscr{P}r_0(G)$.

We also relate the dimension of the space of Whittaker vectors to the asymptotic expansion of the distribution character in the sense of [1].

THEOREM C (Theorem 5.5.1). *For each $\mathcal{O} \in \mathscr{P}r_0(G)$, we fix an admissible unitary character $\psi_{\mathcal{O}} \in i\mathcal{O}$. Let V be a quasi-large Harish-Chandra module and let θ_V be the lift of the distribution character of V to $\mathfrak{g}_0$. Then the Fourier transform of the first term of the asymptotic expansion* (cf. [1]) *of θ_V is*

$$\sum_{\mathcal{O} \in \mathscr{P}r_0(G)} \dim \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi_{\mathcal{O}}}(V)\mu_{\mathcal{O}}.$$

Here, $\mu_{\mathcal{O}}$ is the G-invariant measure on $\mathcal{O}$ with a suitable normalization (see §5).

Our proof of the above results consists of three steps. First, we prove them for real reductive linear Lie groups with only one principal nilpotent orbit using a similar method to that in [48]. For simplicity, we consider the integral infinitesimal character situation. We fix a Cartan subalgebra $\mathfrak{h}$ and let W be the Weyl group for $(\mathfrak{g}, \mathfrak{h})$. We denote by $\mathscr{H}$ the category of Harish-Chandra modules with some fixed regular integral infinitesimal character (say λ) and denote by $K_{\mathbb{C}}(\mathscr{H})$ the complexified Grothendieck group of $\mathscr{H}$. Then, $K_{\mathbb{C}}(\mathscr{H})$ has a structure of W-module via coherent continuations. From a result of Casselman, the functor $\mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}$ is an exact functor from $\mathscr{H}$ to the category of vector spaces. By a result of Vogan, $\dim \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}$ (resp. the multiplicities) induces a W-homomorphism Ψ (resp. Ψ') from $K_{\mathbb{C}}(\mathscr{H})$ to $S(\mathfrak{h}^*)$. Put $k = \operatorname{card} \Delta^+ - \dim \bar{\mathfrak{n}}$. Then, we see $\operatorname{Image} \Phi \subseteq \bigoplus_{i \leqslant k} S^i(\mathfrak{h}^*)$. It holds that an irreducible W-module σ appears as a composition factor of both $K_{\mathbb{C}}(\mathscr{H})$ and $\bigoplus_{i \leqslant k} S^i(\mathfrak{h}^*)$, then σ must coincide with the Springer representation σ_0 corresponding to the complexification of the principal G-nilpotent orbit in $\mathfrak{g}_0$. If we decompose $K_{\mathbb{C}}(\mathscr{H})$ into the direct sum of blocks ([67, 69], then we see σ_0 appears in each block of $K_{\mathbb{C}}(\mathscr{H})$ and in $\bigoplus_{i \leqslant k} S^i(\mathfrak{h}^*)$ with multiplicity one under the assumption that G has only one principal nilpotent orbit. This implies that Φ and Φ' are proportional on each block. The proportionality constants are easily obtained by comparing $\dim \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}$ and the multiplicities for principal series representations. From this, we have Φ is non-trivial and proportional to Φ'. The main results for G with only one principal nilpotent orbit follow from this.

The second step is to prove the main results for discrete series representations. Besides the conclusion of the first step, the main ingredients of this step are:

(1) Theorem A holds for $SL(2, \mathbb{R})$.
(2) Schmid's character identity.
(3) WF and $\mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}$ well-behave under inductions.
(4) Fine structures of principal nilpotent orbits described in Section 2.

Finally, we derive the main results in the general situation using the above (3) and (4) from the conclusions of steps 1 and 2 and the fact that every irreducible Harish-Chandra module is realized as a Langlands quotient.

The plan of this article is as follows. In Section 1 we fix notations. In Section 2 we investigate principal nilpotent G-orbits. In Section 3 we discuss the definition and properties of wave front sets and formulate the main result. In Section 4 we prove the above Theorem A for real reductive linear Lie groups with only one principal nilpotent orbit. In Section 5, we prove the main theorem for the general case.

List of symbols

^sA: 1.2
$^s\mathfrak{a}_0$, $^s\mathfrak{a}$: 1.2
Ad, ad, $\mathrm{Ad}_{\mathfrak{l}}$: 1.2
$\check{\alpha}$: 1.2
$\mathscr{B}$: 4.1
$\mathbf{B}_\mu$: 4.1
BL(μ): 4.1
$b(\sigma)$, $b(V)$: 4.2
b_G: 4.3
$C(r, V)$: 4.2
$c(\mathfrak{g}_0)$: 2.1
$c(G)$, $c(L)$: 2.2
$c(V)$, $c_{\odot}(V)$: 4.5
$\mathscr{C}_V$: 4.1
deg(a): 4.5
Δ, Δ^+, Δ_μ, Δ_μ^+: 1.2
$F^\sharp$, $F^\flat$: 1.2
F^c: 2.1
F_G: 1.3
G, $(G, \mathfrak{g}_0, \theta, \langle\,,\rangle)$: 1.2
$G_{\mathbb{C}}^{\mathrm{ad}}$, $G^\sharp$, $G^\flat$: 1.2
G^Q: 5.1
$\mathfrak{g}_0$, $\mathfrak{g}$: 1.2
$\gamma = (H, \Gamma, \bar{\gamma})$: 4.2
$\check{H}_\alpha$: 1.2
^sH: 1.2
$^sH^\wedge$: 4.1
$\mathscr{H}_G$, $\mathscr{H}_G[\mu]$: 1.4
h_α, $\bar{h}_\alpha$: 1.3
$^s\mathfrak{h}_0$, $^s\mathfrak{h}$: 1.2
h-ind$_P^G(\mathcal{O})$: 2.3
Ind$_P^G(V)$: 3.1
Ind$_G^{G^Q}(V)$: 5.1
$K^\sharp$, $K^\flat$, $K_{\mathbb{C}}^\sharp$, $K_{\mathbb{C}}^\flat$: 1.2
$\mathfrak{k}_0$, $\mathfrak{k}$: 1.2
^sM: 1.2
m_G: 4.3
$^s\mathfrak{m}_0$, $^s\mathfrak{m}$: 1.2
$\mu_{\mathcal{O}}$: 5.1
$\bar{N}$, N: 1.2
$\bar{\mathfrak{n}}_0$, $\bar{\mathfrak{n}}$, $\mathfrak{n}_0$, $\mathfrak{n}$: 1.2
$\mathscr{N}$: 2.1
$\mathcal{O}_\psi$: 3.3
$\mathcal{O}^Q$, $[\mathcal{O}]_Q$: 5.1
Ω_α: 1.2
ω_α: 1.2
$\tilde{\omega}_\alpha$: 5.4
P: 1.2
$\mathscr{P}r_0(G)$: 2.4
p_S: 2.4
$\tilde{\mathfrak{p}}_0$, $\tilde{\mathfrak{p}}_1$, $\tilde{\mathfrak{p}}_2$, $\tilde{\mathfrak{p}}_3$: 2.2
$^s\mathfrak{p}_0$, $^s\mathfrak{p}$: 1.2
Π, Π_μ, Π_{real}, $\Pi_{\mathrm{imaginary}}$, Π_{complex}: 1.2
$\bar{\pi}(\gamma)$, $[\pi(\gamma)]$: 4.1
Φ, Φ_{real}, Φ_{complex}: 1.2
Φ_a: 4.5
ψ_L: 3.4
Q: 1.2
q_S: 2.4
RC(H, μ), RC(μ): 4.1
res$_{G^Q}^G(V)$: 5.1
S_0: 2.2
sgn: 4.2
$\mathfrak{s}_0$, $\mathfrak{s}$: 1.2
Σ, Σ^+: 1.2
θ: 1.2
$\Theta_V(\mu)$: 4.1
$U_B(\mu)$, $U(\mu)$: 4.3
$V_B(\mu)$: 4.1
V_S, V_Φ: 2.4
V^σ: 3.2, 5.1
W, W_μ: 1.2
WF(V), $^{\odot}$WF(V): 3.1
Wh$_{\bar{\mathfrak{n}},\psi}^{\infty}(V)$: 3.3
w_G: 4.5
w_ψ: 4.5
$\bigcup A$: 1.1
$\langle\,,\rangle$: 1.2

1. Notations and preliminaries

1.1. General notations

In this article, we use the following notations.

As usual we denote the complex number field, the real number field, the rational number field, the ring of integers, and the set of non-negative integers by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ respectively. For each set A, we denote by $\operatorname{card} A$ the cardinality of A. We denote the imaginary unit $\sqrt{-1}$ by i.

For a complex vector space V, we denote by V^* the dual vector space and we denote by $S(V)$ (resp. $S^n(V)$) the symmetric algebra (resp. the nth symmetric power) of V. For a real vector space V_0, we denote by V_0' the real dual vector space of V_0. Sometimes, we identify $S(V)$ and the polynomial ring over V^*. For any subspace W of V, put $W^\perp = \{f \in V^* \mid f|_W \equiv 0\}$. Unless we specify, "$\otimes$" means the tensor product over $\mathbb{C}$.

For a real analytic manifold X and a Fréchet space V, we denote by $C^\infty(X)$ (resp. $C^\infty(X, V)$) the space of the $\mathbb{C}$-valued (resp. V-valued) C^∞-functions on X and denote by T^*X the cotangent bundle of X. We denote by μ_2 the (multiplicative) cyclic group of order 2. For any group Q, we denote by $Z(Q)$ the center of Q. We denote by $\varnothing$ the empty set. For a set A whose elements are also sets, we define the union $\bigcup A$ by

$$\bigcup A = \bigcup_{B \in A} B = \{C \mid C \in B \text{ for some } B \in A\}.$$

We denote by $A - B$ the set theoretical difference of A from B.

For an Abelian category $\mathscr{A}$, we denote by $K(\mathscr{A})$ the Grothendieck group of $\mathscr{A}$.

1.2. Notation for reductive Lie groups and algebras

Hereafter we fix a complex reductive Lie algebra $\mathfrak{g}$ and its real form $\mathfrak{g}_0$. Let $\mathfrak{c}_0$ (resp. $\mathfrak{c}$) be the center of $\mathfrak{g}_0$ (resp. $\mathfrak{g}$). We denote by $G_{\mathbb{C}}^{\mathrm{ad}}$ the adjoint group of $\mathfrak{g}$, so it is a connected complex semisimple Lie group. For an Lie group Q with the Lie algebra $\mathfrak{q}$, we denote by $\mathrm{Ad}_{\mathfrak{q}}$ (resp. $\mathrm{ad}_{\mathfrak{q}}$) the adjoint action of Q (resp. $\mathfrak{q}$) on $\mathfrak{q}$. For simplicity, we denote by Ad (resp. ad) the adjoint action of $G_{\mathbb{C}}^{\mathrm{ad}}$ (resp. $\mathfrak{g}$) on $\mathfrak{g}$. Put $G^\sharp = \{g \in G_{\mathbb{C}}^{\mathrm{ad}} \mid \mathrm{Ad}(g)\mathfrak{g}_0 \subseteq \mathfrak{g}_0\}$. We denote by $G^\flat$ the analytic subgroup of $G^\sharp$ with respect to $[\mathfrak{g}_0, \mathfrak{g}_0]$.

Hereafter, we represent by $*$ the one of $\sharp$ or $\flat$.

We fix an involution θ of $\mathfrak{g}_0$ such that its restriction to $[\mathfrak{g}_0, \mathfrak{g}_0]$ is a Cartan involution. We denote the complexification θ on $\mathfrak{g}$ (resp. lifting to $G_{\mathbb{C}}^{\mathrm{ad}}$) by the same letter. Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{s}_0$ (resp. $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$) be the ± 1-eigenspace decomposition with respect to θ. We denote by K^* the maximal compact subgroup of G^*

corresponding to $\mathfrak{k}_0$. So, $K^\flat$ is the identity connected component of $K^\sharp$. Put $K^\sharp_{\mathbb{C}} = \{g \in G^{\mathrm{ad}}_{\mathbb{C}} \mid \theta(g) = g\}$ and denote by $K^\flat_{\mathbb{C}}$ the identity connected component of $K^\sharp_{\mathbb{C}}$. (In [37], $K^\sharp_{\mathbb{C}}$ (resp. $K^\flat_{\mathbb{C}}$) is denoted by K_θ (resp. K).) We fix a non-degenerate real-valued bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{g}_0$ which satisfies the following.

(B1) $\langle\ ,\ \rangle$ is positive definite (resp. negative definite) on $\mathfrak{s}_0$ (resp. $\mathfrak{k}_0$).
(B2) The restrictions of $\langle\ ,\ \rangle$ to $[\mathfrak{g}_0, \mathfrak{g}_0]$ coincides with the Killing form of $[\mathfrak{g}_0, \mathfrak{g}_0]$.

Clearly, such a bilinear form exists and we also denote the complexification of $\langle\ ,\ \rangle$ by the same letter.

We fix a maximal abelian subspace ${}^s\mathfrak{a}_0$ in $\mathfrak{s}_0$ and denote by ${}^s\mathfrak{m}_0$ the centerizer of ${}^s\mathfrak{a}_0$ in $\mathfrak{k}_0$. We also fix a Cartan subalgebra ${}^s\mathfrak{t}_0$ of ${}^s\mathfrak{m}_0$.

We put

$$F^\sharp = K^\sharp \cap \exp({}^s\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]).$$

Here, exp is the exponential map of $[\mathfrak{g}, \mathfrak{g}]$ to $G^{\mathrm{ad}}_{\mathbb{C}}$.

LEMMA 1.2.1 ([37], [42]). *$F^\sharp$ coincides with the group of order two elements in* $\exp({}^s\mathfrak{a})$ *and $F^\sharp$ normalizes $G^\flat$, $K^\flat$, and $K^\flat_{\mathbb{C}}$. Moreover, we have $G^\sharp = G^\flat F^\sharp$, $K^\sharp_{\mathbb{C}} = K^\flat_{\mathbb{C}} F^\sharp$, and $K^\sharp = K^\flat F^\sharp$.*

Put ${}^s\mathfrak{h}_0 = {}^s\mathfrak{t}_0 + {}^s\mathfrak{a}_0$. Then, ${}^s\mathfrak{h}_0$ is a (maximally split) Cartan subalgebra of $\mathfrak{g}_0$. We fix a parabolic subalgebra ${}^s\mathfrak{p}_0$ of $\mathfrak{g}_0$ whose Levi part is ${}^s\mathfrak{m}_0 + {}^s\mathfrak{a}_0$. ${}^s\mathfrak{p}_0$ is a minimal parabolic subalgebra of $\mathfrak{g}_0$.

Let $\mathfrak{n}_0$ be the nilradical of ${}^s\mathfrak{p}_0$ and let $\bar{\mathfrak{n}}_0$ be the opposite nilpotent subalgebra to $\mathfrak{n}_0$, so we have $\mathfrak{g}_0 = \mathfrak{n}_0 + {}^s\mathfrak{m}_0 + {}^s\mathfrak{a}_0 + \bar{\mathfrak{n}}_0$. We denote the complexifications of ${}^s\mathfrak{a}_0$, ${}^s\mathfrak{m}_0$, ${}^s\mathfrak{t}_0$, ${}^s\mathfrak{h}_0$, ${}^s\mathfrak{p}_0, \ldots$ by ${}^s\mathfrak{a}$, ${}^s\mathfrak{m}$, ${}^s\mathfrak{t}$, ${}^s\mathfrak{h}$, ${}^s\mathfrak{p}, \ldots$, respectively. We call $\mathfrak{g}_0$ quasi-split, if ${}^s\mathfrak{p}$ is a Borel subalgebra of $\mathfrak{g}$.

Let Δ be the root system with respect to $(\mathfrak{g}, {}^s\mathfrak{h})$. We fix a positive root system Δ^+ compatible with $\mathfrak{n}$, and denote by Π the set of simple roots in Δ^+. Let W be the Weyl group of the pair $(\mathfrak{g}, {}^s\mathfrak{h})$ and we denote the inner product on ${}^s\mathfrak{h}^*$ which is induced from $\langle\ ,\ \rangle$ by the same letter. For $\alpha \in \Delta$, we denote by s_α the reflection with respect to α. We denote by w_0 the longest element of W.

For $\alpha \in \Delta$, we define the coroot $\check{\alpha}$ by $\check{\alpha} = \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$. We call $\lambda \in {}^s\mathfrak{h}^*$ dominant (resp. antidominant), if $\langle \lambda, \check{\alpha} \rangle$ is not a negative (resp. positive) integer, for each $\alpha \in \Delta^+$. We call $\lambda \in {}^s\mathfrak{h}^*$ regular, if $\langle \lambda, \alpha \rangle \neq 0$, for each $\alpha \in \Delta$. We denote by Q the space of $\mathbb{Z}$-linear span of the roots. We denote by P the integral weight lattice, namely $\mathsf{P} = \{\lambda \in {}^s\mathfrak{h}^* \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}$ for all $\alpha \in \Delta\}$. If $\lambda \in {}^s\mathfrak{h}^*$ is contained in P, we call λ integral. Let P^{--}, P^-, P^{++}, and P^+ be the space of antidominant regular integral weights, antidominant integral weights, dominant regular integral weights, and dominant integral weights, respectively.

For regular $\mu \in {}^s\mathfrak{h}^*$, we define integral root system Δ_μ by

$$\Delta_\mu = \{\alpha \in \Delta \mid \langle \mu, \check{\alpha} \rangle \in \mathbb{Z}\}.$$

Put $\Delta_\mu^+ = \Delta_\mu \cap \Delta^+$, then it is a positive system of Δ_μ. We denote by Π_μ the simple root system of Δ_μ^+. We also define the integral Weyl group by

$$W_\mu = \{w \in W \mid w\mu - \mu \in \mathrm{Q}\}.$$

It is known that W_μ is the Weyl group for the root system Δ_μ.

We define $\rho \in {}^s\mathfrak{h}^*$ by $\rho = \frac{1}{2}\Sigma_{\alpha \in \Delta^+} \alpha$.

As axioms for real reductive linear Lie groups, we choose here those in [67] p. 1. Namely, we say that a quadruplet $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$ is a real reductive linear Lie group, if it satisfies the following conditions. (Often, we simply say G is a real reductive linear Lie group.)

(RL1) G is a real Lie group and the Lie algebra of G is $\mathfrak{g}_0$.

(RL2) $G^\flat \subseteq \mathrm{Ad}(G) \subseteq G^\sharp$.

(RL3) There is an maximal compact subgroup K of G such that the Lie algebra of K is $\mathfrak{k}_0$ and $K \times \exp(\mathfrak{s}_0) \cong G$ by $(k, X) \rightsquigarrow k\exp(X)$ $(k \in K, X \in \mathfrak{s}_0)$, where exp is the exponential map of $\mathfrak{g}_0$ to G.

(RL4) There is a faithful finite-dimensional linear representation of G.

(RL5) Let $\mathfrak{h}_0$ be an arbitrary Cartan subalgebra of $\mathfrak{g}_0$. Let H be the centerizer of $\mathfrak{h}_0$ in G. Then, H is abelian.

Moreover, if $\mathfrak{g}_0$ is semisimple (resp. simple), we call G a real semisimple (resp. simple) linear Lie group.

A real reductive linear Lie group is a real Lie group of matrices in the sense of [30]. Hence, we can apply Knapp-Zuckerman theory ([33]) for tempered representations to real reductive linear Lie groups.

Hereafter, we fix a real reductive linear Lie group G as above.

Let ${}^sA$, N, and $\bar{N}$ be the analytic subgroup in G with respect to ${}^s\mathfrak{a}_0$, $\mathfrak{n}_0$, and $\bar{\mathfrak{n}}_0$, respectively. We denote by ${}^sM$ (resp. ${}^sT$) the centerizer of ${}^s\mathfrak{a}_0$ (resp. ${}^s\mathfrak{h}_0$) in K with respect to the adjoint action. Put ${}^sP = {}^sM{}^sAN$. Then ${}^sP$ is a minimal parabolic subgroup of G. Let ${}^sT_0$ be the identity component of ${}^sT$.

Next we fix the notations on the restricted root system. We always regard ${}^s\mathfrak{a}^*$ as a subspace of ${}^s\mathfrak{h}^*$ by the zero extension to ${}^s\mathfrak{t}$. Since the Cartan involution θ acts on ${}^s\mathfrak{h}$, it also acts on Δ. Following [67], we introduce the following notions. We call a root $\alpha \in \Delta$ real (resp. imaginary) if $\alpha = -\theta(\alpha)$ (resp. $\alpha = \theta(\alpha)$). We call $\alpha \in \Delta$ complex if α is neither real nor complex. Let Π_{real}, $\Pi_{\text{imaginary}}$, and Π_{complex} be the set of real, imaginary, and complex simple roots, respectively.

Let Σ be the set of restricted roots with respect to $(\mathfrak{g}_0, {}^s\mathfrak{a}_0)$, and let Σ^+ be the positive system of Σ compatible to Δ^+. Namely, $\Sigma = \{\beta \in {}^s\mathfrak{a}^* \mid 0 \neq \beta = \alpha|_{{}^s\mathfrak{a}}$ for some $\alpha \in \Delta\}$. Let Φ be the set of simple restricted roots in Σ^+. We identify real roots and corresponding restricted roots. So, we have

$$\Phi = \Phi_{\text{real}} \cup \Phi_{\text{complex}} \quad \text{(disjoint union).}$$

Here, $\Phi_{\text{real}} = \Pi_{\text{real}}$ and $\Phi_{\text{complex}} = \{\frac{1}{2}(\alpha - \theta(\alpha)) \mid \alpha \in \Pi_{\text{complex}}\}$. For $\alpha \in \Sigma$, we denote by $\mathfrak{g}_{0,\alpha}$ the root space in $\mathfrak{g}_0$ with respect to α.

For $\alpha \in \Phi$, we denote by $\check{H}_\alpha$ the element in ${}^s\mathfrak{a}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ such that $\lambda(\check{H}_\alpha) = \langle \lambda, \check{\alpha} \rangle$ for all $\lambda \in {}^s\mathfrak{a}^*$.

For $\alpha \in \Phi$, we denote by Ω_α the element in ${}^s\mathfrak{a}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ such that $\alpha(\Omega_\alpha) = 1$ and $\beta(\Omega_\alpha) = 0$ for all $\beta \in \Phi - \{\alpha\}$. Let $\omega_\alpha = \exp(\pi i \Omega_\alpha)$ for $\alpha \in \Phi$, where exp is the exponential map of $[\mathfrak{g}, \mathfrak{g}]$ to $G_{\mathbb{C}}^{\text{ad}}$. Clearly, we have $\omega_\alpha^2 = 1$ and $\omega_\alpha \in F^\sharp$. Here, 1 is the identity element. Moreover, we can easily see (cf. [55] p. 406) $F^\sharp$ is generated by $\{\omega_\alpha \mid \alpha \in \Phi\}$.

1.3. Fine structures of parabolic subgroups

Let G be a real reductive linear Lie group and let ${}^sP$ (resp. ${}^sH$) be the minimal parabolic (resp. maximally split Cartan) subgroup with respect to ${}^s\mathfrak{p}_0$ (resp. ${}^s\mathfrak{h}_0$), namely ${}^sP$ is the normalizer (resp. centerizer) of ${}^s\mathfrak{p}_0$ (resp. $\mathfrak{h}_0$) in G. Hence, we have ${}^sH = {}^sT{}^sA$. A parabolic subgroup P of G (resp. a parabolic subalgebra $\mathfrak{p}_0$ of $\mathfrak{g}_0$) is called standard if $P \supseteq {}^sP$ (resp. $\mathfrak{p}_0 \supseteq {}^s\mathfrak{p}_0$). We call a Levi decomposition $P = LU$ (resp. $\mathfrak{p}_0 = \mathfrak{l}_0 + \mathfrak{u}_0$) standard, if ${}^sH \subseteq L$ (resp. ${}^s\mathfrak{h}_0 \subseteq \mathfrak{l}_0$). It is known that L is also a real reductive linear Lie group. We denote by $L^\flat$ (resp. L_0) the identity connected component of $\mathrm{Ad}_{\mathfrak{l}}(L)$ (resp. L).

