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Topological equisingularity for isolated complete intersection singularities

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Introduction

To any isolated complete intersection singularity $(X, 0)$ (abbreviated as "icis"), we associate a sequence of (Milnor) numbers called the μ_*-sequence. We prove that in a family of icis, the topological type (cf. Definition 6) of the singularity remains invariant if the sequence μ_* remains constant (cf. Theorem 2). This generalizes a theorem due to Lê Dũng Tráng and C.P. Ramanujam.

The scheme of the paper is as follows. In Section 1 we define the μ_* sequence of an icis (cf. Definition 1), recall some definitions and state a few preliminary lemmas. In Section 2 we prove two crucial propositions, from which the theorems follow fairly easily. In Section 3, we prove the two theorems, one of them concerned with the monodromy fibration and the other with topological equisingularity. At the end of this section we sketch an example to point out that the assumption of constancy of the Milnor number is not sufficient to prove the fibration theorem (cf. Theorem 1). This example is discussed in greater detail in [P], where the fibration theorem is also proved. In Section 4, we comment on some problems naturally arising from the earlier sections.

1. Preliminaries

In this section we prove a few preliminary lemmas and define the μ_*-sequence for an isolated complete intersection singularity (abbreviated as 'icis'). The conventions followed are those of Looijenga [Lo], unless explicitly mentioned otherwise. If (X_0, x) is an icis, then by a *deformation* of (X_0, x) we mean a flat morphism $f:(X, x) \to (S, 0)$ from a complete intersection germ (X, x) (not necessarily isolated) to a smooth germ $(S, 0)$, such that $(f^{-1}(0), x) \cong (X_0, x)$. We call this deformation a *smoothing* if $f^{-1}(s)$ is smooth for some $s \in S - 0$. For any icis (X, x), $\mu = \mu(X, x)$ denotes its Milnor number, and $\nu = \nu(X, x)$ denotes the multiplicity of the discriminant locus of a versal deformation at the base point.

LEMMA 1. *Let $f:(X,x)\to(S,s)$ be a deformation of an icis. Then one can embed this deformation into a versal deformation, i.e. there exists a versal deformation $\tilde{f}:(\tilde{X},x)\to(\tilde{S},\tilde{s})$ of $(f^{-1}(s),x)$ and an embedding $\tilde{\imath}:(S,s)\hookrightarrow(\tilde{S},\tilde{s})$ such that f is the fibre product of $\tilde{\imath}$ and $\tilde{f}$.*

Proof. Let $f':(X',x')\to(S',s')$ be a versal deformation of $f^{-1}(s)$. Then by the versality we obtain a morphism $i':(S,s)\to(S',s')$ such that f is the fibre product of i' and f'. Now consider the embedding $\tilde{\imath}:(S,s)\hookrightarrow(S'\times S, s'\times s)$ given by the graph of i'. Then clearly f is the fibre product of $\tilde{\imath}$ and $\tilde{f}=f'\times id: X'\times S\to S'\times S$. Moreover $\tilde{f}$ is versal because f' is. □

LEMMA 2. *Let $f:(X,x)\to(S,s)$ be a smoothing (general fibre is smooth) of an icis, with* $\dim(S,s)=1$. *Let D_f be the discriminant. Then we have,*

$$\mu(X,x)=\mathrm{mult}(D_f)-\mu(X_s,x),$$

where μ is the Milnor number of the corresponding icis and $\mathrm{mult}(D_f)$ *is the multiplicity of the discriminant D_f at s.*

Proof. For proof we refer to Proposition 3.6.4 of [Le2]. □

DEFINITION 1. Let (X,x) be an icis. Then for each $i\geqslant 0$, we define

$$\mu_i(X,x)=\inf\{\mu(X^i,x)\mid f^i:(X^i,x)\to(S^i,s) \text{ is a deformation of } (X,x) \text{ and } \dim S^i=i\}$$

The sequence $\mu_0=\mu,\ \mu_1,\ \mu_2,\dots$ is called the *μ_*-sequence* of (X,x).

DEFINITION 2. For an icis (X_0,x), am embedding $(X_0,x)\subset(X_i,x)$, $i>0$ in another icis (X_i,x), with $\mathrm{Codim}_{(X_i,x)}(X_0,x)=i$ is called *μ_i-minimal* if $\mu(X_i,x)=\mu_i(X_0,x)$. For $i=1$, instead of saying μ_1-minimal embedding, we just say μ-minimal embedding.

DEFINITION 3. The *monodromy fibrations* of (X_0,x) are defined to be the fibrations obtained from all μ-minimal embeddings, i.e. if $f:(X_1,x)\to(\mathbf{C},0)$ defining (X_0,x) is μ-minimal, then a monodromy fibration is $X_1^*\to\Delta^*$, where $X_1\to\Delta$ is a good representative of f and $X_1^*=X_1-X_0$ and $\Delta^*=\Delta-\{0\}$. Sometimes we also refer to $f^{-1}(\partial\Delta)\to\partial\Delta$ as the monodromy fibration. This depends only on (X_0,x) by Theorem 1.

REMARK. If (X,x) has embedding codimension k, then $\mu_{k+i}(X,x)=0$ for all $i\geqslant 0$, and $\mu_i(X,x)\neq 0$ for $i<k$.

