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On order and metric convexities in Z^n

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Abstract. Of concern here are convexities in the n -dimensional integer lattice Z^n , with respect to two integer valued metrics d_1 and d_2 . The Caratheodory, Helly and Radon type theorems for these two convexities are discussed. Also, Tverberg type Radon number for d_2 convex sets is obtained.

Introduction

In the paper [1], Danzer et al. have discussed in detail the classical theorems of Helly, Radon and Caratheodory for convex sets in Euclidean space and have mentioned various generalized convexities, such as order convexity, metric convexity etc. The study of convexity outside the framework of linear spaces, in particular for discrete sets, was considered among others, by Sampath Kumar [7], for finite graphs and Soltan [10], for finite metric spaces, in which they have defined convex sets and analysed analogue of Helly's theorem. Vijayakumar [11] has defined D convex sets, for the discrete holometric space

$$H = \{(q^m x_0, q^n y_0) \mid m, n \in Z\}, q \in (0, 1)$$

is fixed; and studied concepts like D convex domain, D convex hull, etc. In [2], Doignon has considered, the n dimensional integer lattice Z^n (crystallographical lattice), defined a convex set as intersection of a convex set in R^n with Z^n , and has shown that the Helly number for the family of such convex sets in Z^n is 2^n .

In this paper, we consider Z^n and define order convexity and convexity with respect to two integer valued metrics d_1 and d_2 . The theory discussed here may work well in any discrete set, isometric to Z^n . In particular,

$$H^n = \{(q^{m_1} x_1, q^{m_2} x_2, \dots, q^{m_n} x_n) \mid m_i \in Z\}, q \in (0, 1)$$

is fixed, and (x_1, x_2, \dots, x_n) is fixed in R^n . The d_1 convex sets of Z^n are the generalizations of D convex sets of H , discussed in [11].

In Section 1, we discuss the order convex sets and d_1 convex sets and their relationship and the analogue of Helly, Caratheodory theorems for d_1 convex sets. We note that the family of d_1 convex sets form a subfamily of order convex

sets. We give an example to show that no finite Helly and Radon number exist for the family of order convex sets.

In section 2, d_2 convex sets and its rank are studied. Analogous theorems of above type and Tverberg type theorems are also discussed.

1. Order convexity and d_1 convexity

We consider the n dimensional integer lattice, $Z^n = \{(m_1, m_2, \dots, m_n) \mid m_i \in Z\}$, where Z is the set of integers. If $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n) \in Z^n$, then

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i| \quad \text{and} \quad d_2(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

are two integer valued metrics in Z^n .

A partial order relation ' \leq ' in Z^n can be defined as

$$(x_1, x_2, \dots, x_n) \leq (y_1, y_2, \dots, y_n)$$

if and only if $x_i \leq y_i, \forall i = 1, 2, \dots, n$.

DEFINITION 1.1. A point $z \in Z^n$ is said to be order between $x, y \in Z^n$, if $x \leq z \leq y$ or $y \leq z \leq x$. The set of all points order between x and y is denoted by $[x, y]$. Conventionally, $[x, y] = \emptyset$ if x and y are not comparable.

DEFINITION 1.2. A point $z \in Z^n$ is said to be metrically between $x, y \in Z^n$, if $d(x, z) + d(z, y) = d(x, y)$, where d is a metric in Z^n . The set of all metrically between points of x and y is denoted by $d\langle x, y \rangle$ and is called the metric interval or d interval determined by x and y .

LEMMA 1.3. If $x \leq y$, then $[x, y] = d_1\langle x, y \rangle$.

DEFINITION 1.4. $A \subseteq Z^n$ is said to be order convex, if $[x, y] \subseteq A$, for each pair of points x and $y \in A$.

DEFINITION 1.5. $A \subseteq Z^n$ is said to be metrically convex or d convex, if the metric interval, $d\langle x, y \rangle \subseteq A$, for each pair of points $x, y \in A$.

DEFINITION 1.6. The order (metric) convex hull of a set A is the intersection of all order (metric) convex sets containing A . The order (metric) convex hull of a set A is denoted by order $\text{conv}(A)$ ($d\text{conv}(A)$).

NOTE 1.7. It follows from lemma 1.3 and the definitions that every d_1 convex set is order convex. The relation being not a total order, the converse need not be true.