We say that a standard parabolic subgroup P (resp. subalgebra $\mathfrak{p}_0$) is corresponding to $S \subseteq \Phi$ if S is a simple restricted root system for $(\mathfrak{l}_0, \mathfrak{h}_0)$. It is well-known that this correspondence gives one to one correspondence between the set of standard parabolic subgroups and the set of subsets in Φ. Sometimes, we represent the standard parabolic subalgebra $\mathfrak{p}_0$ corresponding to $S \subseteq \Phi$ as follows. Namely, we replace "○" by "⊙" at the vertices of the Dynkin diagram of Φ corresponding the elements of S

⊙—○—⊙—○—⊙⇒○

○—○—⊙—○—○

We denote by G_0 the identity component of G and denote by $G_{\mathbb{C}}$ the complexification of G_0. For $\alpha \in \Phi$, we put $h_\alpha = \exp(\pi i \check{H}_\alpha) \in G$. Here, exp is the exponential map of $\mathfrak{g}$ to $G_{\mathbb{C}}$. We put $\bar{h}_\alpha = \mathrm{Ad}(h_\alpha)$. We denote by F_G^o (resp. $F^\flat$) the group generated by $\{h_\alpha \mid \alpha \in \Phi\}$ (resp. $\{\bar{h}_\alpha \mid \alpha \in \Phi\}$). So, $F^\flat = \mathrm{Ad}(F_G^o)$. Clearly $F^\flat \subseteq F^\sharp$. We can easily see $h_\alpha \in K \cap G_0$ and $\bar{h}_\alpha \in K^\flat$.

For $\mathfrak{sl}(2, \mathbb{R})$, we fix a Cartan involution θ such that $\mathfrak{so}(2)$ is the $+1$-eigenspace for θ.

For α in Φ_{real}, there is a Lie algebra homomorphism f_α of $\mathfrak{sl}(2, \mathbb{R})$ to $\mathfrak{g}_0$ such that

$$f_\alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \check{H}_\alpha, \quad f_\alpha\left(\left\{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\right\}\right) = \mathfrak{g}_{0,\alpha} \quad \text{and} \quad f_\alpha\left(\left\{\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}\right\}\right) = \mathfrak{g}_{0,-\alpha}.$$

We can (and do) assume that $\theta \circ f_\alpha = f_\alpha \circ \theta$. For α in Φ_{real}, f_α is liftable to a group homomorphism ϕ_α of $SL(2, \mathbb{R})$ to G_0 such that

$$\phi_\alpha\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) = h_\alpha.$$

Put $F_G = \mathrm{Ad}^{-1}(\mathrm{Ad}(K) \cap \exp({}^s\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}]))$. Here, exp means the exponential map of $[\mathfrak{g}, \mathfrak{g}]$ to $G_{\mathbb{C}}^{\text{ad}}$.

The following is known.

LEMMA 1.3.1 (cf. [75] Lemma 1.2.4.5, [42]). *Let G be a real reductive linear Lie group, let P be a standard parabolic subgroup of G, and let $P = LU$ be the standard Levi decomposition of P. Then, we have $L = L_0 F_G$.*

We denote by C the analytic subgroup of G corresponding to $\mathfrak{c}_0$. The following lemma easily follows from the fact $Z(G) \subseteq KC$.

LEMMA 1.3.2 (cf. [33] p. 400). *We assume G, P, and L is as in Lemma 1.3.1. Moreover, we assume G is connected. Then $F_G = Z(G)F_G^\circ$ and $\mathrm{Ad}(F_G) = F^\flat$.*

For $\alpha \in \Phi$, we define

$$N(\alpha) = \{\beta \in \Phi \mid \langle \alpha, \beta \rangle < 0, \langle \beta, \beta \rangle \leqslant \langle \alpha, \alpha \rangle\}.$$

Then, we easily have:

LEMMA 1.3.3. *We assume that $\mathfrak{g}$ does not have a G_2-type factor. Then, for $\alpha \in \Phi$,*

$$\bar{h}_\alpha = \prod_{\beta \in N(\alpha)} \omega_\beta.$$

1.4. Notations for Harish-Chandra modules

We denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ and by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. For $\lambda \in {}^s\mathfrak{h}^*$, let $\chi_\lambda\colon Z(\mathfrak{g}) \to \mathbb{C}$ be the Harish-Chandra homomorphism. We parametrize them so that $\chi_\lambda = \chi_{w\lambda}$ for all $w \in W$. We say a $U(\mathfrak{g})$-module V has an infinitesimal character χ_λ iff $Z(\mathfrak{g})$ acts on V through χ_λ.

For a real reductive linear Lie group $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$. We denote by $\mathscr{H}_G$ (resp. $\mathscr{H}_G[\lambda]$) the category of Harish-Chandra $(\mathfrak{g}, K)$-modules (resp. Harish-Chandra $(\mathfrak{g}, K)$-modules such that all the irreducible constituents have a given infinitesimal character χ_λ).

2. Principal nilpotent orbits

2.1. *Principal nilpotent orbits*

As in Section 1, let $\mathfrak{g}$ be a complex reductive Lie algebra and let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$.

First, we define the nilpotent cone as usual by

$$\mathcal{N} = \{X \in [\mathfrak{g}, \mathfrak{g}] \mid \mathrm{ad}(X) \text{ is nilpotent}\}.$$

We call an element of $\mathcal{N}$ a nilpotent element in $\mathfrak{g}$. Then $\mathcal{N}$ is a closed conic algebraic variety of dimension card Δ. A $G_{\mathbb{C}}^{\mathrm{ad}}$-orbit of $\mathfrak{g}$ under the adjoint action is called a nilpotent orbit if it is contained in $\mathcal{N}$. Similarly, a G^*-orbit (resp. $K_{\mathbb{C}}^*$-orbit) of $\mathfrak{g}_0$ (resp. $\mathfrak{s}$) under the adjoint action is called a nilpotent orbit if it is contained in $\mathcal{N} \cap \mathfrak{g}_0$ (resp. $\mathcal{N} \cap \mathfrak{s}$). Here, G^* (resp. $K_{\mathbb{C}}^*$) means an arbitrary group such that $G^\flat \subseteq G^* \subseteq G^\sharp$ (resp. $K_{\mathbb{C}}^\flat \subseteq K_{\mathbb{C}}^* \subseteq K_{\mathbb{C}}^\sharp$).

A nilpotent orbit of maximal dimension called principal ([34, 35], [37], [54], etc.).

The number of the principal $G_{\mathbb{C}}^{\mathrm{ad}}$-, $K_{\mathbb{C}}^\sharp$-, or $G^\sharp$-nilpotent orbits is just one, but the number of principal nilpotent $G^\flat$- or $K_{\mathbb{C}}^\flat$-orbits need not be one ([34, 35], [37], [55]). In particular, we denote the unique principal nilpotent $G_{\mathbb{C}}^{\mathrm{ad}}$-, $G^\sharp$-, and $K_{\mathbb{C}}^\sharp$-orbit by $\mathcal{O}_{\mathbb{C}}^{\mathrm{reg}}$, $\mathcal{O}_{\mathbb{R}}^\sharp$, and $\mathcal{O}_\theta^\sharp$, respectively. We call an element in $\mathcal{O}_{\mathbb{C}}^{\mathrm{reg}}$ a regular nilpotent element. We have $\mathcal{O}_{\mathbb{C}}^{\mathrm{reg}} \cap \mathfrak{g}_0 \neq \varnothing$ (resp. $\mathcal{O}_{\mathbb{C}}^{\mathrm{reg}} \cap \mathfrak{s} \neq \varnothing$) if and only if $\mathfrak{g}_0$ is quasi-split. If $\mathfrak{g}_0$ is quasi-split, then we have $\mathcal{O}_{\mathbb{C}}^{\mathrm{reg}} \cap \mathfrak{g}_0 = \mathcal{O}_{\mathbb{R}}^\sharp$ and $\mathcal{O}_{\mathbb{C}}^{\mathrm{reg}} \cap \mathfrak{s} = \mathcal{O}_\theta^\sharp$.

Since there is a natural (up to parity) bijection between the set of principal nilpotent $G^\flat$-orbits and that of principal nilpotent $K_{\mathbb{C}}^\flat$-orbits ([55]), the number of principal nilpotent $G^\flat$-orbits coincides with that of principal nilpotent $K_{\mathbb{C}}^\flat$-orbits.

We denote by F^c the subgroup of $F^\sharp$ generated by $\{\omega_\alpha \mid \alpha \in \Phi_{\mathrm{complex}}\}$.

We denote by $c(\mathfrak{g}_0)$ the number of principal nilpotent $G^\flat$-orbits in $\mathfrak{g}_0$. The following result is known.

LEMMA 2.1.1 ([61]). (A) *Let* $\mathfrak{g}_0$ *be a real quasi-split simple Lie algebra. Then the number of principal nilpotent* $K_{\mathbb{C}}^\flat$*-orbits in* $\mathfrak{s}$ (*so, it coincides with* $c(\mathfrak{g}_0)$) *is one except the following cases* (1)–(9).

(1) $\mathfrak{g}_0 = \mathfrak{sl}(2n, \mathbb{R})$ $(n \geqslant 1)$: $c(\mathfrak{g}_0) = 2$.

(2) $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$ $(n \geqslant 2)$: $c(\mathfrak{g}_0) = 2$.
(3) $\mathfrak{g}_0 = \mathfrak{so}(2n+1, 2n+1)$ $(n \geqslant 2)$: $c(\mathfrak{g}_0) = 2$.
(4) $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$ $(n \geqslant 2)$: $c(\mathfrak{g}_0) = 4$.
(5) $\mathfrak{g}_0 = \mathfrak{so}(2n+1, 2n)$ $(n \geqslant 2)$: $c(\mathfrak{g}_0) = 2$.
(6) $\mathfrak{g}_0 = \mathfrak{so}(2n+2, 2n+1)$ $(n \geqslant 1)$: $c(\mathfrak{g}_0) = 2$.
(7) $\mathfrak{g}_0 =$ *a normal real form of* $\mathfrak{e}_7$: $c(\mathfrak{g}_0) = 2$.
(8) $\mathfrak{g}_0 = \mathfrak{su}(n, n)$ $(n \geqslant 2)$: $c(\mathfrak{g}_0) = 2$.
(9) $\mathfrak{g}_0 = \mathfrak{so}(2n+2, 2n)$ $(n \geqslant 2)$: $c(\mathfrak{g}_0) = 2$.

(*Here, the real rank of* $\mathfrak{sp}(2n, \mathbb{R})$ *is* n. $\mathfrak{sl}(2, \mathbb{R})$ *belongs to* (1).)

(B) *Let* $\mathfrak{g}_0$ *be a real non-quasi-split simple Lie algebra. Then the number of principal nilpotent* $K_{\mathbb{C}}^{\flat}$*-orbits in* $\mathfrak{s}$ (*so, it coincides with* $c(\mathfrak{g}_0)$) *is one except the following cases* (10)–(12). *In each case* (10)–(12), $c(\mathfrak{g}_0) = 2$.

(10) $\mathfrak{g}_0 = \mathfrak{so}^*(4n)$ $(n \geqslant 3)$.
(11) $\mathfrak{g}_0 = \mathfrak{so}(2n+k, 2n)$ $(n \geqslant 1, k \geqslant 3)$.
(12) $\mathfrak{g}_0 =$ *a real form of* $\mathfrak{e}_7$ *associated to an Hermitian symmetric space, namely the EVII type.*

For the relation of $c(\mathfrak{g}_0)$ to $F^{\#}$, $F^{\flat}$, and F^c, the following is known.

LEMMA 2.1.2 ([55] Theorem 4.6, 5.3). (1) *For* $\sigma \in F^c$, *we have* $\mathrm{Ad}(\sigma)\mathcal{O} = \mathcal{O}$ *for all principal nilpotent* $G^{\flat}$*-orbit* $\mathcal{O}$. *If* $\mathfrak{g}_0$ *is quasi-split, then* $F^c \subseteq F^{\flat}$.

(2) *Let* $\mathcal{O}$ *be a principal nilpotent* $G^{\flat}$*-orbit in* $\mathfrak{g}_0$ *and let* $\{n_1, \ldots, n_k\}$ *be the set of representative in* $F^{\#}$ *for the quotient group* $F^{\#}/F^{\flat}F^c$. *Then,* $\{\mathrm{Ad}(n_1)\mathcal{O}, \ldots, \mathrm{Ad}(n_k)\mathcal{O}\}$ *is a complete set of the distinct* $G^{\flat}$*-principal nilpotent orbits in* $\mathfrak{g}_0$. *In particular* $\mathrm{card}(F^{\#}/F^{\flat}F^c) = c(\mathfrak{g}_0)$.

Using Lemma 1.3.3, we can describe $F^{\#}$, $F^{\flat}$, and F^c for (1)–(12) in Lemma 2.1.1, as follows. Information on restricted roots is found in [75] p. 30–32.

(1) $\mathfrak{g}_0 = \mathfrak{sl}(2n, \mathbb{R})$ $(n \geqslant 1)$.

In this case, F^c is trivial and h_γ is non-trivial for any $\gamma \in \Phi$. We fix a numeration $\alpha_1, \ldots, \alpha_{2n-1}$ of Φ such that $\langle \alpha_i, \check{\alpha}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant 2n-2$. Then, we have:

(1a) $\omega_{\alpha_i} \in F^{\flat}$ if and only if i is even.
(1b) $F^{\#} = F^{\flat} \cup \omega_{\alpha_i} F^{\flat}$ for any odd i.

(2) $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$ $(n \geqslant 2)$

In this case, F^c is trivial and h_γ is non-trivial for any $\gamma \in \Phi$. Let α be the long simple root. Let $\beta_1, \ldots, \beta_{n-1}$ be the numeration of short simple roots such that $\langle \beta_i, \check{\beta}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant n-2$ and $\langle \beta_{n-1}, \check{\alpha} \rangle = -1$. Then, we have:

(2a) $\omega_{\beta_i} \in F^{\flat}$ for all $1 \leqslant i \leqslant n-1$ and $\omega_\alpha \notin F^{\flat}$.
(2b) $F^{\#} = F^{\flat} \cup \omega_\alpha F^{\flat}$.

(3) $\mathfrak{g}_0 = \mathfrak{so}(2n+1, 2n+1)$ $(n \geqslant 2)$

In this case, F^c is trivial and h_γ is non-trivial for any $\gamma \in \Phi$. Let $\beta_1, \ldots, \beta_{2n-1}$, α_1, α_2 be the numeration of simple roots such that $\langle \beta_i, \check{\beta}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant 2n-2$ and $\langle \beta_{2n-1}, \check{\alpha}_i \rangle = -1$ for $i = 1, 2$. Then, we have:

(3a) $\omega_{\beta_i} \in F^b$ for all $1 \leqslant i \leqslant 2n-1$ and $\omega_{\alpha_i} \notin F^b$ for $i = 1, 2$.
(3b) $F^\# = F^b \cup \omega_{\alpha_i} F^b$ for $i = 1, 2$.

(4) $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$ $(n \geqslant 2)$

In this case, F^c is trivial and h_γ is non-trivial for any $\gamma \in \Phi$. Let $\beta_1, \ldots, \beta_{2n-2}$, α_1, α_2 be the numeration of simple restricted roots such that $\langle \beta_i, \check{\beta}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant 2n-3$ and $\langle \beta_{2n-2}, \check{\alpha}_i \rangle = -1$ for $i = 1, 2$. Then, we have:

(4a) $\omega_{\beta_i} \in F^b$ if and only if i is even. Moreover, $\omega_{\alpha_i} \notin F^b$ for $i = 1, 2$.
(4b) In this case $F^\#/F^b$ is isomorphic to $\mu_2 \times \mu_2$. For any odd $1 \leqslant i < 2n-2$, we have

$$F^\# = F^b \cup \omega_{\beta_i} F^b \cup \omega_{\alpha_1} F^b \cup \omega_{\alpha_2} F^b.$$

(5) $\mathfrak{g}_0 = \mathfrak{so}(2n+1, 2n)$ $(n \geqslant 2)$

In this case, F^c is trivial and h_α is non-trivial for any long simple restricted root $\alpha \in \Phi$. Let β be a unique short simple root in Φ. Then, $h_\beta = 1$. We fix a numeration $\alpha_1, \ldots, \alpha_{2n-1}$ of long simple roots in Φ such that $\langle \alpha_i, \check{\alpha}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant 2n-2$ and $\langle \alpha_{2n-1}, \check{\beta} \rangle = -2$. Then, we have:

(5a) $\omega_{\alpha_i} \in F^b$ if and only if i is even. Moreover, $\omega_\beta \in F^b$.
(5b) $F^\# = F^b \cup \omega_{\alpha_i} F^b$ for all odd $1 \leqslant i \leqslant 2n-1$.

(6) $\mathfrak{g}_0 = \mathfrak{so}(2n+2, 2n+1)$ $(n \geqslant 1)$

In this case, F^c is trivial and h_α is non-trivial for any long simple restricted root $\alpha \in \Phi$. Let β be a unique short simple root in Φ. Then, $h_\beta = 1$. We fix a numeration $\alpha_1, \ldots, \alpha_{2n}$ of long simple roots in Φ such that $\langle \alpha_i, \check{\alpha}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant 2n-1$ and $\langle \alpha_{2n}, \check{\beta} \rangle = -2$. Then, we have:

(6a) $\omega_{\alpha_i} \in F^b$ if and only if i is even. Moreover, $\omega_\beta \notin F^b$.
(6b) $F^\# = F^b \cup \omega_{\alpha_i} F^b$ for all odd $1 \leqslant i \leqslant 2n-1$ and $F^\# = F^b \cup \omega_\beta F^b$.

(7) $\mathfrak{g}_0 =$ a normal real form of $\mathfrak{e}_7$

In this case, F^c is trivial and h_α is non-trivial for any simple restricted root $\alpha \in \Phi$. We fix a numeration $\alpha_1, \ldots, \alpha_7$ of simple roots in Φ such that $\langle \alpha_i, \check{\alpha}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant 5$ and $\langle \alpha_4, \check{\alpha}_7 \rangle = -1$. Then, we have:

(7a) $\omega_{\alpha_i} \in F^b$ if and only if $i = 2, 4, 5, 6$.
(7b) $F^\# = F^b \cup \omega_{\alpha_i} F^b$ for $i = 1, 3, 7$.

(8) $\mathfrak{g}_0 = \mathfrak{su}(n, n)$ $(n \geqslant 2)$

In this case, h_β is non-trivial for all simple restricted root β. Let α be a unique real simple restricted root in Φ. We fix a numeration $\gamma_1, \ldots, \gamma_{n-1}$ of complex simple roots in Φ such that $\langle \gamma_i, \check{\gamma}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant n-2$ and $\langle \gamma_{n-1}, \check{\alpha} \rangle = -1$. Then we have:

(8a) $F^c \subseteq F^\flat$ and $\omega_\alpha \notin F^\flat$.
(8b) $F^\sharp = F^\flat \cup \omega_\alpha F^\flat$.

(9) $\mathfrak{g}_0 = \mathfrak{so}(2n+2, 2n)$ $(n \geqslant 2)$

Let γ be a unique complex simple restricted root in Φ. We fix a numeration $\alpha_1, \ldots, \alpha_{2n-1}$ of long simple real roots in Φ such that $\langle \alpha_i, \check{\alpha}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant 2n-2$ and $\langle \alpha_{2n}, \check{\gamma} \rangle = -2$. In this case $h_\gamma = 1$ and $h_{\alpha_i} \neq 1$ for $1 \leqslant i \leqslant 2n-1$. Moreover, we have:

(9a) $F^c \subseteq F^\flat . \omega_{\alpha_i} \in F^\flat$ if and only if i is even.
(9b) $F^\sharp = F^\flat \cup \omega_{\alpha_i} F^\flat$ for all odd $1 \leqslant i \leqslant 2n-1$.

(10) $\mathfrak{g}_0 = \mathfrak{so}^*(4n)$ $(n \geqslant 3)$

In this case $h_\gamma \neq 1$ for all $\gamma \in \Phi$. Let α be the long simple restricted root. Let $\beta_1, \ldots, \beta_{n-1}$ be the numeration of short simple restricted roots such that $\langle \beta_i, \check{\beta}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant n-2$ and $\langle \beta_{n-1}, \check{\alpha} \rangle = -1$. In this case, $\Phi_{\text{complex}} = \{\beta_1, \ldots, \beta_{n-1}\}$. Moreover, we have:

(10a) $\omega_{\beta_i} \in F^\flat$ for all $1 \leqslant i \leqslant n-1$ and $\omega_\alpha \notin F^\flat$. Hence, $F^c = F^\flat$.
(10b) $F^\sharp = F^\flat \cup \omega_\alpha F^\flat$.

(11) $\mathfrak{g}_0 = \mathfrak{so}(2n+k, 2n)$ $(n \geqslant 1, k \geqslant 3)$

Let γ be a unique complex simple restricted root in Φ. We fix a numeration $\alpha_1, \ldots, \alpha_{2n-1}$ of long simple real restricted roots in Φ such that $\langle \alpha_i, \check{\alpha}_{i+1} \rangle = -1$ for all $1 \leqslant i \leqslant 2n-2$ and $\langle \alpha_{2n}, \check{\gamma} \rangle = -2$. In this case $h_\gamma = 1$ and $h_{\alpha_i} \neq 1$ for $1 \leqslant i \leqslant 2n-1$. Moreover, we have:

(11a) $F^c \subseteq F^\flat$. We have $\omega_{\alpha_i} \in F^\flat$ if and only if i is even.
(11b) $F^\sharp = F^\flat \cup \omega_{\alpha_i} F^\flat$ for all odd $1 \leqslant i \leqslant 2n-1$.

(12) $\mathfrak{g}_0 =$ a real form of $\mathfrak{e}_7$ associated to an Hermitian symmetric space, namely the EVII type

In this case $h_\gamma \neq 1$ for all $\gamma \in \Phi$. Let α be the long simple restricted root. Let β_1 and β_2 be the numeration of the short simple restricted roots such that $\langle \beta_1, \check{\beta}_2 \rangle = -1$ and $\langle \beta_2, \check{\alpha} \rangle = -1$. In this case, $\Phi_{\text{complex}} = \{\beta_1, \beta_2\}$. Moreover, we have:

(12a) $\omega_{\beta_i} \in F^{\flat}$ for all $i = 1, 2$ and $\omega_\alpha \notin F^{\flat}$. Hence, $F^c = F^{\flat}$.
(12b) $F^{\sharp} = F^{\flat} \cup \omega_\alpha F^{\flat}$.

Finally, we remark:

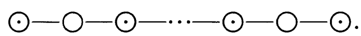
LEMMA 2.1.3. *Let* $\mathfrak{g}_0$ *be a simple real Lie algebra which is one of* (1)–(12) *in Lemma* 2.1.1. *Then we have*:

(1) *The restricted root system* Σ *is reduced.*
(2) $F^c \subseteq F^{\flat}$.
(3) *Any long simple restricted root is real.*

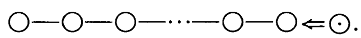
2.2. *Principal nilpotent orbits for Levi subgroups*

Let $\mathfrak{g}_0$ be a simple real Lie algebra which is one of (1)–(12) in Lemma 2.1.1. In this case, we put $S_0 = \{\alpha \in \Phi \mid \omega_\alpha \notin F^{\flat}\}$. We denote by $\tilde{\mathfrak{p}}_0$ the standard parabolic subalgebra of $\mathfrak{g}_0$ such that S_0 is the simple restricted root system of the standard Levi factor $\tilde{\mathfrak{l}}_0$ of $\tilde{\mathfrak{p}}_0$. From the above description of $F^{\flat}$, $\tilde{\mathfrak{p}}_0$ is described as follows.