We recall (cf. [Lo], 2B, p. 25) the definition of a good representative of a deformation of an icis. Let $f:(X,x)\to(S,0)$ be a flat morphism of irreducible analytic germs, where $(S,0)$ is smooth, giving a deformation of the icis $f^{-1}(0)=(X_0,x)$. Let $f:X\to\tilde{S}$ be a morphism of analytic spaces representing this morphism of germs (with $\tilde{S}$ smooth), and let $X_s = f^{-1}(s)$. Let $r: X \to \mathbf{R}_{\geqslant 0}$ be a nonnegative real analytic function such that $r^{-1}(0) \cap X_0 = \{x\}$. Let $B_\varepsilon = \{y \in X \mid r(y) \leqslant \varepsilon\}$, $S_\varepsilon = \{y \in X \mid r(y) = \varepsilon\}$ and $\mathring{B}_\varepsilon = \{y \in X \mid r(y) < \varepsilon\}$. Let S be a contractible neighbourhood of 0 in $\tilde{S}$. Then

$$B_\varepsilon \cap f^{-1}(S) \to S$$

is called a *good representative* of f if

(i) S_ε intersects $f^{-1}(s)$ transversally for all $s \in S$, and
(ii) X_0 intersects S_η transversally, for all $0 < \eta \leqslant \varepsilon$.

(This is referred to in [Lo] as a *good proper representative* of f.) Note that by Sard's theorem, $r^{-1}(\varepsilon) - X_{\text{sing}}$ is smooth for all sufficiently small nonzero ε, if X, S are suitably chosen (cf. [Lo] Prop. 2.4).

If $f: (X, x) \to (S, 0)$ is a deformation of an icis together with a section $\sigma: (S, 0) \to (X, x)$, then we may choose r such that $r^{-1}(0) = \sigma(S)$. More generally if $(\tilde{X}, x) \to (\tilde{S}, 0)$ is a deformation of an icis and $(S, 0) \subset (\tilde{S}, 0)$ is any smooth germ over which we are given a section $\sigma: (S, 0) \to (\tilde{X}, x)$, then we assume that $r^{-1}(0) = \sigma(S)$. This can be done because the germ of any real analytic subset of $(\tilde{X}, x)$ can be defined by a single real analytic function. The advantage of choosing such an r is that the same r (but not necessarily the same ε) can be used to construct a good representative of $(\tilde{X}, \sigma(s)) \to (\tilde{S}, s)$ for $s \in S$ close to 0.

The following will be a useful notion to have.

DEFINITION 4. A *family of icis* is a flat morphism of analytic spaces $f: X \to S$ with S smooth and connected, together with a section $\sigma: S \to X$ (called a singular section) such that $(X_s, \sigma(s))$ is an icis for each $s \in S$ and $\sigma(S)$ is an irreducible component of the critical space C_f of f. A family $f: X \to S$ is said to be a *ν-constant family* if $\nu(X_s, \sigma(s))$ remains constant for all $s \in S$.

Note that any ν-constant family is μ-constant by Lemma 2 and the fact that both μ and ν are semicontinuous.

LEMMA 3. *Let $f: (X, x) \to (S, 0)$ be a μ_*-constant deformation. Then there exists a versal deformation $\tilde{f}: (\tilde{X}, x) \to (\tilde{S}, 0)$, in which f is embedded, and there are submanifolds $(S_1, 0) \subset (S_2, 0) \subset \cdots \subset (S_k, 0)$ with $\dim S_i = i$ such that $(X_0, x) = (\tilde{f}^{-1}(0), x) \subset (\tilde{f}^{-1}(S_i), x)$ is a μ_i-minimal embedding for all i.*

Proof. Let $f_i: (X_i, x) \to (S_i, 0)$ be μ_i-minimal embeddings of (X_0, x). By Lemma 1 there exists a versal deformation $\tilde{f}: (\tilde{X}, x) \to (\tilde{S}, 0)$, in which each f_i and f are embedded such that $T_0 S \cap T_0 S_i = T_0 S_i \cap T_0 S_j = \{0\} \subset T_0 \tilde{S}$, for all $i \neq j$ where T_0 denotes the tangent space at 0. Hence, one can choose a coordinate system on $\tilde{S}$ such that each S_i and S are linear subspaces.

Let $G(i, N)$ denote the Grassmannian of i-dimensional subspaces of $T_0 \tilde{S}$. Then by semicontinuity of μ_i, there is a Zariski open subset Ω_i of $G(i, N)$ such that for each $L_i \in \Omega_i$, the embedding $(X_0, x) \subset (\tilde{f}^{-1}(L_i), x)$ is μ_i-minimal.

Let F_r denote the flag manifold of (linear) subspaces $L_1 \subset L_2 \subset \cdots \subset L_r$ of $T_0\tilde{S}$ with $\dim L_i = i$. Then we have surjective morphisms, $h_{i,r}: F_r \to G(i, N)$ for all $i \leqslant r$. Therefore $\bigcap h_{i,r}^{-1}(\Omega_i)$ is a nonempty open subset of F_r, say U_r. Now for each $L = (L_1 \subset L_2 \subset \cdots \subset L_r) \in U_r$, the embeddings $(X_0, x) \subset (\tilde{f}^{-1}(L_i), x)$ are μ_i-minimal. By taking $r = k$, the embedding codimension of (X_0, x) one obtains the lemma. □

In the next section we also need the following lemma, which is an easy consequence of the existence of a collar for ∂M.

LEMMA 4. *Let $(M, \partial M)$ be a differentiable manifold with boundary. Let $f: (M, \partial M) \to T$ be a fibration of pairs with T contractible. Let $h: \partial M \to \partial M_{t_0} \times T$, $t_0 \in T$ fixed, be a homeomorphism giving a trivialization of $f|_{\partial M}$. Then one can extend this trivialization to the whole of M, i.e. there exists a homeomorphism $H: M \to M_{t_0} \times T$ such that $H|_{\partial M} = h$.* □

2. Nonbifurcation of the critical space

In this section we prove a basic lemma which gives a numerical criterion for the critical space not to bifurcate. Then we construct certain vector fields using which we prove the propositions.

