COMPOSITIO MATHEMATICA

BERND KREUSSLER
HERBERT KURKE

Twistor spaces over the connected sum of 3 projective planes

Compositio Mathematica, tome 82, nº 1 (1992), p. 25-55
<http://www.numdam.org/item?id=CM_1992__82_1_25_0>

Compositio Mathematica **82**: 25–55, 1992.

Twistor spaces over the connected sum of 3 projective planes

BERND KREUßLER and HERBERT KURKE[1]
Fachbereich Mathematik, Humboldt-Universität zu Berlin, PSF 1297, Berlin 1086, Germany

Received 4 January 1989; accepted April 12, 1991

Abstract. We consider the map defined by the half-anticanonical linear system on a simply connected compact twistor space Z under the hypothesis that the natural self-dual conformal structure on the space X of twistor lines of Z contains a metric with positive scalar curvature and that X has signature 3. By well-known results of M. Freedman, X has to be homeomorphic to a connected sum of 3 complex projective planes. Recently, S.K. Donaldson and R. Friedman showed the existence of such twistor spaces in a more general context. Our result leads to a classification of all complex compact 3-folds that can arise as such twistor spaces.

Introduction

Twistor spaces arise naturally in 4-dimensional conformal geometry through the attempt to introduce a complex structure compatible with the given conformal structure. Although neither the existence nor uniqueness of such a structure is evident, like in the analogous 2-dimensional case, this attempt led to the consideration of the collection of all of these complex structures in the tangent spaces to the 4-manifold M (inducing a fixed orientation). One obtains a manifold Z which is fibred over M by Riemann spheres $\mathbb{P}^1(\mathbb{C})$ and which has a natural almost complex structure inducing the almost complex structure on the fibres underlying the natural complex structure of Riemann spheres. This point of view appeared already in a paper of Hirzebruch and Hopf [HH] in 1958, of course not under the name of a twistor space, which was coined much later in a completely different context. In the context of Riemannian geometry the twistor construction was first developed in detail by Atiyah, Hitchin and Singer [AHS]. There are only two compact twistor spaces that can have Kähler structures ([Hit2], [FK]) both are actually algebraic varieties, namely the projective space and the flag variety of the projective plane. Throughout the proof of this result, two more candidates for twistor spaces emerged, the intersection of 2 quadrics in $\mathbb{P}^5$ and double solids with branch locus of degree 4. However, for both of them the value of their Euler number is incompatible with their being a twistor space.

Poon showed in his thesis (1985), among other things, that there are small

(1)Part of this work was done while the second author was a member of the Max–Planck Institut für Mathematik, Bonn, during the academic year 1987/88. He wants to thank this institute for the hospitality and excellent working conditions.

resolutions of certain singular intersections of 2 quadrics in $\mathbb{P}^5$ which are twistor spaces (and therefore non-Kählerian algebraic varieties).

It turned out, from the work of Poon and Hitchin [Hit2], that in addition to self-duality the existence of a Riemannian metric of positive scalar curvature in the given conformal class is necessary for a twistor space to be algebraic (or a Moishezon space) [Poon3].

The reason why positive scalar curvature plays a crucial role for such a kind of questions is the famous vanishing theorem for instanton bundles, proved by several authors, see [Hit1] and the forthcoming paper [K3] for details. It allows to derive certain vanishing theorems which are similar to the Kähler case and could be proved for this case by Kodaira's vanishing theorem and its various generalizations. Positivity of scalar curvature in our context means precisely the following: by a famous result of R. Schoen [Sch] each class of conformally equivalent Riemannian metrics contains metrics with constant scalar curvature (see [Sch]).

If we can find, on a self-dual space, such a metric with positive scalar curvature and compatible with the given conformal structure, we call the space a self-dual space with positive scalar curvature.

If a compact oriented 4-manifold M has a self-dual structure with positive scalar curvature the intersection form on M has to be positive definite [LeBrun]. If M is also simply connected it is therefore homeomorphic to a connected sum of projective planes or to the 4-sphere. The existence of such structures for $\#^n \mathbb{P}^2(\mathbb{C})$ (for $n \geqslant 2$) was shown for $n=2$ in [Poon1] and for arbitrary n by [Don-Free] and [Floer]. After reading Poon's thesis [Poon1] and from the results of [FK] we were convinced that double solids with quartic (singular) ramification locus should also be modifications of twistor spaces.

It is not difficult to check that the number of double points of the ramification locus should be exactly 13 in order to get small resolutions which have the correct Euler number to be a twistor space. Thanks to a hint of H. Knörrer we first found in [Jes] descriptions of such quartics and the geometry of the corresponding double solids was worked out by the first author (published 1989).

The referee of an earlier version of the present paper brought to our attention the preprint [Poon2]. There is an essential overlap of our paper with Poon's preprint concerning the case of twistor spaces as double solids. However Poon does not investigate carefully enough the half-anticanonical linear system and claims that the base locus is empty. This is not true in general. The second author has analyzed carefully the situation in the presence of base points, it leads to a family of twistor spaces which are modifications of conic bundles over a quadric surface and it yields an alternative method to those of LeBrun for an explicit description of self-dual structures on connected sums of projective planes.

A preliminary report appeared in [K4]. A second objection against Poon's preprint is the problem of identifying the twistor lines, which is incomplete. We also could not overcome this difficulty, the problem is that there are too many candidates for possible twistor lines and to determine their normal bundle. Thus we only have an indirect, deformation theoretic argument that generically we get twistor spaces. In spite of these objections we have to admit that methods of Poon's thesis were of significant influence and encouragement for our investigations.

The content of this paper is roughly the following:

In Section 1 we summarize known facts about twistor spaces that are used in this paper and study the case of surface of degree 1 with respect to the twistor figration.

In Section 2 we single out certain distinguished line bundles of degree 1 which could have sections vanishing along a surface of degree 1.

Using this we study surfaces of degree 1 on a twistor space in case of signature 2 in detail, which is the content of Sections 3 and 4.

This leads to two distinct cases according to the existence or non-existence of base points of the 1/2-anticanonical linear system of Z. In the case of the existence of base points, the twistor space is a modification of a certain conic bundle over the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, which is described precisely by Theorem 3 of Section 3. The case of the non-existence of base points is studied in Section 4. It leads to a 6-parameter family of double solids which are described in detail by Theorem 4. They are studied further in detail in the paper [Kr].

We work mainly with techniques of algebraic geometry. The final goal of this philosophy should be an explicit description of the family of self-dual conformal structures on the connected sum of projective planes, starting with the algebraic-geometric description of the twistor space. We add some remarks about this perspective in Section 5.

1. Twistor spaces

We give a summary of main results about twistor spaces, for details we refer to [AHS], [Frie], [K1], [Hit2] and [Poon].

In this paper M will always be a compact oriented 4-manifold with a conformal self-dual structure, i.e. the component W^- of the Weyl tensor of the conformal structure vanishes.

Fixing a Spin^c-structure with half-spinor bundles $\mathbb{F}^{\pm}$ (which always exists on oriented compact 4-manifolds) we get:

(i) a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathbb{F}^-) = Z \xrightarrow{\pi} M$ which is an almost complex manifold.

The condition $W^- = 0$ is equivalent to the integrability of the almost complex structure. The twistor fibres $F = \pi^{-1}(x) \simeq \mathbb{P}^1$ are complex submanifolds.

REMARKS. (1) Z does not depend on the choice of the Spin^c-structure on M since $\mathbb{F}^{\pm}$ are determined up to a twist with a complex line bundle.

(2) Z has a distinguished antiholomorphic involution τ such that Z has no real points and the twistor lines are invariant under τ.

(3) The twistor fibres have the normal bundle $N_{F/Z} \simeq \mathcal{O}_F(1)^{\oplus 2}$.

(ii) A unique holomorphic line bundle L on Z which is diffeomorphic to $T(Z/M)$. It is a square root of the anticanonical bundle K_Z^{-1}.

(iii) A holomorphic bundle H which is of degree 1 on the twistor fibres and which is determined by the Spin^c-structure.

We have the following

THEOREM 1. *If M has positive scalar curvature and E is an instanton bundle on Z, then*

$$H^1(Z, E \otimes H^{-\otimes m}) = 0$$

for $m \geqslant 2$.

A proof can be found in [Hit1] if H arises from a Spin-structure on M, in [K3] for the general case.

THEOREM 2. *If $h = c_1(T(Z/M)) = c_1(L)$ and $[F] \in H^4(Z, \mathbb{Z})$ is the cohomology class of a twistor fibre, then*

$$h^2 = (2e(M) - 3\,\mathrm{sgn}(M))[F], \qquad h \cdot [F] = 2$$

(where we identify $H^6(Z, \mathbb{Z})$ with $\mathbb{Z}$ via the orientation)

$$c_1(Z) = 2h, \qquad c_2(Z) = 3(e(M) - \mathrm{sgn}(M))[F], \qquad c_3(Z) = 2e(M)$$

and the Chern numbers are

$$c_1^3 = 16(2e(M) - 3\,\mathrm{sgn}(M)), \qquad c_1c_2 = 12(e(M) - \mathrm{sgn}(M)), \qquad c_3 = 2e(M).$$

Proof. See [Hit2] or [K1]. Here, $e(M)$ is the Euler number of M and $\mathrm{sgn}(M)$ the signature.

COROLLARY 1. *Under the assumptions of Theorem 1 we have*

$$H^2(Z, L^{\otimes m}) = 0 \quad \textit{for } m \geqslant -1$$
$$H^3(Z, L^{\otimes m}) = 0 \quad \textit{for } m \geqslant -1.$$

Proof. This follows by the Serre duality because of $K_Z \cong L^{-\otimes 2}$.

COROLLARY 2. *Under the assumption of Theorem* 1 *and if* $e(M)-\mathrm{sgn}(M)=2$, *we also have* $H^1(Z, \mathcal{O}_Z)=0$.

This follows because of

$$\chi(\mathcal{O}_Z) = \mathrm{Todd}(Z) = \frac{12(e(M) - \mathrm{sgn}(M))}{24}$$

by Theorem 2.

For the following lemma we refer to [Hit2] and [Poon].

LEMMA 1. *Let N be a holomorphic line bundle on Z,* $k=\deg(N \otimes \mathcal{O}_F)$

(i) $|N| \neq \emptyset \Rightarrow k > 0$.

(ii) *If* $k > 0$, $s \in H^0(Z, N)$ *a section vanishing of order* $k+1$ *on a twistor line, then* $s=0$.

(iii) *If* $k=1$ *and* $S \in |N|$, *the surface S is smooth, connected and it contains at most one twistor line F. If* $F \subset S$ *is a twistor line, then* $(F^2)_S=1$.