LEMMA 1.8. If $A \subseteq Z^n$ is finite then $d_1 \text{conv}(A) = d_1 \langle u, v \rangle$ where $u = \inf A$ and $v = \sup A$.

Proof. We have $u \leq a \leq v \forall a \in A$. Therefore $A \subseteq [u, v] = d_1 \langle u, v \rangle$. Also $d_1 \text{conv}(A) \subseteq d_1 \langle u, v \rangle$, since $d_1 \langle u, v \rangle$ is d_1 convex. Since A is finite both u and v belongs to $d_1 \text{conv}(A)$. Therefore $d_1 \text{conv}(A) = d_1 \langle u, v \rangle$.

DEFINITION 1.9. The Caratheodory number c for the family of order convex (d convex) sets in Z^n is defined as the smallest non-negative integer c such that

$$\text{Conv}(S) = \bigcup \{ \text{conv}(T) \mid T \subseteq S \text{ and } |T| \leq c \}$$

where $\text{conv}(S)$ denotes order $\text{conv}(S)$ or $d \text{conv}(S)$, as the case may be.

In [3] Franklin has proved that the Caratheodory number for the family of order convex sets in any poset is '2'.

THEOREM 1.10. The Caratheodory number for the family of d_1 convex sets in Z^n is n , if $n \geq 2$.

Proof. We have for any $A \subseteq Z^n$

$$d_1 \text{conv}(A) = \bigcup \{ d_1 \text{conv}(B) \mid B \subseteq A \text{ and } |B| < \infty \}$$

By lemma 1.8, if $|B| < \infty$, then $d_1 \text{conv}(B) = d_1 \langle u, v \rangle$, where $u = \inf B$ and $v = \sup B$. Also if $|B| < \infty$, u is the infimum of at most n elements of B and v is the supremum of at most n elements of B .

Let

$$u = \inf \{ a_1, a_2, \dots, a_n \mid a_i \in B \}$$

$$v = \sup \{ b_1, b_2, \dots, b_n \mid b_i \in B \}$$

Therefore, we have $d_1 \text{conv}(B) = d_1 \text{conv} \{ a_1, \dots, a_n, b_1, \dots, b_n \}$. We shall now show that any point $z \in d_1 \text{conv}(B)$ belongs to the d_1 convex hull of at most n points among $a_1, \dots, a_n, b_1, \dots, b_n$. Let $z = (z_1, z_2, \dots, z_n)$. We select the n points a'_1, \dots, a'_n among $a_1, \dots, a_n, b_1, \dots, b_n$, as follows.

a'_i is chosen such that the i th coordinate of a'_i is at most $z_i, \forall i = 1, \dots, n$.

If the j th coordinate of a'_i is at most $z_j, i = 1, \dots, n, i \neq j$, then we delete a'_j and replace it with one among $a_1, \dots, a_n, b_1, \dots, b_n$, whose j th coordinate is greater than or equal to z_j . The points a'_1, \dots, a'_n , selected in this way, satisfies the inequality $u' \leq z \leq v'$, where

$$u' = \inf \{ a'_1, \dots, a'_n \} \quad \text{and} \quad v' = \sup \{ a'_1, \dots, a'_n \}$$

and $z \in d_1 - \langle u', v' \rangle = d_1 \text{conv} \{ a'_1, \dots, a'_n \}$ and hence the theorem. We note that

this theorem can be obtained as a particular case of a product theorem due to Sierksma ([8]).

DEFINITION 1.11. The Helly number h for the family of d convex sets in Z^n is defined as the infimum of all nonnegative integers h such that the intersection of any finite collection of d convex sets is non empty, provided the intersection of each subcollection consisting of at most h members of the family is non empty.

Since for $X \subseteq R$, the natural order and the natural metric of X yield same convexity of Helly number two and the Helly number of a product convexity is the maximum of the Helly numbers of the factors, it follows from [5], [8], [9] that the Helly number for the family of d_1 convex sets in Z^n is two.

DEFINITION 1.12. The Radon number r of a family of d convex sets in Z^n is defined as the infimum of all non-negative integers r such that every set $S \subseteq Z^n$, with $|S| \geq r$ admits a partition, $S = S_1 \cup S_2$, such that $d \text{conv}(S_1) \cap d \text{conv}(S_2) \neq \emptyset$.

The following example illustrates that the family of order convex sets in $Z^n (n \geq 2)$ has an infinite Helly and Radon number.