In order to state the lemma, consider the following situation. Let $f: (X, x) \to (S, 0)$ be a deformation of an icis, with $\dim S = 1$, and let $\sigma: (S, 0) \to (X, x)$ be a section such that f and σ determine a family of icis (cf. Definition 4). Let $f: X \to S$ be a good representative of f, which is embedded in a good representative of another deformation $g: Y \to T$ with $\dim T = 2$.

LEMMA 5. *In the above situation also assume that for a fixed smooth retraction $r: T \to S$, if $h: Y \to S$ is given by $h = r \circ g$, then the embedding $(X_s, \sigma(s)) \subset (Y_s, \sigma(s))$ is μ-minimal for $s \neq 0$. Then critical space of $g: Y \to T$ does not bifurcate (i.e., the reduced critical space coincides with $\sigma(S)$) if and only if $(X_0, x) \subset (Y_0, x)$ is a μ-minimal embedding and $f: X \to S$ is a ν-constant deformation.*

Proof. Let $\tilde{f}: \tilde{X} \to \tilde{S}$ be a good representative of a versal deformation of (X_0, x) in which $g: Y \to T$ is embedded. Such an $\tilde{f}$ exists by Lemma 1. Assume that $(X_0, x) \subset (Y_0, x)$ is μ-minimal and $X \to S$ is ν-constant. The second condition means that

$$S \subset \{\text{Equimultiple stratum of } D_{\tilde{f}}\} := D_{\tilde{f}}^m$$

Since $(X_s, \sigma(s)) \subset (Y_s, \sigma(s))$ is μ-minimal Lemma 2 implies that if l_s is the fibre of the retraction $r: T \to S$ at $s \in S$, then the tangent space $T_s l_s$ is not contained in the tangent cone of $D_{\tilde{f}}$ at $s \in \tilde{S}$. We have the following general formula (see [F], §11.4, Ex. 11.4.4),

$$(D_{\tilde{f}} \cdot l_0)_0 = \sum_{x \in (l_s \cap D_{\tilde{f}})} (D_{\tilde{f}} \cdot l_s)_x \tag{*}$$

for all s sufficiently close to $0 \in S$. Shrinking the base S of the good representative $\tilde{f}$, if necessary, we may assume that (*) holds for all $s \in S$.

Note that we have the following inequalities:

$$\operatorname{mult}(D_{\tilde{f}}, s) \geqslant v(X_s, \sigma(s)) = v(X_0, x) = \operatorname{mult}(D_{\tilde{f}}, 0) \geqslant \operatorname{mult}(D_{\tilde{f}}, s)$$

and also note that $(D_{\tilde{f}} \cdot l_s)_s = v(X_s, \sigma(s))$. Now it follows that for any $s \in S$ the right-side of (*) has only one term, i.e. $D_{\tilde{f}} \cap l_s = \{s\}$. Hence, $T \cap D_{\tilde{f}} = T \cap D_{\tilde{f}}^m = S$. Hence, the discriminant locus does not bifurcate. By Lemma 2, and the semicontinuity of μ, $\sigma(S)$ is contained in the μ-constant stratum. Hence, there is a unique singular point lying over $s \in S$, in $\tilde{X}$. Hence, the critical space does not bifurcate.

Conversely assume that the critical space does not bifurcate. Then the discriminant in T also cannot bifurcate, i.e. $T \cap D_{\tilde{f}} = S$. To prove that $(X_0, x) \subset (Y_0, x)$ is μ-minimal it suffices to prove that $(D_{\tilde{f}} \cdot l_0)_0 = \operatorname{mult}(D_{\tilde{f}}, 0)$; to prove $X \to S$ is v-constant we must show that $\operatorname{mult}(D_{\tilde{f}}, 0) = \operatorname{mult}(D_{\tilde{f}}, s)$ for all $s \in S$. But we have the inequalities:

$$(D_{\tilde{f}} \cdot l_0)_0 \geqslant \operatorname{mult}(D_{\tilde{f}}, 0) \geqslant \operatorname{mult}(D_{\tilde{f}}, s) = (D_{\tilde{f}} \cdot l_s)_s$$

Since $T \cap D_{\tilde{f}} = S$, the extreme terms are equal by (*). Hence, both inequalities are equalities. □

DEFINITION 5. Given any real analytic function φ on an analytic space X, we say that a vector $v \in T_y X$ at a smooth point $y \in X$, *points outward relative to* φ if $v(\varphi) = \langle v, \operatorname{grad} \varphi(y) \rangle > 0$ for some choice of (oriented) local coordinate; however, the property of pointing outwards is independent of this choice. A vector field V defined on a neighbourhood of a subset A of X points outward relative to φ on A if

$$V(\varphi) = \langle V(y), \operatorname{grad} \varphi(y) \rangle > 0, \quad \forall y \in A$$

We recall two results from Looijenga [Lo]. Let (X, x) be a germ of a complete intersection, $f: (X, x) \to (S' \times \mathbf{C}, 0)$ and $f': (X, x) \to (S', 0)$ be deformations of icis, where $\pi: (S' \times \mathbf{C}, 0) \to (S', 0)$ is the projection and $f' = \pi \circ f$. Let r be a non-negative real analytic function on a representative X of (X, x) such that

$r^{-1}(0) \cap f^{-1}(0) = r^{-1}(0) \cap f'^{-1}(0) = \{x\}$. Let $\pi_2: (S' \times \mathbf{C}, 0) \to (\mathbf{C}, 0)$ be the second projection and $f_1 = \pi_2 \circ f$.