(iv) *If* $k=2$, $S \in |N|$ *and* $S = \bar{S}$, *then* S^{sing} *is a union of twistor lines.*

(v) *If* $k=2$, $\tau^*\bar{N} \simeq N$, $F \subset Z$ *a twistor line and* $\Lambda(F)=\{S \in |N|, F \subset S^{\mathrm{sing}}\}$, *then* $\dim \Lambda(F) \leqslant 2$.

LEMMA 2 ([Poon]). *Let N be a holomorphic line bundle on Z,* $\deg(N \otimes \mathcal{O}_F)=2$ *and N real, i.e.*

$$\tau * \bar{N} \simeq N.$$

Let $\Lambda \subset |N|$ *be a real subspace.*

(i) *If* $\dim \Lambda \geqslant 1$, *then* Λ *has no fixed component.*

(ii) *If* $\dim \Lambda \geqslant 3$, *then* Λ *contains smooth connected surfaces.*

Finally we want to discuss surfaces of degree 1, i.e. compact complex surfaces $S \subset Z$ such that $(S \cdot F) = 1$ for twistor lines. We know that they are smooth connected and contain at most one twistor line (Lemma 1, (iii)).

LEMMA 3. *Let S be a surface of degree* 1 *on Z.*

(i) *If S contains no twistor line, then*

$$\begin{aligned}
e(S) &= e(M)\\
b_2^+(S) &= b_2^-(M),\ b_2^-(S) = b_2^+(M)\\
(K_S^2) &= 2e(M) - 3\,\mathrm{sgn}(M)\\
&= 4 - 4b_1(M) + 5b_2^-(M) - b_2^+(M)\\
\chi(\mathcal{O}_S) &= \tfrac{1}{4}(e(M) - \mathrm{sgn}(M))\\
&= \tfrac{1}{2}(1 - b_1(M) + b_2^-(M)).
\end{aligned}$$

(ii) *If S contains a twistor line, then*

$$(K_S^2) = 9 - 4b_1(M) + 5b_2^-(M) - b_2^+(M)$$
$$\chi(\mathcal{O}_S) = 1 + \tfrac{1}{2}(-b_1(M) + b_2^-(M)).$$

Proof. Follows from the fact that the twistor fibration induces a map $\pi: S \to M$ which is diffeomorphic in the case (i) and diffeomorphic on $S\setminus F$ in the case (ii) and contracts F to a point.

Moreover, π changes the orientation. Now the formulas follow from the Noether formula $(K_S^2)+e(S)=12\chi(\mathcal{O}_S)$, the signatur theorem

$$3\,\mathrm{sgn}(S) = (K_S^2) - 2e(S)$$

and in case (ii) by

$$e(S) = e(M) + 1, \qquad \mathrm{sgn}(S) = 1 - \mathrm{sgn}(M).$$

COROLLARY. If *S is chosen as above and if M has positive scalar curvature and* $H^1(Z, \mathcal{O}_Z)=0$ *then S is an algebraic surface with* $p_g=q=0$ $(q=h^1(\mathcal{O}_S),\ p_g=h^2(\mathcal{O}_S))$. *Furthermore* $b_1(M)=0$. *In the case* (i) $b_2^-(M)=1$ *and in the case* (ii) $b_2^-(M)=0$ *and the linear system* $|F|_S$ *defines a birational morphism* $S \to \mathbb{P}^2$.

Proof. By the vanishing theorem and Serre duality

$$H^2(Z, \mathcal{O}_Z) = H^2(Z, \mathcal{O}_Z(-S)) = 0.$$

Also by Serre duality $H^3(Z,\mathcal{O}_Z)=H^3(Z,\mathcal{O}_Z(-S))=0$. Therefore, $H^1(S,\mathcal{O}_S)=H^2(S,\mathcal{O}_S)=0$ by the exact sequence $0 \to \mathcal{O}_Z(-S) \to \mathcal{O}_Z \to \mathcal{O}_S \to 0$. In particular $b_1(S)=2q=0$ and therefore $b_1(M)=0$. The linear system $|F|_S$ defines a birational morphism because of $(F^2)_S=1$ and $|F|_S \cdot F = |\mathcal{O}_S(F) \otimes \mathcal{O}_F| = |\mathcal{O}_F(1)|$ (since $h^1(\mathcal{O}_S)=0$).

2. Linear bundles of degree 1

In this section we assume that M is a compact self-dual simply connected 4-manifold with positive scalar curvature. By $Z \xrightarrow{\pi} M$ we denote the twistor space of M. Then we know that $H^2(M,\mathbb{Z})$ is a positive definite unimodular lattice and $H^q(Z,\mathcal{O}_Z)=0$ for $q>0$. Therefore $\mathrm{Pic}(Z) \simeq H^2(Z,\mathbb{Z})$ and $H^2(M,\mathbb{Z})$ is contained in this group as a primitive sublattice. Then $c_1(\pi * TM)=c_1(T(Z/M))=\tfrac{1}{2}c_1(Z)$ and $T(Z/M)$ has a unique holomorphic structure such that $T(Z/M)^{\otimes 2}$ is the anticanonical class. Furthermore Wu's formula implies that for any orthogonal basis (ξ_ν) of $H^2(M, \mathbb{Z})$ (identified with its image in $H^2(Z, \mathbb{Z})$),

$$\tfrac{1}{2}c_1(Z) - \sum \xi_\nu \equiv 0 \bmod 2.$$

Therefore, each choice of an orthogonal basis ξ of $H^2(M, \mathbb{Z})$ determines a distinguished class

$$\omega = \omega(\xi) \in H^2(Z, \mathbb{Z}), \qquad \tfrac{1}{2}c_1(Z) = 2\omega(\xi) + \sum_\nu \xi_\nu$$

and

$$H^2(Z, \mathbb{Z}) = H^2(M, \mathbb{Z}) \oplus \mathbb{Z}\omega.$$

In this way we get 2^b distinguished cohomology classes $\omega(\xi)$ on Z, where $b = \operatorname{rk} H^2(M, \mathbb{Z})$. By the results of Section 1 we have

$$(\tfrac{1}{2}c_1(Z))^3 = 8 - 2b$$

and, using the Riemann–Roch formula and the vanishing theorem we get

$$\dim |\tfrac{1}{2}c_1(Z)| = 9 - 2b$$

provided $H^1(Z, L) = 0$.

LEMMA 1

(i) *If H is a line bundle on Z of degree* 1 *on the twistor lines, then* $\chi(H) \leqslant 0$ *unless* $c_1(H) = \omega(\xi)$.

(ii) *If $S \subset Z$ is a surface and* $(F \cdot S) = 1$ *for twistor lines F, then*

$$[S] = \omega(\xi).$$

The surface S contains exactly one twistor line F,

$$F = S \cdot \bar{S}, \qquad (F^2)_S = 1, \qquad c_1(S)^2 = 9 - b$$

and the linear system $|F|$ defines a birational morphism $S \to \mathbb{P}^2$.

(iii) *If H is a line bundle on Z, $c_1(H) = \omega(\xi)$, then $\chi(H) = 4 - b$ and $|H| \neq \varnothing$ for* $b \leqslant 3$.

Proof. If $(\xi_1, \ldots, \xi_b)$ is an orthogonal basis of $H^2(M, \mathbb{Z})$ we write $\xi = \xi_1 + \cdots + \xi_b$. This cohomology class determines the basis because of $\xi^2 = b$ and 2nd Stiefel–Whitney class $= \xi \bmod 2$ (by Wu's formula). Any other orthogonal basis is of the form

$$\varepsilon_1\xi_1, \varepsilon_2\xi_2, \ldots, \varepsilon_b\xi_b, \qquad \varepsilon_\nu = \pm 1$$

and

$$\omega\left(\sum \varepsilon_\nu \xi_\nu\right) = \omega(\xi) + \sum_\nu \frac{1-\varepsilon_\nu}{2}\xi_\nu.$$

By the Riemann–Roch formula

$$\begin{aligned}\chi(H) &= \chi(c_1(H))\\ &= \tfrac{1}{6}c_1(H)^3 + \tfrac{1}{4}c_1(Z)c_1(H)^2 + \tfrac{1}{12}(c_1(Z)^2 + c_2(Z))c_1(H) + 1\end{aligned}$$

and

$$c_1(Z) = 4\omega(\xi) + 2\xi, \qquad \omega(\xi)^2 = -\omega(\xi)\xi + [F]$$
$$c_1(Z)^2 + c_2(Z) = (22 - 4b)[F]$$

we get

$$\chi(\omega(\xi)) = 4 - b.$$

If $c_1(H) = \omega(\xi) + \eta$, $\eta \in H^2(M, \mathbb{Z})$ and $\eta = \Sigma_\nu a_\nu \xi_\nu$, then

$$\chi(c_1(H)) - \chi(\omega(\xi)) = -\frac{3}{2}\sum_\nu (a_\nu^2 - a_\nu).$$

Therefore, $\chi(H) \leqslant 0$ unless $c_1(H) = \omega(\varepsilon_1\xi_1 + \cdots + \varepsilon_b\xi_b)$.

Now assume that $S \subset Z$ is a surface and $(S \cdot F) = 1$ for twistor lines. Further, assume $[S] = \omega(\xi) + \eta$, $\eta = \Sigma_\nu a_\nu \xi_\nu$. From Section 1 we know that S is smooth, irreducible, $\chi(\mathcal{O}_S) = 1$, $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$, and because of $b_1(M) + b_2^-(M) = 0$ it contains exactly one twistor line F and $F = S \cdot \bar{S}$. Furthermore,

$$(K_S^2) = 2e(M) - 3\,\mathrm{sgn}(M) + 5 = 9 - b.$$

Furthermore, from Section 1, corollary to Lemma 3 it follows that the linear system $|F|_S$ defines a birational morphism

$$S \to \mathbb{P}^2.$$

It remains to show that $[S]=\omega(\xi)$ for some ξ. We fix ξ and write

$$[S] = \omega(\xi) + \sum_{\nu} a_\nu \xi_\nu = \omega(\xi) + \eta.$$

By the adjunction formula

$$K_S = K_Z(S) \otimes \mathcal{O}_S$$

hence

$$\begin{aligned}(K_S^2) &= (-3\omega(\xi) - 2\xi + \eta)^2(\omega)(\xi) + \eta) \\ &= 9 - b + 5\sum_{\nu} a_\nu(a_\nu - 1) \\ &= 9 - b\end{aligned}$$

and, therefore, $a_\nu = 0$ or 1,

$$[S] = \omega\left(\sum_{\nu} \varepsilon_\nu \xi_\nu\right) \quad \varepsilon_\nu = 1 \quad \text{or } -1.$$

The assertion $|H| \neq \varnothing$ if $b \leqslant 3$ and $c_1(H)=\omega(\xi)$ follows because of $\chi(H) > 0$ and

$$h^2(H) = h^1 K_Z \otimes H^{-1}) = h^1((K_Z \otimes H^{\otimes 4}) \otimes H^{-\otimes 5}) = 0$$

since $K_Z \otimes H^{\otimes 4}$ is an instanton bundle (Theorem 1).