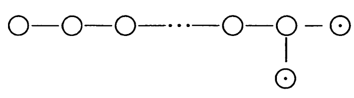
(1) If $\mathfrak{g}_0 = \mathfrak{sl}(2n, \mathbb{R})$, then $\tilde{\mathfrak{p}}_0$ is the standard parabolic subalgebra such that the semisimple part of $\tilde{\mathfrak{l}}_0$ is isomorphic to the direct product of n copies of $\mathfrak{sl}(2, \mathbb{R})$. Namely,



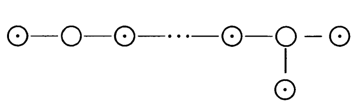
(2) If $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$, then $\tilde{\mathfrak{p}}_0$ is the standard parabolic subalgebra corresponding to $S = \{$the long simple root$\}$. The semisimple part of $\tilde{\mathfrak{l}}_0$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Namely,



(3) If $\mathfrak{g}_0 = \mathfrak{so}(2n+1, 2n+1)$, then the semisimple part of $\tilde{\mathfrak{l}}_0$ is isomorphic to the direct sum of 2 copies of $\mathfrak{sl}(2, \mathbb{R})$. $\tilde{\mathfrak{p}}_0$ is described as follows



(4) If $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$, then $\tilde{\mathfrak{p}}_0$ is the standard parabolic subgroup such that the semisimple part of $\tilde{\mathfrak{l}}_0$ is isomorphic to the direct product of $n+1$ copies of $\mathfrak{sl}(2, \mathbb{R})$. Namely,



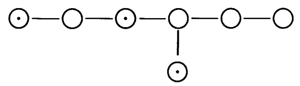
(5) If $\mathfrak{g}_0 = \mathfrak{so}(2n+1, 2n)$, then $\tilde{\mathfrak{p}}_0$ is a standard parabolic subgroup such that the semisimple part of $\tilde{\mathfrak{l}}_0$ is isomorphic to the direct product of n copies of $\mathfrak{sl}(2, \mathbb{R})$. $\mathfrak{p}_0$ is represented as follows

⊙—○—⊙—···—○—⊙⇒○.

(6) If $\mathfrak{g}_0 = \mathfrak{so}(2n+2, 2n+1)$, then $\tilde{\mathfrak{p}}_0$ is the standard parabolic subgroup such that the semisimple part of $\tilde{\mathfrak{l}}_0$ is isomorphic to the direct product of $n+1$ copies of $\mathfrak{sl}(2, \mathbb{R})$. Namely,

⊙—○—⊙—···—○—⊙—○⇒⊙.

(7) If $\mathfrak{g}_0$ = the normal real form of E_7, then $\tilde{\mathfrak{p}}_0$ is represented as follows



(8) If $\mathfrak{g}_0 = \mathfrak{su}(n, n)$, then $\tilde{\mathfrak{p}}_0$ is a unique standard parabolic subalgebra such that the semisimple part of $\tilde{\mathfrak{l}}_0$ is $\mathfrak{sl}(2, \mathbb{R})$. Namely,

○—○—○—···—○—○⇐⊙.

(9) If $\mathfrak{g}_0 = \mathfrak{so}(2n+2, 2n)$, then $\tilde{\mathfrak{p}}_0$ is a standard parabolic subalgebra such that the semisimple part of $\tilde{\mathfrak{l}}_0$ is isomorphic to the direct product of n copies of $\mathfrak{sl}(2, \mathbb{R})$. Namely,

⊙—○—⊙—···—○—⊙⇒○.

(10) If $\mathfrak{g}_0 = \mathfrak{so}^*(4n)$, then $\tilde{\mathfrak{p}}_0$ is the standard parabolic subalgebra corresponding to $S = \{\text{the long simple root}\}$. The semisimple part of $\tilde{\mathfrak{l}}_0$ is the direct product of one copy of $\mathfrak{sl}(2, \mathbb{R})$ and n copies of $\mathfrak{so}(3)$. Namely,

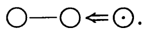
○—○—○—···—○—○⇐⊙.

(11) If $\mathfrak{g}_0 = \mathfrak{so}(2n+k, 2n)$ $(n \geqslant 1, k \geqslant 3)$, then $\tilde{\mathfrak{p}}_0$ is a standard parabolic subalgebra such that the semisimple part of $\tilde{\mathfrak{l}}_0$ is isomorphic to the direct product of n copies of $\mathfrak{sl}(2, \mathbb{R})$ and one copy of $\mathfrak{so}(k)$. Namely,

⊙—○—⊙—···—○—⊙⇒○.

(12) If $\mathfrak{g}_0$ = a real form of $\mathfrak{e}_7$ associated to an Hermitian symmetric space,

then $\tilde{\mathfrak{p}}_0$ is the standard parabolic subalgebra corresponding to $S = \{$the long simple root$\}$. The semisimple part of $\mathfrak{l}_0$ is $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{so}(8)$. Namely,

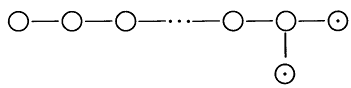


Now, we assume $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$ and G is a connected simple real linear Lie group whose Lie algebra is $\mathfrak{g}_0$. We define subgroups F_i $(i = 1, 2, 3)$ of $F^{\#}$ as follows

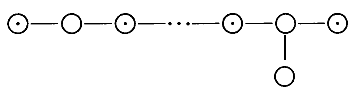
F_1 is generated by $\{\omega_{\beta_i} \mid i$ is odd and $1 \leqslant i < 2n - 2\}$,
F_2 is generated by $\{\omega_{\alpha_1}\}$,
F_3 is generated by $\{\omega_{\alpha_2}\}$.

For $i = 1, 2, 3$, we put $S_i = \{\alpha \in \Phi \mid \omega_\alpha \notin F^\flat F_i\}$. Let $\tilde{\mathfrak{p}}_i$ be the standard parabolic subalgebra of $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$ corresponding to S_i. These are represented as follows.

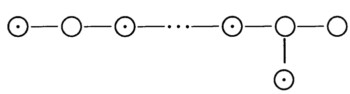
(13) $\tilde{\mathfrak{p}}_1$:



(14) $\tilde{\mathfrak{p}}_2$:



(15) $\tilde{\mathfrak{p}}_3$:



Fix real reductive linear Lie group G and a parabolic subgroup P with the standard Levi decomposition $P = LU$. We denote by $c(L)$ (resp. $c(G)$) the number of principal nilpotent L-orbits in $\mathfrak{l}_0$ (resp. G-orbits in $\mathfrak{g}_0$). P is called type I (resp. type II), if $c(L) > 1$ (resp. $c(L) = 1$). In particular, we say G is of type I (resp. type II) if there exists more than one (resp. exists just one) principal nilpotent G-orbits (resp. G-orbit) in $\mathfrak{g}_0$. Lemma 2.1.1 tells us which connected real simple Lie group is type I. For parabolic subgroups, we have:

LEMMA 2.2.1. (1) *Let G be a type I connected real simple Lie group and let $\mathfrak{g}_0$ be its Lie algebra. Moreover, we assume $\mathfrak{g}_0 \neq \mathfrak{so}(2n, 2n)$ for some $n \geqslant 2$. Namely,* $\mathfrak{g}_0$

is a simple real Lie algebra which is one of (1)–(3) *or* (5)–(12) *in the statement of* Lemma 2.1.1. *Let* $\tilde{\mathfrak{p}}_0$ *be the standard parabolic subalgebra of* $\mathfrak{g}_0$ *which is defined above. Let* $\mathfrak{p}_0$ *be a standard parabolic subalgebra of* $\mathfrak{g}_0$ *and let* P *(resp.* $P = LU$*) be the corresponding parabolic subgroup of* G *(resp. standard Levi decomposition). We denote by* $\mathfrak{l}_0$ *the Lie algebra of* L. *Then, we have*

$$\begin{aligned} c(L) &= 2 \quad \text{if } \tilde{\mathfrak{p}}_0 \subseteq \mathfrak{p}_0 \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

(2) *Let* $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$ *and let* $\tilde{\mathfrak{p}}_i$ $(i = 0, 1, 2, 3)$ *be the standard parabolic subalgebra of* $\mathfrak{g}_0$ *which is defined as above. We denote by* G *a connected real simple linear Lie group whose Lie algebra is* $\mathfrak{g}_0$. *Let* $\mathfrak{p}_0$ *be a standard parabolic subalgebra of* $\mathfrak{g}_0$ *and let* P *(resp.* $P = LU$*) be the corresponding parabolic subgroup of* G *(resp. the standard Levi decomposition). Then, we have*

$$\begin{aligned} c(L) &= 4 \quad \text{if } \tilde{\mathfrak{p}}_0 \subseteq \mathfrak{p}_0, \\ &= 2 \quad \text{if } \tilde{\mathfrak{p}}_i \subseteq \mathfrak{p}_0 \text{ holds for some } i = 1, 2, 3 \text{ and } \tilde{\mathfrak{p}}_0 \not\subseteq \mathfrak{p}_0, \\ &= 1 \quad \text{if } \tilde{\mathfrak{p}}_i \not\subseteq \mathfrak{p}_0 \text{ for any } i = 1, 2, 3. \end{aligned}$$

(3) *Let* $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$ *and let* $\tilde{\mathfrak{p}}_i$ $(i = 0, 1, 2, 3)$ *be the standard parabolic subalgebra of* $\mathfrak{g}_0$ *which is defined as above. We denote by* G_i $(i = 1, 2, 3)$ *a real simple linear Lie group whose Lie algebra is* $\mathfrak{g}_0$ *such that* $\mathrm{Ad}(G_i) = G^\flat F_i$. *Here,* F_i *is the subgroup defined in this section. We fix some* $i = 1, 2, 3$. *Let* $\mathfrak{p}_0$ *be a standard parabolic subalgebra of* $\mathfrak{g}_0$ *and let* P *(resp.* $P = LU$*) be the corresponding parabolic subgroup of* G_i *(resp. standard Levi decomposition). Then, we have*

$$\begin{aligned} c(L) &= 2 \quad \text{if } \tilde{\mathfrak{p}}_i \subseteq \mathfrak{p}_0, \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

Proof. Let $S \subseteq \Phi$ be the set of simple roots in $\mathfrak{l}_0$. Put $S^c = \Phi - S$ and denote by F' the subgroup of $F^\sharp$ generated by $\{\omega_\alpha \mid \alpha \in S^c\}$. From Lemma 1.3.1, Lemma 1.3.2, and Lemma 2.1.2, we have $c(L) = \mathrm{card}(\mathrm{Ad}_{\mathfrak{l}}(F^\sharp)/\mathrm{Ad}_{\mathfrak{l}}(F^\flat))$. Clearly, we have $\mathrm{Ad}_{\mathfrak{l}}(F^\sharp) \cong F^\sharp/F'$ and $\mathrm{Ad}_{\mathfrak{l}}(F^\flat) \cong F^\flat/(F^\flat \cap F')$. Hence $c(L) = \mathrm{card}(F^\sharp/F^\flat F')$. From the definition of $\tilde{\mathfrak{p}}_0$, $\tilde{\mathfrak{p}}_0 \subseteq \mathfrak{p}_0$ if and only if $F' \subseteq F^\flat$. From Lemma 2.1.2, we get (1).

So, we assume $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$ $(n \geqslant 2)$. From the definition, we have:

$\tilde{\mathfrak{p}}_1 \subseteq \mathfrak{p}_0$, if and only if $\omega_{\alpha_i} \notin F'$ for any $i = 1, 2$.
$\tilde{\mathfrak{p}}_2 \subseteq \mathfrak{p}_0$, if and only if $\omega_{\alpha_1} \notin F'$ and $\omega_{\beta_i} \notin F'$ for any odd $1 \leqslant i < 2n-2$.
$\tilde{\mathfrak{p}}_3 \subseteq \mathfrak{p}_0$, if and only if $\omega_{\alpha_2} \notin F'$ and $\omega_{\beta_i} \notin F'$ for any odd $1 \leqslant i < 2n-2$.

Here, we use the notation in 2.1(4). From 2.1(4), we easily see that $\mathfrak{p}_0$ contains $\tilde{\mathfrak{p}}_i$

for some $i = 1, 2, 3$ if and only if $\mathrm{card}(F^{\sharp}/F^{\flat}F') \geqslant 2$. Together with the first half of this proof, we get (2).

(3) is deduced from a similar argument as above. □

2.3. Hyperbolic inductions

Motivation and representation theoretic background of the material in this section are found in [71].

The notion of the induction of the $G_{\mathbb{C}}$-nilpotent orbit is introduced in [39]. (In fact they introduced induced unipotent classes, but the idea is same.) First, we recall the definition. Let $G_{\mathbb{C}}$ be a connected complex reductive Lie group corresponding to $\mathfrak{g}$. Let $\mathfrak{p}$ be a parabolic subalgebra such that $\mathfrak{p} \supseteq \mathfrak{b}$ and let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ be a Levi decomposition such that $\mathfrak{l} \supseteq \mathfrak{h}$ and $\mathfrak{u} \subseteq \mathfrak{n}$. Let $P_{\mathbb{C}}$ (resp. $P_{\mathbb{C}} = L_{\mathbb{C}}U_{\mathbb{C}}$) be the corresponding parabolic subgroup (resp. Levi decomposition). Let $\mathcal{O}_0$ be a nilpotent $L_{\mathbb{C}}$-orbit in $\mathfrak{l}$. Then the induced nilpotent orbit in $\mathfrak{g}$ (say $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_0)$) is the unique nilpotent $G_{\mathbb{C}}$-orbit $\mathcal{O}$ in $\mathfrak{g}$ such that $\mathcal{O} \cap (\mathcal{O}_0 + \mathfrak{u})$ is open in $\mathcal{O}_0 + \mathfrak{u}$. One of the results of [39] is $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_0)$ only depends on $\mathfrak{l}$ and $\mathcal{O}_0$. So, sometimes $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_0)$ is denoted by $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_0)$.

For our purpose, we need an obvious analogy for the real case. In order to introduce such an analogy, we fix several notations. Let $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$ be a real reductive linear Lie group. Let $\mathfrak{p}_0$ be a standard parabolic subalgebra of $\mathfrak{g}_0$, and let $\mathfrak{p}_0 = \mathfrak{l}_0 + \mathfrak{u}_0$ be the standard Levi decomposition. Let P be the corresponding standard parabolic subgroup of G (namely, the normalizer of $\mathfrak{p}_0$ in G), and let $P = LU$ be the standard Levi decomposition. Let $\mathcal{O}_0$ be a nilpotent L-orbit in $\mathfrak{l}_0$. Then we define the hyperbolic induction of $\mathcal{O}_0$ to $\mathfrak{g}$ as follows

$$\text{h-ind}_P^G(\mathcal{O}_0) = \{\mathcal{O} \mid \mathcal{O} \text{ is a } G\text{-orbit in } \mathfrak{g}_0 \text{ such that } \mathcal{O} \cap (\mathcal{O}_0 + \mathfrak{u}_0) \text{ is open in } \mathcal{O}_0 + \mathfrak{u}_0\}.$$

Clearly, $\text{h-ind}_P^G(\mathcal{O})$ consists of G-nilpotent orbits. We can also easily see $\text{h-ind}_P^G(\mathcal{O})$ is non-empty. Contrary to complex cases, $\text{h-ind}_P^G(\mathcal{O}_0)$ may consist of more than one orbit. If $\text{h-ind}_P^G(\mathcal{O}_0)$ happens to be written as $\{\mathcal{O}\}$ by a single G-orbit $\mathcal{O}$, we say $\text{h-ind}_P^G(\mathcal{O}_0)$ is well-posed and we simply write $\text{h-ind}_P^G(\mathcal{O}_0) = \mathcal{O}$.

For a set of L-nilpotent orbits $\mathscr{F}$, we write

$$\text{h-ind}_P^G(\mathscr{F}) = \{\mathcal{O} \mid \mathcal{O} \in \text{h-ind}_P^G(\mathcal{O}') \text{ for some } \mathcal{O}' \in \mathscr{F}\}.$$

REMARK. The terminology "hyperbolic" comes from the fact that $\mathfrak{p}_0$ corresponding to hyperbolic semisimple elements. More precisely, h-ind should be called "hyperbolically parabolic induction". A parabolic subalgebra associated to an elliptic semisimple element is a θ-invariant (complex) parabolic subalgebra $\mathfrak{q}$ in $\mathfrak{g}$. We may define a notion of elliptic inductions for $K_{\mathbb{C}}$-nilpotent orbits with respect to $\mathfrak{q}$.

A G-orbit $\mathcal{O}$ is called hyperbolically rigid, if there is no proper parabolic subgroup $P = LU$ such that there exists a nilpotent L-orbit $\mathcal{O}_0$ in $\mathfrak{l}_0$ which satisfies h-$\mathrm{ind}_P^G(\mathcal{O}_0) = \mathcal{O}$. For example, any nilpotent $SL(2, \mathbb{R})$-orbit is hyperbolically rigid.

2.4. More about principal nilpotent orbits

In this section, we consider the following problem.

PROBLEM. Let G be a real reductive linear Lie group. For a principal nilpotent G-orbit $\mathcal{O}$, find a pair $(P, \mathcal{O}_0)$ such that P is a standard parabolic subgroup of G, $\mathcal{O}_0$ is a hyperbolically rigid nilpotent L-orbit, and $\mathcal{O} =$ h-$\mathrm{ind}_P^G(\mathcal{O}_0)$.

Let $\mathscr{P}r_0(G)$ be the set of principal G-orbits in $\mathfrak{g}_0$ (resp. $\mathfrak{s}$).

Easily we see

$$\mathscr{P}r_0(G) = \text{h-ind}_{^sP}^G(0).$$

So, if $\mathscr{P}r_0$ consists of a single element (namely if G is of type II), the answer to the above question is $({}^sP, \{0\})$. For our purpose, it is important to investigate the real simple linear Lie groups of type I, namely cases (1)–(12) in Lemma 2.1.1.

The answer is:

PROPOSITION 2.4.1. (1) *Let $\mathfrak{g}_0$ be a simple real Lie algebra which is one of* (1)–(12) *in the statement of Lemma* 2.1.1, *and let $\tilde{\mathfrak{p}}_0$ be the standard parabolic subalgebra of $\mathfrak{g}_0$ which is defined in* 2.2. *Let G be a connected simple Lie group whose Lie algebra is $\mathfrak{g}_0$ and let $\tilde{P}$ be the parabolic subgroup corresponding to $\tilde{\mathfrak{p}}_0$. Let $\tilde{\mathfrak{p}}_0 = \tilde{\mathfrak{l}}_0 + \tilde{\mathfrak{u}}_0$ be the standard Levi decomposition and $\tilde{P} = \tilde{L}\tilde{U}$ be the corresponding standard Levi decomposition of the group. Then we have*:

(a) *Every principal nilpotent $\tilde{L}$-orbit in $\tilde{\mathfrak{l}}_0$ is hyperbolically rigid.*

(b) *Let P be an arbitrary parabolic subgroup such that $\tilde{P} \subseteq P$ and let $P = LU$ be the standard Levi decomposition. Then,* h-ind_P^G *gives a bijection of the set of principal nilpotent L-orbits onto the set of principal nilpotent G-orbits.*

(2) *Let $\mathfrak{g}_0 = \mathfrak{so}(2n\, 2n)$ $(n \geqslant 2)$ and let G be one of G_i $(1 = 1, 2, 3)$ in Lemma* 2.2.1(3). *Let $\tilde{P}$ be the parabolic subalgebra of G corresponding to $\tilde{\mathfrak{p}}_i$ in* (13)–(15) *in* 2.2. *Then the above* (a) *and* (b) *hold.*

The sketch of proof the statement (a) is as follows. (We do not use this statement later.) If we assume there is a principal nilpotent $\tilde{L}$-orbit $\mathcal{O}$ which is not hyperbolically rigid, we can easily see $\mathcal{O}$ is invariant under $\mathrm{Ad}(F^\sharp)$. This contradicts Lemma 2.2.1.

In order to give a proof of (b) in the proposition, we introduce some notations. Hereafter G is a real reductive linear Lie group with the Lie algebra $\mathfrak{g}_0$.

For $S \subseteq \Phi$, we put

$$V_S = \bigoplus_{\alpha \in S} \mathfrak{g}_{0,\alpha} \subseteq \mathfrak{g}_0.$$

We fix $S \subseteq \Phi$, hereafter. Let P (resp. $\mathfrak{p}_0$) be a standard parabolic subgroup (resp. algebra) of G corresponding to S and let $P = LU$ and $\mathfrak{p}_0 = \mathfrak{l}_0 + \mathfrak{u}_0$ be the standard Levi decompositions.

We denote by p_α the natural projection of V_S to $\mathfrak{g}_{0,\alpha}$. Clearly, the standard Levi part ${}^sM{}^sA$ of the minimal parabolic subgroup ${}^sP$ acts on V_S via the adjoint action for G. Let ${}^sM_0$ be the identity connected component of ${}^sM$. Using f_α defined in 1.3, we identify $\mathfrak{g}_{0,\alpha}$ for $\alpha \in \Phi_{\text{real}}$ with $\mathbb{R}$ as follows

$$c \rightsquigarrow f_\alpha\left(\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}\right) \in \mathfrak{g}_{0,\alpha}.$$

We denote by Ξ_S the space of mappings of $S \cap \Phi_{\text{real}}$ to $\{\pm 1\}$. For $\varepsilon \in \Xi_S$, we put

$$\mathcal{O}_\varepsilon = \{X \in V_S \mid \varepsilon(\alpha) p_\alpha(X) > 0 \text{ for all } \alpha \in S \cap \Phi_{\text{real}}\}.$$

Since ${}^sM_0$ is compact and trivially acts on real root spaces, we see that $\varepsilon \rightsquigarrow \mathcal{O}_\varepsilon$ gives a bijection of Ξ_S and the set of open ${}^sM_0{}^sA$-orbits in V_S. Since $\mathrm{Ad}_{\mathfrak{l}}({}^sM{}^sA) = \mathrm{Ad}_{\mathfrak{l}}({}^sM_0{}^sA)\,\mathrm{Ad}_{\mathfrak{l}}(F^\flat)$, we can easily deduce the following from the proof of Lemma 2.2.1 and the fact that ω_α acts by -1 on $\mathfrak{g}_{0,\alpha}$ and by 1 on $\mathfrak{g}_{0,\beta}$ for $\beta \in S - \{\alpha\}$.

LEMMA 2.4.2. *The number of open ${}^sM{}^sA$-orbits in V_S coincides with $c(L)$.*

We need:

LEMMA 2.4.3. *Let $\mathcal{O}$ be a principal nilpotent L-orbit in $\mathfrak{l}_0$. Then there exists $u \in \mathcal{O}$ and h, $\bar{u} \in \mathfrak{l}_0$ which satisfy the following* (1)–(3).

(1) *u, h, $\bar{u}$ form an $\mathfrak{sl}(2,\mathbb{R})$-triple. Namely, $[h,u] = 2u$, $[h,\bar{u}] = -2\bar{u}$, and $[u,\bar{u}] = h$.*
(2) *$h \in \mathfrak{s}_0 \cap \mathfrak{l}_0$.*
(3) *If we define $\mathfrak{l}_0(k) = \{X \in \mathfrak{l}_0 \mid \mathrm{ad}(h)X = kX\}$, then we have:*

$$\mathfrak{l}_0 = \bigoplus_{k \in \mathbb{Z}} \mathfrak{l}_0(2k),$$

$$\mathfrak{l}_0(0) = {}^s\mathfrak{m}_0 + {}^s\mathfrak{a}_0,$$

$$\mathfrak{n}_0 \cap \mathfrak{l}_0 = \bigoplus_{k > 0} \mathfrak{l}_0(2k),$$

$$V_S = \mathfrak{l}_0(2).$$

Proof. Fix an arbitrary $u \in \mathcal{O}$. From the Jacobson-Morosov theorem, there exists an $\mathfrak{sl}(2, \mathbb{R})$-triple $\{u, h, \bar{u}\}$ in $\mathfrak{l}_0$ which satisfies (1) above. From [37], we can assume that (2) holds. We define $\mathfrak{l}_0(k)$ as in (3). Then, it is well known that $\mathfrak{l}_0(2k+1)=0$ for all k and $\oplus_{k\geqslant 0}\mathfrak{l}_0(k)$ is a minimal parabolic subalgebra of $\mathfrak{l}_0$. Hence, taking K^b-conjugate, we choose $u \in \mathcal{O}$ such that the above (1)–(3) hold. □

LEMMA 2.4.4. *Let $u \in V_S$. Then u is contained in an open ${}^sM^sA$-orbit, if and only if u is a principal nilpotent element in $\mathfrak{l}_0$.*

Proof. Let $u \in V_S$ and let $\mathcal{O}$ be the ${}^sM^sA$-orbit containing u. First, we assume that $\mathcal{O}$ is not open. Then, clearly $\mathrm{Ad}({}^sP \cap L)u \subseteq \mathcal{O} \times [\mathfrak{l}_0 \cap \mathfrak{n}_0, \mathfrak{l}_0 \cap \mathfrak{n}_0]$. If we denote by $({}^s\mathfrak{p}_0 \cap \mathfrak{l}_0)^u$ (resp. $\mathfrak{g}_0^u$) the centerizer of u in ${}^s\mathfrak{p}_0 \cap \mathfrak{l}_0$ (resp. $\mathfrak{g}_0$), then we have $\dim \mathfrak{g}_0^u \geqslant \dim({}^s\mathfrak{p}_0 \cap \mathfrak{l}_0)^u > \dim({}^s\mathfrak{m}_0 + {}^s\mathfrak{a}_0)$. This means u is not principal nilpotent.

