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Differential operators and rank 2 bundles over elliptic curves

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1. Introduction

Let (L, M) be a commuting pair of linear ordinary differential operators. We suppose that the coefficients of L and M are analytic functions of the variable x, defined (at least) for x in some neighbourhood of the origin in $\mathbb{C}$; we suppose also that the leading coefficients of L and M are invertible. Since the work of Burchnall–Chaundy–Baker [2, 3], recently rediscovered and extended by Krichever [9, 10], it is well known that such pairs can be described by certain algebro-geometric spectral data of which the most important parts are a (complex) algebraic curve X and a holomorphic vector bundle E over X. (More generally (see [14, 16, 17]), if X is singular then E may be a torsion free sheaf: in what follows we shall, for simplicity, assume that X is non-singular.) Translation (in x) of the coefficients of L and M gives us a one-parameter family of commuting pairs of operators: this corresponds to some one-parameter deformation of the bundle E, the curve X remaining fixed. If the rank r of E is 1 (that is the case mainly studied in [3]) the deformation of E is rigidly fixed: E moves along a certain one-parameter subgroup in the Jacobian of X. That leads to well known formulae for the coefficients of L and M in terms of the theta function of X (see [2, 9, 16]). But if $r > 1$, the situation is more complicated: the determinant bundle of E remains fixed, and then the possible ways that E can vary are parametrized by $r-1$ arbitrary functions of x (see [10, 14, 17]). It seems to us that this complexity of the possible movements of E is the main obstacle to giving a useful description of the coefficients of the operators when $r > 1$.

However, the case when the curve X is elliptic is exceptional in this respect. In that case the space of rank r bundles with fixed determinant is itself only of dimension $r-1$ (indeed, almost every bundle is just a direct sum of line bundles), so that there is essentially no restriction on the way E is allowed to move. In concrete terms, the effect of that is that the system of ordinary differential equations given by the commutativity condition $[L, M] = 0$ can (in principle) be solved by elementary methods. In the case of rank 2, the equations are sufficiently simple that this is an attractive problem; it has by now been solved

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more or less completely three times (see [4, 7, 12]). The operators L and M in this case have orders 4 and 6. The first solution of the problem was by Krichever and Novikov (see [12], and the corrections in [6] and [7]). They started off from the curve X and bundle E, the latter described by its 'Tyurin parameters' and proceeded to solve the system of ordinary differential equations for the Tyurin parameters implied by the original system $[L, M] = 0$. This method has the advantage that the role of the algebro-geometric data is clear from the start. The main disadvantage is that the answer comes out in a rather complicated form involving the $\mathfrak{P}$-function of X; also, the fact that the original equations $[L, M] = 0$, and not merely the equations for the Tyurin parameters, are of an elementary nature is somewhat obscured. The paper [4] of Dehornoy, on the other hand, first solves the equations directly, then determines the curve and bundle corresponding to each solution. There is another difference between [12] and [4], namely, these authors normalize the operators L and M in different ways. The point here is that the algebro-geometric data determine L and M only up to automorphisms of the algebra of differential operators, and there are various ways of choosing representatives within each automorphism class. Krichever and Novikov use what we might call the *standard normalization*, in which the leading coefficient of L is 1 and its second coefficient 0. It is this normalization that is required for the most important application of the theory, to partial differential equations of the type of the Kadomtsev–Petviashvili (KP) equation (see [11, 15]). The normalization chosen by Dehornoy has much to commend it, but is not appropriate for this application. The third paper on this subject, that of Grünbaum [7], uses the standard normalization, and solves the equations directly by elementary means. Grünbaum also determines the curve X corresponding to each of his solutions; however, he does not consider the problem of determining the bundle E.