3. The case $b = 3$

We keep the assumptions and notations of the previous section and assume furthermore $b=b_2(M)=3$.

By ξ we will always denote an integral cohomology class of $H^2(M,\mathbb{Z})$ with $(\xi^2)=3$ on M. Each such class decomposes uniquely into $\xi=\xi_1+\xi_2+\xi_3$, where (ξ_1,ξ_2,ξ_3) is an orthogonal basis of $H^2(M,\mathbb{Z})$. We have 8 such classes and correspondingly 8 distinguished holomorphic line bundles $H(\xi)$ characterized by the property

$$2c_1(H(\xi)) + \xi = c_1(L) = c_1(T(Z/M)), \tag{1}$$

where L denotes the uniquely determined holomorphic line bundle with

$c_1(L) = c_1(T(Z/M))$. If we fix one ξ, then

$$\omega = c_1(H(\xi)),\ \xi_1,\ \xi_2,\ \xi_3$$

is a basis of $\mathrm{Pic}(Z) = H^2(Z, \mathbb{Z})$ and the relations in the cohomology ring on Z are

$$\omega^2 + \omega\xi + \xi_1^2 = 0 \tag{2}$$

$$\xi_1^2 = \xi_2^2 = \xi_3^2 (= -[F]) \tag{3}$$

We note the following non-zero products

$$(\omega^3) = -2, \qquad (\omega^2 \xi_j) = 1, \qquad (\omega \xi_j^2) = -1. \tag{4}$$

The linear systems $|H(\xi)|$ are not empty and the surfaces $S \in |H(\xi)|$ are smooth connected rational surfaces containing exactly 1 twistor line $F(S) = F(\bar{S}) = S \cdot \bar{S}$. The linear system $|F(S)|$ on S defines a morphism $\sigma: S \to \mathbb{P}^2$ which is a composition of 3 blowing up. Therefore we have 3 exceptional effective divisors $\Gamma_j(S)$ on S ($j = 1, 2, 3$) which are contracted under σ and satisfy

$$(\Gamma_j(S) \cdot \Gamma_k(S)) = -\delta_{jk}, \qquad (\Gamma_j(S) \cdot F(S)) = 0.$$

LEMMA 1

(i) *The restriction map* $i^*: \mathrm{Pic}(Z) \to \mathrm{Pic}(S)$ *induced by the embedding* $i: S \to Z$ *is an isomorphism. The transfer map* $i_*: \mathrm{Pic}(S) \to H^4(Z, \mathbb{Z})$ *is an isomorphism. The divisors* $\Gamma_j(S)$ *represent the classes* $i^*(\xi_\nu)$, *and the curve* $F(S)$ *represents the class* $i^*(\omega + \xi)$.

(ii) *If* $S' \in |H(\xi')|$ *and*

$$\xi - \xi' = 2 \sum_{j=1}^{3} \delta_j \xi_j, \qquad \delta_j = 0 \quad or\ 1,$$

then, for smooth curves $C/ \subset S \cap S'$, *the self-intersection numbers satisfy*

$$(C^2)_{S'} = (C^2)_S - \sum \delta_j (C \cdot \Gamma_j(S))_S \tag{5}$$

(*if* $\Gamma_j(S)$ *represents the class* $i^*(\xi_j)$).

Proof. Using (3), (4) we see that $(i^*(\omega + \xi), i^*(\xi_1), i^*(\xi_2), i^*(\xi_3))$ is an orthogonal basis of $\mathrm{Pic}(S)$ and using the adjunction formula for K_S and (1), (3), (4) we see $(c_1(K_S) \cdot i^*(\xi_j)) = -1$, hence the classes $i^*(\xi_j)$ are represented by the divisors $\Gamma_j(S)$ (using $i^*(\xi_j) \cdot F = 0$). The proof of (5) follows from the projection formula and the standard exact sequence for normal bundles

$$0 \to N_{C/S} \to N_{C/Z} \to \mathcal{O}_Z(S) \otimes \mathcal{O}_C \to 0$$

and correspondingly for $C \subset S'$.

LEMMA 2. *Let S be a surface of degree* 1 *then either* $\dim|S| = 0$ *or* $\dim|S| = 1$ *and the base curve B of the pencil $|S|$ satisfies*

(i) *B is a (-2)-curve on S, $B \notin \bar{S}$ and $(B \cdot \bar{S}) = 1$,*
(ii) *B is a (-3)-curve on exactly three surfaces S_1, S_2, $S_3 \subset Z$ of degree* 1, $\dim|S_j| = 0$ *and* $B \cap \bar{S}_j = \emptyset$,
(iii) $\dim|L| = 3$.

Proof. Assume $\dim|S| \geqslant 1$ and choose $\xi = \xi_1 + \xi_2 + \xi_3$ such that $|S| = |H(\xi)|$. If $S' \in |S|$, $S' \neq S$ and $B = S \cdot S'$ then

$$(B \cdot F(S))_S = (S \cdot S' \cdot \bar{S}) = 1, \qquad (B \cdot \Gamma_j(S))_S = (\omega^2 \cdot \xi_j) = 1.$$

The relation $(B \cdot S) = (S^3) = -2$ implies $B \notin \bar{S}$. If $S_j \in |H(\xi - 2\xi_j)|$ we have

$$(B \cdot S_j) = (B \cdot S) + (B \cdot \xi_j) = -1$$

hence $B \subset S_j$ and $(B^2)_{S_j} = (B)_S - (B \cdot \Gamma_j(S))_S = -3$ (by formula (5)).

Since for each surface T of degree 1 on Z the map $T \to \mathbb{P}^2$ associated to the linear system $|F(T)|$ is a composition of 3-blowing up there cannot exist both, a (-3)-curve and a (-2)-curve B satisfying $(B \cdot F(T))_T = 1$. Hence, $\dim|S_j| = 0$ (and therefore also $\dim|\bar{S}_j| = 0$) and S does not contain (-3)-curves. From $(B \cdot F(S_j))_{S_j} = 0$ we infer

$$B \cap F(S_j) = B \cap \bar{S}_j = \emptyset.$$

For a surface T of degree 1 we have $T + \bar{T} \in |L|$ and by the adjunction formula

$$\begin{aligned} K_T &\cong L^{-\otimes 2} \otimes \mathcal{O}(T) \otimes \mathcal{O}_T \\ &\simeq (L^{-1} \otimes \mathcal{O}_T) \otimes \mathcal{O}_T(-F(T)) \end{aligned}$$

i.e.

$$L \otimes \mathcal{O}_T \simeq K_T^{-1} \otimes \mathcal{O}_T(-F(T))$$

$$h^2(L \otimes \mathcal{O}_T) = 0, \qquad h^0(L \otimes \mathcal{O}_T) = 3$$

$$h^1(L \otimes \mathcal{O}_T) = 0.$$

We also get an exact sequence

$$0 \to \mathcal{O}_Z(\bar{T}) \to L \to L \otimes \mathcal{O}_T \to 0.$$

Because of $h^0(\mathcal{O}_Z(\bar{T})) - h^1(\mathcal{O}_Z(\bar{T})) = 1$ we infer (for $T = S_j$ or $\dim |T| = 0$)

$$\dim |L| = 3, \qquad H^1(Z, L) = 0$$

and consequently (for $T = \bar{S}$)

$$\dim |S| = 1.$$

LEMMA 3. *If a surface of degree* 1 *on* Z *contains a* (-3)*-curve there exist also a surface* $S \subset Z$ *of degree* 1 *satisfying* $\dim |S| = 1$.

Proof. Choose $\xi' = \xi'_1 + \xi'_2 + \xi'_3$ such that $S' \in |H(\xi')|$ contains a (-3)-curve B. It has to be of the form

$$B = \Gamma_1(S') - \Gamma_2(S') - \Gamma_3(S')$$

(for suitable ordering of ξ'_1, ξ'_2, ξ'_3).

If $S \in |H(\xi' - 2\xi'_1)|$, formula (4) yields $(B \cdot S) = -2$, hence $B \subset S$. By (5) we have

$$(B^2)_S = (B^2)_{S'} - (B \cdot \Gamma_1(S'))_{S'} = -2$$

and by the projection formula and formula (4)

$$(B \cdot F(S))_S = (B \cdot \bar{S}) = 1.$$

Thus,

$$B \in |F(S) - \Gamma_1(S) - \Gamma_2(S) - \Gamma_3(S)|_S$$

and

$$\mathcal{O}_S(B) \simeq \mathcal{O}_Z(S) \otimes \mathcal{O}_S, \ \dim |\mathcal{O}_Z(S)| = 1.$$

COROLLARY 1. $\dim |L| = 3$ *and the base locus of the linear system* $|L|$ *is empty or the disjoint union* $B \cup \bar{B}$, *where* B *is a smooth rational curve.*

COROLLARY 2. *If* $Bs|L| = B \cup \bar{B}$ *there exists a distinguished class* $\pm\xi = \pm(\xi_1 + \xi_2 + \xi_3)$ *such that* $\dim |H(\pm\xi)| = 1$. *If* $S, S' \in (H(\xi))$ *then* $B = S \cdot S'$. *The isomorphism* $H(\xi) \otimes H(-\xi) \simeq L$ *induces an isomorphism*

$$H^0(Z, H(\xi)) \otimes H^0(Z, H(-\xi)) \simeq H^0(Z, L).$$

Proof of Corollaries 1 and 2

If no surface S of degree 1 contains (-3)-curves it follows $\dim|L|=3$ and $H^1(Z, \mathcal{O}_Z(S))=0$ (by Lemma 2). Therefore $|L|\cdot S = |L\otimes\mathcal{O}_S|$ which is base point free. Since $S+\bar{S}\in|L|$ it follows that $|L|$ is base point free. If there exist (-3)-curves we apply Lemma 3 and then Lemma 2 and only the last assertion of Corollary 2 remains to be checked. With notations from Lemma 2 consider the exact sequence

$$0\to\mathcal{O}_Z(-S)\to\mathcal{O}_Z\otimes H^0(\mathcal{O}_Z(S))\to I_B(S)\to 0$$

where $I_B\subset\mathcal{O}_Z$ is the sheaf of ideals corresponding to $B\subset Z$. Tensoring with $\bar{S}$ yields the exact sequence

$$0\to\mathcal{O}_Z(\bar{S}-S)\to\mathcal{O}_Z(\bar{S})\otimes H^0(\mathcal{O}_Z(S))\to L\otimes I_B\to 0$$

and therefore

$$H^0(Z,\ H(-\xi))\otimes H^0(Z,\ H(\xi))\subseteq H^0(Z,\ I_B\otimes L)\subseteq H^0(Z)$$

comparing dimensions we get equality and a surjection

$$\mathcal{O}_Z\otimes H^0(Z,L)\to I_B I_{\bar{B}}\otimes L.$$

COROLLARY 3. *With the notations of Corollary* 2

$$|H(\xi)|\cdot S_j = B + |F(S_j) - \Gamma_j(S_j)|$$
$$|H(-\xi)|\cdot S_j = |F(S_j) - \Gamma_j(S_j)|$$

and for $S\in|H(\xi)|$ *there holds*

$$S\cdot S_j = B + \Gamma_j(S)$$
$$S_j\cdot S_k - B = \Gamma_j(S_k)+\Gamma_k(S_j)\subset|F(S_j)-\Gamma_j(S_j)|+\Gamma_k(S_j).$$