Next we assume $\mathcal{O}$ is open and u is not principal nilpotent. Since any open orbit is contained in a unique $L_{\mathbb{C}}^{\mathrm{ad}}$-orbit in $\mathfrak{l}$, the first half of this proof implies any element in V_S is not principal nilpotent. However this contradicts Lemma 2.4.3. □

Lemma 2.4.2, Lemma 2.4.3 and Lemma 2.4.4 imply:

COROLLARY 2.4.5. *Let $\mathcal{O}$ be a principal nilpotent L-orbit in $\mathfrak{l}_0$. Then $\mathcal{O} \cap V_S$ is a non-empty open ${}^sM^sA$-orbit of V_S. Moreover $\mathcal{O} \rightsquigarrow \mathcal{O} \cap V_S$ gives a bijection of the set of principal nilpotent L-orbits in $\mathfrak{l}_0$ onto the set of open ${}^sM^sA$-orbits in V_S.*

We denote by p'_S the projection of V_Φ to V_S with respect to the direct sum $V_\Phi = V_S \oplus V_{\Phi-S}$. Put $V_{\Phi-S}^{\sim} = \{(x_\alpha)_{\alpha\in\Phi-S} \in V_{\Phi-S} \mid x_\alpha \neq 0 \ (\alpha \in \Phi - S)\}$ and denote by p_S the restriction of p'_S to $V_S \oplus V_{\Phi-S}^{\sim}$. The following is clear from the definition of $\tilde{\mathfrak{p}}_i$ $(i = 0, 1, 2, 3)$.

LEMMA 2.4.6. (1) *We assume that $\mathfrak{g}_0$ is a simple real Lie algebra which is one of* (1)–(12) *in the statement of Lemma* 2.1.1 *and G is connected. Let $\tilde{\mathfrak{p}}_0$ be the standard parabolic subalgebra of $\mathfrak{g}_0$ which is defined in 2.2. We assume $\tilde{\mathfrak{p}}_0 \subseteq \mathfrak{p}_0$. Then $p_S^{-1}(\mathcal{O})$ is an open ${}^sM^sA$-orbit in V_Φ for any open ${}^sM^sA$-orbit $\mathcal{O}$ in V_S and $\mathcal{O} \rightsquigarrow p_S^{-1}(\mathcal{O})$ gives a bijection of the set of open ${}^sM^sA$-orbits in V_S onto the set of open ${}^sM^sA$-orbits in V_Φ.*

(2) *We assume that $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$ $(n \geqslant 2)$ and G is one of G_i $(i = 1, 2, 3)$ in Lemma* 2.2.1(3). *Let $\tilde{\mathfrak{p}}_i$ $(i = 1, 2, 3)$ be the standard parabolic subalgebra of $\mathfrak{g}_0$ which is defined in 2.2. We assume $\tilde{\mathfrak{p}}_i \subseteq \mathfrak{p}_0$. Then the same conclusions as those in* (1) *hold.*

Next, we consider open ${}^sP \cap L$-orbits in $\mathfrak{n}_0 \cap \mathfrak{l}_0$. We choose an $\mathfrak{sl}(2, \mathbb{R})$-triple $\{u, h, \bar{u}\}$ as in Lemma 2.4.3 and let $\mathfrak{l}_0 = \oplus_{k\in\mathbb{Z}}\mathfrak{l}_0(2k)$ be the corresponding graded structure. Let q_S be the projection of $\mathfrak{n}_0 \cap \mathfrak{l}_0$ to V_S with respect to a direct sum $\mathfrak{n}_0 \cap \mathfrak{l}_0 = V_S \oplus [\mathfrak{n}_0 \cap \mathfrak{l}_0, \mathfrak{n}_0 \cap \mathfrak{l}_0]$.

LEMMA 2.4.7. *For an open ${}^sM^sA$-orbit $\mathcal{O}$ in V_S, $q_S^{-1}(\mathcal{O})$ is an open ${}^sP \cap L$-orbit in $\mathfrak{n}_0 \cap \mathfrak{l}_0$. Moreover, $\mathcal{O} \rightsquigarrow q_S^{-1}(\mathcal{O})$ gives a bijection of the set of open ${}^sM^sA$-orbits in V_S onto the set of open ${}^sP \cap L$-orbits in $\mathfrak{n}_0 \cap \mathfrak{l}_0$.*

Proof. Let $\mathcal{O}$ be an open ${}^sM^sA$-orbit in V_S and let $\mathcal{O}'$ be a unique principal nilpotent L-orbit in $\mathfrak{l}_0$. From Lemma 2.4.3 and Corollary 2.4.5, we can easily see that for any $u \in \mathcal{O}$ there exists an $\mathfrak{sl}(2, \mathbb{R})$-triple $\{u, h, \bar{u}\} \subseteq \mathfrak{l}_0$ which satisfies the conditions in Lemma 2.4.3. Let $\mathfrak{l}_0 = \bigoplus_{k \in \mathbb{Z}} \mathfrak{l}_0(2k)$ be the corresponding graded structure. We can easily see that, under suitable choice of an $\mathfrak{sl}(2, \mathbb{R})$-triple for each $u \in \mathcal{O}$, this graded structure is independent of $u \in \mathcal{O}$. For $u \in \mathcal{O}$, $\mathrm{Ad}(u)$ gives a surjection of $\mathfrak{n}_0 \cap \mathfrak{l}_0$ to $[\mathfrak{n}_0 \cap \mathfrak{l}_0, \mathfrak{n}_0 \cap \mathfrak{l}_0]$. This fact immediately follows from the corresponding statement in complex case which is famous and an easy consequence of the representation theory of $\mathfrak{sl}(2, \mathbb{C})$. Using this, we can easily see that $\mathrm{Ad}(N \cap L)u = u + [\mathfrak{n}_0 \cap \mathfrak{l}_0, \mathfrak{n}_0 \cap \mathfrak{l}_0]$. The lemma easily follows from this fact. □

COROLLARY 2.4.8. *Let $\mathcal{O}$ be a principal nilpotent L-orbit in $\mathfrak{l}_0$. Then $\mathcal{O} \cap (\mathfrak{n}_0 \cap \mathfrak{l}_0)$ is a non-empty open $L \cap {}^sP$-orbit of $\mathfrak{n}_0 \cap \mathfrak{l}_0$. Moreover $\mathcal{O} \rightsquigarrow \mathcal{O} \cap (\mathfrak{n}_0 \cap \mathfrak{l}_0)$ gives a bijection of the set of principal nilpotent L-orbits in $\mathfrak{l}_0$ onto the set of open $L \cap {}^sP$-orbits in $\mathfrak{n}_0 \cap \mathfrak{l}_0$.*

Proof of Proposition 2.4.1(b). Assume $\tilde{\mathfrak{p}}_0 \subseteq \mathfrak{p}_0$. Let $\mathcal{O}$ be a principal nilpotent L-orbit and we assume $\mathcal{O}' \in \text{h-ind}_P^G(\mathcal{O})$. Namely, $\mathcal{O}' \cap (\mathcal{O} + \mathfrak{u}_0)$ is open in $\mathcal{O} + \mathfrak{u}_0$. Choose $u \in \mathcal{O}$ such that $\mathcal{O}' \cap (u + \mathfrak{u}_0) \neq \varnothing$. Taking conjugacy by L, we can assume that $u \in \mathcal{O} \cap (\mathfrak{n}_0 \cap \mathfrak{l}_0)$. Moreover, from Lemma 2.4.7 and Corollary 2.4.8, we can assume $u \in \mathcal{O} \cap V_S$. Hence $\mathcal{O}' \cap q_\Phi^{-1}(p_S^{-1}(V_S \cap \mathcal{O}))$ is non-empty. Since $q_\Phi^{-1}(p_S^{-1}(V_S \cap \mathcal{O})) \subseteq \mathcal{O} + \mathfrak{u}_0$, we have $\mathcal{O}' \cap q_\Phi^{-1}(p_S^{-1}((V_S \cap \mathcal{O}))$ is open in $q_\Phi^{-1}(p_S^{-1}(V_S \cap \mathcal{O}))$. From Corollary 2.4.5, Lemma 2.4.6, and Lemma 2.4.7, $q_\Phi^{-1}(p_S^{-1}(V_S \cap \mathcal{O}))$ is an open ${}^sP$-orbit in $\mathfrak{n}_0$. Hence, $\mathcal{O}' \supseteq q_\Phi^{-1}(p_S^{-1}(V_S \cap \mathcal{O})) \neq \varnothing$. This means $\text{h-ind}_P^G(\mathcal{O}) = \mathcal{O}'$ and $\mathcal{O}' \cap V_\Phi \supseteq p_S^{-1}(V_S \cap \mathcal{O})$. From Corollary 2.4.5, we have $\mathcal{O}' \cap V_\Phi = p_S^{-1}(V_S \cap \mathcal{O})$. Hence, Lemma 2.4.6 implies h-ind is a bijection. □

Finally, from Lemma 2.4.7 and a similar argument in the proof of Lemma 2.4.4 we can easily deduce:

LEMMA 2.4.9. *Let G be a real reductive linear Lie group. Let P be an arbitrary standard parabolic subgroup of G and let $P = LU$ be its standard Levi decomposition. Let $\mathcal{O}$ be a non-principal nilpotent L-orbit in $\mathfrak{l}_0$. Then,* $\text{h-ind}_P^G(\mathcal{O})$ *does not contain any principal nilpotent G-orbit.*

Finally, we show:

LEMMA 2.4.10. *Let G be a real reductive linear Lie group of type II. Then, for any (standard) parabolic subgroup of G, its (standard) Levi part is type II.*

Proof. Let P be a standard parabolic subgroup of G whose standard Levi part L is not of type II. Let S be the subset of Φ corresponding to P. There exists two distinct principal nilpotent L-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathfrak{l}_0$. So, $\mathcal{O}_1 \cap V_S$ and $\mathcal{O}_2 \cap V_S$ are distinct ${}^sM^sA$-orbits in V_S (Corollary 2.4.5). Clearly $p_S^{-1}(\mathcal{O}_1 \cap V_S)$ and $p_S^{-1}(\mathcal{O}_2 \cap V_S)$ are disjoint open ${}^sM^sA$-subsets in V_Φ. This means G is not of type II. □

3. Whittaker vectors and wave front sets

3.1. Wave front sets

For real reductive cases, the study of singularities of representations are, as far as I know, originated by Kashiwara and Vergne ([26]). The notion of wave front sets of unitary representations of general Lie groups is introduced and studied by Howe ([17]). For real reductive cases his definition is also applicable to admissible Hilbert representations. He also studied the relation between the wave front sets of the representations and the wave front sets of distribution characters. For the definitions of wave front sets of distributions, see [16], [65]. Another approach to the wave front sets was proposed by Barbasch and Vogan ([1]), namely the asymptotic supports of distribution characters.

In order to describe the relations among them, we fix several notations. Let $(G, \mathfrak{g}_0, \theta, \langle\, ,\rangle)$ be a real reductive linear Lie group and let V be a Harish-Chandra module in $\mathscr{H}_G$. We denote by Θ_V the distribution character of V. An admissible continuous Hilbert G-representation H is called a Hilbert globalization of V if the K-finite part of H coincides with V as a compatible $(\mathfrak{g}, K)$-module. Choose a Hilbert globalization H of V (the existence of Hilbert globalization is established by Casselman and Wallach ([72])). We denote by $\mathrm{WF}(H)$ the wave front set of H ([17]), so $\mathrm{WF}(H)$ is a closed conic subset of the cotangent bundle T^*G of G. For our purpose, we consider the fiber $\mathrm{WF}^\circ(H)$ of $\mathrm{WF}(H)$ at the identity element e of G. Since we can identify T_e^*G with $\mathfrak{g}_0'$, $\mathrm{WF}^\circ(H)$ is a closed conical subset in $\mathfrak{g}_0'$. Here, we denote by $\mathfrak{g}_0'$ the real dual vector space of $\mathfrak{g}_0$. Using the bilinear form $\langle\, ,\rangle$, we identify $\mathfrak{g}_0'$ and $\mathfrak{g}_0$. This means that using the Killing form we identify $[\mathfrak{g}_0, \mathfrak{g}_0]'$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$. Then, it is known that $\mathrm{WF}^\circ(H)$ is a union of some nilpotent G-orbits ([17] Proposition 2.4). In particular, $\mathrm{WF}^\circ(H) \subseteq [\mathfrak{g}_0, \mathfrak{g}_0]$ under the above identification. Similarly, the fiber $\mathrm{WF}(\Theta_V)_e$ at e of the wave front set (resp. the asymptotic support $\mathrm{AS}(\theta_V)$ at 0) of the distribution character Θ_V is a closed conic subset of $\mathfrak{g}_0$ and a union of some nilpotent G-orbits (cf. [1]). Here, θ_V is the lifting of Θ_V to the neighbourhood of 0 in $\mathfrak{g}_0$ via the exponential mapping (cf. [1]).

From [17], [1, 2], we have the following inclusions

$$\mathrm{AS}(\theta_V) \subseteq \mathrm{WF}(\Theta_V)_e \subseteq \mathrm{WF}^\circ(H).$$

From [22] and [1] Theorem 4.1, if G is complex, then all the above inclusions turn out to be equations. If H is unitary, then $\mathrm{WF}(\Theta_V)_e = \mathrm{WF}^\circ(H)$ ([17] Theorem 1.8).

According to Kashiwara (1985/86 personal communication), a deep theory of asymptotic expansions of solutions of regular holonomic systems [24] implies a conjecture in [1], namely $\mathrm{AS}(\theta_V)$ coincides with $\mathrm{WF}(\Theta_V)_e$. I understand this result of Kashiwara has not been published.

We can ask if $\mathrm{WF}(\Theta_V)_e = \mathrm{WF}^\circ(H)$ holds or not in general situation. It seems likely that the affirmative answer holds, but I understand this is an open problem at this point. In particular, $\mathrm{WF}^\circ(H)$ seems likely to be independent of the choice of a Hilbert globalization H of V, but I do not know a proof.

Before giving the definition of the wave front set of a Harish-Chandra module in this article, I remark that it is more natural to consider the wave front set as a closed conic set in $i\mathfrak{g}_0'$ than that of $\mathfrak{g}_0'$. Here, i means the imaginary unit $\sqrt{-1}$. This is a viewpoint of [56]. They defined the singular spectrum of a hyperfunction, which is a real-analytic counterpart of the wave front set, as a closed conic set of i times the cotangent bundle. The wave front set and the singular spectrum (analytic wave front set) of a distribution character coincide, since it is a solution of a regular holonomic system ([23] Theorem 1).

In this article, we define the wave front set $\mathrm{WF}(V)$ of a Harish-Chandra module V in $\mathscr{H}_G$ by $\mathrm{WF}(V) = i\mathrm{AS}(\theta_V)$. (From the unpublished result of Kashiwara above, it also equals $i\mathrm{WF}(\Theta_V)_e$. However, we do not use this in this article.) Hence, $\mathrm{WF}(V)$ is a closed conic subset in $i\mathfrak{g}_0'$. Using the identification $\mathfrak{g}_0 \cong \mathfrak{g}_0'$, we regard $\mathrm{WF}(V)$ as a closed conic subset in $i[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq i\mathfrak{g}_0$.

For our purpose, the important thing is which principal nilpotent G-orbit is contained in $\mathrm{WF}(V)$. So, for a given Harish-Chandra module $V \in \mathscr{H}_G$, we define the principal part ${}^\odot\mathrm{WF}(V)$ of $\mathrm{WF}(V)$ as follows

$$ {}^\odot\mathrm{WF}(V) = \{\mathcal{O} \mid \mathcal{O} \text{ is a principal nilpotent } G\text{-orbit such that } i\mathcal{O} \subseteq \mathrm{WF}(V)\}. $$

If ${}^\odot\mathrm{WF}(V)$ is written by $\{\mathcal{O}\}$ by a single principal nilpotent G-orbit $\mathcal{O}$, we sometimes write ${}^\odot\mathrm{WF}(V) = \mathcal{O}$.

We recall some fundamental properties of wave front sets. First, the following is immediately deduced from the definition (cf. [1]).

LEMMA 3.1.1. *Let G and G' be real reductive linear Lie groups. For $V \in \mathscr{H}_G$ and $V' \in \mathscr{H}_{G'}$, we denote by $V \otimes V'$ the external tensor product of V and V'. Clearly $V \otimes V' \in \mathscr{H}_{G \times G'}$. In this situation, the principal part of the wave front of $V \otimes V'$ is expressed by those of V and V' as follows*

$$ {}^\odot\mathrm{WF}(V \otimes V') = \{\mathcal{O} \times \mathcal{O}' \mid \mathcal{O} \in {}^\odot\mathrm{WF}(V), \mathcal{O}' \in {}^\odot\mathrm{WF}(V')\}. $$

The following is also immediately deduced from the definition (cf. [1]).

LEMMA 3.1.2. *Let $V \in \mathscr{H}$ and let V_1 be an irreducible subquotient of V. If $\mathcal{O} \in {}^{\odot}\mathrm{WF}(V_1)$ is not contained in ${}^{\odot}\mathrm{WF}(V)$, then there exists another irreducible subquotient V_2 of V which satisfies the following* (1), (2).

(1) *V_2 is not isomorphic to V_1.*
(2) $\mathcal{O} \in {}^{\odot}\mathrm{WF}(V_2)$.

The following is proved by the same argument as the proof of [1], Theorem 4.1.

LEMMA 3.1.3. *Let $V \in \mathscr{H}$ and let V_1 be an irreducible subquotient of V. If ${}^{\odot}\mathrm{WF}(V_1) \neq \varnothing$, then ${}^{\odot}\mathrm{WF}(V) \neq \varnothing$.*

REMARK. Together with Casselman's result Theorem 3.3.1, the following stronger result is immediately deduced from the main theorem of this article (Theorem 3.3.3).

COROLLARY 3.1.4. *Let $V_1, V_2 \in \mathscr{H}_G$ be Harish-Chandra modules such that V_1 is a subquotient of V_2. Then,*

$$^{\odot}\mathrm{WF}(V_1) \subseteq {}^{\odot}\mathrm{WF}(V_2).$$

This statement is not at all clear from the definition. Hence, we do not use this for proving the main theorem, although the corresponding statement holds for Howe's wave front sets ([17] Proposition 1.3).

In order to describe another important property of ${}^{\odot}\mathrm{WF}$, we fix the notations on parabolic induction. Let $(G, \mathfrak{g}_0, \theta, \langle\,,\rangle)$ be a real reductive linear Lie group and let P be a standard parabolic subgroup of G with the standard Levi decomposition $P = LU$. Let $V \in \mathscr{H}_L$ and we fix a Hilbert globalization H of V such that V is unitary as $K \cap L$-representation. Let $\mathrm{Ind}_P^G(H)$ be the normalized induced representation (for example see [73] p. 31). So, if H is unitary, so is Ind_P^G (H). We denote by $\mathrm{Ind}_P^G(V)$ the K-finite part of $\mathrm{Ind}_P^G(H)$. We have $\mathrm{Ind}_P^G(V) \in \mathscr{H}_G$ and $\mathrm{Ind}_P^G(V)$ does not depend on the choice of H (Casselman, cf. [73] p.141).

LEMMA 3.1.5 ([1] Theorem 3.5 and Lemma 2.4.9). *Let $(G, \mathfrak{g}_0, \theta)$ be a real reductive linear Lie group and let P be a standard parabolic subgroup of G with the standard Levi decomposition $P = LU$. Let $V \in \mathscr{H}_L$. Then we have ${}^{\odot}\mathrm{WF}(\mathrm{Ind}_P^G(V)) \subseteq \text{h-ind}_P^G({}^{\odot}\mathrm{WF}(V))$.*

REMARK. Strictly speaking, Barbasch and Vogan assumed G is connected semisimple in [1]. However, we can immediately see their argument works for the general reductive linear case. In this article, we shall quote several results in [1] as statements for real reductive linear Lie groups.

3.2. Quasi-large Harish-Chandra module and type II envelope

Let $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$ be a real reductive linear Lie group. For $V \in \mathscr{H}_G$, we denote by $\mathrm{Dim}(V)$ (resp. $c(V)$) the Gelfand-Kirillov dimension (resp. the multiplicities or Bernstein degree) of V (cf. [66], pp. 76–77).

We call $V \in \mathscr{H}_G$ quasi-large if ${}^{\odot}\mathrm{WF}(V) \neq \emptyset$. $V \in \mathscr{H}_G$ is quasi-large if and only if $\mathrm{Dim}(V) = \dim N$. (This follows from [66] Lemma 3.4 and [1] Theorem 4.1.)

If G is quasi-split, then we call a quasi-large Harish-Chandra module in $\mathscr{H}_G$ large ([66]).

For a real reductive linear Lie group $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$, a real reductive linear Lie group $(G^+, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$ which satisfies the following properties is called a type II envelope of G.

(E1) G is a subgroup of G^+.
(E2) G^+ is type II. (Namely, G^+ acts transitively on the set of principal nilpotent G-orbits.)

Clearly a type II envelope exists and it is not unique. If G^+ is a type II envelope of G, then we can easily see G^+ normalizes G (the linearity of G is crucial). We denote by K^+ the maximal compact subgroup of G^+ associated to θ by the condition (RL3). Hence G^+ is generated by G and K^+ and K^+ normalizes K. Let $\sigma \in K^+$ and $V \in \mathscr{H}_G$. For $v \in V$, $k \in K$, and $X \in \mathfrak{g}$, we define new $(\mathfrak{g}, K)$-module structure "$*$" of V^σ by $k * v = (\sigma k \sigma^{-1})v$ and $X * v = \mathrm{Ad}(\sigma)(X)v$. This new action $*$ defines another Harish-Chandra module $V^\sigma \in \mathscr{H}_G$.

3.3. Whittaker vectors and the formulation of the main result

Let $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$ be a real reductive linear Lie group. For a Harish-Chandra module $V \in \mathscr{H}_G$, we fix a Hilbert globalization H of V. Let V_∞ be the space of C^∞-vectors in H. It is known that V_∞ does not depend on the choice of H ([9], and [72]). We call V_∞ the Casselman-Wallach globalization of V. V_∞ is a nuclear Fréchet G-representation and the functor $V \rightsquigarrow V_\infty$ is exact ([9]).

We consider two real reductive linear Lie groups G and G' and the external tensor product $V \otimes V' \in \mathscr{H}_{G \times G'}$ for $V \in \mathscr{H}_G$ and $V' \in \mathscr{H}_{G'}$. Then easily we have $(V \otimes V')_\infty = V_\infty \hat{\otimes} V'_\infty$. Here, the topological tensor product $\hat{\otimes}$ is well-defined from the nuclearity.