The purpose of the present paper, then, is to complete the picture by determining the bundle corresponding to a commuting pair (of orders 4 and 6) given in the standard normalization. This turns out to be more interesting than might be expected. At first we thought that nothing needed to be done except to adapt Dehornoy's work [4] to the desired normalization. However, we have found that Dehornoy's analysis is in fact incomplete; this incompleteness then leads him to what seems to us a fundamentally erroneous conclusion. Namely, he claims ([4], Proposition 2′) that rank 2 bundles over an elliptic curve that are a direct sum of a line bundle with itself do not correspond to any commuting pair (L, M). As we pointed out in [15], that contradicts the basic construction of the theory, which obtains commuting operators starting off from any bundle of the appropriate degree (see §2 below). There is a confusing circumstance here that might seem to lend some plausibility to Dehornoy's assertion, namely, that Krichever's original formulation [10] of his construction made certain assumptions of general position that excluded (among other things) bundles of the type just mentioned. However, as Krichever himself indicates, these assumptions are not an essential part of the theory, but are introduced in order to obtain a more down-to-earth version (involving the Tyurin parameters mentioned above) of the list of algebro-geometric data corresponding to a pair of commuting operators. Indeed, in [15] we gave a reformulation of Krichever's construction that does not require any assumptions of general position. Nevertheless, since the elliptic case, though untypical, is of some importance as the only case in which (so far) the higher rank commuting pairs can be found effectively, it seems desirable to understand it very thoroughly. For that reason we have thought it worthwhile to reexamine this question.

Let us formulate our main result. We are given a rank 2 commuting pair of operators (L, M) of orders 4 and 6 (with coefficients regular near the origin), normalized in the standard way, so that L can be written in the form

$$L = (\partial^2 + \tfrac{1}{2}c_2(x))^2 + 2c_1(x)\partial + c_1'(x) + c_0(x), \tag{1.1}$$

where $\partial \equiv \partial/\partial x$. Thus L is (formally) self-adjoint if and only if $c_1 \equiv 0$: this case often requires separate treatment in what follows (for example, in the definition of the number ν occurring in the statement of Theorem 1.2 below). For each point $P \in X$ we denote by $\mathcal{O}_P$ the line bundle[2] (of degree one) corresponding to the divisor P, and by E_P the unique (up to isomorphism) indecomposable rank 2 bundle that is an extension of $\mathcal{O}_P$ by itself. We define the non-negative integer $\nu = \nu(L, M)$ to be the order to which $c_1(x)$ vanishes at the origin, if this function is not identically zero; or the order to which $c_0'(x)$ vanishes at the origin if $c_1 \equiv 0$.

THEOREM 1.2. *The integer ν defined above is* 0, 1, 2 *or* 3. *If $\nu = 0$, then the vector bundle corresponding to the commuting pair (L, M) may be either $\mathcal{O}_P \oplus \mathcal{O}_Q$ (for some points $P \neq Q$ of X) or E_P (for some point P of X). If $\nu > 0$, the type of the bundle is entirely determined by ν, as follows:*

(i) *if $\nu = 1$, the bundle is $\mathcal{O}_P \oplus \mathcal{O}_Q$ (with $P \neq Q$);*
(ii) *if $\nu = 2$, the bundle is E_P;*
(iii) *if $\nu = 3$, the bundle is $\mathcal{O}_P \oplus \mathcal{O}_P$.*

REMARK. We have not worked out the analogue of (1.2) in the case when the curve X is singular, that is, when X is a nodal or cuspidal cubic. It would probably be rather complicated (because of the need to consider torsion free sheaves).

Using the results of [7], it is easy to give concrete examples of each of the cases in (1.2). The formulae of [7] are simplest if L is self-adjoint, so that $c_1 \equiv 0$. In that case, if we choose the coefficient c_0 arbitrarily and set

$$c_2 = \frac{K_2 + 2K_3c_0 + c_0^3 - c_0'c_0''' + \frac{1}{2}c_0''^2}{c_0'^2} \tag{1.3}$$

[2]Here and elsewhere, we write 'bundle' when strictly speaking we mean 'isomorphism class of bundles', or perhaps 'bundle defined only up to isomorphism'. We hope this will not cause confusion.