Proof. Since $\dim|H(\xi)|\cdot S_j=\dim|F(S_j)-\Gamma_j(S_j)|=1$ and B is a fixed component of $|H(\xi)|\cdot S_j$, we obtain

$$|H(\xi)|\cdot S_j = B + |F(S_j) - \Gamma_j(S_j)|$$

since both sides represent the same element in $\mathrm{Pic}(S_j)$. In the same way one proves

$$|H(-\xi)|\cdot S_j = |F(S_j) - \Gamma_j(S_j)|$$

and

$$S_j \cdot S_k - B \in |F(S_j) - \Gamma_j(S_j) + \Gamma_k(S_j)|.$$

The divisors $S \cdot S_j - B$ are effective divisors on S representing the cohomology classes $i^*(\xi_j)$, hence $S \cdot S_j - B = \Gamma_j(S)$. Because of

$$((F(S_j) - \Gamma_j(S_j) + \Gamma_k(S_j)) \cdot \Gamma_k(S_j)) = -1$$

it follows

$$|F(S_j) - \Gamma_j(S_j) + \Gamma_k(S_j)| = |F(S_j) - \Gamma_j(S_j)| + \Gamma_k(S_j)$$
$$S_j \cdot S_k - B = F_0 + \Gamma_k(S_j) \quad \text{with } F_0 \in |F(S_j) - \Gamma_j(S_j)|.$$

F_0 is also a divisor on S_k and satisfies

$$\begin{aligned}(F_0 \cdot F(S_k)) &= (F_0 \cdot \bar{S}_k)\\ &= 1 + \xi_j(\omega + \xi_j)(\omega + \xi_i + \xi_j)\\ &= 0\\ (F_0 \cdot \Gamma_k(S_k)) &= -(F_0 \cdot \xi_k) = -(\omega + \xi_j)(\omega + \xi_i + \xi_k)\xi_k\\ &= 0\\ (F_0 \cdot \Gamma_j(S_k)) &= (F_0 \cdot \xi_j)\\ &= -1\end{aligned}$$

$$(F_0 \cdot \Gamma_i(S_k)) = (F_0 \cdot \xi_i) = 0$$

hence

$$S_j \cdot S_k - B = \Gamma_j(S_k) + \Gamma_k(S_j).$$

Now we can prove

THEOREM 3. *Let Z be the twistor space of a 4-manifold M with positive scalar curvature and signature 3 such that the linear system $|L|$ has base points. Then $Bs|L|$ is a disjoint union of smooth rational curves B, $\bar{B}$. If $\hat{\sigma}: \hat{Z} \to Z$ is the blowing up of Z along B, $\bar{B}$ and $E = \hat{\sigma}^{-1}(B)$, $\bar{E} = \hat{\sigma}^{-1}(\bar{B})$, then the linear system $|\hat{\sigma}^* L(-E - \bar{E})|$ defines a morphism*

$$\hat{Z} \xrightarrow{\pi} Q = \mathbb{P}^1 \times \mathbb{P}^1$$

equivariant with respect to the given antiholomorphic involution on $\hat{Z}$ and the involution

$$\tau_Q: (x_0: x_1, y_0: y_1) \mapsto (\bar{y}_0: \bar{y}_1, \bar{x}_0: \bar{x}_1) \quad \textit{on } \mathbb{P}^1 \times \mathbb{P}^1.$$

The surfaces E and $\bar{E}$ are disjoint sections of π. $\hat{Z}$ is a small resolution of a conic bundle with isolated singularities

$$\hat{Z}_0 \subset \mathbb{P}_Q(\mathscr{E}), \; \mathscr{E} = \mathcal{O}_Q z_0 \oplus \mathcal{O}_Q(-1, -2)z_1 \oplus \mathcal{O}_Q(-2, -1)z_2$$

which is defined by an equation

$$\begin{aligned} \phi \in H^0(\mathbb{P}_Q(\mathscr{E}), \quad & \mathcal{O}_{\mathbb{P}_Q(\mathscr{E})}(2) \otimes \pi^* \mathscr{Q}_Q(3,3)) \\ = H^0(Q, (S^2\mathscr{E})(3,3)) & \end{aligned}$$

$$\phi = \varphi z_0^2 + 2z_1 z_2$$

$\varphi \in H^0(Q, \mathcal{O}_Q(3,3))$, real with respect to the involution τ_Q. The sections E, $\bar{E}$ are defined by $z_0 = z_1 = 0$ and $z_0 = z_2 = 0$. The antiholomorphic involution on $\hat{Z}$ induces the antiholomorphic involution on $\hat{Z}_0$ given by

$$(q, z_0, z_1, z_2) \mapsto (\tau_Q(q), \bar{z}_0, \bar{z}_2, \bar{z}_1).$$

Proof. We determine $\pm\xi$ as in Corollary 2 then $|\hat{\sigma}^* H(\xi)(-E)|$, $|\hat{\sigma}^* H(-\xi)(-\bar{E})|$ are base point free pencils on $\hat{Z}$ and define a morphism $\pi: \hat{Z} \to Q = \mathbb{P}^1 \times \mathbb{P}^1$. Together with the Segre-embedding $Q \subset \mathbb{P}^3$ this is the morphism defined by $|\hat{\sigma}^* L(-E-\bar{E})|$. There exists an antilinear isomorphism

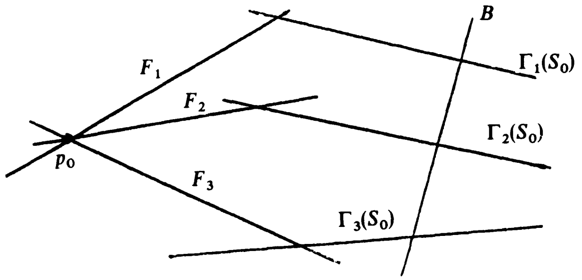
$$\psi: H^0(Z, H(\xi)) \xrightarrow{\sim} H^0(Z, H(-\xi))$$

such that $\operatorname{div}(\psi(\lambda)) = \overline{\operatorname{div}(\lambda)}$ holds for sections λ of $H(\xi)$. This isomorphism is unique up to a constant and induces the real structure τ_Q on $\mathbb{P}^1 \times \mathbb{P}^1$. If $S_0 \in |H(\xi)|$, then $\bar{B}$ intersects S_0 in one point p_0 transversally and $\Lambda_0 = |H(-\xi)| \cdot S_0 \subset |F(S_0)|_{S_0}$ corresponds to the pencil of lines through $\sigma(p_0) \in \mathbb{P}^2$ under the morphism $\sigma: S_0 \to \mathbb{P}^2$. The strict transform $\hat{S}_0 \subset \hat{Z}$ of S_0 is the blowing up of S_0 in the point p_0. Similarly, for $\bar{S} \in |H(-\xi)|$ the strict transform $\hat{\bar{S}} \subset \hat{Z}$ of $\bar{S}$ is the blowing up of $\bar{S}$ in the point $B \cap \bar{S}$. The fibres of π are the curves $\hat{S}_0 \cdot \hat{\bar{S}}$. If $\bar{S}$ moves, they sweep out a pencil $\hat{\Lambda}_0$ on $\hat{S}_0$ which is the strict transform of the pencil Λ_0. In general, $\sigma: S_0 \to \mathbb{P}^2$ is the blowing up of 3 distinct colinear points on a line in $\mathbb{P}^2$ and B is the strict transform of this line under σ. There are at most 6 surfaces $S_0 \in |(H(\xi)|$, where 2 or 3 of these centres are infinitely near (Corollary 3). Namely, if S moves in $|H(\xi)|$, we obtain a pencil

$|H(\xi)| \cdot S_j - B = \Gamma_j(S)$ on S_j which is $|F(S_j) - \Gamma_j(S_j)|$. If the centres of $\sigma: S \to \mathbb{P}^2$ are infinitely near, there exist $j \neq k$ and $\Gamma_j(S) \subset \Gamma_k(S)$, i.e. $\Gamma_j(S) \subset S_j \cdot S_k - B$, hence $\Gamma_j(S) = \Gamma_j(S_k)$ by Corollary 2. Therefore the divisors $\Gamma_i(S)$, $i = 1, 2, 3$, are (-1)-curves on S if $\Gamma_j(S) \neq \Gamma_j(S_k)$ for each pair (j, k), $j \neq k$. Thus, if $S_0 \in |H(\xi)|$ is general in this sense, the pencil $|H(-\xi)| \cdot S_0$ contains exactly three divisors, which split into

$$F = F_j + \Gamma_j(S)$$

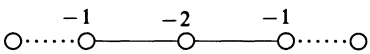
as illustrated in the following picture.