Let $\psi: \bar{N} \to \mathbb{C}^\times$ be a unitary character. We denote the differential character of $\bar{\mathfrak{n}}_0$ to $i\mathbb{R}$ by the same letter ψ. We also denote the complexification of ψ on $\bar{\mathfrak{n}}$ by the same letter. We have $\psi \in i\bar{\mathfrak{n}}'_0$. Using the Killing form of $[\mathfrak{g}_0, \mathfrak{g}_0]$, we identify $\bar{\mathfrak{n}}'_0$ and $\mathfrak{n}_0$. So, we can regard $\psi \in i\mathfrak{n}_0$. Since ψ is a character, we have $\psi \in iV_\Phi$ (see 2.4).

We call a unitary character ψ admissible if ψ is contained in an open ${}^sM^sA$-

orbit in iV_Φ (cf. [40]). (In [13], etc., such characters are called non-degenerate.) For an admissible unitary character ψ, we denote by $\mathcal{O}_\psi$ the unique principal nilpotent G-orbit such that $i\mathcal{O}_\psi$ contains $\psi \in iV_\Phi$. For $V \in \mathcal{H}_G$, we have $\psi \in \mathrm{WF}(V)$ if and only if $\mathcal{O}_\psi \in {}^{\odot}\mathrm{WF}(V)$.

Let $V \in \mathcal{H}_G$. We define a space of $C^{-\infty}$-Whittaker vectors $\mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V)$ as follows (cf. [48]).

$$\mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) = \{v \in V'_\infty \mid Xv = \psi(X)v \quad (X \in \bar{\mathfrak{n}})\}.$$

Here, V'_∞ means the continuous dual of V_∞.

Since the Casselman-Wallach globalization behaves well under external tensor products, so does the functor $\mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}$.

The following result of Casselman is important.

THEOREM 3.3.1 ([8], see also [48]). *Assume that ψ is an admissible unitary character on $\bar{\mathfrak{n}}_0$. Then, $V \rightsquigarrow \mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V)$ defines an exact functor from $\mathcal{H}_G$ to the category of finite-dimensional complex vector spaces.*

Actually, Casselman proved more, namely vanishing of all the higher degree of the twisted $\bar{\mathfrak{n}}$-cohomology. Using a standard argument, we see this implies the following (cf. [48]).

LEMMA 3.3.2. *Assume that ψ is an admissible unitary character on $\bar{\mathfrak{n}}_0$ and let E be a finite dimensional G-module. Then, for $V \in \mathcal{H}_G$, we have a non-canonical isomorphism of vector spaces:*

$$\mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V \otimes E) \cong \mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) \otimes E.$$

Now we can state our main result.

THEOREM 3.3.3. *Let G be a real reductive linear Lie group. Assume that ψ is an admissible unitary character on $\bar{\mathfrak{n}}_0$ and V is a Harish-Chandra module in $\mathcal{H}_G$. Then, $\mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) \neq 0$ if and only if $\psi \in \mathrm{WF}(V)$.*

REMARK 1. In the above theorem, we do not put the assumption that V is irreducible.

REMARK 2. In the above theorem, we can relax the conditions (RL2, 3, 5) on G. However, the linearity of G (i.e. (RL4)) is crucial for our proof.

From the following result, we have only to consider quasi-large representations.

LEMMA 3.3.4. *Assume that ψ is an admissible unitary character on $\bar{\mathfrak{n}}_0$ and V is an irreducible Harish-Chandra module in $\mathcal{H}_G$ such that $\mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) \neq 0$. Then V is quasi-large.*

For a quasi-split G, this result is proved by Kostant ([36] Theorem D). For non-quasi-split case, it follows, for example, from the main result of [44].

Next, we remark:

LEMMA 3.3.5. *Let G be a real reductive linear Lie group. Then, we have*:

(1) *If the statement of Theorem* 3.3.3 *holds for a finite covering group of G, the statement also holds for G.*
(2) *If the statement of Theorem* 3.3.3 *holds for the identity connected component G_0 of G, the statement also holds for G.*
(3) *If the statement of Theorem* 3.3.3 *holds for real reductive linear Lie groups G_1 and G_2, then the statement also holds for $G_1 \times G_2$.*
(4) *In order to show Theorem* 3.3.3, *it suffices to show the statement of Theorem* 3.3.3 *holds for any connected real simple linear Lie group G.*

In the above lemma, (1) is trivial. If we denote by $\mathrm{res}_G^{G_0}(V)$ the restriction of $V \in \mathscr{H}_G$ to $\mathscr{H}_{G_0}$, then we have $\mathrm{WF}(V) = \mathrm{WF}(\mathrm{res}_G^{G_0}(V))$ and $\mathrm{Wh}_{\bar{\mathfrak{n}},\psi}^{\infty}(V) = \mathrm{Wh}_{\bar{\mathfrak{n}},\psi}^{\infty}(\mathrm{res}_G^{G_0}(V))$. So, (2) is clear. (3) is easy, since $\mathrm{Wh}_{\bar{\mathfrak{n}},\psi}^{\infty}$ and WF behave well under the external tensor product. (4) is easily proved using (1)–(3).

We shall show:

LEMMA 3.3.6. *If G is a real reductive linear Lie group of type II, then the statement of Theorem* 3.3.3 *holds.*

If G is quasi-split, this is essentially due to Kostant. (Although Theorem L in [36] is apparently a little different from this, we can immediately deduce the lemma for quasi-split case from the results in [36] and [66] using the Casselman-Wallach theory.) For general G, the above lemma is proved in Section 4.

As a conclusion, we have only to prove Theorem 3.3.3 for connected real simple linear Lie groups of type I which are classified in Lemma 2.1.1. We perform this in Section 5.

3.4. A result of Hashizume

In this section, we fix a real reductive linear Lie group G and a standard parabolic subgroup P of G with the standard Levi decomposition $P = LU$. Let S be the subset of Π corresponding to P. We also fix an admissible unitary character ψ on $\bar{\mathfrak{n}}$. Then ψ is regarded as an element in iV_{Φ} (cf. 3.3). Under this identification, put $\psi_L = p_S(\psi)$ (cf. 2.4). Then, ψ_L is regarded as an admissible unitary character on $\bar{\mathfrak{n}} \cap \mathfrak{l}$.

The following result is one of the main ingredients of the proof of the main result.

THEOREM 3.4.1 ([13], Theorem 1). *Let G be a real reductive linear Lie group* *and let P be a standard parabolic subgroup of G with the standard Levi decomposition $P = LU$. Let $V \in \mathscr{H}_L$. Then, we have*

$$\dim \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(\mathrm{Ind}^G_P(V)) \leqslant \dim \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}}\cap \mathfrak{l},\psi_L}(V).$$

In fact, the statement of [13] Theorem 1 is not the same as the above theorem, since the above statement involving Casselman-Wallach globalizations. So, we give here a proof. (The main idea of the proof is the same as that of Hashizume's.)

Proof of Theorem 3.4.1. From Theorem 3.3.1 and the exactness of the parabolic induction, we can assume V is irreducible. Let V_∞ be the Casselman-Wallach globalization of V. We define the smooth induction (cf. [9]) as follows

$${}^{\infty}\mathrm{Ind}^G_P(V_\infty) = \{f \in C^\infty(G, V) \mid f(glu) = l^{-1} f(g) \quad (g \in G,\ l \in L,\ u \in U)\}.$$

Under the left translation, ${}^{\infty}\mathrm{Ind}^G_P(V_\infty)$ has a structure of Fréchet G-module. Then, from the Casselman-Wallach theory ([9]), ${}^{\infty}\mathrm{Ind}^G_P(V_\infty) = \mathrm{Ind}^G_P(V)_\infty$.

From Casselman's subrepresentation theorem, there exists some irreducible finite dimensional ${}^sM^sA$-module E such that V is a quotient of $\mathrm{Ind}^L_{L\cap {}^sP}(E)$. Let X be the kernel of the projection $\mathrm{Ind}^L_{L\cap {}^sP}(E)$ to V. Hence, we have an exact sequence

$$0 \to \mathrm{Ind}^G_P(X) \to \mathrm{Ind}^G_{{}^sP}(E) \to \mathrm{Ind}^G_P(V) \to 0.$$

From the Casselman-Wallach theory, we have the following exact sequence

$$0 \to {}^{\infty}\mathrm{Ind}^G_P(X_\infty) \to {}^{\infty}\mathrm{Ind}^G_{{}^sP}(E) \to \mathrm{Ind}^G_P(V)_\infty \to 0.$$

Moreover, the image of ${}^{\infty}\mathrm{Ind}^G_P(X_\infty)$ into ${}^{\infty}\mathrm{Ind}^G_{{}^sP}(E)$ is closed. Put $Z = {}^{\infty}\mathrm{Ind}^G_P(X_\infty)$. Hence we have

$$\mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(\mathrm{Ind}^G_P(V)) = \{w \in {}^{\infty}\mathrm{Ind}^G_{{}^sP}(E)' \mid Yw = \psi(Y)w,\ (Y \in \bar{\mathfrak{n}}),\ w|_Z = 0\}.$$

Here, "′" means the topological dual. Let $\bar{U}$ be the opposite unipotent subgroup of G to U. We consider an open subset $\bar{U}P$ in G. Put

$$J_1 = \{f \in {}^{\infty}\mathrm{Ind}^G_{{}^sP}(E) \mid \mathrm{supp}(f) \subseteq \bar{U}P\} \cong C_0^\infty(\bar{U}) \hat{\otimes}\, {}^{\infty}\mathrm{Ind}^L_{{}^sP\cap L}(E),$$
$$J_2 = \{f \in Z \mid \mathrm{supp}(f) \subseteq \bar{U}P\}.$$

Here, $C_0^\infty(\bar{U})$ means the space of C^∞-functions on $\bar{U}$ with compact support. Then we have canonical injection $q: J_2 \hookrightarrow J_1$ induced from $X_\infty \hookrightarrow {}^{\infty}\mathrm{Ind}^L_{{}^sP\cap L}(E)$. Restriction of $w \in \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(\mathrm{Ind}^G_P(V))$ to J_1 induces a map $F: \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(\mathrm{Ind}^G_P(V)) \to J_1'$.

Using the Bruhat theory and stratifying $G-\bar{U}P$ by Bruhat cells, we can conclude F is injective just the same way as the proof of [13] Theorem 1.

Let $w \in \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(\mathrm{Ind}^G_P(V))$. Then we have $F(w)$ takes value zero on $q(J_2)$. Since $F(w)$ is also an ψ-eigenvector. The restriction $F(w)|_L$ of $F(w)$ to L is well-defined. Namely we have

$$F(w)(\varphi \otimes \phi) = \int_{\bar{U}} \psi(\bar{u})^{-1} \varphi(\bar{u})\, d\bar{u} F(w)|_L(\phi),$$

for $\varphi \in C_0^{\infty}(\bar{U})$ and $\phi \in {}^{\infty}\mathrm{Ind}^L_{{}^sP\cap L}(E)$. Here, $d\bar{u}$ is a Haar measure on $\bar{U}$. Moreover, we easily have $F(w)|_L \in \mathrm{Wh}_{\bar{\mathfrak{n}}\cap \mathfrak{l},\psi_L}(\mathrm{Ind}^L_{{}^sP\cap L}(E))$ and $F(w)|_L$ takes value zero on X_{∞}. This means $F(w)|_L \in \mathrm{Wh}_{\bar{\mathfrak{n}}\cap \mathfrak{l},\psi_L}(V)$. Since $w \rightsquigarrow F(w)|_L$ is an injective linear map, we have the theorem. □

4. Proof of the main theorem (type II case)

4.1. Collection of some notions from [67, 69] *and* [64]

In this section, we recall several important notions introduced in [67, 69] or [64]. Fix a real reductive linear Lie group $(G, \mathfrak{g}_0, \theta, \langle\, , \rangle)$.

First, we recall the notion of coherent continuation of characters (or coherent families of characters) of Harish-Chandra modules ([67], Definition 7.2.5, also see [64]. A prototype of this notion is introduced by Hecht and Schmid for discrete series characters. The corresponding notion for the category $\mathcal{O}$ is due to Jantzen). We denote by ${}^sH^{\vee}$ the character group of the fixed maximally split Cartan subgroup ${}^sH$ of G (1.3). We define a subgroup ${}^sH^{\wedge}$, called the weight lattice, of ${}^sH^{\vee}$ ([67]) as follows.

$${}^sH^{\wedge} = \{\lambda \in {}^sH^{\vee} \,|\, \text{there is a finite-dimensional } G\text{-module which has } \lambda \text{ as a weight}\}.$$

Let $d: {}^sH^{\wedge} \to \mathsf{P}$ be the differential homomorphism. Since $\mathrm{Ad}(G) \subseteq G^{\mathrm{ad}}_{\mathbb{C}}$, the root lattice Q is canonically embedded into ${}^sH^{\wedge}$ such that the composition of this embedding and d is identity on Q. For $\lambda \in {}^sH^{\wedge}$, there exists a unique finite-dimensional irreducible G-representation V_{λ} with extremal weight λ (cf. [67], Proposition 0.4.12). Conversely, any irreducible finite-dimensional G-representation V has a highest ${}^sH$-weight $\lambda \in {}^sH^{\wedge}$ and an arbitrary ${}^sH$-weight in V is contained in $\lambda - \mathsf{Q}$ (cf. [67], Proposition 0.4.9). For an irreducible finite-dimensional G-module E and $\mu \in {}^sH^{\wedge}$, we denote by $m_{\mu}(E)$ the multiplicity of μ in E.

We fix a regular weight $\lambda \in {}^s\mathfrak{h}^*$. For $\mu \in {}^sH^{\wedge}$, we consider a symbol "$\lambda + \mu$". Let

$\lambda + {}^sH^\wedge$ be the set of the symbols "$\lambda+\mu$" ($\mu \in {}^sH^\wedge$). A map Θ of $\lambda + {}^sH^\wedge$ to the Grothendieck group $K(\mathscr{H}_G)$ of $\mathscr{H}_G$ is called a coherent family, if it satisfies the following (C1, 2).

(C1) For $\mu \in {}^sH^\wedge$, $\Theta(\lambda+\mu) \in K(\mathscr{H}_G[\lambda + d\mu])$.

(C2) For any irreducible finite-dimensional G-representation E and all $\nu \in {}^sH^\wedge$, we have

$$\Theta(\lambda+\nu) \otimes E = \sum_{\mu \in {}^sH^\wedge} m_\mu(E)\Theta(\lambda+\mu+\nu).$$

Let $\mu \in {}^s\mathfrak{h}^*$ be regular and let $V \in \mathscr{H}_G[\mu]$, then there exists a unique coherent family Θ_V on $\mu + {}^sH^\wedge$ such that $\Theta_V(\mu) = V$ (Schmid cf. [67] Theorem 7.2.7). Here, we denote the image of V into the Grothendieck group by the same letter.

For regular $\mu \in {}^s\mathfrak{h}^*$ and $w \in W_\mu$, we see $w\mu \in \mu + Q$. Using the canonical inclusion of Q into ${}^sH^\wedge$, we regard $w\mu$ as an element in $\mu + {}^sH^\wedge$. For regular $\mu \in {}^s\mathfrak{h}^*$, we define a W_μ-module structure on $K(\mathscr{H}[\mu])$ by

$$w \cdot V = \Theta_V(w^{-1}\mu).$$

This W_μ-module $K(\mathscr{H}_G[\mu])$ is called a coherent continuation representation ([67], [33] Appendix).

We denote the basis of $K(\mathscr{H}_G[\mu])$ consisting of the irreducible modules by $\mathbf{B}_\mu$. Following [69], we introduce a preorder $\leqslant$ on $\mathbf{B}_\mu$ as follows. Let V_1, $V_2 \in \mathbf{B}_\mu$ and write $wV_1 = \Sigma_{V \in \mathbf{B}_\mu} \phi_V^w V$ for $w \in W_\mu$, where ϕ_V^w ($V \in \mathbf{B}_\mu$) are integers. We say $V_1 \leqslant V_2$ if $\phi_{V_2}^w \neq 0$ for some $w \in W_\mu$. We say $V_1 \sim V_2$ if both the $V_1 \leqslant V_2$ and $V_2 \leqslant V_1$ hold. Following [5] and [69], we introduce the notion of cones for Harish-Chandra modules. For $V \in \mathbf{B}_\mu$, we put

$$\mathscr{C}_V = \sum_{V \leqslant V_1} \mathbb{C}V_1.$$

This is a W_μ-submodule of $K(\mathscr{H}_G[\mu]) \otimes_{\mathbb{Z}} \mathbb{C}$. We call $\mathscr{C}_V$ the cone (representation) of V. If $V \in \mathbf{B}_\mu$ is quasi-large, we call $\mathscr{C}_V$ the big cone.

Next, we introduce the notion of regular characters and block relations [67, 69]. Let $\mathfrak{h}_0$ be a θ-stable Cartan subalgebra of $\mathfrak{g}_0$ and put $\mathfrak{a}_0 = \mathfrak{s}_0 \cap \mathfrak{h}_0$. Let H be the Cartan subgroup of G corresponding to $\mathfrak{h}_0$. (RL5)(1, 2) assures H is abelian. We put

$$T = K \cap H, \qquad A = \exp(\mathfrak{a}_0).$$

Then $H = TA$. Let MA be the Langlands decomposition of the centerizer of A in G. So, T is a compact Cartan subgroup of M. Let $\Delta(\mathfrak{m}, \mathfrak{t})$ be the root system with respect to $(\mathfrak{m}, \mathfrak{t})$. Here, $\mathfrak{m}$ (resp. $\mathfrak{t}$) is the complexified Lie algebra of M (resp. T).

We also denote by $\mathfrak{t}_0$ the Lie algebra of T.

We define the notion of a regular character of H as in [69] (also see [64], [67]). Namely, we say a pair $\gamma = (H, \Gamma, \bar{\gamma})$ is a regular character of H if Γ is a character of H and $\bar{\gamma}$ is a element of $\mathfrak{h}^*$ such that they satisfy the following two assumptions. The first assumption is that $\bar{\gamma} \in i\mathfrak{t}'_0$ is regular for $\Delta(\mathfrak{m}, \mathfrak{t})$. We define a positive system $\Delta^+(\mathfrak{m}, \mathfrak{t})$ of $\Delta(\mathfrak{m}, \mathfrak{t})$ such that $\langle \bar{\gamma}, \alpha \rangle > 0$ for all $\alpha \in \Delta^+(\mathfrak{m}, \mathfrak{t})$. We denote by $\rho_{\mathfrak{m}}$ (resp. $\rho_{\mathfrak{m}\cap\mathfrak{k}}$) half the sum of the roots in $\Delta^+(\mathfrak{m}, \mathfrak{t})$ (resp. $\Delta^+(\mathfrak{m}\cap\mathfrak{k}, \mathfrak{t})$). The second assumption is

$$d\Gamma = \bar{\gamma} + \rho_{\mathfrak{m}} - 2\rho_{\mathfrak{m}\cap\mathfrak{k}}.$$

$\bar{\gamma}$ defines a Harish-Chandra homomorphism $\chi_{\bar{\gamma}}$ of $Z(\mathfrak{g})$ to $\mathbb{C}$ via 1.4. For $\mu \in {}^s\mathfrak{h}^*$, we define

$$\mathrm{RC}(H, \mu) = \{\gamma \mid \gamma \text{ is a regular character of } H \text{ and } \chi_{\bar{\gamma}} = \chi_\mu\},$$

$$\mathrm{RC}(\mu) = \bigcup \{\gamma = (H', \Gamma, \bar{\gamma}) \mid H' \text{ is a } \theta\text{-stable Cartan subgroup of } G \text{ and } \gamma \in \mathrm{RC}(H', \mu)\}.$$

The conjugations define a K-action on $\mathrm{RC}(\mu)$.

As in [64], for a regular character $\gamma = (H, \Gamma, \bar{\gamma})$ of H, we attach (a Harish-Chandra module of) a discrete series representation σ of M and a character μ of A. Fix a cuspidal parabolic subgroup P of G whose Levi part is MA. We denote by $\pi(\gamma)$ the induced Harish-Chandra module $\mathrm{Ind}_P^G(\sigma \otimes \nu \otimes 1)$. For $\gamma \in \mathrm{RC}(H, \mu)$, $\pi(\gamma)$ may depend on the choice of P, but its image $[\pi(\gamma)]$ in $K(\mathscr{H}_G[\mu])$ is independent of the choice of P. If μ is regular, then we have the irreducible Langlands subquotient $\bar{\pi}(\gamma)$. We have:

THEOREM 4.1.1 (Langlands, see [69] Theorem 2.13). *Let $\mu \in {}^s\mathfrak{h}^*$ be regular. For any irreducible Harish-Chandra module V in $\mathscr{H}_G(\mu)$, there exists some $\gamma \in \mathrm{RC}(\mu)$ such that $V \cong \bar{\pi}(\gamma)$. Moreover, for $\gamma^1, \gamma^2 \in \mathrm{RC}(\mu)$, $\bar{\pi}(\gamma^1) \cong \bar{\pi}(\gamma^2)$ if and only if γ^1 is K-conjugate to γ^2. Also, we have $[\pi(\gamma^1)] = [\pi(\gamma^2)]$ if and only if γ^1 is K-conjugate to γ^2.*

For $\gamma^1, \gamma^2 \in \mathrm{RC}(\mu)$, we denote by $m(\bar{\pi}(\gamma^1), \pi(\gamma^2))$ the multiplicity of $\bar{\pi}(\gamma^1)$ in $\pi(\gamma^2)$ as a composition factor. One of the definition of block relation $\overset{B}{\sim}$ on $\mathbf{B}_\mu$ is the equivalence relation generated by

$$\bar{\pi}(\gamma^1) \overset{B}{\sim} \bar{\pi}(\gamma^2) \quad \text{if } m(\bar{\pi}(\gamma^1), \pi(\gamma^2)) \neq 0.$$

This definition is found in [69] Definition 1.14, another (original) definition is in [67] Chapter 9. Following [69], we also introduce block relation $\overset{B}{\sim}$ on $\mathrm{RC}(\mu)$ such that $\gamma^1 \overset{B}{\sim} \gamma^2$ if and only if $\bar{\pi}(\gamma^1) \overset{B}{\sim} \bar{\pi}(\gamma^2)$. We denote by $\mathrm{BL}(\mu)$ the set of equivalence class in $\mathrm{RC}(\mu)$ under the block relation. We identify $\mathrm{BL}(\mu)$ with the set of equivalence class in $\mathbf{B}_\mu$ under the block relation.

For $B \in \mathrm{BL}(\mu)$, we consider the $\mathbb{C}$-subspace $V_B(\mu)$ in $K(\mathscr{H}_G[\mu]) \otimes_{\mathbb{Z}} \mathbb{C}$ spanned by the elements in B. The following is important.

THEOREM 4.1.2 (Vogan [67], [69]). *Fix a regular weight $\mu \in {}^s\mathfrak{h}^*$. For all $B \in \mathrm{BL}(\mu)$, we have*

(1) *$V_B(\mu)$ is a W_μ-submodule of $K(\mathscr{H}_G[\mu]) \otimes_{\mathbb{Z}} \mathbb{C}$.*
(2) *For any irreducible $V \in B$, we have $\mathscr{C}_V \subseteq V_B(\mu)$.*

Fix regular $\mu \in {}^s\mathfrak{h}^*$. Next, we introduce the notion of cross product ([67] Chapter 8, [69] Definition 4.1). Actually, Vogan defined it in a much more general situation, but we only define it in the case which we need. Let $\gamma = ({}^sH, \Gamma, \bar{\gamma}) \in \mathrm{RC}({}^sH, \mu)$ and $w \in W_{\bar{\gamma}}$. Here ${}^sH$ is the fixed maximally split Cartan subgroup of G(1.3). For all regular $\lambda \in {}^s\mathfrak{h}^*$, we choose a positive system $\Delta_\lambda^+({}^s\mathfrak{m}, {}^s\mathfrak{t})$ of $\Delta({}^s\mathfrak{m}, {}^s\mathfrak{t})$ such that $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta_\lambda^+({}^s\mathfrak{m}, {}^s\mathfrak{t})$. We denote by $\rho_{\mathfrak{m}}[\lambda]$ half the sum of the roots in $\Delta_\lambda^+({}^s\mathfrak{m}, {}^s\mathfrak{t})$. Since $w \in W_{\bar{\gamma}}$, $\eta = w\bar{\gamma} - \rho_{\mathfrak{m}}[w\bar{\gamma}] - (\bar{\gamma} - \rho_{\mathfrak{m}}[\bar{\gamma}])$ is contained in the root lattice $\mathfrak{Q}$. Using the canonical embedding $\mathfrak{Q} \hookrightarrow {}^sH^\wedge$, put $w \times \Gamma = \Gamma + \eta \in {}^sH^\wedge$. We define the cross action $w \times \gamma = ({}^sH, w \times \Gamma, w\bar{\gamma})$. Any block is closed under this cross action ([69] Theorem 8.8).