where K_2 and K_3 are constants, then the operator L in (1.1) is part of a rank 2 commuting pair (L, M). More precisely, this formula gives all self-adjoint L such that the operator M can be taken to be the approximate fractional power $L_+^{3/2}$: see Section 5 below for a brief discussion of this minor point. Now, recall that we are considering only operators L with coefficients that are regular at $x=0$. If $c_0'(0) \neq 0$, that is, if $v = 0$, it is clear from (1.3) that c_2 is regular at $x=0$. It seems to us that Dehornoy [4] implicitly confines himself to that case when determining the bundle E: (1.2) then agrees with his result. However, if c_0' vanishes to orders 1, 2, or 3 at $x=0$, it is still possible for the zero in the denominator of (1.3) to be cancelled by a zero in the numerator, so that c_2 is still regular. For that, c_0, K_2 and K_3 must satisfy certain constraints which the reader will easily calculate from (1.3). In the most interesting case $v=3$, the simplest example is obtained by taking $c_0=x^4$, $K_2=0$, $K_3=12$, so that $c_2=x^6/16$. The operator

$$L = (\partial^2 + \tfrac{1}{32}x^6)^2 + x^4$$

is thus of the type asserted in [4] not to exist. According to [7], the curve X here has equation $\mu^2=\lambda^3+6\lambda$; and by (1.2), the bundle E has the form $\mathcal{O}_P \oplus \mathcal{O}_P$. In Section 5 we shall see that the point P in this example is the origin.

The paper is arranged as follows. Section 2 summarizes the material we need from the general theory of commuting pairs. Details can be found in [10] (see also [3, 15, 18], and, for a slightly different version of the theory [5, 14, 17]). In Section 3 we review some facts about rank 2 bundles over an elliptic curve; and in Section 4 we show that, given a rank 2 commuting pair (L, M), we can determine the corresponding bundle E in terms of the exponents at a singular point of the greatest common divisor of $L - \lambda$ and $M - \mu$. Finally, in Section 5 we prove Theorem 1.2 by calculating these exponents (or at least their sum, which is all that is needed) in terms of the order of vanishing of c_1 or c_0'. In contrast to the preceding sections, we are obliged here to do extensive computations, and to use the results of [7].

2. The general theory of commuting operators

Suppose we are given two commuting ordinary differential operators L and M, of the kind considered in the introduction. We consider the joint eigenfunctions of L and M, that is, the solutions of the system

$$\left.\begin{aligned} L\psi &= \lambda\psi \\ M\psi &= \mu\psi \end{aligned}\right\}. \tag{2.1}$$

The pairs (λ, μ) such that (2.1) has non-zero solutions ψ form an irreducible affine algebraic curve $X_0 \subset \mathbb{C}^2$, and the space of solutions of (2.1) has the same dimension r at every smooth point $(\lambda, \mu) \in X_0$. We call r the *rank* of the pair (L, M): another characterization of it is that it is the greatest common divisor of the orders of all the operators in the algebra generated by L and M. If the orders of L and M are rn and rm, respectively, then the equation of the curve X_0 has the form

$$\mu^n - \lambda^m + (\text{lower order terms}) = 0. \tag{2.2}$$

The affine curve X_0 can be completed by adding a single smooth point 'at infinity' x_∞: we denote by $X = X_0 \cup \{x_\infty\}$ this complete curve. If (as we assume from now on) X is non-singular, the space of solutions of (2.1) has dimension r for *every* point (λ, μ) of X_0. These solution spaces then form a rank r holomorphic vector bundle over X_0: we denote the dual bundle by E_0. For each $i \geqslant 0$ we have a holomorphic section s_i of E_0, which assigns to each point $P \equiv (\lambda, \mu)$ of X_0 the linear functional $s_i(P)$ on the space of solutions of (2.1) defined by

$$s_i(P)(\psi) = \psi^{(i)}(0) \in \mathbb{C}. \tag{2.3}$$

The sections $s_0, \ldots, s_{r-1}$ are linearly independent near x_∞, hence define a trivialization of E_0 near x_∞. We use this trivialization to extend E_0 to a holomorphic bundle E over the complete curve X. This bundle E then has the properties[3] $h^0(E) = r$, $h^1(E) = 0$, and $\{s_0, \ldots, s_{r-1}\}$ is a basis for its space of global holomorphic sections. The values of these global sections span the fibre of E except at a finite number of points of X. More precisely, they give us a preferred holomorphic section $s_0 \wedge \cdots \wedge s_{r-1}$ of the determinant bundle det E, so that if $\mathscr{D}$ denotes the divisor of this section, then $\mathscr{D}$ is an effective divisor of degree rg (where g is the genus of X), and $s_0, \ldots, s_{r-1}$ are linearly independent at every point of X outside $\mathscr{D}$.