EXAMPLE 1.13. Suppose that there exists finite Helly number h and Radon number r , for the family of order convex sets in Z^n . Consider the set

$$A = \{(x, y, 0, \dots, 0), (x - 1, y + 1, 0, \dots, 0), \dots, (x - h, y + h, 0, \dots, 0)\} \subseteq Z^n.$$

Then $|A| = h + 1$ and A is trivially order convex.

Now consider subsets of A defined as

$$A_0 = A \setminus (x, y, 0, \dots, 0), A_1 = A \setminus (x - 1, y + 1, 0, \dots, 0), \dots$$

$$A_h = A \setminus (x - h, y + h, 0, \dots, 0)$$

Then $\{A_0, A_1, \dots, A_h\}$ is a family of $h + 1$ order convex sets such that each h members of the family have non empty intersection. But $\bigcap_{i=0}^h A_i = \emptyset$, which is a contradiction to the assumption that h is the Helly number. Since $h \leq r - 1$ for any convexity [6], it follows that family of order convex sets in Z^n do not have finite Radon number also.

2. d_2 convexity

In this section we shall discuss the properties of d_2 convex sets in Z^n where d_2 is the metric defined by

$$d_2(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|,$$

where $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

DEFINITION 2.1. Let $A = \{x_1, x_2, \dots, x_r\}$ be a set of r points in Z^n . A is said to be convexly independent, if no element of A lies in the convex hull of the remaining elements of A . Otherwise A is said to be convexly dependent.

In the following discussion, we consider only d_2 convexly independent sets.

EXAMPLE.

$$A = \{(0, \dots, 0), (1, 0, \dots, 0), \dots, (1, 1, \dots, 0), \dots, (1, 1, \dots, 1, 0), \dots, (1, 1, \dots, 1)\}$$

is an independent set with 2^n elements in Z^n . The maximal cardinality of convexly independent set is called its rank [4].

LEMMA 2.2. Let $A \subseteq Z^n$ be a set with $r = 2^n$ independent points. Let $\pi_j: Z^n \rightarrow Z$ denote the projection to the j th factor. Then for each $x \in A$ and $j = 1, \dots, n$, there is a point $y \in A$ with $d_2(x, y) = |\pi_j(x) - \pi_j(y)|$.

Proof. We prove the lemma by induction on the dimension n of Z^n .

For $n = 1$, it is trivially true.

For $n = 2$, let $A = \{x_1, x_2, x_3, x_4\}$ be a set of $2^2 = 4$ independent points in Z^2 . Required to show that, for each $x_i \in A$, and $j = 1, 2$, there is a point $x_k \in A$, with

$$d_2(x_i, x_k) = |\pi_j(x_i) - \pi_j(x_k)|$$

Suppose not. That is, for at least one $x_i \in A$, say x_1 ,

$$d_2(x_1, x_k) = |\pi_1(x_1) - \pi_1(x_k)| \text{ or } |\pi_2(x_1) - \pi_2(x_k)|$$

for all $x_k \in A$.

Let

$$m = \min_{x_k \in A \setminus x_1} d_2(x_1, x_k),$$

and

$$A' = \{x_k \in A \mid d_2(x_1, x_k) = m\}.$$

Then $A' \neq \emptyset$ and $x_k \in d_2 \text{ conv}(A \setminus x_1)$, for every $x_k \in A'$, which is a contradiction to the assumption that A is a d_2 convexly independent set, hence the lemma for $n = 2$. Now assume the result for $n - 1$. Let $A = \{x_1, \dots, x_r\}$, $r = 2^n$ be an independent set in Z^n . For each $x_i \in A$, any $(n - 1)$ dimensional projection A' of A containing x_i , contains 2^{n-1} independent points. So by the induction

assumption, for each $j = j_1, j_2, \dots, j_{n-1}$, there is a point $x_k \in A'$, with

$$d_2(x_i, x_k) = |\pi_j(x_i) - \pi_j(x_k)| \text{ where } j_1, j_2, \dots, j_{n-1} \in \{1, \dots, n\}.$$

Consider another $(n - 1)$ dimensional projection B' of A , containing x_i and again by inductive assumption, there is a point $x'_k \in B'$, such that

$$d_2(x_i, x'_k) = |\pi_j(x_i) - \pi_j(x'_k)|, \text{ for each } j = j_2, \dots, j_n;$$

where $j_2, \dots, j_n \in \{1, \dots, n\}$.