The dual graph of these divisors on $\hat{S}_0$ (with selfintersection numbers on $\hat{S}_0$ *indicated*) is

For the remaining at most 6 surfaces $\hat{S}_0$ we have one of the following degenerate divisors:

(a) 

and 

(b) 

The curve $B = E \cap \hat{S}_0$ respectively $\bar{B} = \bar{E} \cap \hat{S}^0$ intersects these curves as indicated by the dotted edges. The line bundle $\mathcal{M} = \mathcal{O}_Z(E + \bar{E})$ on $\hat{Z}$ restricted to the fibres of π has degree 2, is generated by its global sections and has trivial higher cohomology. Therefore, if

$$\mathcal{E} = \pi_* \mathcal{M},$$

we get a vector bundle of rank 3 on Q and a Q-morphism

$$\hat{Z} \to \mathbb{P}_Q(\mathcal{E}) \xrightarrow{\pi_0} Q$$

which maps the fibres of π onto conics, isomorphically except for the fibres with 3 or 4 components for which the (-2)-curves are contracted. Therefore, the image $\hat{Z}_0 \subset \mathbb{P}_Q(\mathscr{E})$ of $\hat{Z}$ is a conic bundle with isolated singularities and $\hat{Z} \to \hat{Z}_0$ is a small resolution. Now consider the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{j} & \hat{Z} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\hat{\sigma}} \\ B & \xrightarrow{i} & Z \end{array} \tag{*}$$

By standard arguments for blowing up we obtain

$$N^*_{B/S} = \mathcal{O}_B(2)^{\oplus 2}$$
$$E \cong B \times \mathbb{P}^1$$

and the relatively tautological bundle of $E = \mathbb{P}_B(N^*_{B/Z})$ is

$$\mathcal{O}_E(1) \simeq \mathcal{O}_E(2E_0 + B_0) = \mathcal{O}_E(2, 1),$$

where E_0 (resp. B_0) is a fibre of the projection $E \xrightarrow{\rho} B$ (respectively $E \to \mathbb{P}^1$), hence

$$\mathcal{O}_E \otimes \mathcal{O}_{\hat{Z}}(E) \simeq \mathcal{O}_E(-2, \ -1) \tag{6}$$

From the diagram (*) we infer

$$\mathcal{O}_E \otimes \sigma^* H(\xi) \simeq \rho^*(\mathcal{O}_B \otimes H(\xi)) \simeq \mathcal{O}_E(-2, 0) \tag{7}$$

$$\mathcal{O}_E \otimes \sigma^* H(-\xi) \simeq \rho^*(\mathcal{O}_B \otimes H(-\xi)) \simeq \mathcal{O}_E(1, 0) \tag{8}$$

$$\mathcal{O}_E \otimes \mathcal{O}_{\hat{Z}}(\bar{E}) \simeq \mathcal{O}_E \tag{9}$$

Thus,

$$\mathcal{O}_E \otimes \hat{\sigma}^* H(\xi)(-E) \cong \mathcal{O}_E(0, 1)$$
$$\mathcal{O}_E \otimes \hat{\sigma}^* H(-\xi)(-\bar{E}) \simeq \mathcal{O}_E(1, 0),$$

which yields the following commutative diagram

$$\begin{array}{ccc} E & \overset{\pi}{\twoheadrightarrow} & Q = \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle pr_2} \\ B & \overset{\sim}{\to} & \mathbb{P}^1 \end{array}$$

In the same way we get a corresponding diagram for $\bar{B}$

$$\begin{array}{ccc} \bar{E} & \overset{\pi}{\twoheadrightarrow} & Q=\mathbb{P}^1\times\mathbb{P}^1 \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle pr_1} \\ \bar{B} & \overset{\sim}{\rightarrow} & \mathbb{P}^1 \end{array}$$

We also obtain an exact sequence

$$0\rightarrow \mathcal{O}_{\hat{Z}}\rightarrow \mathcal{M}\rightarrow \begin{array}{c} \mathcal{O}_E\otimes\mathcal{O}_{\hat{Z}}(E) \\ \oplus \\ \mathcal{O}_{\bar{E}}\otimes\mathcal{O}_{\hat{Z}}(\bar{E}) \end{array}\rightarrow 0$$

and therefore, using (6), we infer an exact sequence

$$0\rightarrow\mathcal{O}_Q\rightarrow\mathscr{E}\rightarrow\mathcal{O}_Q(-2,\ -1)\oplus\mathcal{O}_Q(-1,\ -2)\rightarrow 0.$$

Choosing non-zero sections z_0 of $\mathscr{E}$, z_1 of $\mathscr{E}(2,1)$, z_2 of $\mathscr{E}(1,2)$ we can write

$$\mathscr{E}\cong\mathcal{O}z_0\oplus\mathcal{O}(-2,\ -1)z_1\oplus\mathcal{O}(-1,\ -2)z_2$$

for a suitable choice of z_1, z_2.

The bundle $\hat{Z}_0\subset\mathbb{P}_Q(\mathscr{E})=:P$ is defined by section ϕ of a line bundle $\mathcal{O}_P(2)\otimes\pi_0^*\mathcal{N}$ on P. By the adjunction formula

$$\begin{aligned} K_{\hat{Z}} &\simeq \sigma_0^*(K_P\otimes\mathcal{O}_P(2)\otimes\pi_0^*\mathcal{N}) \\ &\simeq \sigma_0^*K_P\otimes\mathcal{M}^{\otimes 2}\otimes\pi^*\mathcal{N} \end{aligned}$$

and since

$$K_P\simeq\mathcal{O}_P(-3)\otimes\pi_0^*(\det\mathscr{E}\otimes K_Q),$$

we obtain

$$K_{\hat{Z}}\simeq\mathcal{M}^{-1}\otimes\pi^*\mathcal{N}(-5,\ -5)$$

outside a set of codimension $\geqslant 2$. On the other hand,

$$\begin{aligned} K_{\hat{Z}} &\simeq \hat{\sigma}^*K_Z\otimes\mathcal{M} \\ &\simeq \hat{\sigma}^*L^{-\otimes 2}\otimes\mathcal{M}\simeq\pi^*\mathcal{O}_Q(-2,\ -2)\otimes\mathcal{M}^{-1} \end{aligned}$$

consequently

$$\pi^*\mathcal{N} \simeq \pi^*\mathcal{O}_Q(3, 3)$$
$$\mathcal{N} \simeq \mathcal{O}_Q(3, 3)$$
$$\phi \in H^0(P, \mathcal{O}_P(2) \otimes \pi_0^*\mathcal{N}) = H^0(Q, (S^2\mathscr{E})(3, 3))$$
$$\phi = \varphi z_0^2 + 2\varphi_{01}z_0z_1 + 2\varphi_{02}z_0z_2 + 2z_1z_2.$$

If we change the splitting of $\mathscr{E}$ by

$$z_1 \mapsto z_1 - \varphi_{02}z_0, \qquad z_2 \mapsto z_2 - \varphi_{01}z_2$$

the equation has the form

$$\phi = \varphi z_0^2 + 2z_1z_2$$
$$\varphi \in H^0(\mathcal{O}_Q(3, 3)).$$

This form φ defines a reduced curve Δ on Q, which is the discriminant of the conic bundle. The map $\hat{Z}_0^{\text{sing}} \to Q$ provides a bijection $\hat{Z}_0^{\text{sing}} \to \Delta^{\text{sing}}$. The sections $E, \bar{E}$ are defined by $z_0 = z_1 = 0$ respectively $z_0 = z_2 = 0$.

REMARK. We shall discuss varieties obtained by this construction in the forthcoming paper [K4].

4. Double solids

We continue to study the situation of Section 3 under the additional assumption that the linear system $|L|$ has no base points. It defines a morphism

$$\phi : Z \to \mathbb{P}^3$$

of degree 2 that is real with respect to the standard real structure on $\mathbb{P}^3$ (because of $\tau^*\bar{L} \simeq L$). For surfaces $S \subset Z$ of degree 1 we know

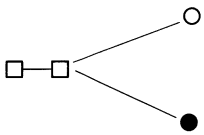
$$|L| \cdot S = |L \otimes \mathcal{O}_S|$$

and $(S \cdot L \cdot L) = 1$. Therefore, ϕ maps S birationally onto a plane $H(S) = H \subset \mathbb{P}^3$ and because of $S + \bar{S} \in |L| = |\phi^{-1}(H)|$ we infer

$$\phi^{-1}(H) = S + \bar{S}.$$

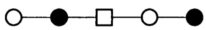
In particular, H is a real plane. Since $(F(S)\cdot L)=2$, the twistor fibre $F(S)$ is mapped isomorphically onto a conic $C(S)\subset H(S)$ which has no real points. The map $\phi: S\to H$ is a composition of 3 blowing up and since $(F(S)^2)_S=1$, the following dual graphs of curves with negative self-intersections are possible:

(i) (only infinitely near centres)



(a dot denotes a smooth rational curve contracted under $|F|$, a small circle indicates a smooth rational curve contracted under $|L\otimes \mathcal{O}_S|$, and a small square denotes a smooth rational curve contracted under both, $|F|$ and $|L\otimes \mathcal{O}_S|$),

(ii) (2 infinitely near centres)



(iii) (del Pezzo surfaces)

The surfaces S and $\bar{S}$ are always of the same type. If $S\in|H(\xi)|$, an orthogonal basis of exceptional divisor classes with respect to $|L\otimes\mathcal{O}_S|$ is represented by the divisors

$$D_j(S) = S\cdot S_j, \qquad S_j\in|H(\xi-2\xi_j)|$$

since

$$(L\cdot S\cdot S_j)=0, \qquad (D_j(S)\cdot D_k(S))_S = -\delta_{jk}.$$

Since $(F(S)\cdot D_j(S))=1$, the curves $D_j(S)$, $D_j(\bar{S})$ are both contracted under ϕ to the points P, $\bar{P}$, which are the intersection of the real line $H(S)\cdot H(S_j)$ in $\mathbb{P}^3$ with the conic $C(S)$.

LEMMA 1. *If $Z \xrightarrow{\alpha} Z_0 \xrightarrow{\phi_0} \mathbb{P}^3$ is the Stein factorization of ϕ, then Z_0 is a normal algebraic 3-fold with isolated singularities and $Z\xrightarrow{\alpha} Z_0$ is a small resolution.*

Proof. We have to show that ϕ does not contract surfaces $E\subset Z$. This follows because of

$$0<(E\cdot L\cdot L) = (E\cdot\phi^*\mathcal{O}(1)\cdot\phi^*\mathcal{O}(1)) = (\phi_*(E)\cdot\mathcal{O}(1)\cdot\mathcal{O}(1)).$$

LEMMA 2. *The 4 planes H_0, H_1, H_2, H_3 determined by the surfaces of degree* 1 *are the only planes H for which $\phi^{-1}(H)$ splits.*

Proof. $\phi^{-1}(H)$ can only split into surfaces of degree 1.