We shall need the notion of *greatest common divisor* (gcd) of two operators. It is easy to see that in the algebra of (ordinary) differential operators with coefficients in the field of germs (at 0) of *meromorphic* functions of x, we can perform the Euclidean algorithm; that is, given operators L and M with (say) ord $M \geqslant$ ord L, we can find operators Q_i, R_i such that

$$M = Q_1 L + R_1, \quad \text{ord } R_1 < \text{ord } L$$

$$L = Q_2 R_1 + R_2, \quad \text{ord } R_2 < \text{ord } R_1,$$

[3]Conversely (see [15]), we can obtain in this way any bundle such that $h^0 = r$, $h^1 = 0$, and the global sections span the fibres near the point at infinity. These restrictions correspond exactly to our assumption that the coefficients of L and M are regular at the origin: if we allowed them to have poles, then any bundle of Euler characteristic r could arise. That is implicit in the results of [16], since any bundle of Euler characteristic r can be used to construct points of the infinite dimensional Grassmannian studied in that paper, and any point of the Grassmannian can be used to construct a commutative algebra of operators. Cf. [15], Prop. 2C.1 and Remark 3.3.

and so on. If $R_i \neq 0$, $R_{i+1} = 0$, then we call $R \equiv R_i$ the gcd of L and M. Clearly, the kernel of R is exactly the intersection of the kernels of L and M. The gcd is well defined up to left multiplication by a unit in the algebra of differential operators, that is, by a function: we may normalize it so that its leading coefficient is 1. Returning to the case when L and M commute, we can now form the gcd of $L - \lambda$ and $M - \mu$ for any $(\lambda, \mu) \in \mathbb{C}^2$. This will be 1 unless $(\lambda, \mu) \in X_0$, and an operator of order r if $(\lambda, \mu) \equiv P$ is a point of X_0. We thus obtain an operator of the form

$$R(x, P) = \partial^r + (\text{lower order terms}),$$

whose coefficients are meromorphic functions of x and P, and whose kernel at each point $P \in X_0$ is the fibre at P of the dual bundle to E. It follows that if P is not a point of the divisor $\mathscr{D}$ above, then the coefficients of $R(x, P)$ are regular at $x = 0$; and if P is one of the points involved in $\mathscr{D}$, then $R(x, P)$ has a regular singular point at $x = 0$. The reason the singular points are regular is that ker R is a subspace of the kernel of the regular operator $L - \lambda$. More precisely, we see from this that for any $P \in X_0$ the exponents of R are distinct integers satisfying $0 \leqslant \rho_i < \operatorname{ord} L$. (We recall (see [8]) that the *exponents* are the r numbers ρ_i such that the equation $R\psi = 0$ has a solution of the form $\psi(x) = x^{\rho_i}\psi_0(x)$ with ψ_0 regular and non-vanishing at $x = 0$.)

Finally, we shall need to know the behaviour of the gcd near the point x_∞. For this we suppose that L is normalized in the standard way (leading coefficient 1, second coefficient 0). From (2.2) we see that λ and μ, thought of as functions on X, have poles at x_∞ of orders n and m, respectively. Thus we can introduce a local parameter z^{-1} near x_∞ such that

$$\lambda = z^n, \qquad \mu = z^m + (\text{lower order terms}).$$

Then near x_∞, the gcd of $L - \lambda$ and $M - \mu$ has the form

$$R(x, z) = \partial^r + v_{r-2}(x)\partial^{r-2} + \cdots + v_0(x) - z + O(z^{-1}).$$

3. Rank 2 bundles over an elliptic curve

Let X be a (non-singular) elliptic curve, E a rank 2 vector bundle over X of the kind that arose in the previous section, that is, such that $h^0(E) = 2$, $h^1(E) = 0$, and the global sections span the fibre of E at almost all points of X. Then E must be one of the bundles occurring in the statement of Theorem 1.2. Indeed, if $E = E_1 \oplus E_2$ is decomposable, it is easy to see that the conditions just stated force both summands E_i to be of degree 1, hence of the form $\mathcal{O}_P$ for some $P \in X$; on the other hand, it is well known (see [1]) that every indecomposable bundle of rank 2 and degree 2 is one of the bundles E_P of Section 1, and that these bundles satisfy the above conditions. If $\{s_0, s_1\}$ is a basis for the sections of E, then $s_0 \wedge s_1$ is a holomorphic section of the degree 2 line bundle $\det E$, so its divisor has the form