LEMMA 6 (cf. [Lo], proof of Proposition 5.4, pp. 69–70). *There exist an $\varepsilon > 0$ and good representatives $f: f^{-1}(S' \times \Delta) \cap B_\varepsilon \to S' \times \Delta$ where Δ is a closed disc centered at $0 \in \mathbf{C}$, and $f': f'^{-1}(S') \cap B_\varepsilon \to S'$ of f and f' respectively, and a $\Delta_1 \subset \Delta$ a smaller concentric disc, such that*

$$D_f \subset S' \times \Delta_1 \quad \text{and} \quad f^{-1}(S' \times \Delta) \cap B_\varepsilon \subset f'^{-1}(S') \cap B_\varepsilon.$$

Moreover there exists a vector field V_1 in a neighbourhood of

$$f'^{-1}(S') \cap B_\varepsilon - f^{-1}(S' \times \Delta) \cap B_\varepsilon$$

which points outward relative to r and $|f_1|^2$ and preserves the fibers of f'. □

LEMMA 7 (cf. [Lo] Proposition 5.4, pp. 69–70). *If the good representatives are chosen as in Lemma 6, then there exists a homeomorphism*

$$H: f^{-1}(S' \times \Delta) \cap B_\varepsilon \to f'^{-1}(S') \cap B_\varepsilon$$

induced by the vector field V_1 such that $f' \circ H = \pi \circ f$ and H is the identity on a neighbourhood of $f^{-1}(S' \times 0)$. □

Now we note that there is a natural inclusion of $f^{-1}(S' \times 0)$ in $f'^{-1}(S')$. Under this natural inclusion, we have

PROPOSITION 1. (i) *The map $f^{-1}(S' \times \Delta) \cap S_\varepsilon \to S' \times \Delta$ is a trivial fibration and, hence, any trivialization induces a homeomorphism $h: f^{-1}(0) \cap S_\varepsilon \to f^{-1}(s) \cap S_\varepsilon$ for all $s \in S' \times \Delta$.*

(ii) *Given any trivialization as in* (i), *there exists a trivialization of $f'^{-1}(S') \cap S_\varepsilon \to S'$ such that it coincides with the trivialization in* (i) *on $f^{-1}(S' \times 0)$. Hence, for any $s \in S'$ one obtains a homeomorphism $h': f'^{-1}(0) \cap S_\varepsilon \to f'^{-1}(s) \cap S_\varepsilon$ which restricts to the homeomorphism h on $f^{-1}(0) \cap S_\varepsilon$.*

Proof. Choose S', Δ and $\varepsilon > 0$ as in Lemma 6. Also fix a trivialization over $S' \times \Delta$ of f,

$$H_1: f^{-1}(S' \times \Delta) \cap S_\varepsilon \to (f^{-1}(0) \cap S_\varepsilon) \times S' \times \Delta.$$

Now $f: f^{-1}(S' \times \partial\Delta) \to S' \times \partial\Delta$ is a fibration and, hence, trivial over S'. Notice that the boundary of $f^{-1}(S' \times \partial\Delta)$ is $f^{-1}(S' \times \partial\Delta) \cap S_\varepsilon$, on which we have a trivialization (induced by H_1) over S', namely

$$H_1: f^{-1}(S' \times \partial\Delta) \cap S_\varepsilon \to f^{-1}(0 \times \partial\Delta) \cap S_\varepsilon \times S'.$$

By Lemma 4, this trivialization can be extended to the whole of $f^{-1}(S' \times \partial\Delta)$ i.e., there exists a homeomorphism over S', $H_2: f^{-1}(S' \times \partial\Delta) \to f^{-1}(0 \times \partial\Delta) \times S'$. The homeomorphism H of Lemma 7 identifies

$$f^{-1}(S' \times \partial\Delta) \cup \{f^{-1}(S' \times \Delta) \cap S_\varepsilon\} \quad \text{with} \quad f'^{-1}(S') \cap S_\varepsilon.$$

Hence, one obtains a topological trivialization of $f'^{-1}(S') \cap S_\varepsilon \to S'$ over S', obtained from H_1 and H_2, which we denote as $H_3: f'^{-1}(S') \cap S_\varepsilon \to f'^{-1}(0) \cap S_\varepsilon \times S'$, such that $H_3|_{f^{-1}(S \times 0)} = H_1$. This trivialization gives the homeomorphism as in (ii). □

Now fix good representatives, f, f' etc. as in Lemma 6, and let V_1 be the resulting vector field on $f'^{-1}(S') \cap B_\varepsilon - f^{-1}(S' \times \Delta) \cap B_\varepsilon$. Assume that we are given a section σ of $f: f^{-1}(S' \times 0) \to S' \times 0$, with $\sigma(0) = x$ making f a ν-constant family, such that $(f^{-1}(s), \sigma(s)) \subset (f'^{-1}(s), \sigma(s))$ is μ-minimal for all $s \in S'$. We also assume that $\dim f^{-1}(s) \neq 2$. Fix $s \in S'$. Choose $\varepsilon > \varepsilon_s > 0$, an open neighbourhood K' of $s \in S'$, and a concentric disc $\Delta'' \subset \Delta \subset \mathbf{C}$ of smaller radius, so that $K = K' \times \Delta'' \subset S' \times \Delta$ is a neighbourhood of $(s, 0)$ such that $f^{-1}(K) \cap B_{\varepsilon_s} \to K$ and $f'^{-1}(K') \cap B_{\varepsilon_s} \to K'$ are good representatives of f and f' at $\sigma(s)$, respectively. Further assume ε_s is chosen so that the conditions of Lemma 6 are again satisfied (with K' in place of S' and Δ'' in place of Δ). Let $\Delta_s = s \times \Delta$ and $\Delta' = K \cap \Delta_s \subset \Delta_s$.