Therefore, for each $x_i \in A$, and $j = 1, \dots, n$, there is a point $x_k \in A$ with $d_2(x_i, x_k) = |\pi_j(x_i) - \pi_j(x_k)|$ and hence the lemma for all n .

LEMMA 2.3. *The rank of the family of d_2 convex set in Z^n is 2^n .*

Proof. We prove that every set with cardinality $2^n + 1$ is dependent. Let $B = \{x_1, x_2, \dots, x_{r+1}\}$, $r = 2^n$ be any subset of Z^n . Let $A = \{x_1, \dots, x_r\}$, be any subset of B , containing 2^n independent points. If $x_{r+1} \in d_2 \text{ conv}(A)$, then we are done. If not, let $m = \min(d_2(x_i, x_{r+1}) | x_i \in A)$. Define

$$C = \{x_j \in A | d_2(x_j, x_{r+1}) = m\}.$$

Then $C \neq \emptyset$, and for $x_j \in C$, let $d_2(x_j, x_{r+1})$ be the difference between the k th coordinates ($1 \leq k \leq n$). By lemma 2.2, there exists a point $x_p \in A$ such that $d_2(x_p, x_j)$ is also the difference between the k th coordinates. Since $x_{r+1} \notin d_2 \text{ conv}(A)$ and $d_2(x_j, x_{r+1})$ is the minimum, $d_2(x_p, x_{r+1})$ is also the difference between the k th coordinates. Therefore we have $d_2(x_p, x_{r+1}) = d_2(x_p, x_j) + d_2(x_j, x_{r+1})$. That is, $x_j \in d_2 < x_p, x_{r+1}$.

Therefore $x_j \in d_2 \text{ conv}(B \setminus x_j)$, which completes the proof.

COROLLARY 2.4. *Let $S \subseteq Z^n$ be finite, with $|S| \geq 2^n$. Then there exists an independent subset A of S , $|A| \leq 2^n$, such that $d_2 \text{ conv}(S) = d_2 \text{ conv}(A)$.*

We note that if $A = \{x_1, \dots, x_r\}$, $r \leq 2^n$, is a set of r independent points in Z^n , then for any point $x \in d_2 \text{ conv}(A)$, there is an $(n - 1)$ dimensional submodule of Z^n , containing x . In Z^{n-1} there are at most 2^{n-1} independent points of A , the d_2 convex hull of which contains x . Thus any point $x \in d_2 \text{ conv}(A)$, belongs to the d_2 convex hull of a subset of A , containing at most 2^{n-1} points of A .

Therefore, $d_2 \text{ conv}(A) = \bigcup \{d_2 \text{ conv}(T) | T \subseteq A \text{ and } |T| < 2^{n-1}\}$. In fact this bound is sharp. For example, consider the subset $A = \{(x_1, \dots, x_n) \in Z^n | x_i = 0 \text{ or } 2, \forall i = 1, \dots, n\}$. Define subsets A_j and A'_j of A with cardinality 2^{n-1} as

$$A_j = \{(x_1, \dots, x_j, \dots, x_n) \in A | x_j = 0\} \quad \text{and}$$

$$A'_j = \{(x_1, \dots, x'_j, \dots, x_n) \in A | x'_j = 2\}, \quad \text{for } j = 1, \dots, n.$$

We have $d_2 \text{conv}(A) = \cup \{d_2 \text{conv}(B) \mid B = A_j \text{ or } A'_j\}$. Now consider the point $z = (4, 1, \dots, 1) \in d_2 \text{conv}(A)$. Then $z \in d_2 \text{conv}(A'_1)$ and it can be verified easily that, z cannot lie in the d_2 convex hull of a subset of A of cardinality less than 2^{n-1} . Thus we have,

THEOREM 2.5. *The Caratheodory number for the family of d_2 convex sets in Z^n is 2^{n-1} .*

THEOREM 2.6. *Let $\mathcal{F} = \{B_1, B_2, \dots, B_r\}$ be a family of r non empty d_2 convex sets in Z^n , with $r \geq 2^n$, then the intersection of the family \mathcal{F} is non-empty, provided each sub-collection of \mathcal{F} consisting of 2^n members has non-empty intersection.*

Proof. The proof is by induction. For $r = 2^n$, the result is trivial.