$$\operatorname{div}(s_0 \wedge s_1) = P + Q,$$

for some points P and Q of X. Clearly, if $P \neq Q$ then $s_0 \wedge s_1$ must vanish exactly to order 1 at each of the points P and Q, and in that case E is $\mathcal{O}_P \oplus \mathcal{O}_Q$. If $P = Q$, then $s_0 \wedge s_1$ vanishes to order 2 at P, and E may be either $\mathcal{O}_P \oplus \mathcal{O}_P$ or E_P. It will be important for us that these cases too can be distinguished by the behaviour of the sections at P.

PROPOSITION 3.1. (i) *All the global sections of the bundle $\mathcal{O}_P \oplus \mathcal{O}_P$ vanish at P.*

(ii) *The bundle E_P has a section that does not vanish at P.*

Proof. The first statement is trivial. To prove (ii), suppose both sections s_0 and s_1 of a basis vanish at P. Then they define sections of the degree zero bundle $E(-P)$ which span the fibre at every point of X except P, hence, at P also, otherwise the determinant would have the wrong degree. Thus $E(-P)$ is a trivial bundle, and E is $\mathcal{O}_P \oplus \mathcal{O}_P$.

4. Orders 4 and 6

We now specialize the general theory of Section 2 to the case when L and M have orders 4 and 6, respectively, and the rank is 2. The Equation (2.2) of the curve X_0 in this case is cubic, so X is an elliptic curve; we assume it is non-singular, so that the bundle E is of one of the three types discussed in Section 3. We have the basis $\{s_0, s_1\}$ (defined by (2.3)) for the sections of E; recall that s_0 and s_1 are linearly independent near x_∞, so that the divisor $P + Q$ of $s_0 \wedge s_1$ consists of *finite* points P and Q of X. If (λ, μ) is one of these points the exponents of the gcd of $L - \lambda$ and $M - \mu$ are distinct integers between 0 and 3; thus there are 5 possibilities, namely $(0, 2)$, $(0, 3)$, $(1, 2)$, $(1, 3)$ and $(2, 3)$. (The case $(0, 1)$ would give a regular point.) We shall show that, except in the first case, the knowledge of the exponents at one of the singular points, say P, is enough to determine the type of the bundle E (and also the exponents at Q if $P \neq Q$). In detail, we have the following.

THEOREM 4.1. (i) *If the exponents at P are $(0, 2)$, then the bundle E is either $\mathcal{O}_P \oplus \mathcal{O}_Q$ (if $P \neq Q$) or E_P (if $P = Q$). In the first case, the exponents at Q are also* $(0, 2)$.

(ii) *If the exponents at P are $(1, 2)$, then $P \neq Q$ and the exponents at Q are $(0, 3)$.*

(iii) *If the exponents at P are $(0, 3)$, then again $P \neq Q$ and the exponents at Q are $(1, 2)$.*

(iv) *If the exponents at P are* $(1, 3)$, *then $P = Q$ and the bundle E is E_P.*

(v) *If the exponents at P are* $(2, 3)$, *then again $P = Q$ and the bundle E is $\mathcal{O}_P \oplus \mathcal{O}_P$.*

Perhaps the most interesting part of (4.1), namely that the bundle E is $\mathcal{O}_P \oplus \mathcal{O}_P$ exactly when the exponents are (2, 3), follows at once from (3.1), which shows that $\mathcal{O}_P \oplus \mathcal{O}_P$ is the only allowable bundle such that both the sections s_0 and s_1 vanish at P. Indeed, the vanishing of both these sections means that all the joint eigenfunctions of L and M, together with their first derivatives, vanish at P; but if we recall that having an exponent ρ means that there is a joint eigenfunction of the form

$$\psi(x) = x^\rho + (\text{higher order terms}), \tag{4.2}$$

it is clear that this occurs only when both exponents are greater than 1.