LEMMA 3. *Only the following combinations of types of surfaces of degree* 1 *are possible*

(a) 2 *pairs* $\{S_0, \bar{S}_0\}$, $\{S_1, \bar{S}_1\}$ *of type* (i) *and* 2 *pairs* $\{S_2, \bar{S}_2\}$, $\{S_3, \bar{S}_3\}$ *of type* (ii). *In this case* $H_j \cap H_k \cap C(S_j) = \{P_{01}, \bar{P}_{01}\}$ *for each* $j \neq k$,

$$\phi^{-1}(P_{01}) = S_0 \cdot S_1 = E_1 + E_2 + E_3 \supset S_0 \cdot S_2 = E_2 + E_3 \supset S_0 \cdot S_3 = E_3$$

(*and conjugate for* $\bar{P}_{01}$). *We have*

$$E_1 \subseteq S_0 \cap S_1 \cap \bar{S}_2 \cap \bar{S}_3$$

$$E_2 \subseteq S_0 \cap S_1 \cap S_2 \cap \bar{S}_3$$

$$E_3 \subseteq S_0 \cap S_1 \cap S_2 \cap S_3$$

(b) 2 *pairs* $\{S_0, \bar{S}_0\}$, $\{S_1, \bar{S}_1\}$ *of type* (ii) *and* 2 *pairs* $\{S_2, \bar{S}_2\}$, $\{S_3, \bar{S}_3\}$ *of type* (iii). *In this case*

$$H_j \cap H_k \cap C(S_j) = \{P_{01}, \bar{P}_{01}\} \quad \text{for } \{j, k\} = \{0, 1\} \quad \text{or } \{0, 2\} \quad \text{or } \{1, 2\}$$

$$H_j \cap H_k \cap C(S_j) = \{P_{jk}, \bar{P}_{jk}\} \quad \text{for } \{j, k\} = \{0, 3\} \quad \text{or } \{1, 3\} \quad \text{or } \{2, 3\},$$

$$\phi^{-1}(P_{01}) = E_{01} + E_{02}, \qquad \phi^{-1}(P_{03}) = E_{03}$$

$$\phi^{-1}(P_{13}) = E_{13}, \qquad \phi^{-1}(P_{23}) = E_{23}$$

$$E_{01} \subset S_0 \cap S_1 \cap \bar{S}_2, \qquad E_{02} \subset S_0 \cap S_1 \cap S_2$$

$$E_{jk} \subset S_j \cap \bar{S}_k \quad \text{for } (j, k) = (1, 3), (2, 3) \quad \text{and} \quad E_{03} \subset S_0 \cap S_3.$$

(c) *All pairs of type* (iii) (*del Pezzo surfaces*). *In this case all pairs*

$$\{P_{jk}, \bar{P}_{jk}\} = H_j \cap H_k \cap C(S_j) \quad (j \neq k)$$

are distinct and

$$\phi^{-1}(P_{jk}) = E_{jk} \subset S_0 \cap S_k \quad \text{for } j = 0$$

$$\text{resp. } E_{jk} \subset S_j \cap \bar{S}_k \quad \text{for } 0 < j < k.$$

In each of the cases (a), (b), (c) *the points P_{jk}, $\bar{P}_{jk}$ are the only points on $H_0 \cup H_1 \cup H_2 \cup H_3$ where ϕ is not finite and the indicated inclusions between curves and surfaces cover all possible inclusions. The E_{jk} and E_j are irreducible exceptional curves on some S_i, contracted under $|L \otimes \mathcal{O}_{S_i}|$.*

Proof. We fix ξ as usually such that $S_0 \in |H(\xi)|$ is of type with smallest occurring number. We choose $S_j \in |H(\xi - 2\xi_j)|$, $j = 1, 2, 3$, in a suitable ordering and denote $D_j = S_0 \cdot S_j$, $\omega = c_1(H(\xi))$. If S_0 is of type (i) the situation on this surface is as follows:

$$D_1 \supset D_2 \supset D_3$$

$$D_1 = E_1 + E_2 + E_3, \quad D_2 = E_2 + E_3, \quad D_3 = E_3.$$

The exceptional divisors on S_1 are

$$S_1 \cdot S_0 = D_1, \qquad S_1 \cdot \bar{S}_2, \qquad S_1 \cdot \bar{S}_3.$$

Since $E_2 \subset S_2$, $E_3 \subset S_2 \cdot S_3$ it follows that $E_2 \not\subseteq \bar{S}_2$, $E_3 \not\subseteq \bar{S}_2$, $E_3 \not\subseteq \bar{S}_3$ and consequently

$$S_1 \cdot \bar{S}_2 = E_1, \qquad S_1 \cdot \bar{S}_3 = E_1 + E_2.$$

Therefore $E_2 \subset S_2 \cdot \bar{S}_3$, but $E_1 \not\subset S_2 \cdot \bar{S}_3$ (since $E_1 \subseteq S_1$, but $E_1 \not\subseteq S_1 \cdot S_2$). The exceptional divisors on S_2 are then

$$S_2 \cdot S_0 = E_2 + E_3, \quad S_2 \cdot \bar{S}_1 = \bar{E}_1, \quad S_2 \cdot \bar{S}_3 = E_2$$

and on S_3

$$S_3 \cdot S_0 = E_3, \quad S_3 \cdot \bar{S}_1 = \bar{E}_1 + \bar{E}_2, \quad S_3 \cdot \bar{S}_2 = \bar{E}_2.$$

Hence, $S_0, S_1, \bar{S}_0, \bar{S}_1$ are of type (i), $S_2, S_3, \bar{S}_2, \bar{S}_3$ of type (ii) and there are 2 points $P_{01}, \bar{P}_{01}$ on $H_0 \cap H_1 \cap H_2 \cap H_3 \cap C(S_j)$ such that

$$\phi^{-1}(P_{01}) = E_1 + E_2 + E_3, \quad \phi^{-1}(\bar{P}_{01}) = \bar{E}_1 + \bar{E}_2 + \bar{E}_3.$$

These are the only points on $H_0 \cup H_1 \cup H_2 \cup H_3$ where ϕ is not finite. The assumption that S_0 is of type (ii) leads to the following:

$$D_1 \supset D_2, \quad D_3 \text{ disjoint to } D_1$$

$$D_1 = E_{01} + E_{02}, \quad D_2 = E_{02}, \quad D_3 = E_{03}.$$

On S_1 we have the exceptional divisors

$$S_1 \cdot S_0 = D_1 = E_{01} + E_{02}, \quad S_1 \cdot \bar{S}_2, \quad S_1 \cdot \bar{S}_3.$$

The cohomology class of $S_1 \cdot \bar{S}_2$ is

$$(\omega + \xi_1)(\omega + \xi_1 + \xi_3) = \omega(\xi_1 - \xi_2)$$

and that E_{01} is

$$\omega(\omega + \xi_1) - \omega(\omega + \xi_2) = \omega(\xi_1 - \xi_2),$$

hence

$$S_1 \cdot S_0 = E_{01} + E_{02}, \quad S_1 \cdot \bar{S}_2 = E_{01}, \quad S_1 \cdot \bar{S}_3 = E_{13}.$$

In a similar way one finds the exceptional divisors on S_2

$$S_2 \cdot S_0 = E_{02}, \quad S_2 \cdot \bar{S}_1 = \bar{E}_{01}, \quad S_2 \cdot \bar{S}_3 = E_{23}$$

and on S_3

$$S_3 \cdot S_0 = E_{03}, \quad S_3 \cdot \bar{S}_1 = \bar{E}_{13}, \quad S_3 \cdot \bar{S}_2 = \bar{E}_{23}.$$

Hence S_0, S_1, $\bar{S}_0$, S_1 are of type (ii), S_2, S_3, $\bar{S}_2$, $\bar{S}_3$ of type (iii). There are 4 pairs of conjugate points

$$P_{01}, \bar{P}_{01} \in H_0 \cap H_1 \cap H_2, \qquad P_{03}, \bar{P}_{03} \in H_0 \cap H_3$$
$$P_{13}, \bar{P}_{13} \in H_1 \cap H_3, \qquad P_{23}, \bar{P}_{23} \in H_2 \cap H_3$$

such that

$$\phi^{-1}(P_{01}) = E_{01} + E_{02}, \qquad \phi^{-1}(\bar{P}_{01}) = \bar{E}_{01} + \bar{E}_{02}$$
$$\phi^{-1}(P_{03}) = E_{03}, \qquad \phi^{-1}(\bar{P}_{03}) = \bar{E}_{03}$$
$$\phi^{-1}(P_{13}) = E_{13}, \qquad \phi^{-1}(\bar{P}_{13}) = \bar{E}_{13}$$
$$\phi^{-1}(P_{23}) = E_{23}, \qquad \phi^{-1}(\bar{P}_{23}) = \bar{E}_{23}$$

and these are the only points on $H_0 \cup H_1 \cup H_2 \cup H_3$ where ϕ is not finite. The remaining case that all surfaces are del Pezzo is obvious.

LEMMA 4. *The ramification locus of ϕ (or ϕ_0) is a normal quartic surface B in $\mathbb{P}^3$.*

Proof. On the complement of the finite number of points on $\mathbb{P}^3$ where ϕ is not finite we get

$$K_Z \otimes \phi^* K_{\mathbb{P}^3}^{-1} \simeq L^{-\otimes 2} \otimes \phi^* \mathcal{O}_{\mathbb{P}^3}(4) \simeq \phi^* \mathcal{O}_{\mathbb{P}^3}(2).$$

It follows that B is a quartic surface with only isolated singularities.

LEMMA 5. $\mathrm{Pic}(Z_0) \simeq H^2(Z_0, \mathbb{Z}) \subseteq H^2(Z, \mathbb{Z})$.

Proof. $\mathrm{Pic}(Z_0) \subset \mathrm{Pic}(Z) \simeq H^2(Z, \mathbb{Z})$ follows from $H^0(Z, \alpha^*E) = H^0(Z_0, E)$ for line bundles E on Z_0. $\mathrm{Pic}(Z_0) \simeq H^2(Z_0, \mathbb{Z})$ comes from the exponential sequence on Z_0 and

$$\begin{aligned} H^q(Z_0, \mathcal{O}_{Z_0}) &= H^q(\mathbb{P}^3, \phi_0 * \mathcal{O}_{Z_0}) \\ &\cong H^q(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2)) \\ &= H^q(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}). \end{aligned}$$

LEMMA 6. *For each singular point $x \in Z_0$ let $E_x \subset Z$ be the fibre over x, $c(x)$ the number of components of E_x and $\mu(x)$ the Milnor number of B at $\phi(x)$. Then the dual graph of E_x is a tree of curves homeomorphic to smooth rational curves and*

$$\sum_x (\mu(x) + c(x)) = 26.$$

Proof. The exact sequence arising from the Cartan–Leray spectral sequence for α

$$0 \to E_2^{10} \to H^1(Z, \mathbb{Z}) \to E_2^{01} \to E_2^{20} \to H^2(Z, \mathbb{Z})$$

and

$$E_2^{p0} = H^p(Z_0, \mathbb{Z}), \qquad E_2^{01} = \bigoplus_x H^1(E_x, \mathbb{Z})$$

entails

$$H^1(Z_0, \mathbb{Z}) = 0, \qquad H^1(E_x, \mathbb{Z}) = 0 \quad \text{(by Lemma 5)}.$$

Therefore, the dual graph of the curve E_x is a tree and each component is a curve homeomorphic (under normalization) to $\mathbb{P}^1$. Thus $e(E_x) = 1 + c(x)$ and the formula is a special case of [Kr] Theorem 4.6.