There exists unique $w_\gamma \in W$ such that $\bar{\gamma} = w_\gamma \mu$. Since $W_{\bar{\gamma}} = w_\gamma W_\mu w_\gamma^{-1}$, we can define W_μ-cross action by $y \times \gamma = (w_\gamma y^{-1} w_\gamma^{-1}) \times \gamma$. Actually this defines left action of W_μ on $\mathrm{RC}({}^sH, \mu)$.

The following is one of the main ingredients of the proof.

LEMMA 4.1.3 ([69] Theorem 8.5). *Fix a regular weight $\mu \in {}^s\mathfrak{h}^*$. Assume $\mathrm{RC}(\mu) \neq \emptyset$ and let $B \in \mathrm{BL}(\mu)$. Then B contains an element in $\mathrm{RC}({}^sH, \mu)$ and it is uniquely determined up to the cross action and the K-conjugacy.*

4.2. MacDonald representations

We fix a regular weight $\mu \in {}^s\mathfrak{h}^*$. Let R be a subroot system of the integral root system Δ_μ. We fix a positive system R^+. Let $d_R = \mathrm{card}\, R^+$. We define an element P_R in the d_Rth symmetric power $S^{d_R}({}^s\mathfrak{h}^*)$ of ${}^s\mathfrak{h}^*$ as follows

$$P_R = \prod_{\alpha \in R^+} \alpha.$$

We consider the W_μ-module structure on $S^{d_R}({}^s\mathfrak{h}^*)$ induced from that on ${}^s\mathfrak{h}^*$. We denote by $\sigma(R)$ the W_μ-subrepresentation of $S^{d_R}({}^s\mathfrak{h}^*)$ generated by P_R. Let W_R be the Weyl group of R, which is regarded as a subgroup of W_μ. We denote by $l(w)$ the length of $w \in W_R$. We define the sign-representation sgn of W_R by $\mathrm{sgn}(w) = (-1)^{l(w)}$ for $w \in W_R$. The representation $\sigma(R)$ is studied by MacDonald ([41]) and he proved the following.

THEOREM 4.2.1 ([41]). *In the above situation the following hold.*

(1) $\sigma(R)$ *is irreducible.*

(2) *If* $f \in S({}^s\mathfrak{h}^*)$ *satisfies* $wf = \mathrm{sgn}(w)f$ *for all* $w \in W_R$, *then* P_R *divides* f.

For an irreducible W_μ-representation σ, we denote by $b(\sigma)$ the minimal number in the set

$$\{n \in \mathbb{N} \mid \sigma \text{ is an irreducible constituent of } S^n({}^s\mathfrak{h}^*)\}$$

The above theorem implies that $b(\sigma(R)) = d_R$.

We can reduce the following immediately from (2) in the above theorem.

COROLLARY 4.2.2. *We have*

(1) *The multiplicity of the sign-representation of* W_R *in* $\sigma(R)|_{W_R}$ *is one.*

(2) *The multiplicity of* $\sigma(R)$ *in* $\mathrm{Ind}_{W_R}^{W_\mu}(\mathrm{sgn})$ *is one.*

(3) *Let* τ *be an irreducible constituent of* $\mathrm{Ind}_{W_R}^{W_\mu}(\mathrm{sgn})$ *which is not isomorphic to* $\sigma(R)$. *Then,* $b(\tau) > d_R$.

For a (possibly non-irreducible) W_μ-representation V, we define $b(V)$ by

$$b(V) = \min\{b(\tau) \mid \tau \text{ is an irreducible constituent of } V\}.$$

We define $b(0) = \infty$ for the trivial module 0. For a W_μ-representation V and a non-negative integer l, we denote by $C(l, V)$ the sum of the multiplicities $[\tau : V]$ in V of irreducible representations τ such that $b(\tau) = l$.

4.3. Coherent families of principal series

We fix a regular weight $\mu \in {}^s\mathfrak{h}^*$ and let $d: {}^sH^\wedge \to \mathsf{P}$ be the differential map. Put $R_i = \Delta({}^s\mathfrak{m}, {}^s\mathfrak{t})$ and $R_i^+(\eta) = \Delta_\eta^+({}^s\mathfrak{m}, {}^s\mathfrak{t})$ (4.1). Let W_i be the Weyl group for R_i. Let $\gamma = ({}^sH, \Gamma, \bar{\gamma})$ be a regular character in $\mathrm{RC}({}^sH, \mu)$. So, there exists a unique $w_\gamma \in W$ such that $\bar{\gamma} = w_\gamma \mu$. First, we define a coherent family Θ'_γ on $\bar{\gamma} + {}^sH^\wedge$ as follows. For $v \in {}^sH^\wedge$, we define $\Theta'_\gamma(\bar{\gamma} + v) = 0$ if $\bar{\gamma} + dv$ is not regular with respect to $\Delta({}^s\mathfrak{m}, {}^s\mathfrak{t})$. Assume that $\bar{\gamma} + dv$ is regular with respect to $\Delta({}^s\mathfrak{m}, {}^s\mathfrak{t})$. Then there exists a unique element $w_v \in W_i$ such that $w_v(\bar{\gamma} + dv)$ is dominant with respect to $R_i^+(\bar{\gamma})$. Then we have $\rho_{{}^s\mathfrak{m}}[\bar{\gamma}] - \rho_{{}^s\mathfrak{m}}[\bar{\gamma} + dv] = \rho_{{}^s\mathfrak{m}}[\bar{\gamma}] - w_v \rho_{{}^s\mathfrak{m}}[\bar{\gamma}] \in \mathsf{Q}$. Let $l(w_v)$ be the length of w_v in W_i. Put $\Gamma_v = \Gamma + (\rho_{{}^s\mathfrak{m}}[\bar{\gamma}] - w_\mu \rho_{{}^s\mathfrak{m}}[\bar{\gamma}]) + v \in {}^sH^\wedge$. Then, $({}^sH, \Gamma_v, \bar{\gamma} + dv) \in \mathrm{RC}({}^sH, \mu)$ and we denote this by $\gamma + v$. We define $\Theta'_\gamma(\bar{\gamma} + v) = (-1)^{l(w_v)}[\pi(\gamma + v)] \in K(\mathscr{H}_G[\bar{\gamma} + dv])$. Then, we easily see that Θ'_γ is a coherent family on $\bar{\gamma} + {}^sH^\wedge$ such that $\Theta'_\gamma(\bar{\gamma}) = [\pi(\gamma)]$. Finally, we define $\Theta_\gamma(\mu + v) = \Theta'_\gamma(\bar{\gamma} + w_\gamma v)$. Θ_γ is a coherent family on $\mu + {}^sH^\wedge$ such that $\Theta_\gamma(\mu) = [\pi(\gamma)]$.

From the definition of the coherent continuation representation and the cross action, we have:

LEMMA 4.3.1. *For* $\gamma \in \mathrm{RC}({}^sH, \mu)$ *and* $w \in W_\mu$, *we have* $w[\pi(\gamma)] = \pm[\pi(w \times \gamma)]$. *Here, the sign depends on w.*

Let $U(\mu)$ be the subspace of $K(\mathscr{H}_G[\mu]) \otimes_{\mathbb{Z}} \mathbb{C}$ spanned by $\{[\pi(\gamma)] \mid \gamma \in \mathrm{RC}({}^sH, \mu)\}$. The above lemma says $U(\mu)$ is a W_μ-submodule of $K(\mathscr{H}_G[\mu]) \otimes_{\mathbb{Z}} \mathbb{C}$. For $B \in \mathrm{BL}(\mu)$, put $U_B(\mu) = U(\mu) \cap V_B(\mu)$. From the definition of the block relation, we immediately have:

$$U(\mu) = \bigoplus_{B \in \mathrm{BL}(\mu)} U_B(\mu).$$

Moreover, $\{[\pi(\gamma)] \mid \gamma \in B \cap \mathrm{RC}({}^sH, \mu)\}$ is a basis of $U_B(\mu)$.

From Lemma 4.1.3 and Lemma 4.3.1, we have:

LEMMA 4.3.2. *Let* $B \in \mathrm{BL}(\mu)$.

(1) $U_B(\mu) \neq 0$.

(2) *For any* $\gamma \in B \cap \mathrm{RC}({}^sH, \mu)$, $[\pi(\gamma)]$ *is a cyclic element of* $U_B(\mu)$ *as a* W_μ-*module.*

We denote by m_G the number of the positive roots in $R_i = \Delta({}^s\mathfrak{m}, {}^s\mathfrak{t})$. Fix $\gamma = ({}^sH, \Gamma, \bar{\gamma}) \in \mathrm{RC}({}^sH, \mu)$ and $w_\gamma \in W$ such that $\bar{\gamma} = w_\gamma \mu$. Then $R_i(\gamma) = w_\gamma^{-1} R_i$ is a subroot system in the integral root system Δ_μ. Let $W_i(\gamma)$ be the Weyl group for $R_i(\gamma)$. This is a subgroup of W_μ and $W_i(\gamma) = w_\gamma^{-1} W_i w_\gamma$. The construction of Θ_γ and the definition of the coherent continuation representation implies that $w[\pi(\gamma)] = \mathrm{sgn}(w)[\pi(\gamma)]$ for any $w \in W_i(\gamma)$. This fact, Lemma 4.3.2 and Corollary 4.2.2 implies:

LEMMA 4.3.3. (1) $b(U_B(\mu)) \geqslant m_G$. (2) $C(m_G, U_B(\mu)) \leqslant 1$.

4.4. Big cones for type II groups

Fix a regular weight $\mu \in {}^s\mathfrak{h}^*$. For $n \in \mathbb{N}$, we define as follows

$\mathrm{RC}_{(n)}(\mu) = \{(H, \Gamma, \bar{\gamma}) \in \mathrm{RC}(\mu) \mid$ the dimension of the vector part of H is greater than or equal to $n\}$.

Let $W_{(n)}(\mu)$ be the subspace of $K(\mathscr{H}_G[\mu]) \otimes_{\mathbb{Z}} \mathbb{C}$ spanned by $\{[\pi(\gamma)] \mid \gamma \in \mathrm{RC}_{(n)}(\mu)\}$. It is known that $\{W_{(n)}(\mu) \mid n \geqslant 0\}$ defines a finite decreasing filtration of W_μ-submodules of $K(\mathscr{H}_G[\mu]) \otimes_{\mathbb{Z}} \mathbb{C}$ (cf. [5]). If r_G is the real rank of G, we have $W_{(r_G)}(\mu) = U(\mu)$.

First, we assume that G has a compact Cartan subgroup T. Put $W(G, T) = N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. Let $\mathfrak{t}$ be the complexified Lie algebra of T and let $W(\mathfrak{g}, \mathfrak{t})$ (resp. $W(\mathfrak{k}, \mathfrak{t})$ the Weyl group for $(\mathfrak{g}, \mathfrak{t})$ (resp. $(\mathfrak{k}, \mathfrak{t})$). Then, $W(\mathfrak{k}, \mathfrak{t}) \subseteq W(G, T) \subseteq W(\mathfrak{g}, \mathfrak{t})$.

Fix a regular integral weight $\mu \in \mathsf{P}$. The following is more or less well-known (cf. [5], [50]). $K(\mathscr{H}_G[\mu]) \otimes_{\mathbb{Z}} \mathbb{C}/W_{(i)}(\mu)$ can be regarded as the space of invariant eigendistribution on the regular elliptic elements in G with infinitesimal character μ. Hence, we can derive the following lemma from the well-known formula of a discrete series character on (the regular part of) a compact Cartan subgroup due to Harish-Chandra.

LEMMA 4.4.1. *Let $\gamma \in \mathrm{RC}(T, \mu)$. Namely, $\pi(\gamma)$ is a discrete series representation. If we identify $W = W(\mathfrak{g}, {}^s\mathfrak{h})$ and $W(\mathfrak{g}, \mathfrak{t})$ suitably (this identification depends on γ), then for any reflection $s \in W(G, T)$ we have*

$$s[\pi(\gamma)] = -[\pi(\gamma)] \pmod{W_{(1)}(\mu)}.$$

Let m_K be the dimension of the nilradical of a Borel subalgebra of $\mathfrak{k}$. Put $b_G = b(K(\mathscr{H}_G[\mu]) \otimes_{\mathbb{Z}} \mathbb{C})/W_{(1)}(\mu))$. From the translation principle, b_G is independent of the choice of μ. Theorem 4.2.1(2) implies:

COROLLARY 4.4.2. *We use notations in 4.2. Let R be a subroot system of $\Delta(\mathfrak{g}, \mathfrak{t})$ such that $W_R \subseteq W(G, T)$. Then we have $b_G \geqslant d_R$. In particular, $b_G \geqslant m_K \geqslant m_G$.*

We have:

LEMMA 4.4.3. *If G is of type II and has a non-compact semisimple part, then $b_G > m_G$.*

Proof. First, we assume that G contains a non-compact reductive linear Lie group other than $SL(2, \mathbb{R})$ or $PSL(2, \mathbb{R})$. Then, we can easily see $\operatorname{rank} K > \operatorname{rank} {}^sM$. From the Corollary 4.4.2, we have $b_G \geqslant m_K > m_G$. If one of the direct product factors of G is a type II envelope of $SL(2, \mathbb{R})$ or $PSL(2, \mathbb{R})$, we can easily check $b_G > m_G$. Assume $b_G = m_G$ and G is of type II. So, the identity connected component of G is written as a product of compact groups, abelian groups, copies of $SL(2, \mathbb{R})$ or $PSL(2, \mathbb{R})$. Since $W(G, T) = W(\mathrm{Ad}(G), \mathrm{Ad}(T))$, we see $b_{\mathrm{Ad}(G)} = b_G = m_G = m_{\mathrm{Ad}(G)}$. However, clearly, $\mathrm{Ad}(G)$ is written as a direct product of a compact group and at least one copy of a type II envelope of $PSL(2, \mathbb{R})$. This is a contradiction. □

We can deduce:

LEMMA 4.4.4. *Assume G is of type II and $\mu \in {}^s\mathfrak{h}^*$ is regular. Then, we have*

(1) $b(K(\mathscr{H}_G[\mu]) \otimes_{\mathbb{Z}} \mathbb{C}) \geqslant m_G$.
(2) *For all $B \in \mathrm{BL}(\mu)$, $C(m_G, V_B(\mu)) \leqslant 1$.*

Proof. From Lemma 4.3.3, we have only to show that $b(W_{(i)}(\mu)/W_{(i+1)}(\mu)) > m_G$ for all $i < r_G$. Here, r_G is the real rank of G. Let $\gamma = (H, \Gamma, \bar{\gamma}) \in \mathrm{RC}_{(i)}(\mu)$ and let P be any cuspidal parabolic subgroup of G associated to H and let $P=LU$ be a Levi decomposition. From Lemma 2.4.10, L is also of type II. We choose any $\gamma=(H,\Gamma,\bar{\gamma})\in \mathrm{RC}_{(i)}(\mu)-\mathrm{RC}_{(i+1)}(\mu)$. Let P be a cuspidal parabolic subgroup of G associated to H and let $P=LU$ be a Levi decomposition such that $H\subseteq L$. γ is also regarded as a regular character for L and the corresponding standard L-representation $\pi^L(\gamma)$ is an external tensor product of discrete series representation and one dimensional representation. $\pi(\gamma)$ is the parabolic induction of $\pi^L(\pi)$, and the coherent family commutes with the parabolic induction (this fact is a conclusion of the Mackey's tensor product theorem, see [64], p. 262). Hence, the action of $W(L,H)$ on $[\pi(\gamma)]$ and $[\pi^L(\gamma)]$ coincide. (Strictly speaking, we should take care of some shift of the infinitesimal character coming from the definition of parabolic induction. However, such shift does not affect on $W(L,H)$ action anyway.) Hence, if L is of type II, Lemma 4.4.3 implies $b(W_{(i)}(\mu)/W_{(i+1)}(\mu))>m_L=m_G$. Therefore the lemma follows from Lemma 2.4.10. □

The following is the main result of this section.

PROPOSITION 4.4.5. *Let $\mu\in{}^s\mathfrak{h}^*$ be a regular weight such that $\mathscr{H}_G[\mu]$ is non-trivial. Then the following* (1)–(3) *is the same number.*

(1) card $\mathrm{BL}(\mu)$.
(2) *The numbers of big cones in* $K(\mathscr{H}_G[\mu])\otimes_{\mathbb{Z}}\mathbb{C}$ *(say* D_2*).*
(3) $C(m_G, K(\mathscr{H}_G[\mu])\otimes_{\mathbb{Z}}\mathbb{C})$ *(say* D_3*).*

Proof. (The idea of this proof is due to D. A. Vogan.) From Lemma 4.1.3, each $B\in\mathrm{BL}(\mu)$ contains the Langlands subquotient of some principal series representation π (namely, π is induced from a minimal parabolic subgroup). From the definition of the block relation, an arbitrary irreducible subquotient of π is also contained in B. Since π contains at least one quasi-large irreducible subquotient, since π itself is quasi-large. This means $V_B(\mu)$ contains at least one big cone. So, we have card $\mathrm{BL}(\mu)\leqslant D_2$.

$D_2\leqslant D_3$ is proved by King ([28, 29]).

$D_3\leqslant$ card $\mathrm{BL}(\mu)$ follows from Lemma 4.4.4. □

COROLLARY 4.4.6. *Assume G is of type II. Let $\mu\in{}^s\mathfrak{h}^*$ be a regular weight such that $\mathscr{H}_G[\mu]$ is non-trivial. Then, we have*

(1) $C(m_G, V_B(\mu))=1$ *and* $b(V_B(\mu))\geqslant m_G$ *for all* $B\in\mathrm{BL}(\mu)$.
(2) *If a cone* $\mathscr{C}$ *is not big,* $b(\mathscr{C})>m_G$.

REMARK 1. Proposition 4.4.5 and Corollary 4.4.6 fail for groups of type I.
REMARK 2. (2) in Corollary 4.4.6 also follows from a Vogan's unpublished result which is stated as a conjecture in [69], p. 1055.

Finally, we state the following fact. Since we do not use it later, we omit the proof. (The proof is easy case-by-case check.)

PROPOSITION 4.4.7. *Let* $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$ *be a connected simple linear Lie group and let P be a standard parabolic subgroup of G with standard Levi decomposition* $P = LU$. *Let* $\mathfrak{p}_0$ *be the Lie algebra of P. If* $b_L = m_L (= m_G)$ *and P is not minimal, then* $(\mathfrak{g}_0, \mathfrak{p}_0)$ *is one of the following.*

(1) $\mathfrak{g}_0$ *is one of* (1)–(3), (5)–(12) *in Lemma* 2.1.1 *and* $\mathfrak{p}_0$ *is* $\tilde{\mathfrak{p}}_0$ *defined in* 2.2.
(2) $\mathfrak{g}_0 = \mathfrak{so}(2n, 2n)$ $(n \geqslant 2)$ *and* $\mathfrak{p}_0$ *is one of* $\tilde{\mathfrak{p}}_i$ $(i = 1, 2, 3)$ *defined in* 2.2.

4.5. Additive invariants

Let $(G, \mathfrak{g}_0, \theta, \langle\ ,,\ \rangle)$ be a real reductive linear Lie group. In this section, if necessary, taking a finite covering and throwing away the vector part of the center, we assume the differential map ${}^sH^\wedge \to \mathsf{P}$ is surjective. Clearly, there is no harm in supposing this assumption in proving Lemma 3.3.5.

A map $a: \mathscr{H}_G \to \mathbb{C}$ is called a ($\mathbb{C}$-valued) additive invariant if a satisfies the following (A1) and (A2) ([66], [48]).

(A1) For all exact sequence in $\mathscr{H}_G$

$$0 \to V_1 \to V_2 \to V_3 \to 0,$$

we have

$$a(V_2) = a(V_1) + a(V_3).$$

(A2) For any $V \in \mathscr{H}_G$ and any finite dimensional G-module E, we have

$$a(V \otimes E) = \dim E \cdot a(V).$$

Thanks to the condition (A1), a extends uniquely to a homomorphism of $K(\mathscr{H}_G)$ to $\mathbb{C}$. We denote this homomorphism by the same letter a.

EXAMPLE 1 ([66]). For $V \in \mathscr{H}_G$, we denote by $c(V)$ the multiplicity (or the Bernstein degree) of V ([66]). We define a map $c_{\odot}: \mathscr{H}_G \to \mathbb{N}$ as follows

$$c_{\odot}(V) = c(V) \quad \text{if } V \in \mathscr{H}_G \text{ is quasi-large,}$$
$$= 0 \quad \text{otherwise.}$$

Then, $c_{\odot}$ is an additive invariant.

EXAMPLE 2 (cf. Theorem 3.3.1, Lemma 3.3.2). Let ψ be an admissible unitary character on $\bar{\mathfrak{n}}$. Put $w_\psi(V) = \dim \mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V)$ for $V \in \mathscr{H}_G$. Then, w_ψ is an additive invariant.

The following is important.

THEOREM 4.5.1 ([66]). *Let a be an additive invariant on $\mathscr{H}_G$ and let $\mu \in {}^s\mathfrak{h}^*$ be a regular weight such that $\mathscr{H}_G[\mu]$ is non-trivial. Then, we have*

(1) *Let Θ be a coherent family on $\mu + {}^sH^\wedge$. Then $a(\Theta(\mu+\nu))$ only depends on $\mu + d\nu \in \mu + \mathsf{P}$. We denote it by $a(\Theta)(\mu+d\nu)$.*

(2) *Let Θ be a coherent family on $\mu + {}^sH^\wedge$. Then the map*

$$\mu + \mathsf{P} \ni \eta \rightsquigarrow a(\Theta)(\eta) \in \mathbb{Z}$$

extends uniquely to a W-harmonic polynomial $p[a;\Theta]$ on ${}^s\mathfrak{h}^*$.

(3) *For $V \in K(\mathscr{H}_G[\mu])$, let Θ_V be a unique coherent family on $\mu + {}^sH^\wedge$ such that $\Theta_V(\mu) = V$. Then the map*

$$\Phi_a : K(\mathscr{H}_G[\mu]) \ni V \rightsquigarrow p[a;\,\Theta_V] \in S({}^s\mathfrak{h})$$

is W_μ-equivariant.

Using $\langle\,,\rangle$, we identify $S({}^s\mathfrak{h})$ with $S({}^s\mathfrak{h}^*)$ as W_μ-module. For additive invariant a, we define the degree of a as follows (cf. [48]).

$$\deg(a) = \min\left\{ n \in \mathbb{N} \,\middle|\, \Phi_a(K(\mathscr{H}_G[\mu])) \subseteq \bigoplus_{0 \leqslant i \leqslant n} S^i({}^s\mathfrak{h}^*) \text{ for all regular } \mu \in {}^s\mathfrak{h}^* \right\}.$$

For example, $\deg(c_\odot) = m_G$.

Theorem 4.5.1 and Corollary 4.4.6 implies:

PROPOSITION 4.5.2. *Assume G is a real reductive linear Lie group of type II. Let a be a non-trivial additive invariant on $\mathscr{H}_G$ such that $\deg(a) \leqslant m_G$, and let $\mu \in {}^s\mathfrak{h}^*$ be a regular weight such that $\mathscr{H}_G[\mu]$ is non-trivial. Then, we have*

(1) $\deg(a) = m_G$.

(2) *Let $B \in \mathrm{BL}(\mu)$. Then there is a non-negative constant h_B, which depends only on B, such that*

$$a(V) = h_B c_\odot(V)$$

for all $V \in V_B(\mu)$.