We base our investigation of the other cases on the asymptotic form

$$R(x, z) = \partial^2 + v_0(x) - z + O(z^{-1})$$

of the gcd of $L - \lambda$ and $M - \mu$ near x_∞ (where z^{-1} is a local parameter near x_∞ such that $\lambda = z^2$). Bearing in mind that the kernel of R consists of the joint eigenfunctions of L and M, this gives for the section $s_2(\psi) = \psi''(0)$ the asymptotic behaviour

$$s_2(z) = (z - v_0(0) + O(z^{-1}))s_0(z) + O(z^{-1})s_1(z).$$

From that we can read off

LEMMA 4.3. (i) *$s_0 \wedge s_2$ is a holomorphic section of* $\det E$ *that vanishes at x_∞.*

(ii) *$s_1 \wedge s_2$ is a meromorphic section of* $\det E$ *whose only singularity is a simple pole at x_∞.*

COROLLARY 4.4. *The section $s_0 \wedge s_2$ of* $\det E$ *is identically zero unless the exponents at P are* $(0, 2)$, *in which case it is not.*

Proof. It is easy to check case by case that $s_0 \wedge s_2$ vanishes at P unless the exponents are (0, 2), and that in that case it does not vanish at P. But if $s_0 \wedge s_2$ vanishes at P and is not identically zero, then its divisor must be $x_\infty + P$. But then we should have

$$\operatorname{div}(s_0 \wedge s_2) = x_\infty + P \sim \operatorname{div}(s_0 \wedge s_1) = P + Q$$

(where $\sim$ denotes linear equivalence of divisors) and hence a linear equivalence $x_\infty \sim P$, which is impossible (since $P \neq x_\infty$).

Part (i) of (4.1) is now clear, since if $P \neq Q$, (4.4) is equally valid with Q instead of P, and in the case $P = Q$ we already know that the bundle must be E_P.

We have a natural basis for the fibre at P of the dual bundle to E, given by the eigenfunctions of the form (4.2) (with ρ an exponent). For the arguments that follow, we fix a local trivialization of E which at P is given by the dual basis to this one. We fix also a local parameter w near P, so that sections of E are written locally as $\mathbb{C}^2$-valued functions of w.

Suppose first that the exponents at P are $(1, 2)$. Thus $s_0(P)=0$, and near P we have (say) $s_0(w) \sim (p, q)w$ (where here $\sim$ means equality modulo terms of higher order in w). On the other hand, $s_1(P)=(1, 0)$, so that $s_0 \wedge s_1 \sim -qw$. Now, if $P=Q$, this has to vanish to order 2 at P, so that $q=0$. But then since $s_2(P)=(?, 1)$, we have $s_0 \wedge s_2 \sim pw$. Since by (4.4) we have $s_0 \wedge s_2 \equiv 0$, that implies $p = 0$ too, which is impossible, since s_0 cannot vanish to order 2 at P (for then s_1 would be a nowhere vanishing section of E, so that E would contain a trivial one dimensional subbundle, and $h^1(E)$ would not be zero). Hence, $P \neq Q$, and E is $\mathcal{O}_P \oplus \mathcal{O}_Q$. Since s_0 vanishes at P, it cannot vanish at Q, which means that one of the exponents at Q is 0. We have seen that exponents $(0, 2)$ at Q would imply exponents $(0, 2)$ at P too; hence, the exponents at Q must be $(0, 3)$. That proves (ii) of Theorem 4.1.

Next, suppose the exponents at P are $(0, 3)$, so that

$$s_0(P) = (1, 0), \qquad s_1(P) = (\alpha, 0), \qquad s_2(P) = (\beta, 0)$$

for some α, β. Then $s_1 - \alpha s_0$ vanishes at P, so we have, say,

$$(s_1 - \alpha s_0)(w) \sim (p, q)w$$

near P. Thus $s_0 \wedge s_1 \sim qw$ and $s_1 \wedge s_2 \sim -q\beta w$ near P (here we have used that $s_0 \wedge s_2 \equiv 0$). We claim that $q \neq 0$. For otherwise $s_1 \wedge s_2$ and $s_0 \wedge s_1$ would both vanish to order (at least) 2 at P, and the divisor of $s_1 \wedge s_2$ would have the form $2P + R - x_\infty$ for some finite point R of X. But then we should have a linear equivalence

$$2P + R - x_\infty \sim \operatorname{div}(s_0 \wedge s_1) = 2P,$$

and, hence, $R \sim x_\infty$, a contradiction. Hence, $q \neq 0$, and $s_0 \wedge s_1$ vanishes only to order 1 at P. So again $P \neq Q$, and E is $\mathcal{O}_P \oplus \mathcal{O}_Q$. To complete the proof of (4.1) we use the following lemma.