Now we use these results to determine the possible ramification loci. Recall that $B \subset \mathbb{P}^3$ has the properties

(a) B has only finitely many singular points.
(b) B is defined by a real equation but $B \backslash B^{\mathrm{sing}}$ has no real points (since Z has no real points).
(c) There are 4 distinguished real planes, $H \subset \mathbb{P}^3$. These are the only planes for which $\phi^{-1}(H)$ splits. Each of these planes cuts B along a smooth conic without real point (the image of the twistor line).

Let $F \subset H^0(\mathcal{O}_{\mathbb{P}^3}(4))$ be a real equation of the quartic B which is non-negative in the real points of $H^0(\mathcal{O}_{\mathbb{P}^3}(1))^*$. The double covering Z_0 is defined by the equation

$$y^2 + F = 0$$

as a subvariety of the weighted projective space $\mathbb{P}(1,1,1,1,2)$. For each of the 4 distinguished planes H_0, H_1, H_2, H_3 there exists a unique positive definite real quadratic form $Q_j \subset H^0(H_j, \mathcal{O}_{H_j}(2))$ such that

$$F - Q_j^2 = 0 \quad \text{on } H_j.$$

Then $(Q_i - Q_j)(Q_i + Q_j) = 0$ on $H_i \cap H_j$ and $Q_i + Q_j$ is positive definite on the line $H_i \cap H_j$, hence

$$Q_i = Q_j \quad \text{on } H_i \cap H_j.$$

Therefore there exists a unique real quadratic form $Q \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ such that $Q = Q_j$ on H_j. Then the form $F - Q^2$ vanishes on $H_0 \cup H_1 \cup H_2 \cup H_3$ and we can choose real linear forms $L_0, \ldots, L_3$ defining $H_0, \ldots, H_3$ such that

$$F = Q^2 - L_0 L_1 L_2 L_3. \tag{1}$$

The image of the twistor line on S_j under ϕ is the conic C_j defined by

$$Q = L_j = 0$$

and for $0 \leqslant j < k \leqslant 3$ we get 2 singular points of B, $C_j \cap C_k = \{P_{jk}, \bar{P}_{jk}\}$.

According to Lemma 3 we have to distinguish the cases (a), (b), (c).

Case (a). $H_0 \cap H_1 \cap H_2 \cap H_3$ is a line. Then we can choose coordinates x_0, x_1, x_2, x_3 on $\mathbb{P}^3$ such that

$$L_0 = x_0, \quad L_1 = x_1, \quad L_2 = x_0 + x_1, \quad L_3 = L(x_0, x_1).$$
$$Q = x_2^2 + x_3^2 + Q_0(x_0, x_1)$$
$$F = (x_2^2 + x_3^2 + Q_0(x_0, x_1))^2 - x_0 x_1 (x_0 + x_1) L(x_0, x_1).$$

The points $P_{01}, \bar{P}_{01}$ are $(0:0:\pm i:1)$. In each of these points we can use

$$Z = \frac{x_2^2 + x_3^2 + Q_0}{x_3^2}, \qquad x = \frac{x_0}{x_3}, \qquad y = \frac{x_1}{x_3}$$

as local coordinates. Singularities of this type $z^2 - xy(x+y)(ax+by)$ have the Milnor number $\mu = 9$. Since the inverse image of each of these singularities under ϕ splits into 3 components and $2 \cdot (9+3) = 24$, there must be precisely one more singularity. This has to be an ordinary double point $P_0 \in B \backslash (H_0 \cup H_1 \cup H_2 \cup H_3)$, $P_0 = (1:\alpha:0:0)$, where α is a real double root of the polynomial $Q_0(1,x)^2 - x(x+1)L(1,x)$.

Case (b). $H_0 \cap H_1 \cap H_2$ is a line. Then we can choose coordinates such that

$$L_0 = x_0, \quad L_1 = x_1, \quad L_2 = x_0 + x_1, \quad L_3 = x_2$$
$$Q = x_3^2 + Q_0(x_0, x_1, x_2)$$
$$F = [x_3^2 + Q_0(x_0, x_1, x_2)]^2 - x_0 x_1 x_2 (x_0 + x_1).$$

The singularities P_{01}, $\bar{P}_{01}$ are $(0:0:1:i\sqrt{Q_0(0,0,1)})$

$$P_{03}, \bar{P}_{03}: (0:1:0:\pm i\sqrt{Q_0(0,1,0)})$$
$$P_{13}, \bar{P}_{13}: (1:0:0:\pm i\sqrt{Q_0(1,0,0)})$$
$$P_{23}, \bar{P}_{23}: (1:-1:0:\pm i\sqrt{Q_0(1,-1,0)}).$$

The last 3 pairs are ordinary double points which give Milnor number 1. In P_{01}, $\bar{P}_{01}$ we can use local coordinates

$$x = \frac{x_0}{x_2}, \qquad y = \frac{x_1}{x_2}, \qquad z = \frac{x_3^2 + Q_0(x_0, x_1, x_2)}{x_2^2}.$$

Thus, these are singularities of the type $z^2 - xy(x+y) = 0$ which have the Milnor number 4. The inverse image under ϕ of P_{01}, $\bar{P}_{01}$ splits into 2 components each, and for the remaining 6 singularities it is irreducible. Since $2 \cdot (2+4) + 6 \cdot (1+1) = 24$, there is again exactly one ordinary double point P_0 missing.

Case (c). Each 3 of the 4 hyperplanes are in general position. Then F has the form

$$F = [x_3^2 + Q_0(x_0, x_1, x_2)]^2 - x_0 x_1 x_2 (x_0 + x_1 + x_2)$$

or

$$F = Q^2 - x_0 x_1 x_2 x_3.$$

Here we get 12 ordinary double points defined by

$$P_{jk}, \bar{P}_{jk}: x_j = x_k = 0, \quad Q = 0$$

(respectively $x_j = x_0 + x_1 + x_2 = 0$, $Q = 0$ in the first case, for $k = 3$) and, again, precisely one ordinary double point $P_0 \in B \setminus (H_0 \cup H_1 \cup H_2 \cup H_3)$ is missing. Using geometric arguments one can derive a specific form of equation of the quartic B [cf. [Kr] §5]. However, here we present a shorter way to obtain these equations.

If $F = Q^2 - L_0 L_1 L_2 L_3$ we define linear forms $y_0, y_1, y_2, y_3 \in \mathbb{C}[x_0, x_1, x_2, x_3]$ by

$$L_0 = y_3 - y_0 + y_1 + y_2, \qquad L_1 = y_3 + y_0 - y_1 + y_2$$
$$L_2 = y_3 + y_0 + y_1 - y_2, \qquad L_3 = y_3 - y_0 - y_1 - y_2, \tag{2}$$

where L_0, L_1, L_2, L_3 normalized, such that

$$(L_1 : L_0) = (L_2 : L_0) = (L_3 : L_0) \quad \text{at } P_0.$$

If we define a quadratic form R by $2R = Q + y_0^2 + y_1^2 + y_2^2 - y_3^2$, we get

$$F = 4 \cdot (Ry_3^2 + 2y_0 y_1 y_2 y_3 + R^2 - R(y_0^2 + y_1^2 + y_2^2) + (y_0^2 y_1^2 + y_1^2 y_2^2 + y_0^2 y_1^2)).$$

Now we choose (real) coordinates $z_0, \ldots, z_3$ such that $z_0 = z_1 = z_2 = 0$ in P_0 and $z_3 = y_3$. By (2) we get $y_0 = y_1 = y_2 = 0$ in P_0, hence these y_j ($j \leqslant 2$) are linear forms in z_0, z_1, z_2. Since P_0 is an ordinary double point it follows that R has to be a quadratic form in z_0, z_1, z_2, non-degenerate and, therefore, positive definite (since F takes only non-negative values on $\mathbb{R}^4$). Thus, we have an alternative form of our equation (up to factor 4) by

$$F = Rz_3^2 + 2y_0 y_1 y_2 z_3 + R^2 - R(y_0^2 + y_1^2 + y_2^2) + y_0^2 y_1^2 + y_0^2 y_2^2 + y_1^2 y_2^2.$$

The discriminant is

$$\Delta = (R - y_0^2)(R - y_1^2)(R - y_2^2)$$

and since B has no real point except P_0, also the quadratic forms $R - y_j^2$ have to be positive definite. If we use R to introduce a metric on $V = \mathbb{R}^3$, this can be expressed as follows: we write the linear form y_j as $\langle v_j, z \rangle$ using the metric. Since $R(z) = \langle z, z \rangle$, the condition $R - y_j^2$ is positive definite is equivalent to

$$\langle v_j, v_j \rangle < 1.$$

The condition that the 4 planes H_j are distinct can be expressed (using (2)) by the conditions

$$v_j \pm v_k \neq 0 \quad \text{for } j < k.$$

The point P_0 corresponds to a smooth rational curve E_0 which is invariant under conjugation and which is homologically equivalent to zero (since it is disjoint to each of the surfaces of degree 1). The vector space V corresponds to the space of real sections of L that vanish on E_0. Summarizing, we have the following result:

THEOREM 4. *Let Z be the twistor space of a simply connected 4-manifold with positive scalar curvature, of signature 3. If the 1/2-anticanonical linear system $|L|$ has no base point, it holds*

(i) *There exist exactly 8 surfaces of degree 1 and a unique smooth rational curve E_0 which is homologically equivalent to zero.*