In order to apply this proposition to w_ψ, we need:

THEOREM 4.5.3 ([74], Corollary 7.3(1)). *Let ψ be an admissible unitary character on $\bar{\mathfrak{n}}$ and let σ be a finite dimensional ${}^sM{}^sA$-module. Then, we have*

$$\dim \mathrm{Wh}^\infty_{\bar{\mathfrak{n}},\psi}(\mathrm{Ind}^G_{{}^sP}(\sigma)) = \dim \sigma.$$

For the multiplicity, the following is well-known. (For example, this follows from [40] Theorem 6.4 and [46] Corollary B.) We denote by w_G the cardinality of the little Weyl group (i.e. $N_K({}^s\mathfrak{a}_0)/Z_K({}^s\mathfrak{a}_0)$) of G.

PROPOSITION 4.5.4. *Let σ be a finite dimensional ${}^sM^sA$-module. Then, we have*

$$c_{\odot}(\mathrm{Ind}_{{}^sP}^G(\sigma)) = w_G \dim \sigma.$$

Finally, we have the following result which implies Lemma 3.3.5.

PROPOSITION 4.5.5. *Assume that G is a real reductive linear Lie group of type II. Let ψ be an admissible unitary character on $\bar{\mathfrak{n}}$. Then, for all $V \in \mathscr{H}_G$, we have*

$$\dim \mathsf{Wh}_{\bar{\mathfrak{n}},\psi}^{\infty}(V) = w_G^{-1} c_{\odot}(V).$$

Proof. First, we show that $\deg(w_\psi) \leqslant m_G$. Let $\mu \in {}^s\mathfrak{h}^*$ be regular. For $\gamma \in \mathrm{RC}({}^sH, \mu)$, let Θ_γ be a coherent family such that $\Theta_\gamma(\mu) = [\pi(\gamma)]$. Then, Weyl's dimension formula and Theorem 4.5.3 implies $\deg(p[w_\psi; \Theta_\gamma]) = m_G$. For any irreducible $V \in \mathscr{H}_G[\mu]$, Harish-Chandra's subquotient theorem assures the existence of some $\gamma \in \mathrm{RC}({}^sH, \mu)$ such that V appears in $[\pi(\gamma)]$ as an irreducible constituent. This implies $\deg(\Phi_{w_\psi}(V)) \leqslant \deg(\Phi_{w_\psi}([\pi(\gamma)])) = m_G$. (For details, see the proof of Lemma 2.7.2 in [48].) Hence, $\deg(w_\psi) \leqslant m_G$.

Using this we show the proposition. From the additivity, we can assume that V is irreducible. We assume that $V \in \mathscr{H}[\mu]$ for regular μ. Let $B \in \mathrm{BL}(\mu)$ be the block which contains V. From the Casselman's subrepresentation theorem, there exists some $\gamma \in \mathrm{RC}({}^sH, \mu)$ such that V is an irreducible constituent of $[\pi(\gamma)]$. From the definition of block relation, all the irreducible constituents of $[\pi(\gamma)]$ are also contained in B. Hence, Theorem 4.5.3 and Corollary 4.5.4 imply h_B in Proposition 5.4.2 equals w_G^{-1}. So, we proved the regular infinitesimal character case.

For an irreducible $V \in \mathscr{H}_G$ with non-regular infinitesimal character η, the conclusion of the proposition follows immediately from the conclusion for regular infinitesimal character case and the following result; there exists a coherent family Θ on $\eta + {}^sH^\wedge$ such that $\Theta(\eta) = V$ (Zuckerman, cf. [67], Theorem 7.2.7). □

5. Proof of the main theorem (type I case)

5.1. *Envelopes and extendabilities*

In this section, we assume that $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$ is a real reductive linear Lie group such that $F^c \subseteq \mathrm{Ad}(G)$ (2.1). Let Q be a subgroup of $F^\sharp$ such that $\mathrm{Ad}(G) \cap F^\sharp \subseteq Q$.

A real reductive linear Lie group $(G^Q, \mathfrak{g}_0, \theta, \langle\, ,\rangle)$ is called a Q-envelope of G if $G \subseteq G^Q$ and $\mathrm{Ad}(G^Q) = \mathrm{Ad}(G)Q$. We denote by K^Q a maximal compact subgroup of G^Q defined by (RL3). We easily see a Q-envelope always exists, but, in general, it is not unique. An $F^\sharp$-envelope is nothing but type II envelope (3.2). Let Q be a subgroup of $F^\sharp$ such that $\mathrm{Ad}(G) \cap F^\sharp \subseteq Q$ and let G^Q be a Q-envelope of G. As in 2.4, we denote by $\mathscr{P}r_0(G)$ (resp. $\mathscr{P}r_0(G^Q)$) the set of principal G- (resp. G^Q)-nilpotent orbits. For $\mathcal{O} \in \mathscr{P}r_0(G)$, we define

$$\mathcal{O}^Q = \bigcup_{\sigma \in Q} \mathrm{Ad}(\sigma)\mathcal{O}.$$

Clearly, $\mathcal{O}^Q \in \mathscr{P}r_0(G^Q)$ and $\mathcal{O} \rightsquigarrow \mathcal{O}^Q$ is a surjection of $\mathscr{P}r_0(G)$ to $\mathscr{P}r_0(G^Q)$. Conversely, for $\mathcal{O} \in \mathscr{P}r_0(G^Q)$, we denote by $[\mathcal{O}]_Q$ the set of principal nilpotent G-orbits contained in $\mathcal{O}$.

For a Q-envelope G^Q of G, we can define restriction functor

$$\mathrm{res}^G_{G^Q} : \mathscr{H}_{G^Q} \to \mathscr{H}_G.$$

Let $V \in \mathscr{H}_G$. We call V Q-extendable if there exists some Q-envelope G^Q of G and some $\tilde{V} \in \mathscr{H}_{G^Q}$ such that $V = \mathrm{res}^G_{G^Q}(\tilde{V})$. If V is $F^\sharp$-extendable, then we say that V is extendable to some type II envelope.

There is the following right adjoint functor of $\mathrm{res}^G_{G^Q}$.

$$\mathrm{Ind}^{G^Q}_G : \mathscr{H}_G \to \mathscr{H}_{G^Q}.$$

An explicit definition of Ind is found in [67] Definition 0.3.25.

For $\sigma \in K^Q$ and $V \in \mathscr{H}_G$, we define the twisting V^σ as in 3.2. Fix $\tau \in Q/\mathrm{Ad}(G) \cap Q$ and let $\tilde{\tau}$ be a representative of τ in K^Q. We put $V^\tau = V^{\tilde{\tau}}$ for $V \in \mathscr{H}_G$. Clearly, the definition of V^τ does not depend on the choice of $\tilde{\tau}$. The following is clear.

LEMMA 5.1.1. *Let Q be a subgroup of $F^\sharp$ such that* $\mathrm{Ad}(G) \cap F^\sharp \subseteq Q$ *and let $V \in \mathscr{H}_G$. Let G^Q be a Q-envelope of G. Then we have*

$$\mathrm{res}^G_{G^Q} \mathrm{Ind}^{G^Q}_G(V) = \bigoplus_{\tau \in Q/\mathrm{Ad}(G) \cap Q} (V^\tau)^{\oplus m}.$$

Here, $m = \mathrm{card}(G^Q/G)/\mathrm{card}(Q/\mathrm{Ad}(G) \cap Q) \geqslant 1$.

A quasi-large irreducible Harish-Chandra module $V \in \mathscr{H}_G$ is called pure iff $\mathrm{card}\, {}^{\odot}\mathrm{WF}(V) = 1$. A quasi-large irreducible Harish-Chandra module $V \in \mathscr{H}_G$ is called purely Q-extendable, if there exists some Q-envelope G^Q of G and some pure $\tilde{V} \in \mathscr{H}_{G^Q}$ such that $V = \mathrm{res}^G_{G^Q}(\tilde{V})$.

From Lemma 2.1.2 and $F^c \subseteq \mathrm{Ad}(G)$, we have:

LEMMA 5.1.2. *Let $V \in \mathscr{H}_G$ be a pure quasi-large irreducible Harish-Chandra module. Then, we have*

(1) *For all $\tau \in F^{\#}/G \cap F^{\#}$, V^{τ} is pure.*
(2) ${}^{\odot}\mathrm{WF}(V^{\tau}) \cap {}^{\odot}\mathrm{WF}(V^{\sigma}) = \emptyset$ *for all $\tau, \sigma \in F^{\#}/\mathrm{Ad}(G) \cap F^{\#}$ such that $\tau \neq \sigma$.*
(3) $\bigcup_{\tau \in F^{\#}/\mathrm{Ad}(G) \cap F^{\#}} {}^{\odot}\mathrm{WF}(V^{+})$ *coincides with the set of principal nilpotent G-orbits.*

Next we consider the following condition for a Harish-Chandra module V.

(SW) Assume that ψ is an admissible unitary character on $\bar{\mathfrak{n}}_0$. Then, $\mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) \neq 0$ if and only if $\mathcal{O}_{\psi} \in {}^{\odot}\mathrm{WF}(V)$.

We also consider the following weaker condition.

(WW) Assume that ψ is an admissible unitary character on $\bar{\mathfrak{n}}_0$. Then, $\mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) \neq 0$ implies $\mathcal{O}_{\psi} \in {}^{\odot}\mathrm{WF}(V)$.

The following lemma is clear from Lemma 3.3.6, Lemma 5.1.1, and Lemma 5.1.2.

LEMMA 5.1.3. *Let $V \in \mathscr{H}_G$. Then, we have*

(1) *If V is extendable to some type II envelope, then V satisfies (SW).*
(2) *If V is pure and V^{τ} satisfies (WW) for all $\tau \in F^{\#}/G \cap F^{\#}$, then V satisfies (SW).*

5.2. Preliminaries on asymptotic expansions of distribution characters

We fix a real reductive linear Lie group $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$. For a principal nilpotent G-orbit $\mathcal{O}$, we denote by $\mu_{\mathcal{O}}$ an invariant measure on $\mathcal{O}$. From [51], $\mu_{\mathcal{O}}$ is a tempered distribution on $\mathfrak{g}_0$. As in 3.1, we denote by θ_V the lift of the distribution character of $V \in \mathscr{H}_G$ to $\mathfrak{g}_0$. The following is a special case of [1] Theorem 4.1.

LEMMA 5.2.1 (Barbasch-Vogan). *Let $V \in \mathscr{H}_G$ be quasi-large. Then the Fourier transform of first term of the asymptotic expansion of θ_V is a linear combination of invariant measures on principal nilpotent G-orbits.*

For a quasi-large Harish-Chandra module V, if $\Sigma_{\mathcal{O} \in \mathscr{P}r_0(G)} a_{\mathcal{O}} \mu_{\mathcal{O}}$ ($a_{\mathcal{O}} \in \mathbb{C}$) is the Fourier transform of the first term of the asymptotic expansion of θ_V, then we write as follows

$$V \sim \sum_{\mathcal{O} \in \mathscr{P}r_0(G)} a_{\mathcal{O}} \mu_{\mathcal{O}}.$$

If V is not quasi-large, we write $V \sim 0$. From the definition of $\mathrm{AS}(\theta_V)$ ([1]), we have

$$\{\mathcal{O} \in \mathscr{P}r_0(G) \mid a_{\mathcal{O}} \neq 0\} \subseteq {}^{\odot}\mathrm{WF}(V).$$

The following is immediately deduced from [1] Lemma 3.3 and Theorem 3.5.

LEMMA 5.2.2. *Let σ be a finite dimensional ${}^sM^sA$-module. If we normalize invariant measures $\mu_{\mathcal{O}}$ ($\mathcal{O} \in \mathscr{P}r_0(G)$) suitably, we have*

$$\mathrm{Ind}_{{}^sP}^G(\sigma) \sim \dim(\sigma) \sum_{\mathcal{O} \in \mathscr{P}r_0(G)} \mu_{\mathcal{O}}.$$

Hereafter, we fix the normalization of invariant measures in Lemma 5.2.2.

REMARK. Assume $F^c \subseteq \mathrm{Ad}(G)$. Let Q be a subgroup of $F^\sharp$ such that $\mathrm{Ad}(G) \cap F^\sharp \subseteq Q$ and let G^Q be a Q-envelope of G. Then, under the normalization in Lemma 5.2.2, we have

$$\mu_{\mathcal{O}} = \sum_{\mathcal{O}' \in [\mathcal{O}]_Q} \mu_{\mathcal{O}'}.$$

For a quasi-large irreducible Harish-Chandra module $V \in \mathscr{H}_G$ such that $V \sim \Sigma_{\mathcal{O} \in \mathscr{P}r_0(G)} a_{\mathcal{O}} \mu_{\mathcal{O}}$, we consider the following condition on V.

(P) $a_{\mathcal{O}}$ is a non-negative real number for all $\mathcal{O} \in \mathscr{P}r_0(G)$ and we have

$$\{\mathcal{O} \in \mathscr{P}r_0(G) \mid a_{\mathcal{O}} \neq 0\} = {}^{\odot}WF(V).$$

REMARK. Let $\{V_1, \ldots, V_k\}$ be the set of all the quasi-large irreducible constituent of $V \in \mathscr{H}_G$. If $V_1, \ldots, V_k$ satisfy the condition (P), then we clearly have

$$\odot\mathrm{WF}(V) = \bigcup_{i=1}^{k} {}^{\odot}\mathrm{WF}(V_k).$$

Hereafter, we often use this fact. In particular, if (P) holds for all irreducible quasi-large $V \in \mathscr{H}_G$, then the statement of Corollary 3.1.4 holds for G.

For $V \in \mathscr{H}_G$, we write

$$V \sim \sum_{\mathcal{O} \in \mathscr{P}r_0(G)} a_{\mathcal{O}}(V) \mu_{\mathcal{O}}.$$

Then $V \rightsquigarrow a_{\mathcal{O}}(V)$ extends to a linear map $a_{\mathcal{O}}: K(\mathscr{H}_G) \to \mathbb{C}$ for any $\mathcal{O} \in \mathscr{P}r_0(G)$. From [1] Proposition 4.7 and its proof, we immediately see that a is a $\mathbb{C}$-valued additive invariant of degree m_G. Since there is no harm in assuming $d: {}^sH^\wedge \to \mathsf{P}$ is surjective, from Theorem 4.5.3, Proposition 4.5.2, and Lemma 5.2.2, we have:

LEMMA 5.2.3. *Assume that G is a real reductive linear Lie group of type II. Let ψ be an admissible unitary character on $\bar{\mathfrak{n}}$. Then, we have*

$$V \sim \dim \mathrm{Wh}_{\bar{\mathfrak{n}},\psi}^{\infty}(V) \mu_{\mathcal{O}_0}, \quad (V \in \mathscr{H}_G).$$

Here, $\mathcal{O}_0$ is a unique principal nilpotent G-orbit. In particular, any irreducible quasi-large Harish-Chandra module satisfies the condition (P).

Hereafter, we fix for all $\mathcal{O} \in \mathcal{P}r_0(G)$ an admissible unitary character $\psi_{\mathcal{O}}$ such that $\mathcal{O}_{\psi_{\mathcal{O}}} = \mathcal{O}$, and we assume that $F^c \subseteq \mathrm{Ad}(G)$. The following is clear.

LEMMA 5.2.4. *Let Q be a subgroup of $F^\sharp$ such that* $\mathrm{Ad}(G) \cap F^\sharp \subseteq Q$ *and let G^Q be a Q-envelope of G. Fix an admissible unitary character ψ on $\bar{\mathfrak{n}}$ and the corresponding principal G^Q-nilpotent orbit $\mathcal{O}_\psi$. We assume that $V \in \mathcal{H}$ is extendable to G^Q, namely there exists some $\tilde{V} \in \mathcal{H}_{G^Q}$ such that* $\mathrm{res}_G^{G^Q}(\tilde{V}) = V$. *Then we have*

$$\dim \mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi_{\mathcal{O}}}(V) = \dim \mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(\tilde{V}),$$

for all $\mathcal{O} \in [\mathcal{O}_\psi]_Q$.

From Lemma 5.1.1, Lemma 5.1.2, Lemma 5.1.3, Lemma 5.2.3, and Lemma 5.2.4, we easily have:

LEMMA 5.2.5. *Let Q be a subgroup of $F^\sharp$ such that* $\mathrm{Ad}(G) \cap F^\sharp \subseteq Q$. *We assume that an irreducible quasi-large Harish-Chandra module $V \in \mathcal{H}_G$ is purely Q-extendable. We also assume that V^σ satisfies (WW) for all $\sigma \in F^\sharp$. Then, we have*

$$V \sim \sum_{\mathcal{O} \in {}^{\odot}\mathbf{WF}(V)} \dim \mathsf{Wh}_{\bar{\mathfrak{n}},\psi_{\mathcal{O}}}(V) \mu_{\mathcal{O}}.$$

Moreover, V satisfies the condition (SW) and (P).

Let $\mu \in {}^s\mathfrak{h}^*$ be a regular dominant weight and let $V \in \mathcal{H}_G$ be irreducible. Then, there exists a coherent family Θ_V on $\mu + {}^sH^\wedge$ such that $\Theta_V(\mu) = V$. Assume $\lambda \in {}^sH^\wedge$ satisfies that $\mu + d\lambda$ is dominant and $\Theta_V(\mu+\lambda) \neq 0$. Then, $\Theta_V(\mu+\lambda)$ is the character of some irreducible $V_0 \in \mathcal{H}[\mu + d\lambda]$ ([6]). In this situation, we say V_0 is a limit of V. It is not difficult to see (and well known) that V is quasi-large if and only if V_0 is quasi-large. We have:

LEMMA 5.2.6. *Let $V \in \mathcal{H}_G[\mu]$ be a quasi-large irreducible Harish-Chandra module and let V_0 be its limit. Then we have*

(1) *Let ψ be an admissible unitary character on $\bar{\mathfrak{n}}$. Then, $\mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) \neq 0$ if and only if $\mathsf{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V_0) \neq 0$.*
(2) *Assume that V satisfies the condition (P). Then, V_0 also satisfies (P) and* ${}^{\odot}\mathrm{WF}(V) = {}^{\odot}\mathrm{WF}(V_0)$.

Proof. (1) follows from Theorem 3.3.1, Lemma 3.3.2, and [67] Proposition 7.3.10. (2) follows from results of Barbasch and Vogan [1] Proposition 4.7 and [10] Theorem 3.4. □

5.3. Quasi-large discrete series

We fix a real reductive linear Lie group $(G, \mathfrak{g}_0, \theta, \langle\, , \rangle)$ with a compact Cartan subgroup. If $V \in \mathscr{H}_G$ is the Harish-Chandra module of a discrete series representation of G, then we simply say that V is a discrete series module. A limit of a discrete series module is nothing but the Harish-Chandra module of a limit of discrete series representation ([77] and [33]).

The goal of this section is to prove:

PROPOSITION 5.3.1. *Let* $V \in \mathscr{H}_G$ *be a limit of a quasi-large discrete series module. Then,* V *is pure and satisfies the conditions* (*P*) *and* (*SW*).

REMARK. The following statement is stronger than the purity of V.

$\mathrm{AS}(\theta_V)$ *is the closure of a nilpotent* G*-orbit.*

This is proved by Rossman [53] for large discrete series modules of quasi-split reductive groups. Barbasch and Vogan proved independently this for (not necessarily quasi-large) discrete series modules of classical groups (unpublished).

Now, we are going into the proof of Proposition 5.3.1. We prove the proposition by the induction on $\dim G$.

First we remark:

LEMMA 5.3.2. (1) *The statement of Proposition* 5.3.1 *holds for type II groups.*

(2) *Assume that the statement of Proposition* 5.3.1 *holds for some finite covering reductive linear Lie group of* G. *Then, the statement also holds for* G.

(3) *Assume that the statement of Proposition* 5.3.1 *holds for real reductive linear Lie groups* G_1 *and* G_2. *Then, the statement also holds for* $G_1 \times G_2$.

(4) *The statement of Proposition* 5.3.1 *holds for* $SL(2, \mathbb{R})$.

(5) *Assume that the statement of Proposition* 5.3.1 *holds for the identity component* G_0 *of* G. *Then, the statement also holds for* G.

Proof. (1) follows from Lemma 5.2.3. (2) and (3) are easy. (4) is more or less known. For a limit of discrete series module V of $SL(2, \mathbb{R})$, $\mathrm{WF}(V)$ is known from its K-type asymptotics ([17], [1] Theorem 3.6). Fix an admissible unitary character ψ of $\bar{\mathfrak{n}}$. For $w \in \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V)$ and $v \in V$, the generalized matrix coefficient [72]) with respect to v and w is written by the classical Whittaker function (this is a famous fact). An explicit calculation is found in [12] Chapter 7. Using these, we can know when $\mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) \neq 0$ holds and we can prove (4).

Now, we prove (5). If $V \in \mathscr{H}_G$ is a limit of discrete series module, then the restriction $\mathrm{res}^{G_0}_G(V)$ is a direct sum of discrete series modules of G_0. We fix an irreducible constituent V_0 of $\mathrm{res}^{G_0}_G(V)$. Since V_0 is pure, we put $\mathrm{WF}(V_0) = \mathcal{O}$. Then, we have $V \hookrightarrow \mathrm{Ind}^G_{G_0}(V_0)$. Since all the irreducible constituents of $\mathrm{res}^{G_0}_G \mathrm{Ind}^G_{G_0}(V_0)$ satisfies the condition (P), we have V satisfies (P) and

$\mathrm{WF}(V)=\mathrm{Ad}(G)\mathcal{O}$; namely V is pure. Now that we can easily check V satisfies (SW), since $\mathrm{WF}(V)=\bigcup_{V'\in D}\mathrm{WF}(V')$, where D is the set of irreducible direct summand of $\mathrm{res}_G^{G_0}(V)$. □

The argument in [1] pp. 47–48 proves the following result. (The point is that $f \rightsquigarrow \varphi_f^j$ is a continuous map between the space of Schwartz class functions. So, the leading term of the expansion in [1] p. 46 l. 27 is a sum of integrals of f under the invariant measures for any Schwartz class f. The case we need is that f is the Fourier transform of a compact supported C^∞-function.)