LEMMA 4.5. *If $P \neq Q$, then* 3 *cannot be an exponent at both P and Q.*

Indeed, if that were so, then $s_1 \wedge s_2$ would vanish at both P and Q, so that, using (4.3)(ii), we should have

$$\operatorname{div}(s_1 \wedge s_2) = P + Q + R - x_\infty \sim \operatorname{div}(s_0 \wedge s_1) = P + Q,$$

and, hence, $R \sim x_\infty$, for some finite point R of X, a contradiction. Thus if the exponents at P are $(0, 3)$, we have $(1, 2)$ as the only possibility left for the exponents at Q, which finishes the proof of part (iii) of Theorem 4.1. Finally, (iv) of the theorem is now trivial: if the exponents at P are $(1, 3)$, we must have $P = Q$, otherwise the results already proved leave no possibility for the exponents at Q.

5. Proof of Theorem 1.2

Let us assign an integer μ to each point P of the divisor $\mathrm{div}(s_0 \wedge s_1)$ by setting

$$\mu(P) = \rho_1 + \rho_2 - 2,$$

where ρ_1 and ρ_2 are the exponents at P. Thus $0 \leqslant \mu(P) \leqslant 3$. In this section we shall prove the following.

THEOREM 5.1. *The number $\mu(P)$ coincides with the number v occurring in the statement of Theorem* 1.2; *that is, if L is written in the form* (1.1), *then $\mu(P)$ is the order of vanishing at $x = 0$ of the coefficient c_1 (or of c_0' if L is self-adjoint, so that $c_1 \equiv 0$).*

Granting (5.1), we can check at once that everything in (1.2) follows from the corresponding statements in (4.1). Note that (as we could have read off from (4.1)) in the case $P \neq Q$ the numbers $\mu(P)$ and $\mu(Q)$ must be the same, since they are both equal to v.

It does not seem possible to prove (5.1) without a certain amount of calculation. For these calculations it is convenient to assume (as is done in [7]) that M is exactly the 'approximate fractional power' $L_+^{3/2}$ (see, for example, [14, 16, 18] for the machinery of fractional powers). That is harmless, because of the next lemma.

LEMMA 5.2. *Let A be the algebra generated by L and M. Then we can always find generators $\hat{L}$ and $\hat{M}$ for A, of the form*

$$\hat{L} = L + a, \qquad \hat{M} = M + bL + c$$

(*for constant a, b, c*) *so that $\hat{M} = \hat{L}_+^{3/2}$.*

The lemma shows that it is enough to prove (5.1) in the case $M = L_+^{3/2}$, because the curve X, the point x_∞, the bundle E and the exponents ρ_i depend only on the algebra A, not on the choice of generators; and adding a constant to L clearly does not change v. We sketch the proof of (5.2). It is a general fact that any order 6 operator that commutes with L must have the form

$$M = \sum_0^6 c_i L_+^{i/4},$$

for some constants c_i. We have supposed $c_6 = 1$. A necessary condition for the pair to be rank 2 is that $c_1 = c_3 = c_5 = 0$. We can get rid of c_2 by adding a constant to L, since by the binomial theorem we have

$$(L + a)_+^{3/2} = L_+^{3/2} + \tfrac{3}{2} a L_+^{1/2},$$
$$(L + a)_+^{1/2} = L_+^{1/2}.$$

We can then get rid of c_4 and c_0 by incorporating these terms into M.