LEMMA 8. *Suppose $f: f^{-1}(S' \times \{0\}, 0) \to (S' \times \{0\}, 0)$ is ν-constant along the section $\sigma: (S' \times \{0\}, 0) \to (X, x)$ and $(f^{-1}(s), \sigma(s)) \subset (f'^{-1}(s), \sigma(s))$ is μ-minimal for all $s \in S'$. Fix $s \in S'$ and choose ε, ε_s and Δ' as above. Then there exists a vector field on $f'^{-1}(s) \cap B_\varepsilon - f^{-1}(\Delta') \cap B_\varepsilon$ which points outward relative to $|f_1|^2$ everywhere and points outward relative to r on $f'^{-1}(s) \cap S_\varepsilon - f^{-1}(\Delta') \cap S_\varepsilon$ and $f'^{-1}(s) \cap S_{\varepsilon_s} - f^{-1}(\Delta') \cap S_{\varepsilon_s}$.*

Proof. The vector field V_1 given by Lemma 6 preserves the fibers of f'. Hence, it restricts to a vector field $\overline{V_1}$ on $f'^{-1}(s) \cap B_\varepsilon - f^{-1}(\Delta_s) \cap B_\varepsilon$. Moreover V_1 points outward relative to $|f_1|^2$ and r. Lemma 5 implies that $f^{-1}(\Delta_s - \Delta') \to \Delta_s - \Delta'$ is a locally trivial fibration over the annulus $\Delta_s - \Delta'$. Hence, the vector field $\overline{V_1}$ on $f'^{-1}(s) \cap B_\varepsilon - f^{-1}(\Delta_s) \cap B_\varepsilon$ can be extended to a vector field V_2 on

$$\{f^{-1}(\Delta_s - \Delta') \cap B_\varepsilon\} \cup \{f'^{-1}(s) \cap B_\varepsilon - f^{-1}(\Delta_s) \cap B_\varepsilon\}$$

such that it points outward relative to $|f_1|^2$ everywhere and relative to r on $f'^{-1}(s) \cap S_\varepsilon - f^{-1}(\Delta') \cap S_\varepsilon$. Here note that V_2 may not point outward relative to r everywhere on $f^{-1}(\Delta_s - \Delta')$, because of the possible presence of a vanishing fold.

Again applying Lemma 6 to the good representatives

$$f: f^{-1}(K) \cap B_{\varepsilon_s} \to K \quad \text{and} \quad f': f'^{-1}(K') \cap B_{\varepsilon_s} \to K'$$

we obtain a vector field V_3 in a neighbourhood U of $f'^{-1}(K') \cap B_{\varepsilon_s} - f^{-1}(K) \cap B_{\varepsilon_s}$, such that it points outward relative to $|f_1|^2$ and r on U and preserves the fibres of f'. Hence, V_3 induces a vector field on $f'^{-1}(s) \cap B_{\varepsilon_s} - f^{-1}(\Delta') \cap B_{\varepsilon_s}$. Let φ be a C^∞ function on $f'^{-1}(s) \cap B_\varepsilon$ with values in $[0, 1]$, supported in U and $\varphi \equiv 1$ on B_{ε_s}. Then the vector field $V_4 = (1-\varphi)V_2 + \varphi V_3$ is nowhere vanishing on $f'^{-1}(s) \cap B_\varepsilon - f^{-1}(\Delta') \cap B_\varepsilon$. Moreover, since V_2 and V_3 point outward relative to $|f_1|^2$ so does V_4. Since $V_4 \equiv V_2$ on $f'^{-1}(s) \cap S_\varepsilon - f^{-1}(\Delta')$ and $V_4 \equiv V_3$ on $f'^{-1}(s) \cap S_{\varepsilon_s} - f^{-1}(\Delta') \cap S_{\varepsilon_s}$, it follows that V_4 points outward relative to r on these sets. □

PROPOSITION 2. *Suppose* $f: f^{-1}(S' \times \{0\}, 0) \to (S' \times \{0\}, 0)$ *be v-constant along*

$$\sigma: (S' \times \{0\}, 0) \to (X', x) \quad \textit{and} \quad (f^{-1}(s), \sigma(s)) \subset (f'^{-1}(s), \sigma(s))$$

is μ*-minimal for all* $s \in S'$. *Fix* $s \in S'$ *and choose* ε *and* ε_s *as in Lemma* 8. *Then*

(i) *if* $\dim f^{-1}(s) \neq 2$ *then there exists a vector field* V *on* $f^{-1}(s) \cap B_\varepsilon - f^{-1}(s) \cap \mathring{B}_{\varepsilon_s}$, *which is nowhere vanishing and points outward relative to* r *on the boundary components* $f^{-1}(s) \cap S_\varepsilon$ *and* $f^{-1}(s) \cap S_{\varepsilon_s}$.