LEMMA 5.3.3 (Barbasch-Vogan). *Any quasi-large discrete series module $V\in\mathscr{H}_G$ satisfies the condition (P).*

From this lemma and Lemma 5.2.6, we have:

LEMMA 5.3.4. *Let $V\in\mathscr{H}_G$ be a quasi-large discrete series module and let V_0 be its limit. Then V_0 also satisfies the condition (P) and ${}^{\odot}\mathrm{WF}(V)={}^{\odot}WF(V_0)$. Moreover, V satisfies the condition (SW) (resp. (WW)) if and only if V_0 satisfies (SW) (resp. (WW)).*

From the above lemma and Lemma 2.1.1, we have only to prove the statement of Proposition 5.3.1 for a connected real simple linear Lie group $(G, \mathfrak{g}_0, \theta, \langle\,,\rangle)$ of type I such that $\mathfrak{g}_0 \neq \mathfrak{sl}(2,\mathbb{R})$. The possibility of $\mathfrak{g}_0$ is (2) or (4)–(12) in Lemma 2.1.1. So, hereafter we assume that $(G, \mathfrak{g}_0, \theta, \langle\,,\rangle)$ is a connected real simple linear Lie group such that $\mathfrak{g}_0$ is one of (2) or (4)–(12) in Lemma 2.1.1. Moreover, we assume that the differential $d\colon {}^sH^\wedge \to \mathsf{P}$ is surjective. We fix a compact Cartan subgroup T of G which is given by successive applications of Cayley transforms (for example, see [33] p. 402) to ${}^sH$. Let $\mathfrak{t}$ be the complexified Lie algebra of T. We consider the root system $\Delta(\mathfrak{g},\mathfrak{t})$ for $(\mathfrak{g},\mathfrak{t})$. A root $\alpha\in\Delta(\mathfrak{g},\mathfrak{t})$ is called long if there is no root $\beta\in\Delta(\mathfrak{g},\mathfrak{t})$ such that the length of β is strictly smaller than α. Hence, if the Dynkin diagram of $\Delta(\mathfrak{g},\mathfrak{t})$ is simply-laced, every root is long. We call $\alpha\in\Delta(\mathfrak{g},\mathfrak{t})$ short, if α is not long. As usual, we call $\alpha\in\Delta(\mathfrak{g},\mathfrak{t})$ compact (resp. non-compact) if the root space of α is contained (resp. not contained) in $\mathfrak{k}$. We show:

LEMMA 5.3.5. *Assume that G is not quasi-split. (Namely, $\mathfrak{g}_0$ is one of (10)–(12) in Lemma 2.1.1.) Let $\Delta^+(\mathfrak{g},\mathfrak{t})$ be any positive root system of $\Delta(\mathfrak{g},\mathfrak{t})$. Then there exists some long noncompact simple root for $\Delta^+(\mathfrak{g},\mathfrak{t})$.*

Proof. The only non-trivial (i.e. non-simply-laced) case is $\mathfrak{g}_0=\mathfrak{so}(2n+2k+1,2n)$ ($n\geqslant 1, k\geqslant 1$). If we assume that all the long simple roots are compact, we must have $\mathfrak{sl}(2n+k,\mathbb{C})\subseteq\mathfrak{k}$. However, this is impossible. □

We fix a non-compact root $\alpha\in\Delta(\mathfrak{g},\mathfrak{t})$. Let H_α be the Cartan subgroup of G which is the Cayley transform of T with respect to α. We denote by $\mathfrak{h}_{\alpha,0}$ the Lie algebra of H_α. Put $T_\alpha=K\cap H_\alpha$ and $A_\alpha=\exp(\mathfrak{s}_0\cap\mathfrak{h}_{\alpha,0})$. Then, we have $H_\alpha=T_\alpha A_\alpha$, From the assumption of T, we have $A_\alpha \subseteq {}^sA$. We denote by L_α the centerizer of A_α in G and let $L_\alpha = M_\alpha A_\alpha$ be a Langlands decomposition. There exists a standard parabolic subgroup P_α whose standard Levi part is L_α. We denote by U_α the nilradical of P_α. We also denote by $\mathfrak{p}_{\alpha,0}$ the Lie algebra of P_α. P_α is a maximal parabolic subgroup and it is also cuspidal. If the Dynkin diagram of $\Delta(\mathfrak{g}, \mathfrak{t})$ is not of type B, then there is only one possibility of P_α. For type B case, there are two possibilities. It is very easy to check:

LEMMA 5.3.6. *If* $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$ *is a long non-compact root, then* $\mathfrak{p}_{\alpha,0}$ *contains* $\tilde{\mathfrak{p}}_0$, *which is defined in* 2.2(2), (4)–(12).

Now, we are going to finish the proof of Proposition 5.3.1. From Lemma 5.3.4, we can assume V is a quasi-large discrete series module. As is well known, for V we can associate some positive system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ up to $W(\mathfrak{k}, \mathfrak{t})$. If G is quasi-split, then V is large. So, from [66] the τ-invariant of V is empty. This means that every simple root of $\Delta^+(\mathfrak{g}, \mathfrak{t})$ is non-compact. Together with Lemma 5.3.5, $\Delta^+(\mathfrak{g}, \mathfrak{t})$ has a long non-compact simple root in both quasi-split and non-quasi-split cases. Let V' be a discrete series module corresponding to $s_\alpha\Delta^+(\mathfrak{g}, \mathfrak{t})$, here s_α is the reflection with respect to α.

Then, we can apply Schmid's character identity ([58, 59]) for V and α. Namely, there exists some limit V_0 (resp. V_0') of V (resp. V') and a discrete series module $\sigma \in \mathscr{H}_{M_\alpha}$ such that

$$V_0 \oplus V_0' = \mathrm{Ind}_{P_\alpha}^G(\sigma \otimes 1).$$

From Lemma 5.3.4, V_0 satisfies the condition (P). Also, we have either V_0' is not quasi-large, or V_0' is quasi-large and satisfies the condition (P). Hence, from Lemma 5.3.4, Lemma 3.1.5, we have

$$^{\circ}\mathrm{WF}(V) = {}^{\circ}\mathrm{WF}(V_0) \subseteq {}^{\circ}\mathrm{WF}(\mathrm{Ind}_{P_\alpha}^G(\sigma \otimes 1)) \subseteq \text{h-ind}_{P_\alpha}^G({}^{\circ}\mathrm{WF}(\sigma)).$$

This means σ is also quasi-large. From the assumption of the induction (on $\dim G$), Proposition 2.4.1, and Lemma 5.3.6, we have V is pure. Let ψ be an admissible unitary character on $\bar{\mathfrak{n}}$ such that $\mathcal{O}_\psi \notin {}^{\circ}\mathrm{WF}(V_0)$. Then, we easily see $\psi_{L_\alpha} \notin \mathrm{WF}(\sigma \otimes 1)$. From Theorem 3.4.1 and the assumption of the induction, we have $\dim \mathrm{Wh}_{\bar{\mathfrak{n}},\psi}^{\infty}(V_0) \leqslant \dim \mathrm{Wh}_{\bar{\mathfrak{n}}\cap \mathfrak{l}_{\alpha},\psi_{L_\alpha}}(\sigma \otimes 1) = 0$, where $\mathfrak{l}_\alpha$ is the complexified Lie algebra of L_α. This means that V_0 satisfies (WW). Again from Lemma 5.3.4, V satisfies (WW). Therefore, Lemma 5.1.3(2) implies V satisfies (SW), since all the V^σ's are also quasi-large discrete series modules. □

5.4. The final step

First, we assume that $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$ is a connected real simple linear Lie group of type I. Hence, $\mathfrak{g}_0$ is one of (1)–(12) in Lemma 2.1.1, so we have $F^c \subseteq \mathrm{Ad}(G)$ (Lemma 2.1.3). We denote by $G_{\mathbb{C}}$ the complexification of G. Fix $\alpha \in \Phi$. Let $\exp: \mathfrak{g} \to G_{\mathbb{C}}$ be the exponential map and define $\tilde{\omega}_\alpha = \exp(\pi i \Omega_\alpha) \in \exp({}^s\mathfrak{a}) \subseteq G_{\mathbb{C}}$ (cf. 1.2). Clearly, we have $\mathrm{Ad}(\tilde{\omega}_\alpha) = \omega_\alpha \in F^\sharp$. As in 2.2, we put $S_0 = \{\alpha \in \Phi \mid \omega_\alpha \notin F^\flat = \mathrm{Ad}(G) \cap F^\sharp\}$. For $\alpha \in \Phi$ (resp. $S \subseteq \Phi$), we denote by G_α (resp. G_S) the subgroup of $G_{\mathbb{C}}$ generated by G and $\tilde{\omega}_\alpha$ (resp. $\{\tilde{\omega}_\beta \mid \beta \in S\}$). For $\alpha \in \Phi$ (resp. $S \subseteq \Phi$), we denote by Q_α (resp. Q_S) the subgroup of $F^\sharp$ generated by $F^\flat$ and ω_α (resp. $\{\omega_\beta \mid \beta \in S\}$). Then, clearly G_α (resp. G_S) is a Q_α (resp. Q_S)-envelope of G.

Let P be a standard parabolic subgroup of G and let $P = MAU$ be the Langlands decomposition associated to the Cartan involution θ (we call such a decomposition the standard Langlands decomposition). Then, $A \subseteq {}^sA$ and if we put $L = MA$ then $P = LU$ is the standard Levi decomposition.

The following is clear from the definition of Ω_α (1.2).

LEMMA 5.4.1. *Let P be a standard parabolic subgroup of G and let S be the subset of Φ corresponding to P. Let $P = MAU$ be the standard Langlands decomposition. Assume $S' \subseteq S_0$ satisfies $S' \cap S = \emptyset$. We denote by $M_{S'}^+$ the subgroup in $G_{\mathbb{C}}$ generated by M and $\{\tilde{\omega}_\alpha \mid \alpha \in S'\}$. Then, for all $\alpha \in S'$, $\tilde{\omega}_\alpha$ is contained in the center of $M_{S'}^+$. Hence, any $V \in \mathscr{H}_L$ is extendable to $M_{S'}^+$. If we put $P_{S'}^+ = M_{S'}^+ AU$, then this is a standard parabolic subgroup of $G_{S'}$.*

REMARK. We use crucially the condition (RL4) for the formulation and proof of the above lemma.

We prove:

LEMMA 5.4.2. *Let $(G, \mathfrak{g}_0, \theta, \langle\ ,\ \rangle)$ be a connected real quasi-split simple linear Lie group of type I. (Hence $\mathfrak{g}_0$ is one of* (1)–(9) *in Lemma* 2.1.1.) *Let $V \in \mathscr{H}_G$ be a irreducible large Harish-Chandra module. Then, V is purely Q-extendable for some subgroup Q of $F^\sharp$ such that $F^\flat \subseteq Q$. Moreover, V satisfies the condition* (*WW*).

Proof. From [66] Theorem 6.2, there is a standard cuspidal parabolic subgroup P with the standard Langlands decomposition $P = MAU$ such that $V = \mathrm{Ind}_P^G(\sigma \otimes \nu)$, where σ is a limit of discrete series module of M and ν is a character on A. Let $\mathfrak{p}_0$ be the Lie algebra of P and let S be the subset of Φ corresponding to P.

(1) First, we assume $\mathfrak{g}_0$ is not $\mathfrak{so}(2n, 2n)$ $(n \geqslant 2)$. In this case, we have $\mathrm{card}\, F^\sharp/F^\flat = 2$ and for any $\alpha \in S_0$, G_α is a type II envelope of G. There are two cases.

Case 1.1. $\tilde{\mathfrak{p}}_0 \not\subseteq \mathfrak{p}_0$. There exists some $\alpha \in S_0$ which is not contained in S. From Lemma 5.4.1, σ is extendable to $\tilde{\sigma} \in \mathscr{H}_{M_{\{\alpha\}}^+}$. (Here, $\tilde{\sigma}$ is a limit of discrete series module for M_α^+.) If we put $\tilde{V} = \mathrm{Ind}_{P_{\{\alpha\}}^+}^{G_\alpha}(\tilde{\sigma} \otimes \nu)$, then clearly $\tilde{V}$ is an extension of V to G_α. So, V is (purely) extendable to some type II envelope. From Lemma 3.3.6, V satisfies (SW).

Case 1.2. $\tilde{\mathfrak{p}}_0 \subseteq \mathfrak{p}_0$. From Proposition 2.4.1(1), Proposition 5.3.1, and Lemma

3.1.5, we have V is pure. More precisely, if we put $\{\mathcal{O}\} = {}^{\odot}\mathrm{WF}(\sigma \otimes \nu) = {}^{\odot}\mathrm{WF}(\sigma)$, then ${}^{\odot}\mathrm{WF}(V) = \text{h-ind}(\mathcal{O})$. Hence, from Proposition 5.3.1 and Theorem 3.4.1, we see that V satisfies (WW) (cf. the proof of Proposition 5.3.1).

(2) Finally, we assume that $\mathfrak{g}_0 = \mathfrak{so}(2n\ 2n)$ $(n \geqslant 2)$. In this case, $F^{\sharp}/F^{\flat}$ is Klein's four group $\mu_2 \times \mu_2$. There are three cases.

Case 2.1. $\mathfrak{p}_0$ does not contain any $\tilde{\mathfrak{p}}_i$ $(i = 1, 2, 3)$ defined in (13)—(15) in 2.2. In this case, there are two simple restricted roots $\alpha, \beta \in S_0$ such that $\{\alpha, \beta\} \cap S \neq \emptyset$ and $G_{\{\alpha,\beta\}}$ is a type II envelope of G (cf. 2.1(4)). Replacing $\{\alpha\}$ in the argument of Case 1.1 by $\{\alpha, \beta\}$, we have the desired conclusion.

Case 2.2. $\mathfrak{p}_0$ does not contain $\tilde{\mathfrak{p}}_0$, but contains one of $\tilde{\mathfrak{p}}_i$ $(i = 1, 2, 3)$. In this case, there is some $\alpha \in S_0$ which is not contained in S. Then, G_α is a Q_α-envelope of G and satisfies the definition of G_i in Lemma 2.2.1(3). From the same argument in Case 1.1, we have V is G_α-extendable. Let $\tilde{V}$ be an extension of V to G_α. Applying the same argument in Case 1.2 (in this case, we use Proposition 2.4.1(2) instead of (1)), we get the purity of $\tilde{V}$ and we have $\tilde{V}$ satisfies (WW). This means V is purely G_α-extendable and satisfies (WW).

Case 2.3. $\mathfrak{p}_0$ contains $\tilde{\mathfrak{p}}_0$. In this case, the same argument as Case 1.2 works. □

Next, we consider non-quasi-split cases. Hence, we assume that $(G, \mathfrak{g}_0, \theta, \langle\, ,\rangle)$ is a connected real non-quasi-split simple linear Lie group. Namely, $\mathfrak{g}_0$ is one of (10)–(12) in Lemma 2.1.1. In this case card $F^{\sharp}/F^{\flat} = 2$, so card $\mathscr{P}r_0(G) = 2$.

Let $V \in \mathscr{H}_G$ be irreducible. From [32] Theorem 5, there exists some standard parabolic subgroup P_V with the standard Langlands decomposition $P_V = M_V A_V U_V$, a limit of discrete series module σ_V and a character ν_V on A_V such that V is a unique quotient module of $\mathrm{Ind}_{P_V}^{G}(\sigma_V \otimes \nu_V)$. Moreover, we can choose $\Re\nu$ is contained in the closure of the positive Weyl chamber (for details, see [32]). We fix such a triple (P_V, σ_V, ν_V) for each irreducible $V \in \mathscr{H}_V$. Following [70], we introduce "lambda norm" as follows. For an irreducible $V \in \mathscr{H}_G$, we denote the norm of the infinitesimal character of σ_V by $\|V\|_{\mathbf{lambda}}$. (In the notation of [70], if we write $V = \bar{X}(\gamma)$ for $\gamma \in \mathscr{P}_f(H)$, then $\|V\|_{\mathbf{lambda}}$ is equal to $\|\gamma\|_{\mathbf{lambda}}$ in [70] Definition 3.22.) Theorem 2.9(a) in [70] means $\|V\|_{\mathbf{lambda}}$ does not depend on the choice of (P_V, σ_V, ν_V).

Now we prove:

LEMMA 5.4.3. *Let $V \in \mathscr{H}_G$ be an irreducible quasi-large Harish-Chandra module. Then, we have either V is extendable to some type II envelope or V is pure. In both cases, V satisfies (SW).*

Proof. We prove the lemma by the descending induction on $\|V\|_{\mathbf{lambda}}$. (We fix the infinitesimal character of V.) We denote by $\mathfrak{p}_{V,0}$ the Lie algebra of P_V. So, $\mathfrak{p}_{V,0}$ is a standard parabolic subalgebra. There are two cases.

Case 1. $\tilde{\mathfrak{p}}_0 \nsubseteq \mathfrak{p}_{V,0}$. Let S be the subset of Φ corresponding to P_V. We choose $\alpha \in S_0$ which is not contained in S. Then, from Lemma 5.4.1, we can extend σ_V to a limit of discrete series module $\tilde{\sigma}_V \in \mathscr{H}_{M^+_{\{\alpha\}}}$. Moreover, $\mathrm{Ind}^{G_\alpha}_{(P_V)^+_{\{\alpha\}}}(\tilde{\sigma}_V \otimes \nu_V)$ is an extension of $\mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V)$ to a type II envelope G_α of G. Therefore, we have $(\mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V))^{\tilde{\omega}_\alpha} \cong \mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V)$. Hence, $V^{\tilde{\omega}_\alpha} \cong V$.

Let $R: \mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V) \to V$ be the canonical projection. We denote by B the set of submodules of $\mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V)$ such that V does not appear in X as an irreducible constituent. Since the multiplicity of V in $\mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V)$ as a composition factor is one, we have Kernel $(R) = \Sigma_{X \in B} X$. Fix $X \in B$ and we regard X as a subspace of $\mathrm{Ind}^{G_\alpha}_{(P_V)^+_{\{\alpha\}}}(\tilde{\sigma}_V \otimes \nu_V)$. Since $V^{\tilde{\omega}_\alpha} \cong V$ holds, $\tilde{\omega}_\alpha X$ is also contained in B. This means Kernel (R) is a submodule in $\mathrm{Ind}^{G_\alpha}_{(P_V)^+_{\{\alpha\}}}(\tilde{\sigma}_V \otimes \nu_V)$ as an object of $\mathscr{H}_{G_\alpha}$. Hence, V is extendable to G_α. From Lemma 3.3.6, V satisfies (SW).

Case 2. $\tilde{\mathfrak{p}}_0 \subseteq \mathfrak{p}_{V,0}$. In this case, from Lemma 2.2.1 and Lemma 2.4.1, we have card $\mathscr{P}r_0(M_V A_V) = 2$ and h-$\mathrm{ind}^G_{P_V}$ gives a bijection of $\mathscr{P}r_0(M_V A_V)$ onto $\mathscr{P}r_0(G)$. From Lemma 3.1.3 and Lemma 3.1.5, σ_V is quasi-large, since V is quasi-large. From Proposition 5.3.1, σ_V is pure. So, we put ${}^{\bigcirc}\mathrm{WF}(\sigma_V) = \mathscr{O}$. From Lemma 3.1.5, we have ${}^{\bigcirc}\mathrm{WF}(\mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V)) \subseteq$ h-$\mathrm{ind}^G_{M_V A_V}(\mathscr{O})$. Since $\mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V))$ is quasi-large, we have ${}^{\bigcirc}\mathrm{WF}(\mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V)) =$ h-$\mathrm{ind}^G_{M_V A_V}(\mathscr{O})$. We denote by $\mathscr{O}'$ the other element in $\mathscr{P}r_0(M_V A_V)$ than $\mathscr{O}$.

We need:

CLAIM 1. *Let X be a quasi-large irreducible constituent of* $\mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V)$ *different from V. Then* h-$\mathrm{ind}^G_{P_V}(\mathscr{O}') \notin {}^{\bigcirc}\mathrm{WF}(X)$.

Proof. For simplicity, put $L = M_V A_V$ and denote by $\mathfrak{l}$ its complexified Lie algebra. We fix an admissible unitary character ψ' on $\bar{\mathfrak{n}}$ such that $\mathscr{O}_{\psi'} =$ h-$\mathrm{ind}^G_{P_V}(\mathscr{O}')$. So, ψ'_L is not contained in $\mathscr{O}$. From [70] Corollary 3.25, we have $\|X\|_{\mathbf{lambda}} > \|V\|_{\mathbf{lambda}}$. From the assumption of the induction, X satisfies (SW). Hence $\mathsf{Wh}^\infty_{\bar{\mathfrak{n}},\psi'}(X) \neq 0$. From Theorem 3.3.1 and Theorem 3.4.1, we have

$$0 < \dim \mathsf{Wh}^\infty_{\bar{\mathfrak{n}},\psi'}(X) \leqslant \dim \mathsf{Wh}^\infty_{\bar{\mathfrak{n}},\psi'}(\mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V)) \leqslant \dim \mathsf{Wh}^\infty_{\bar{\mathfrak{n}}\cap\mathfrak{l},\psi'_L}(\sigma_V \otimes \nu_V).$$

However, from Proposition 5.3.1, $\sigma_V \otimes \nu_V$ satisfies (SW). Hence, $\mathsf{Wh}^\infty_{\bar{\mathfrak{n}}\cap\mathfrak{l},\psi'_L}(\sigma_V \otimes \nu_V) = 0$. This is a contradiction. □

We are going back to the proof of Lemma 5.4.3. Since ${}^{\bigcirc}\mathrm{WF}(\mathrm{Ind}^G_{P_V}(\sigma_V \otimes \nu_V)) =$ h-$\mathrm{ind}^G_{M_V A_V}(\mathscr{O})$, using Lemma 3.1.2 and Claim 1 above, we have ${}^{\bigcirc}\mathrm{WF}(V) =$ h-$\mathrm{ind}^G_{M_V A_V}(\mathscr{O})$. Hence V is pure. We choose ψ' same as the proof of Claim 1. Then, we have $\mathsf{Wh}^\infty_{\bar{\mathfrak{n}},\psi'}(V) = 0$ by the same argument in the proof of Claim 1. This means V satisfies (WW). In order to prove that V satisfies (SW), from Lemma 5.1.3, we have only to prove:

CLAIM 2. *Let $\alpha \in S_0$. Then, we have*

$$\|V\|_{\mathbf{lambda}} = \|V^{\tilde{\omega}_\alpha}\|_{\mathbf{lambda}}.$$

Proof. We easily see $\tilde{\omega}_\alpha$ normalize M_V. So, we can consider $(\sigma_V)^{\tilde{\omega}_\alpha}$ and this is also a limit of discrete series module. We can easily see $(\mathrm{Ind}_{P_V}^G(\sigma_V \otimes \nu_V))^{\tilde{\omega}_\alpha} = \mathrm{Ind}_{P_V}^G((\sigma_V)^{\tilde{\omega}_\alpha} \otimes \nu_V)$. Hence $V^{\tilde{\omega}_\alpha}$ is a unique quotient of $\mathrm{Ind}_{P_V}^G((\sigma_V)^{\tilde{\omega}_\alpha} \otimes \nu_V)$. Since the infinitesimal characters of σ_V and $(\sigma_V)^{\tilde{\omega}_\alpha}$ coincide, we have the claim. Hence, we complete the proof of Lemma 5.4.3. □

From Lemmas 3.3.6, 5.2.3, 5.4.2, 5.2.5, and 5.4.3, we have:

LEMMA 5.4.4. *Let G be a connected simple linear Lie group. Then, all the irreducible quasi-large Harish-Chandra module $V \in \mathscr{H}_G$ satisfy the conditions (P) and (SW). Moreover, we have*

$$V \sim \sum_{\mathcal{O} \in {}^{\mathcal{O}}\mathrm{WF}(V)} \dim \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi_{\mathcal{O}}}(V)\mu_{\mathcal{O}}.$$

Here, $\psi_{\mathcal{O}}$ is an admissible unitary character on $\bar{\mathfrak{n}}$ such that $\mathcal{O}_{\psi_{\mathcal{O}}} = \mathcal{O}$.

From the remark after the definition of the condition (P) in 5.2, we easily have:

LEMMA 5.4.5. *Let G be a connected simple linear Lie group and $V \in \mathscr{H}_G$. Let ψ be an admissible unitary character on $\bar{\mathfrak{n}}$. Then, $\mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) \neq 0$ if and only if $\psi \in \mathrm{WF}(V)$.*

REMARK. In this lemma, V is not necessarily irreducible.

As we explained in 3.3, this lemma and Lemma 3.3.5 imply our main result Theorem 3.3.3. □

5.5. On the dimension of the space of Whittaker vectors

In the same way as Theorem 3.3.3, we can deduce easily the following result from Lemma 5.4.4.

THEOREM 5.5.1. *Let G be a real reductive linear Lie group. For each $\mathcal{O} \in \mathscr{P}r_0(G)$, we fix an admissible unitary character $\psi_{\mathcal{O}}$ such that $\mathcal{O}_{\psi_{\mathcal{O}}} = \mathcal{O}$. Then, for any irreducible Harish-Chandra module $V \in \mathscr{H}_G$, we have:*

$$V \sim \sum_{\mathcal{O} \in {}^{\mathcal{O}}\mathrm{WF}(V)} \dim \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi_{\mathcal{O}}}(V)\mu_{\mathcal{O}}.$$

Finally, we describe the relation between the dimension of the space of Whittaker vectors and the multiplicity.

THEOREM 5.5.2. *Let G be a connected real reductive linear Lie group and let $V \in \mathscr{H}_G$ be an irreducible Harish-Chandra module. Let ψ be an admissible unitary character on $\bar{\mathfrak{n}}$ such that $\psi \in \mathrm{WF}(V)$. Then, there exists a non-negative integer s such that $2^s \leqslant c(G)$ which only depends on V such that*

$$\dim \mathrm{Wh}^{\infty}_{\bar{\mathfrak{n}},\psi}(V) = 2^s w_G^{-1} c(V).$$

If V is extendable to some type II envelope, then $s = 0$. If V is pure (e.g. discrete series), then $2^s = c(G)$.

Proof. We have seen that the theorem holds for connected real simple linear Lie groups. The theorem is easily deduced from this. □

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