From now on, then, we assume $M = L_+^{3/2}$. As in Section 4, we normalize the gcd of $L - \lambda$ and $M - \mu$ to have the form

$$R(x; \lambda, \mu) = \partial^2 - a_1 \partial + a_2.$$

When (λ, μ) is our point P, the coefficients $a_1(x; \lambda, \mu)$ and $a_2(x; \lambda, \mu)$ have poles at $x = 0$ of orders (at most) 1 and 2, respectively. We recall [8] that the exponents ρ_1 and ρ_2 are the roots of the *indicial equation* of R; that means, in particular, that the residue at $x = 0$ of a_1 is $\rho_1 + \rho_2 - 1$. Our task is therefore to calculate a_1 and prove that this residue is $\nu + 1$. We consider first the self-adjoint case $c_1 \equiv 0$. In that case, when we calculate R by the Euclidean algorithm, we find that the first remainder R_1 is already of order 2, hence, is the gcd. For the coefficient a_1 we find

$$a_1(x; \lambda, \mu) = c_0'(x)/(2\lambda + c_0(x)). \tag{5.3}$$

At the singular point P we must have[4] $2\lambda + c_0(0) = 0$, and the residue at $x = 0$ of (5.3) is indeed just the order of vanishing of c_0, that is, $\nu + 1$. In the non-self-adjoint case the calculation is more complicated. First, we have to go to the second remainder R_2 to reach the gcd. For the coefficient a_1 we find this time (cf. [13])

$$a_1(x; \lambda, \mu) = [\tfrac{1}{2} V_x(x, \lambda) - c_1(x)\mu]/V(x, \lambda) \tag{5.4}$$

where

$$V(x, \lambda) = \lambda^2 + c_0\lambda + \tfrac{1}{2}c_1^2 c_2 + \tfrac{1}{2}c_1 c_1'' - \tfrac{1}{4}(c_1')^2 + \tfrac{1}{4}c_0^2.$$

It remains to show that the order of vanishing at $x = 0$ of the expression (5.4) is one more than the order of vanishing of c_1. That seems to be a happy coincidence, and cannot be deduced merely by inspecting (5.4). We are forced to use the explicit solution of the equations $[L, L_+^{3/2}] = 0$ found in [7]. Grünbaum's result is that in the non-self-adjoint case every rank 2 solution of these equations

[4] This shows that P is the origin in the example given in the Introduction.

is given by the formulae

$$c_1 = g'$$
$$c_0 = -g^2 + K_{11}g + K_{12}$$
$$c_2 = (K_{14}+6g^2K_{12}+2g^3K_{11}-2gK_{10}-g^4+g''^2-2g'g''')/2g'^2,$$

where g is an arbitrary function of x, which we shall normalize so that $g(0)=0$, and K_{10}, K_{11}, K_{12} and K_{14} are constants parametrizing the data (X, x_∞, E) (which do not depend on g). The polynomial $V(x, \lambda)$ in (5.4) now takes the form

$$V(x, \lambda) = \alpha(\lambda) + \beta(\lambda)g(x) + \gamma(\lambda)g(x)^2,$$

where

$$\alpha(\lambda) = (\lambda + \tfrac{1}{2}K_{12})^2 + \tfrac{1}{4}K_{14}$$
$$\beta(\lambda) = (\lambda+\tfrac{1}{2}K_{12})K_{11}-\tfrac{1}{2}K_{10}$$
$$\gamma(\lambda) = -\lambda+K_{12}+\tfrac{1}{4}K_{11}^2.$$

At the point $P \equiv (\lambda, \mu)$, a_1 has to have a pole when $x=0$, so we have $\alpha(\lambda)=0$, and a_1 takes the form

$$a_1(x; \lambda, \mu) = \frac{[\frac{1}{2}\beta(\lambda)-\mu]g'+\gamma(\lambda)gg'}{\beta(\lambda)g+\gamma(\lambda)g^2}.$$

Further, from the explicit equation of the curve X_0 given in [7] we can calculate that at P we have $\mu = \pm\frac{1}{2}\beta(\lambda)$. There are now two cases. First, if $\beta(\lambda)=0$, then $\mu=0$ too, and a_1 becomes simply g'/g, with residue the order of vanishing of g, as required. And if $\beta(\lambda) \neq 0$, then necessarily $\mu = -\frac{1}{2}\beta(\lambda)$, for if we had $\mu = +\frac{1}{2}\beta(\lambda)$ then a_1 would have no singularity at $x=0$. So the above expression for a_1 reduces to

$$a_1 = (g'/g)\frac{\beta+\gamma g}{\beta+\gamma g^2},$$

which again has residue the order of vanishing of g. That completes the proof of (5.1).

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