(ii) *Any vector field as in* (i) *can be extended to a vector field on* $f'^{-1}(s) \cap B_\varepsilon - f'^{-1}(s) \cap \mathring{B}_{\varepsilon_s}$, *which is again nowhere vanishing and points outward relative to* r *on the boundary components.*

Proof. (i) The semicontinuity of μ and v and the fact that $\mu + \mu_1 = v$, implies $f^{-1}(S' \times 0) \to S' \times 0$ is μ-constant. Then for each $s \in S'$, $f^{-1}(s) \cap B_\varepsilon$ is contractible. This in particular implies that the inclusions of $f^{-1}(s) \cap S_\varepsilon$ and $f^{-1}(s) \cap S_{\varepsilon_s}$ in $f^{-1}(s) \cap B_\varepsilon - f^{-1}(s) \cap \mathring{B}_{\varepsilon_s}$ are homotopy equivalences. For $\dim f^{-1}(s) \geqslant 3$, it follows that the links $f^{-1}(s) \cap S_\varepsilon \cong f^{-1}(0) \cap S_\varepsilon$ and $f^{-1}(s) \cap S_{\varepsilon_s}$ are simply connected (cf. [H]. In fact the link of an icis of dimension n is $n-2$ connected). Hence, by the h-cobordism theorem there exists a vector field V on $f^{-1}(s) \cap B_\varepsilon - f^{-1}(s) \cap \mathring{B}_{\varepsilon_s}$ which is nowhere vanishing and transversal to the boundaries. For $\dim f^{-1}(s) = 1$, again the existence of such a vector field can be obtained from classification of surfaces. Replacing V by $-V$, if necessary, we may assume that it points outward relative to r on $f^{-1}(s) \cap S_\varepsilon$ and $f^{-1}(s) \cap S_{\varepsilon_s}$.

(ii) Since the critical space of f does not bifurcate, by Lemma 5,

$$f: f^{-1}(\Delta') \cap B_\varepsilon - f^{-1}(\Delta') \cap \mathring{B}_{\varepsilon_s} \to \Delta'$$

is a trivial fibration. Hence, the vector field V as given in (i) on

$$f^{-1}(s) \cap B_\varepsilon - f^{-1}(s) \cap \mathring{B}_{\varepsilon_s},$$

can be extended to a vector field V_5 on the whole of

$$f^{-1}(\Delta') \cap B_\varepsilon - f^{-1}(\Delta') \cap \mathring{B}_{\varepsilon_s}$$

such that V_5 points outward relative to r on S_ε and S_{ε_s} and it preserves the fibres of f. Again the vector fields V and V_5 may not point outward relative to r everywhere because of the possible presence of a vanishing fold.

The vector field V_4 constructed in Lemma 8 is transversal to the fibres of f and V_5 is tangential to the fibres of f over $\partial\Delta'$ and both are nowhere vanishing, hence they are linearly independent on $f^{-1}(\partial\Delta')$. Moreover both V_4 and V_5 points outward relative to r on S_ε and S_{ε_s}. Hence by using partitions of unity one can construct a vector field V_6 on $f'^{-1}(s) \cap B_\varepsilon - f'^{-1}(s) \cap \mathring{B}_{\varepsilon_s}$ satisfying

(i) V_6 is nowhere vanishing,
(ii) it points outward relative to r on $f'^{-1}(s) \cap S_\varepsilon$ and $f'^{-1}(s) \cap S_{\varepsilon_s}$,
(iii) it coincides with V_5 on a neighbourhood of $f^{-1}(s) \cap B_\varepsilon - f^{-1}(s) \cap \mathring{B}_{\varepsilon_s}$, and hence it is an extension of V.

This proves the proposition. □

3. Topological equisingularity

In this section we prove the main theorems. At the end we also give an example to point out that Theorem 1 is false without the assumption ν-constant. We begin with the following definitions,

DEFINITION 6. The *topological type* of an icis (X, x) is defined to be the homeomorphism type of the sequence of germs

$$(X, x) = (X_0, x) \subset (X_1, x) \subset \cdots \subset (X_k, x),$$

where k = the embedding codimension of (X, x) and the embedding $(X, x) \subset (X_i, x)$ is μ_i-minimal for all i. This depends only on (X, x) and not on the particular choice of the nested sequence of μ_i-minimal embeddings, by Theorem 2. A family of icis $f: X \to S$ with a singular section σ is said to be *topologically equisingular* if the topological types of the singularities $(X_s, \sigma(s))$ are the same.

REMARK. If L_i denotes the link of (X_i, x) in the definition above, then the topological type of (X, x) is determined by the homeomorphism type of the nested sequence of links $L = L_0 \subset L_1 \subset \cdots \subset L_k \cong S^{2n+2k-1}$. The topological type determines the μ_*-sequence (cf. Definition 1). This is easily proved along the same lines as the proof for hypersurfaces (cf. [T], Theorem 1.4, p. 295).

THEOREM 1. *If $f: X \to S$ is a ν-constant family then the monodromy fibrations of $(f^{-1}(s), \sigma(s))$ are isomorphic.*

Proof. Fix $0 \in S$; it suffices to show that the monodromy fibrations of $(f^{-1}(0), \sigma(0))$ and $(f^{-1}(s), \sigma(s))$ are equivalent for all s in a neighbourhood of 0 in S. Choose a μ-minimal embedding $(X_0, \sigma(0)) \subset (Y_0, x)$ and a deformation $g: (Y, x) \to (S \times \mathbf{C}, (0, 0))$ such that the deformation $g': (Y, x) \to (S, 0)$ is μ-constant and $(X_s, \sigma(s)) \subset (Y_s, \sigma(s))$ are μ-minimal embeddings. Choose good representatives $g: Y \to S \times \Delta$ and $g': Y' \to S$ satisfying the conditions of Propositions 1 and 2. Also assume that $X \cong g^{-1}(S)$ and $f = g|_X$. Then by definition, a monodromy fibration of $f^{-1}(0)$ is $g^{-1}(0 \times \partial\Delta) \to 0 \times \partial\Delta$ and that of $f^{-1}(s)$ is $g^{-1}(s \times \partial\Delta') \cap B_{\varepsilon_s} \to s \times \partial\Delta'$. By Proposition 1, $g^{-1}(s \times \partial\Delta') \cap B_\varepsilon \to s \times \partial\Delta'$ and $g^{-1}(0 \times \partial\Delta) \cap B_\varepsilon \to 0 \times \partial\Delta$ are isomorphic fibrations. By Proposition 2, $g^{-1}(s \times \partial\Delta') \cap B_\varepsilon \to s \times \partial\Delta'$ and $g^{-1}(s \times \partial\Delta') \cap B_{\varepsilon_s} \to s \times \partial\Delta'$ are isomorphic fibrations if $\dim f^{-1}(s) \neq 2$. Hence, for $\dim f^{-1}(s) \neq 2$ the monodromy fibrations of a ν-constant family are isomorphic. For $\dim f^{-1}(0) = 2$ they are fibre homotopy equivalent, again by Proposition 2. □