(ii) *For each pair of conjugate surfaces of degree 1, $\{S_j, \bar{S}_j\}$, choose a real section λ_j of L, normalized by $\lambda_i/\lambda_j = 1$ on E_0, and define $z_3 = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3$. Define $V_{\mathbb{C}} = \mathrm{Ker}(H^0(Z, L) \to H^0(E_0, \mathcal{O}_{E_0} \otimes L))$ and let $V \subset V_{\mathbb{C}}$ be the subspace of real sections. Then V has a unique Euclidian structure and there exist 3 vectors $v_0, v_1, v_2 \in V$,* unique up to sign and satisfying

$$\langle v_j, v_j \rangle < 1, \qquad v_j \pm v_k \neq 0 \quad (j \neq k)$$

such that the ramification locus of the map $\phi: Z \to \mathbb{P}(V_{\mathbb{C}} \oplus \mathbb{C}z_3)$ is defined by the equation

$$Rz_3^2 + 2y_0y_1y_2z_3 + R^2 - R\sum_{j=0}^{2} y_j^2 + \sum_{j<k} y_j^2y_k^2 = 0$$

with quadratic respectively linear forms on $V_{\mathbb{C}}$

$$R = \langle z, z \rangle, \qquad y_j = \langle v_j, z \rangle.$$

(iii) *The singular points on B are the point P_0 with $\phi^{-1}(P_0) = E_0$ and the following points (where E_{jk} denotes a smooth rational curve).*

(a) *If v_0, v_1, v_2 are on a line through the origin of V and, say, $v_0 \neq 0$ the points $P_{01}, \bar{P}_{01}$ defined by*

$$y_0 = 0, \quad z_3 + y_2 = 0, \quad R - y_0^2 = 0.$$

Then $\phi^{-1}(P_{01}) = E_{01} + E_{02} + E_{03}$

(b) *If v_0, v_1, v_2 are not on a line through the origin but if v_2 is on one of the 4 affine lines through $\pm v_0$, $\pm v_1$, say on the line through v_0, v_1. We have* *two singular points* P_{01}, $\bar{P}_{01}$ *defined by*

$$y_0 - y_1 = 0, \quad R - y_0^2 = 0, \quad z_3 + y_2 = 0$$

with $\phi^{-1}(P_{01}) = E_{01} + E_{02}$ *and* 6 *singular points*

$$P_{03}, \bar{P}_{03}\colon y_1 + y_2 = 0, \quad z_3 = y_0, \quad R - y_0^2 = 0$$

$$P_{13}, \bar{P}_{13}\colon y_0 + y_2 = 0, \quad z_3 = y_1, \quad R - y_1^2 = 0$$

$$P_{23}, \bar{P}_{23}\colon y_0 + y_1 = 0, \quad z_3 = y_2, \quad R - y_2^2 = 0$$

with fibres $\phi^{-1}(P_{jk}) = E_{jk}$,

(c) *In the general case we have* 12 *singular points* P_{jk}, $\bar{P}_{jk}\colon L_j = L_k = 0$, $R - y_j^2 = 0$, *where* $0 \leqslant j < k \leqslant 3$ *and* L_j *is defined by* (2) *with* $y_3 = z_3$. *The fibres are* $\phi^{-1}(P_{jk}) = E_{jk}$.

REMARK. In [Kr] Lemma 5.5, the case where the linear forms L_0, L_1, L_2 fulfil a relation $(1-a-b)L_0 \pm aL_1 \pm bL_2 = 0$ must be excluded, because then two of the lines $L_i \pm L_j = 0$ would coincide. This corresponds to our case (iii)(b) in Theorem 4. Furthermore in [Kr] Lemma 5.5, we only need to assume that two of the linear forms L_0, L_1, L_2 are linearly independent, but the third can be a multiple of one of the other two forms. Then the quartic B contains a conjugate pair of lines through the real singular point P_0 containing five of the thirteen nodes.

5. Concluding remarks

If we start with an Euclidean vector space V of dimension 3 and 3 vectors satisfying the conditions in Theorem 4, (ii) we can construct a double covering Z_0 of $\mathbb{P}(V_{\mathbb{C}} \otimes \mathbb{C}z_3)$ with a ramification locus defined as in Theorem 4, (ii). Then it is easy to describe all small resolutions Z with a real structure without real points for which condition (iii) of Theorem 4 is satisfied. It is very likely that the resulting 3-folds are twistor spaces. This description is done in the paper [Kr]. There, it is also shown that the cohomology ring of these 3-folds Z is the same as that for a twistor space. The problem to show that these are indeed twistor spaces is to single out the family of twistor lines. There are too many candidates, more precisely 8 pencils in each of the smooth real surfaces of the 1/2-anticanonical linear system, which could serve as twistor lines. Finally, we want to remark that in case Z is a twistor space it is easy to see directly that it is a twistor space of a manifold M diffeomorphic to a connected sum of 3 projective planes. One simply has to restrict the twistor fibration to one of the surfaces S of degree 1. Since it contracts on S a smooth rational curve F with normal bundle

$\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ and since S is the blowing up of 3 points of $\mathbb{P}^2$, S is diffeomorphic to $\mathbb{P}^2 \# 3(-\mathbb{P}^2)$ and, therefore, M is diffeomorphic to $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$.

References

[AHS] Atiyah, M.F., Hitchin, N.J., Singer, I.M.: Self-duality in four-dimensional Riemannian geometry. *Proc. R. Soc. Lond.*, Ser. A 362 (1978), 425–461.

[BS] Bănică, C., Stănasilă, O.: Méthodes Algébriques dans la Théorie Globale des Espaces Complexes I, II. Paris: Gauthier Villars, 1977.

[Bes] Besse, A.: Géométrie Riemannienne en Dimension 4. Paris: Cedic, 1981.

[BH] Bombieri, E., Husemoller, D.: Classification and embeddings of surfaces. In: *Algebraic Geometry, 1974*. Providence, R.I., *American Math. Soc.* (1975), 329–420 (*Proc. of Symp. Pure Math.* 29).

[Cay1] Cayley, A.: First memoir on quartic surfaces. *Proc. Lond. Math. Soc.*, I Ser. 3 (1871), 19–69.

[Cay2] Cayley, A.: Sketch of recent researches upon quartic and quintic surfaces. *Proc. Lond. Math. Soc.*, I Ser. 3 (1871), 186–195.

[Cay3] Cayley, A.: Second memoir on quartic surfaces. *Proc. Lond. Math. Soc.*, I Ser. 3 (1871), 198–202.

[Cay4] Cayley, A.: Third memoir on quartic surfaces. *Proc. Lond. Math. Soc.*, I Ser. 3 (1871), 234–266.

[Cle] Clemens, H.: Double solids. *Adv. Math.* 47 (1983), 107–230.

[Dem] Demazure, M.: Surfaces de Del Pezzo II. V. In: *Séminaire sur les Singularités des Surfaces.* Berlin, Heidelberg, New York: Springer-Verlag, 1980 (*Lect. Notes Math.* 77).

[Dold] Dold, A.: Lectures on Algebraic Topology. Berlin, Heidelberg, New York: Springer-Verlag, 1972.

[Don] Donaldson, S.K.: An application of gauge theory to the topology of 4-manifolds. *J. Differ. Geom.* 18 (1983), 269–316.

[DonF] Donaldson, S., Friedmann, R.: Connected Sums of Self-Dual Manifolds and Deformations of Singular Spaces. Oxford, Mathematical Institute, 1988 (Preprint).

[Floer] Floer, A.: Selfdual Conformal Structures on $l\mathbb{C}\mathbb{P}^2$ (Preprint 1987).

[Free] Freedman, M.: The topology of four-dimensional manifolds. *J. Differ. Geom.* 17 (1982), 357–454.

[Frie] Friedrich, Th.: Self-duality of Riemannian manifolds and connections. In: *Self-dual Riemannian Geometry and Instantons.* Leipzig: Teubner-Verlag, 1981 (*Teubner-Texte zur Mathematik*, 34).

[FK] Friedrich, Th., Kurke, H.: Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature. *Math. Nachr.* 106 (1982), 271–299.

[GH] Griffiths, P., Harris, J.: Principles of Algebraic Geometry. New York: Wiley, 1978.

[Hir] Hirzebruch, F.: Topological Methods in Algebraic Geometry. Berlin, Heidelberg, New York: Springer-Verlag, 1966.

[HH] Hirzebruch, F., Hopf, H.: Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten. *Math. Ann.* 136 (1958), 156–172.

[Hit1] Hitchin, N.J.: Linear field equations on self-dual spaces. *Proc. R. Soc. Lond.*, Ser. A 370 (1980), 173–191.

[Hit2] Hitchin, N.J.: Kählerian twistor spaces. *Proc. Lond. Math. Soc.*, III Ser. 43 (1981), 133–150.

[Hur] Hurtubise, J.: The intersection of two quadrics in $\mathbb{P}_5(\mathbb{C})$ as a twistor space. *Ann. Global Anal. Geom.* 3 (1985), 185–195.

[Jes] Jessop, C.M.: Quartic Surfaces. Cambridge: University Press, 1916.

[Knut] Knutson, D.: Algebraic Spaces. Berlin, Heidelberg, New York: Springer-Verlag, 1971 (*Lect. Notes Math.* 203).

[Kr] Kreußler, B.: Small resolutions of double solids, branched over a 13-nodal quartic surface. *Ann. Global Anal. Geom.* 7 (1989), 227–267.

[Kum] Kummer, E.E.: Über die Flächen vierten Grades mit sechzehn singulären Punkten. Monatsberichte der Königlichen PreuØischen Akademie der Wissenschaften zu Berlin (1864), 246–260.

[K1] Kurke, H.: Applications of algebraic geometry to twistor spaces. In: Badescu, L., Kurke, H. (eds) *Week of Algebraic Geometry*) Leipzig: Teubner-Verlag, 1981 (*Teubner-Texte zur Mathematik* 40).

[K2] Kurke, H.: Vorlesungen über algebraische Flächen. Leipzig: Teubner-Verlag, 1982 (*Teubner-Texte zur Mathematik* 43).

[K3] Kurke, H.: Vanishing theorem for instanton bundles. To appear.

[K4] Kurke, H.: A family of self-dual structures on the connected sum of projective planes. Preprint, Forschungsschwerpunkt komplexe Mannigfaltigkeiten, Nr. 60, Erlangen 1990.

[Mil2] Milnor, J.: Singular Points of Complex Hypersurfaces. Princeton: University Press, 1968.

[PTh] Poon, Y.S.: Compact Self-dual Manifolds with Positive Scalar Curvature. Oxford, St. Catherine's College, Thesis 1985.

[Poon1] Poon, Y.S.: Compact self-dual manifolds with positive scalar curvature. *J. Differ. Geom.* 24 (1986), 97–132.

[Poon2] Poon, Y.S.: Small Resolutions of Double Solids as Twistor Spaces (Preprint).

[Poon3] Poon, Y.S.: Algebraic Dimension of Twistor Spaces. *Math. Ann.* 282 (1988), 621–627.

[Sch] Schoen, R.: Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differ. Geom.* 20 (1984), 479–495.

[Wer] Werner, J.: Kleine Auflösungen spezieller dreidimensionaler Varietäten. Bonn, Max-Planck-Institut, 1987 (Preprint MPI 87-34).