Let $r = 2^n + 1$, then $\mathcal{F} = \{B_1, \dots, B_r\}$ is a family of $2^n + 1$, d_2 convex sets, satisfying the conditions of the theorem. Therefore, there exists x_1, x_2, \dots, x_r such that

$$x_i \in \bigcap_{\substack{j=1 \\ j \neq i}}^r B_j, \quad \text{for } i = 1, \dots, r.$$

Now $A = \{x_1, x_2, \dots, x_r\}$ is a set of $2^n + 1$ points in Z^n , which by lemma 2.3 is dependent.

Therefore $x_i \in d_2 \text{conv}\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r\}$, for some i .

Clearly $x_i \in B_i, \forall i = 1, \dots, r$, completing the proof for $r = 2^n + 1$.

Assume the result for $r = 2^n + m$.

Let $\mathcal{F} = \{B_1, \dots, B_{r+1}\}$ $r = 2^n + m$, be a family of non-empty convex sets, such that every 2^n members of \mathcal{F} has non-empty intersection.

Define $B'_i = B_i \cap B_{r+1}$, for $i = 1, \dots, r = 2^n + m$. Now B'_1, \dots, B'_r is a family of r non empty d_2 convex sets, satisfying the induction hypothesis.

Therefore

$$\bigcap_{i=1}^r B'_i \neq \emptyset, \text{ i.e. } \bigcap_{i=1}^{r+1} B_i \neq \emptyset, \quad r = 2^n + m,$$

which completes the proof by induction. Thus the Helly number for the family of d_2 convex sets in Z^n is 2^n .

THEOREM 2.7. *The Radon number for the family of d_2 convex sets in Z^n is $2^n + 1$.*

Proof. By the lemma 2.3, any $2^n + 1$ points in Z^n is dependent and therefore any set $S, |S| \geq 2^n + 1$, has a partition into two disjoint sets S_1 and S_2 , whose d_2 convex hulls contain at least one common point.

DEFINITION 2.8. The Tverberg type Radon number P_m , for the family of d

convex sets in Z^n is defined to be the infimum of all positive integers k with the property that, each set S in Z^n with $|S| \geq k$, admits a Radon m -partition. That is, a partition of S into m disjoint sets S_1, \dots, S_m such that

$$\bigcap_{i=1}^m d \operatorname{conv}(S_i) \neq \emptyset.$$

THEOREM 2.9. *Tverberg type Radon number P_m for the family of d_2 convex sets in Z^n is $(m - 1)2^n + 1$.*

Proof. We shall now show that every subset $S \subseteq Z^n$, with $|S| = (m - 1)2^n + 1$ have a Radon m -partition and there exists subsets S , with $|S| = (m - 1)2^n$, having no Radon m -partition. Let $S \subseteq Z^n$ be such that $|S| = (m - 1)2^n + 1$. Choose $F_1 \subseteq S$, so that $|F_1| \leq 2^n$ and $d_2 \operatorname{conv}(F_1) = d_2 \operatorname{conv}(S)$. This is possible, since rank of the family of d_2 convex sets in Z^n is 2^n . Again choose $F_2 \subseteq S \setminus F_1$ with $|F_2| \leq 2^n$ and $d_2 \operatorname{conv}(F_2) = d_2 \operatorname{conv}(S \setminus F_1) \subseteq d_2 \operatorname{conv}(F_1)$. Proceeding in this way, we get a partition of S into $(m - 1)$ sets F_i with $|F_i| \leq 2^n$, for each i , and there remains at least one point z in S and

$$z \in d_2 \operatorname{conv}(F_{m-1}) \subseteq d_2 \operatorname{conv}(F_{m-2}) \subseteq \dots \subseteq d_2 \operatorname{conv}(F_1) = d_2 \operatorname{conv}(S).$$

Thus we have a Radon- m -partition. Further the set

$$S = \{ \begin{array}{lll} (0, \dots, 0), & (2m - 3, \dots, 0), & (0, 2m - 3, \dots) \dots (0, \dots, 2m - 3), \\ \dots & \dots & (2m - 3, 2m - 3, \dots, 2m - 3), \\ (1, 1, \dots, 1) & \dots & (2m - 2, 2m - 2, \dots, 2m - 2), \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ (m - 2, m - 2, \dots, m - 2) & \dots & (m - 1, m - 1, \dots, m - 1) \end{array} \}$$

has cardinality $(m - 1)2^n$ and has no Radon m -partition.

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