THEOREM 2. *If $f: X \to S$ is a μ_*-constant family with* $\dim f^{-1}(s) \neq 2$, *then f is a topologically equisingular family.*

Proof. Let $0 \in S$, we prove the theorem for all s in a neighbourhood of 0 in S. By Lemma 3 there exists a versal deformation $\tilde{f}: (\tilde{X}, x) \to (\tilde{S}, 0)$ of $(X_0, \sigma(0))$ containing f and a flag $(S_1, 0) \subset (S_2, 0) \subset \cdots \subset (S_k, 0)$ in $(\tilde{S}, 0)$ such that the embeddings $(\tilde{f}^{-1}(0), x) \subset (\tilde{f}^{-1}(S_i), x)$ are μ_i-minimal. Consider any smooth retraction $(\tilde{S}, 0) \to (S, 0)$, whose special fibre containing all the linear spaces of the flag. Write $(\tilde{S}, 0) = (\tilde{S}' \times S, 0)$. Then by semicontinuity of $\mu_i(f^{-1}(s), \sigma(s))$, we obtain that $(f^{-1}(s), \sigma(s)) \subset (\tilde{f}^{-1}(s \times S_i), \sigma(s))$ is also μ_i-minimal for all i and for all s in a neighbourhood of 0 in S.

This implies that each $(\tilde{f}^{-1}(s \times S_i), \sigma(s)) \subset (\tilde{f}^{-1}(s \times S_{i+1}), \sigma(s))$ are μ-minimal. So one can choose a smooth retraction $(S_{i+1}, 0) \to (S_i, 0)$ such that $(S_{i+1}, 0) \cong (S_i \times \Delta, 0)$, $(\Delta, 0)$ a smooth one dimensional germ, and the morphism $\tilde{f}^{-1}(S \times S_i \times \Delta, 0) \to (S \times \Delta, 0)$ will have reduced discriminant S and reduced critical space $\sigma(S)$ by Lemma 5.

Let $g: \tilde{f}^{-1}(S \times S_i \times \Delta) \cap B_\varepsilon \to S \times \Delta$ be a good representative, where S, Δ and $\varepsilon > 0$ are chosen as in Proposition 1. Inductively we may assume that there is a homeomorphism given on $g^{-1}(0) \cap S_\varepsilon$ with $g^{-1}(s \times 0) \cap S_\varepsilon$, which is obtained by a topological local trivialization of $g^{-1}(S \times \Delta) \cap S_\varepsilon \to S \times \Delta$ such that it maps $\tilde{f}^{-1}(0 \times S_j) \cap S_\varepsilon$ onto $\tilde{f}^{-1}(s \times S_j) \cap S_\varepsilon$ for all $j < i$. Then by Proposition 1(ii), this trivialization can be extended to obtain a homeomorphism of $g'^{-1}(0)$ onto $g'^{-1}(s)$, where $\pi\colon S \times \Delta \to S$ is the projection and $g' = \pi \circ g$.

By Proposition 2(i) and induction on i, we may assume that we are given an $\varepsilon_s > 0$ and a nowhere vanishing vector field on $g^{-1}(s) \cap B_\varepsilon - g^{-1}(s) \cap \mathring{B}_{\varepsilon_s}$ which points outward relative to r on the boundary components, and maps $\tilde{f}^{-1}(s \times S_j)$ into itself for all $j < i$. By Proposition 2(ii) this can be extended to a vector field on $g'^{-1}(s) \cap B_\varepsilon - g'^{-1}(s) \cap \mathring{B}_{\varepsilon_s}$ which points outward relative to r on the boundary components, and nowhere vanishing. But for each j, $\tilde{f}^{-1}(s \times S_j) \cap S_{\varepsilon_s}$ is isomorphic to the link of the germ $(\tilde{f}^{-1}(s \times S_j), \sigma(s))$. Moreover $\tilde{f}^{-1}(0 \times S_j) \cap S_\varepsilon$ is isomorphic to the link of $(\tilde{f}^{-1}(0 \times S_j), x)$. Hence, combining the homeomorphisms constructed above with the homeomorphisms obtained by the vector fields, we get a homeomorphism of the $(i+1)$-tuple

$$\begin{aligned} &S_\varepsilon \cap (\tilde{f}^{-1}(0 \times S_{i+1}), \tilde{f}^{-1}(0 \times S_i), \ldots, \tilde{f}^{-1}(0)) \to \\ &\quad \to S_{\varepsilon_s} \cap (\tilde{f}^{-1}(s \times S_{i+1}), \tilde{f}^{-1}(s \times S_i), \ldots, \tilde{f}^{-1}(s \times 0)) \end{aligned}$$

By taking $i = k - 1$ one obtains that the link sequence of the topological types of the singularities $(X_s, \sigma(s))$ are homeomorphic. Hence, by the remark after the definition of the topological equisingularity, we proved that any μ_*-constant family is topologically equisingular. □

EXAMPLE. Consider a deformation of a smooth hyperelliptic curve of genus 3 to smooth planar quartics, and look at the corresponding family of canonical rings. The general member X_1 has a hypersurface singularity, while the special member X_0 is a complete intersection of embedding codimension 2; hence, the multiplicity of the discriminant is not constant. However, μ is clearly constant.

Given a smoothing of an icis, the eigenvalues of monodromy which are $\neq 1$ can be computed by "compactifying" the deformation (to a smoothing of a complete variety with an isolated singularity), and computing the eigenvalues of monodromy for the compactified family. In our situation there is an obvious compactification of the singularity as a projective cone (in $\mathbf{P}^3$ for X_1, and as a cone in weighted projective space for X_0 – which we can uniformly describe as the result of taking Proj of the graded ring obtained by adjoining a variable of degree 1).

In each case the projective cone is smoothed by an embedded deformation. The monodromy for each compactified family can be computed by first making a semistable reduction (blow up to get a normal crossing divisor in the special fibre, and base change by a degree 4 map). Then one observes that in the special fibre of the semistable family, all but one component blow down to smooth curves resulting in a smooth family; thus the original family has a monodromy transformation of order 4. The smooth special fibre of the family obtained from the semistable reduction has an automorphism of order 4; to compute the dimensions of the eigenspaces for ± 1, $\pm i$ we need only compute the Betti numbers of quotients of this special fibre by powers of the automorphism. In the plane quartic case, the quotient modulo the automorphism of order 4 is $\mathbf{P}^2$, while it is $\mathbf{F}_4$ (the Hirzebruch surface, $\cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(4))$) in the other case.

The referee has pointed out to us that there is a local example of a μ-constant family of plane curves with two Puiseux pairs which specialize to a monomial curve which is a complete intersection, hence cannot be ν-constant. There is also an example given in [B-G].

4. Further remarks

In [cf. B-G] Buchweitz and Greuel defined a notion of Milnor number for arbitrary isolated curve singularities. They proved that any μ-constant deformation of curves is topologically equisingular in the classical sense (two singular germs embedded in a smooth germ are said to be topologically equivalent (in the classical sense) if there is a homeomorphism of the smooth germ into itself carrying one singular germ on to the other. Using this one can define a notion of equisingularity – cf. [B-G], §5, p. 261).

If $f: X \to S$ is a μ-constant family of icis of dimension n embedded in $\mathbf{C}^N \times S$, then the topological equisingularity (in the classical sense) follows if one can show that $(S_\varepsilon, X_s \cap S_\varepsilon)$ and $(S_{\varepsilon_s}, X_s \cap S_{\varepsilon_s})$ are homeomorphic pairs (here the real analytic function r is chosen to be the square of the distance function in $\mathbf{C}^N$; $\varepsilon > \varepsilon_s > 0$ is chosen so that S_η intersects $f^{-1}(0)$ transversally for all $\eta < \varepsilon$ and S_η intersects $f^{-1}(s)$ transversally for all $\eta < \varepsilon_s$). Now if $N = n+1$, this follows from Theorem 2. If $N > n+1$ then $\mathrm{Codim}_{S_\varepsilon}(S_\varepsilon \cap X_s) \geqslant 4$, where the codimension is taken in the real sense. Hence if $n > 2$, $S_\varepsilon - S_\varepsilon \cap X_s$ and $S_{\varepsilon_s} - S_{\varepsilon_s} \cap X_s$ are simply connected by the homotopy exact sequence of pairs. Then the h-cobordism theorem of Smale (cf. [Sm], Theorem 1.4) for pairs implies that $(S_\varepsilon, S_\varepsilon \cap X_s)$ is diffeomorphic to $(S_{\varepsilon_s}, S_{\varepsilon_s} \cap X_s)$. Hence, any μ-constant deformation is topologically equisingular in the classical sense.

Theorem 2 has been recently proved in the case of hypersurface singularities for $n=2$ with the additional hypotheses that the fundamental group of the links of the singularities is constant, with the possible exception of singularities with a link which is a torus bundle over the circle (cf. [Sz] Theorem B). The methods employed there are quite different and do not extend to the complete intersection case.

We pose the following problem whose affirmative answer would extend Theorem 2 to the case $n=2$.

PROBLEM 1. *Let $f: X \to S$ be μ_*-constant family of icis of* dim 2. *Can one find a μ_*-constant family of curves $g: Y \to S$ which is embedded in f such that the embeddings $(g^{-1}(s), \sigma(s)) \subset (f^{-1}(s), \sigma(s))$ are μ-minimal for all $s \in S$?*

More generally one can ask;

PROBLEM 2. *Given any μ_*-constant family of icis $f: X \to S$, can one find a function $g: X \to \Delta$ where Δ is a smooth one dimensional complex analytic space, such that $(f, g): X \to S \times \Delta$ has reduced critical space as $\sigma(S)$?*

Another problem regarding the topological type is related to the Zariski multiplicity conjecture (cf. [Z]);

PROBLEM 3. *If $f: X \to S$ is a μ_*-constant family, then does the multiplicity of $(X_s, \sigma(s))$ remain constant?*

This has been proved for quasi-homogenous hypersurface singularities by Greuel in [G], using a deep result of Varchenko [Var].

Massey (cf. [M1]) and Vannier (cf. [Van]) have given a criterion for the Milnor fibrations to be constant in a family of hypersurfaces with one dimensional singular locus. Massey has also studied the case when the singularities are arbitrary in the case of hypersurfaces and conjectured a criterion for the constancy of Milnor fibrations (cf. [M2], Conjecture 5.1). It would be interesting to investigate these cases for complete intersection singularities with arbitrary singular locus.

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